Filomat 38:25 (2024), 8849–8860
https://doi.org/10.2298/FIL2425849K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A note on the Browder's theorem and a Cline's formula for generalized Drazin-g-meromorphic inverses

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Abstract. In this paper, we give a new characterization of Browder's theorem by means of the generalized Drazin-g-meromorphic Weyl spectrum and the generalized Drazin-g-meromorphic spectrum. Also, for operators A and B satisfying $A^kB^kA^k = A^{k+1}$ for some positive integer k, we generalize Cline's formula to the case of generalized Drazin-g-meromorphic invertibility.

1. Introduction and Preliminaries

Throughout this paper, let $\mathbb{N}$ and $\mathbb{C}$ denote the set of natural numbers and complex numbers, respectively. Let $B(X)$ denote the Banach algebra of all bounded linear operators acting on a complex Banach space X. For $T \in B(X)$, we denote the adjoint of T, null space of T, range of T and spectrum of T by T^*, $N(T)$, $R(T)$ and $\sigma(T)$, respectively. For a subset A of $\mathbb{C}$, the set of interior points of A and the set of accumulation points of A are denoted by $\text{int}(A)$ and $\text{acc}(A)$, respectively. For $T \in B(X)$, let $\alpha(T)$ be the nullity of T, defined as the dimension of $N(T)$ and $\beta(T)$ be the deficiency of T, defined as codimension of $R(T)$. An operator $T \in B(X)$ is called a lower semi-Fredholm operator if $\beta(T) < \infty$. An operator $T \in B(X)$ is called an upper semi-Fredholm operator if $\alpha(T) < \infty$ and $R(T)$ is closed. The class of all lower semi-Fredholm operators (upper semi-Fredholm operators, respectively) is denoted by $\phi_-(X)$ ($\phi_+(X)$, respectively). An operator T is called semi-Fredholm if it is upper or lower semi-Fredholm. For a semi-Fredholm operator $T \in B(X)$, the index of T is defined by $\text{ind}\,(T) = \alpha(T) - \beta(T)$. The class of all Fredholm operators is defined by $\phi(X) = \phi_+(X) \cap \phi_-(X)$. The class of all lower semi-Weyl operators (upper semi-Weyl operators, respectively) is defined by $W_-(X) = \{T \in \phi_-(X) : \text{ind}\,(T) \geq 0\}$ ($W_+(X) = \{T \in \phi_+(X) : \text{ind}\,(T) \leq 0\}$, respectively). An operator $T \in B(X)$ is said to be Weyl if $T \in \phi(X)$ and $\text{ind}\,(T) = 0$. The spectra for *upper semi-Fredholm operator*, *lower semi-Fredholm operator*, *Fredholm operator*, *upper semi-Weyl operator*, *lower semi-Weyl operator* and *Weyl*

2020 *Mathematics Subject Classification*. Primary 47A10; Secondary 47A53.
Keywords. g-meromorphic operators, generalized Drazin-g-meromorphic invertible, Cline's formula.
Received: 09 December 2023; Revised: 26 April 2024; Accepted: 28 April 2024
Communicated by Snežana Č. Živković-Zlatanović
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operator are defined by

$$\sigma_{uf}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Fredholm}\},$$
$$\sigma_{lf}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi-Fredholm}\},$$
$$\sigma_{f}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Fredholm}\},$$
$$\sigma_{uw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Weyl}\},$$
$$\sigma_{lw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi-Weyl}\},$$
$$\sigma_{w}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Weyl}\}, \text{ respectively.}$$

A bounded linear operator T is said to be bounded below if $R(T)$ is closed and T is injective. The *approximate point* and *surjective spectra* are defined by

$$\sigma_{a}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\},$$
$$\sigma_{s}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not surjective}\}, \text{ respectively.}$$

For an operator $T \in B(X)$, the ascent $p(T)$ is the smallest non negative integer p such that $N(T^p) = N(T^{p+1})$. If no such integer exists, we set $p(T) = \infty$. For an operator $T \in B(X)$, the descent $q(T)$ is the smallest non negative integer q such that $R(T^q) = R(T^{q+1})$. If no such integer exists, we set $q(T) = \infty$. By [1, Theorem 1.20] we know that if both $p(T)$ and $q(T)$ are finite, then $p(T) = q(T)$.

An operator $T \in B(X)$ is said to have the single-valued extension property (SVEP) at $\mu_0 \in \mathbb{C}$ if for every neighborhood U of μ_0 the only analytic function $f : U \to X$ satisfying $(\mu I - T)f(\mu) = 0$ is the function $f = 0$. An operator T is said to have SVEP if T has SVEP at every $\mu \in \mathbb{C}$. It is known that if $p(\mu I - T)$ is finite, then T has SVEP at μ and if $q(\mu I - T)$ is finite, then T^* has SVEP at μ.

An operator $T \in B(X)$ is said to be Drazin invertible if there exist $S \in B(X)$ and a positive integer n such that

$$ST = TS, \;\; T^{n+1}S = T^n \text{ and } STS = S.$$

By [1, Theorem 1.132] T is Drazin invertible if and only if $p(T) = q(T) < \infty$. An operator $T \in B(X)$ is said to be left Drazin invertible if $p(T) < \infty$ and $R(T^{p+1})$ is closed. An operator $T \in B(X)$ is said to be lower semi-Browder if it is a lower semi-Fredholm and $q(T) < \infty$. An operator $T \in B(X)$ is said to be right Drazin invertible if $q(T) < \infty$ and $R(T^q)$ is closed. An operator $T \in B(X)$ is said to be upper semi-Browder if it is an upper semi-Fredholm and $p(T) < \infty$. We say that an operator $T \in B(X)$ is Browder if it is lower semi-Browder and upper semi-Browder. The spectra for *lower semi-Browder operator*, *upper semi-Browder operator* and *Browder operator* are defined by

$$\sigma_{lb}(T) : = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi-Browder}\},$$
$$\sigma_{ub}(T) : = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Browder}\},$$
$$\sigma_{b}(T) : = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Browder}\}, \text{ respectively.}$$

Clearly, every Browder operator is Drazin invertible.

An operator $T \in B(X)$ is said to be semi-regular if $R(T)$ is closed and $N(T) \subset R(T^n)$ for every $n \in \mathbb{N}$. An operator $T \in B(X)$ is said to be nilpotent if $T^n = 0$ for some $n \in \mathbb{N}$. An operator $T \in B(X)$ is said to be quasi-nilpotent if $\lambda I - T$ is invertible for all $\lambda \in \mathbb{C} \setminus \{0\}$. An operator $T \in B(X)$ is said to be Riesz if $\lambda I - T$ is Browder for all $\lambda \in \mathbb{C} \setminus \{0\}$. An operator $T \in B(X)$ is said to be meromorphic if $\lambda I - T$ is Drazin invertible for all $\lambda \in \mathbb{C} \setminus \{0\}$. Clearly, every Riesz operator is meromorphic.

A subspace M of X is said to be T-*invariant* if $T(M) \subset M$. For a T-invariant subspace M of X, we define $T_M : M \to M$ by $T_M(x) = T(x), x \in M$. We say that T is completely reduced by the pair (M, N) (denoted by $(M, N) \in Red(T)$) if M and N are two closed T-invariant subspaces of X such that $X = M \oplus N$.

An operator T is said to possess a *generalized Kato decomposition* (*GKD*) if there exists a pair $(M, N) \in Red(T)$ such that T_M is semi-regular and T_N is quasi-nilpotent. Here, if we assume that T_N to be nilpotent, then T is said to be of Kato type. An operator is said to possess a *Kato-Riesz decomposition* (*GKRD*), if there exists a pair $(M, N) \in Red(T)$ such that T_M is semi-regular and T_N is Riesz (see [20]). Živković-Zlatanović and Duggal

[22] introduced the notion of generalized Kato-meromorphic decomposition. An operator $T \in B(X)$ is said to possess a *generalized Kato-meromorphic decomposition* (*GKMD*), if there exists a pair $(M, N) \in Red(T)$ such that T_M is semi-regular and T_N is meromorphic. Živković-Zlatanović[19] generalized Kato-g-meromorphic decomposition and introduced the notion of g-meromorphic operators. An operator $T \in B(X)$ is called g-meromorphic if every nonzero spectral point is an isolated point. Clearly, every meromorphic operator is g-meromorphic. An operator $T \in B(X)$ is said to possess a *generalized Kato-g-meromorphic decomposition* ($GK(gM)D$), if there exists a pair $(M, N) \in Red(T)$ such that T_M is semi-regular and T_N is g-meromorphic. For $T \in B(X)$, the *generalized Kato spectrum*, *generalized Kato Riesz spectrum*, *generalized Kato meromorphic spectrum* and *generalized Kato-g-meromorphic spectrum* are defined by

$$\begin{aligned}
\sigma_{gKD}(T) &: = \{\lambda \in \mathbb{C} : \lambda I - T \text{ does not admit a GKD}\},\\
\sigma_{gKRD}(T) &: = \{\lambda \in \mathbb{C} : \lambda I - T \text{ does not admit a GKRD}\},\\
\sigma_{gKMD}(T) &: = \{\lambda \in \mathbb{C} : \lambda I - T \text{ does not admit a GKMD}\},\\
\sigma_{gK(gM)}(T) &: = \{\lambda \in \mathbb{C} : \lambda I - T \text{ does not admit a GK(gM)D}\}, \text{ respectively.}
\end{aligned}$$

For $T \in B(X)$ and a non negative integer n, define $T_{[n]}$ to be the restriction of T to $T^n(X)$. If for some non negative integer n, the range space $T^n(X)$ is closed and $T_{[n]}$ is Fredholm (an upper semi Fredholm, a lower semi Fredholm, an upper semi Browder, a lower semi Browder, Browder, respectively) then T is said to be B-Fredholm (an upper semi B-Fredholm, a lower semi B-Fredholm, an upper semi B-Browder, a lower semi B-Browder, B-Browder, respectively). For a semi B-Fredholm operator T (see [8]), the index of T is defined as index of $T_{[n]}$. The spectra for *upper semi B-Fredholm operator*, *lower semi B-Fredholm operator*, *B-Fredholm operator*, *upper semi B-Browder operator*, *lower semi B-Browder operator* and *B-Browder operator* are defined by

$$\begin{aligned}
\sigma_{usbf}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi B-Fredholm}\},\\
\sigma_{lsbf}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi B-Fredholm}\},\\
\sigma_{bf}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not B-Fredholm}\},\\
\sigma_{usbb}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi B-Browder}\},\\
\sigma_{lsbb}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi B-Browder}\},\\
\sigma_{bb}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not B-Browder}\}, \text{ respectively.}
\end{aligned}$$

By [1, Theorem 3.47] we know that an operator $T \in B(X)$ is upper semi B-Browder (lower semi B-Browder, B-Browder, respectively) if and only if T is left Drazin invertible (right Drazin invertible, Drazin invertible, respectively).

An operator $T \in B(X)$ is said to be an upper semi B-Weyl (a lower semi B-Weyl, respectively) if it is an upper semi B-Fredholm (a lower semi B-Fredholm, respectively) having ind $(T) \leq 0$ (ind $(T) \geq 0$, respectively). An operator $T \in B(X)$ is said to be B-Weyl if ind $(T) = 0$ and T is B-Fredholm. The spectra for *upper semi B-Weyl operator*, *lower semi B-Weyl operator* and *B-Weyl operator* are defined by

$$\begin{aligned}
\sigma_{usbw}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi B-Weyl}\},\\
\sigma_{lsbw}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi B-Weyl}\},\\
\sigma_{bw}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not B-Weyl}\}, \text{ respectively.}
\end{aligned}$$

By [8, Theorem 2.7], it is known that $T \in B(X)$ is B-Fredholm (B-Weyl, respectively) if there exists $(M, N) \in Red(T)$ such that T_M is Fredholm (Weyl, respectively) and T_N is nilpotent.

An operator $T \in B(X)$ is called Drazin invertible if there exists a pair $(M, N) \in Red(T)$ such that T_M is invertible and T_N is nilpotent. This definition aligns with the assertion that there exists $S \in B(X)$ such that $TS = ST$, $STS = S$ and $TST - T$ is nilpotent. Koliha [17] replaced the third condition with $TST - T$ is quasi-nilpotent and generalized this concept. An operator is called generalized Drazin invertible if there exist a pair $(M, N) \in Red(T)$ such that T_M is invertible and T_N is quasi-nilpotent. Cvetković and Živković-Zlatanović [11] introduced the concept of operators which are direct sum of a quasi-nilpotent and a bounded below (surjective, upper (lower) semi-Fredholm, Fredholm, upper (lower) semi-Weyl, Weyl). An operator $T \in B(X)$

is said to be generalized Drazin bounded below (surjective, upper (lower) semi-Fredholm, Fredholm, upper (lower) semi-Weyl, Weyl, respectively) if there exists a pair $(M, N) \in Red(T)$ such that T_M is bounded below (surjective, upper (lower) semi-Fredholm, Fredholm, upper (lower) semi-Weyl, Weyl, respectively) and T_N is quasi-nilpotent. The *generalized Drazin*, *generalized Drazin bounded below*, *generalized Drazin surjective spectra*, *generalized Drazin lower (upper) semi-Fredholm*, *generalized Drazin Fredholm*, *generalized Drazin upper (lower) semi-Weyl* and *generalized Drazin Weyl spectra* are defined by

$$\begin{aligned}
\sigma_{gD}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin invertible}\},\\
\sigma_{gD\mathcal{J}}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin bounded below}\},\\
\sigma_{gD\mathcal{Q}}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin surjective}\},\\
\sigma_{gD\phi_+}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin upper semi-Fredholm}\},\\
\sigma_{gD\phi_-}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin lower semi-Fredholm}\},\\
\sigma_{gD\phi}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin Fredholm}\},\\
\sigma_{gDW_+}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin upper semi-Weyl}\},\\
\sigma_{gDW_-}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin lower semi-Weyl}\},\\
\sigma_{gDW}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin Weyl}\}, \text{ respectively.}
\end{aligned}$$

By [11], it is known that

$$\begin{aligned}
&\sigma_{gD\phi}(T) = \sigma_{gD\phi_+}(T) \cup \sigma_{gD\phi_-}(T),\\
&\sigma_{gKD}(T) \subset \sigma_{gD\phi_+}(T) \subset \sigma_{gDW_+}(T) \subset \sigma_{gD\mathcal{J}}(T),\\
&\sigma_{gKD}(T) \subset \sigma_{gD\phi_-}(T) \subset \sigma_{gDW_-}(T) \subset \sigma_{gD\mathcal{Q}}(T),\\
&\sigma_{gKD}(T) \subset \sigma_{gD\phi}(T) \subset \sigma_{gDW} \subset \sigma_{gD}(T).
\end{aligned}$$

Recently, Živković-Zlatanović and Cvetković [20] introduced the notion of generalized Drazin-Riesz invertible operators by substituting the third condition with $TST - T$ is Riesz. They established that an operator $T \in B(X)$ is generalized Drazin-Riesz invertible if and only if there exists a pair $(M, N) \in Red(T)$ such that T_M is invertible and T_N is Riesz. An operator $T \in B(X)$ is said to be generalized Drazin-Riesz bounded below (surjective, upper (lower) semi-Fredholm, upper (lower) semi-Weyl, Weyl, respectively) if there exists a pair $(M, N) \in Red(T)$ such that T_M is bounded below (surjective, upper (lower) semi-Fredholm, upper (lower) semi-Weyl, Weyl, respectively) and T_N is Riesz. The *generalized Drazin-Riesz bounded below*, *generalized Drazin-Riesz surjective*, *generalized Drazin-Riesz invertible*, *generalized Drazin-Riesz upper (lower) semi-Fredholm*, *generalized Drazin-Riesz Fredholm*, *generalized Drazin-Riesz upper (lower) semi-Weyl* and *generalized Drazin-Riesz Weyl spectra* are defined by

$$\begin{aligned}
\sigma_{gDR\mathcal{J}}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-Riesz bounded below}\},\\
\sigma_{gDR\mathcal{Q}}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-Riesz surjective}\},\\
\sigma_{gDR}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-Riesz invertible}\},\\
\sigma_{gDR\phi_+}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-Riesz upper semi-Fredholm}\},\\
\sigma_{gDR\phi_-}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-Riesz lower semi-Fredholm}\},\\
\sigma_{gDR\phi}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-Riesz Fredholm}\},\\
\sigma_{gDRW_+}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-Riesz upper semi-Weyl}\},\\
\sigma_{gDRW_-}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-Riesz lower semi-Weyl}\},\\
\sigma_{gDRW}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-Riesz Weyl}\}, \text{ respectively.}
\end{aligned}$$

Recently, Živković-Zlatanović and Duggal [22] replaced the third condition with $TST - T$ is meromorphic and introduced the notion of generalized Drazin-meromorphic invertible operators. They established that an operator $T \in B(X)$ is generalized Drazin-meromorphic invertible if and only if there exists a pair

$(M, N) \in Red(T)$ such that T_M is invertible and T_N is meromorphic. An operator $T \in B(X)$ is said to be generalized Drazin-meromorphic bounded below (surjective, upper (lower) semi-Fredholm, Fredholm, upper (lower) semi-Weyl, Weyl, respectively) if there exists a pair $(M, N) \in Red(T)$ such that T_M is bounded below (surjective, upper (lower) semi-Fredholm, Fredholm, upper (lower) semi-Weyl, Weyl respectively) and T_N is meromorphic. The *generalized Drazin-meromorphic bounded below*, *generalized Drazin-meromorphic surjective*, *generalized Drazin-meromorphic invertible spectra*, *generalized Drazin-meromorphic upper (lower) semi-Fredholm*, *generalized Drazin-meromorphic Fredholm*, *generalized Drazin-meromorphic upper (lower) semi-Weyl* and *generalized Drazin-meromorphic Weyl spectra* are defined by

$$\begin{aligned}
\sigma_{gDM\mathcal{J}}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-meromorphic bounded below}\},\\
\sigma_{gDM\mathcal{Q}}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-meromorphic surjective}\},\\
\sigma_{gDM}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-meromorphic invertible}\},\\
\sigma_{gDM\phi_+}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-meromorphic upper semi-Fredholm}\},\\
\sigma_{gDM\phi_-}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-meromorphic lower semi-Fredholm}\},\\
\sigma_{gDM\phi}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-meromorphic Fredholm}\},\\
\sigma_{gDMW_+}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-meromorphic upper semi-Weyl}\},\\
\sigma_{gDMW_-}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-meromorphic lower semi-Weyl}\},\\
\sigma_{gDMW}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-meromorphic Weyl}\}, \text{ respectively.}
\end{aligned}$$

Also, Živković-Zlatanović [19] introduced the notion of generalized Drazin-g-meromorphic invertible operators by substituting the third condition with $TST - T$ is g-meromorphic. They established that an operator $T \in B(X)$ is generalized Drazin-g-meromorphic invertible if and only if there exists a pair $(M, N) \in Red(T)$ such that T_M is invertible and T_N is g-meromorphic. An operator $T \in B(X)$ is said to be generalized Drazin-g-meromorphic bounded below (surjective, upper (lower) semi-Fredholm, Fredholm, upper (lower) semi-Weyl, Weyl, respectively) if there exists a pair $(M, N) \in Red(T)$ such that T_M is bounded below (surjective, upper (lower) semi-Fredholm, Fredholm, upper (lower) semi-Weyl, Weyl, respectively) and T_N is g-meromorphic. The *generalized Drazin-g-meromorphic bounded below*, *generalized Drazin-g-meromorphic surjective*, *generalized Drazin-g-meromorphic invertible*, *generalized Drazin-g-meromorphic lower (upper) semi-Fredholm*, *generalized Drazin-g-meromorphic Fredholm*, *generalized Drazin-g-meromorphic lower (upper) semi-Weyl* and *generalized Drazin-g-meromorphic Weyl spectra* are defined by

$$\begin{aligned}
\sigma_{gD(gM)\mathcal{J}}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-}g\text{-meromorphic bounded below}\},\\
\sigma_{gD(gM)\mathcal{Q}}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-}g\text{-meromorphic surjective}\},\\
\sigma_{gD(gM)}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-}g\text{-meromorphic invertible}\},\\
\sigma_{gD(gM)\phi_+}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-}g\text{-meromorphic upper semi-Fredholm}\},\\
\sigma_{gD(gM)\phi_-}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-}g\text{-meromorphic lower semi-Fredholm}\},\\
\sigma_{gD(gM)\phi}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-}g\text{-meromorphic Fredholm}\},\\
\sigma_{gD(gM)W_+}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-}g\text{-meromorphic upper semi-Weyl}\},\\
\sigma_{gD(gM)W_-}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-}g\text{-meromorphic lower semi-Weyl}\},\\
\sigma_{gD(gM)W}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-}g\text{-meromorphic Weyl}\}, \text{ respectively.}
\end{aligned}$$

By [19, 20, 22], it is known that

$$\begin{aligned}
&\sigma_{gD*\phi}(T) = \sigma_{gD*\phi_+}(T) \cup \sigma_{gD*\phi_-}(T),\\
&\sigma_{gK*}(T) \subset \sigma_{gD*\phi_+}(T) \subset \sigma_{gD*W_+}(T) \subset \sigma_{gD*\mathcal{J}}(T),\\
&\sigma_{gK*}(T) \subset \sigma_{gD*\phi_-}(T) \subset \sigma_{gD*W_-}(T) \subset \sigma_{gD*\mathcal{Q}}(T),\\
&\sigma_{gK*}(T) \subset \sigma_{gD*\phi}(T) \subset \sigma_{gD*W} \subset \sigma_{gD*}(T),
\end{aligned}$$

where $*$ stands for Riesz or meromorphic or g-meromorphic operators.

Recall that an operator T satisfies Browder's theorem if $\sigma_b(T) = \sigma_w(T)$ and generalized Browder's theorem if $\sigma_{bb}(T) = \sigma_{bw}(T)$. Amouch et al. [6] and Karmouni and Tajmouati [16] provided a novel characterization of Browder's theorem using the spectra derived from Drazin invertibilty and Fredholm theory. Gupta and Kumar [14] gave a new characterization of generalized Browder's theorem by taking equality between the generalized Drazin-meromorphic spectrum and the generalized Drazin-meromorphic Weyl spectrum. Motivated by them, we give a new characterization of operators satisfying Browder's theorem. We prove that an operator T satisfies Browder's theorem if and only if $\sigma_{gD(gM)W}(T) = \sigma_{gD(gM)}(T)$. In the last section, for operators A and B satisfying $A^kB^kA^k = A^{k+1}$ for some positive integer k, we generalize Cline's formula to the case of generalized Drazin-g-meromorphic invertibility.

2. Main Results

In this section, we will utilize the following result:

Theorem 2.1. [19, Theorem 3.7] *Let $T \in B(X)$, then T is generalized Drazin-g-meromorphic upper semi-Weyl (generalized Drazin-g-meromorphic lower semi-Weyl, generalized Drazin-g-meromorphic upper semi-Fredholm, generalized Drazin-g-meromorphic lower semi-Fredholm, generalized Drazin-g-meromorphic Weyl, respectively) if and only if T admits a GK(gM)D and* $0 \notin \mathrm{acc}\sigma_{gDW_+}(T)$ $(\mathrm{acc}\sigma_{gDW_-}(T), \mathrm{acc}\sigma_{gD\phi_+}(T), \mathrm{acc}\sigma_{gD\phi_-}(T), \mathrm{acc}\sigma_{gDW}(T)$, *respectively).*

Theorem 2.2. [11, Theorem 3.4] *Let $T \in B(X)$, then T is generalized Drazin upper semi-Weyl (generalized Drazin lower semi-Weyl, generalized Drazin upper semi-Fredholm, generalized Drazin lower semi-Fredholm, generalized Drazin Weyl, respectively) if and only if T admits a GKD and* $0 \notin \mathrm{acc}\sigma_{uw}(T)$ $(\mathrm{acc}\sigma_{lw}(T)$, $\mathrm{acc}\sigma_{uf}(T)$, $\mathrm{acc}\sigma_{lf}(T)$, $\mathrm{acc}\sigma_w(T)$, *respectively).*

The following example illustrates that the inclusions $\sigma_{gD(gM)W_-}(T) \subset \sigma_{gD(gM)\mathcal{Q}}(T)$ and $\sigma_{gD(gM)W_+}(T) \subset \sigma_{gD(gM)\mathcal{J}}(T)$ can be proper.

Example 2.3. [20, Example 3.3] Let $X = c(\mathbb{N})$, $c_0(\mathbb{N})$, $l^p(\mathbb{N})$ $(p \geq 1)$ or $l^\infty(\mathbb{N})$. Let U and V be the forward and the backward unilateral shifts on X, respectively. Let $T = U \oplus V$. Then $\sigma_a(T) = \sigma_s(T) = \mathbb{D}$, where $\mathbb{D}$ denotes the closed unit disc. Therefore, $0 \in \mathrm{int}\sigma_a(T)$ and $0 \in \mathrm{int}\sigma_s(T)$. Thus, by [19, Theorems 3.13 and 3.14] $0 \in \sigma_{gD(gM)\mathcal{J}}(T)$ and $0 \in \sigma_{gD(gM)\mathcal{Q}}(T)$. Since $0 \notin \sigma_{gDRW_+}(T)$ and we know that $\sigma_{gD(gM)W_+}(T) \subset \sigma_{gDRW_+}(T)$, $0 \notin \sigma_{gD(gM)W_+}(T)$. Thus, $0 \in \sigma_{gD(gM)\mathcal{J}}(T) \setminus \sigma_{gD(gM)W_+}(T)$. Similarly, $0 \in \sigma_{gD(gM)\mathcal{Q}}(T) \setminus \sigma_{gD(gM)W_-}(T)$.

In the following results we obtain necessary and sufficient conditions to get equality.

Proposition 2.4. *Let $T \in B(X)$, then $\sigma_{gD(gM)\mathcal{J}}(T) = \sigma_{gD(gM)W_+}(T)$ if and only if T has SVEP at every $\lambda \notin \sigma_{gD(gM)W_+}(T)$.*

Proof. Assume that $\sigma_{gD(gM)\mathcal{J}}(T) = \sigma_{gD(gM)W_+}(T)$. Let $\lambda \notin \sigma_{gD(gM)W_+}(T)$, then $\lambda I - T$ is generalized Drazin-g-meromorphic bounded below. Therefore, by [19, Theorem 3.13] T has SVEP at λ. Conversely, assume that T has SVEP at every $\lambda \notin \sigma_{gD(gM)W_+}(T)$. It is sufficient to show that $\sigma_{gD(gM)\mathcal{J}}(T) \subset \sigma_{gD(gM)W_+}(T)$. Let $\lambda \notin \sigma_{gD(gM)W_+}(T)$ which implies that $\lambda I - T$ is generalized Drazin-g-meromorphic upper semi-Weyl. Therefore, by Theorem 2.1 $\lambda I - T$ admits a $GK(gM)D$. Thus, there exists $(M, N) \in Red(\lambda I - T)$ such that $(\lambda I - T)_M$ is semi-regular and $(\lambda I - T)_N$ is g-meromorphic. Since T has SVEP at every $\lambda \notin \sigma_{gD(gM)W_+}(T)$, $(\lambda I - T)$ has SVEP at 0. As SVEP at a point is transmitted to the restrictions on closed invariant subspaces, $(\lambda I - T)_M$ has SVEP at 0. Therefore, by [1, Theorem 2.91] $(\lambda I - T)_M$ is bounded below. Thus, by [19, Theorem 3.13] we have $\lambda I - T$ is generalized Drazin-g-meromorphic bounded below. Hence, $\lambda \notin \sigma_{gD(gM)\mathcal{J}}(T)$. □

Proposition 2.5. *Let $T \in B(X)$, then $\sigma_{gD(gM)\mathcal{Q}}(T) = \sigma_{gD(gM)W_-}(T)$ if and only if T^* has SVEP at every $\lambda \notin \sigma_{gD(gM)W_-}(T)$.*

Proof. Assume that $\sigma_{gD(gM)\mathcal{Q}}(T) = \sigma_{gD(gM)W_-}(T)$. Let $\lambda \notin \sigma_{gD(gM)W_-}(T)$, then $\lambda I - T$ is generalized Drazin-g-meromorphic surjective. Therefore, by [19, Theorem 3.14] T^* has SVEP at λ. Conversely, assume that T^* has SVEP at every $\lambda \notin \sigma_{gD(gM)W_-}(T)$. It is sufficient to show that $\sigma_{gD(gM)\mathcal{Q}}(T) \subset \sigma_{gD(gM)W_-}(T)$. Let

$\lambda \notin \sigma_{gD(gM)W_-}(T)$ which implies that $\lambda I - T$ is generalized Drazin-g-meromorphic lower semi-Weyl. Then by Theorem 2.1 $\lambda I - T$ admits a $GK(gM)D$ and $\lambda \notin \mathrm{acc}\sigma_{gDW_-}(T)$. Since T^* has SVEP at every $\lambda \notin \sigma_{gD(gM)W_-}(T)$ and $\sigma_{gD(gM)W_-}(T) \subset \sigma_{lw}(T)$ then T^* has SVEP at every $\lambda \notin \sigma_{lw}(T) = \sigma_{uw}(T^*)$. Therefore, by [1, Theorem 5.27] we have $\sigma_{lw}(T) = \sigma_{uw}(T^*) = \sigma_{ub}(T^*) = \sigma_{lb}(T)$. Now we prove that $\sigma_{gDW_-}(T) = \sigma_{gDQ}(T)$. Clearly, $\sigma_{gDW_-}(T) \subset \sigma_{gDQ}(T)$. Let $\mu \notin \sigma_{gDW_-}(T)$, then by Theorem 2.2, we have $\mu I - T$ has GKD and $\mu \notin acc\sigma_{lw}(T) = acc\sigma_{lb}(T)$. Therefore, by [11, Theorem 3.7] $\mu \notin \sigma_{gDQ}(T)$. Thus, $\sigma_{gDW_-}(T) = \sigma_{gDQ}(T)$. This implies that $\lambda \notin \mathrm{acc}\sigma_{gDQ}(T)$. Therefore, by [19, Theorem 3.14] $\lambda I - T$ is generalized Drazin-g-meromorprhic surjective and it follows that $\lambda \notin \sigma_{gD(gM)Q}(T)$. □

Corollary 2.6. *Let $T \in B(X)$, then $\sigma_{gD(gM)}(T) = \sigma_{gD(gM)W}(T)$ if and only if T and T^* have SVEP at every $\lambda \notin \sigma_{gD(gM)W}(T)$.*

Proof. Suppose that $\sigma_{gD(gM)}(T) = \sigma_{gD(gM)W}(T)$. Let $\lambda \notin \sigma_{gD(gM)W}(T)$, then $\lambda I - T$ is generalized Drazin-g-meromorphic invertible. Therefore, by [19, Theorem 3.10] T and T^* have SVEP at λ. Conversely, let $\lambda \notin \sigma_{gD(gM)W}(T) = \sigma_{gD(gM)W_+}(T) \cup \sigma_{gD(gM)W_-}(T)$. Then by proofs of Proposition 2.4 and Proposition 2.5 we have $\lambda \notin \sigma_{gD(gM)\mathcal{J}}(T) \cup \sigma_{gD(gM)Q}(T) = \sigma_{gD(gM)}(T)$. □

Theorem 2.7. *Let $T \in B(X)$, then following statements are equivalent:*
(i) $\sigma_{gD(gM)}(T) = \sigma_{gD(gM)W}(T)$,
(ii) T or T^* have SVEP at every $\lambda \notin \sigma_{gD(gM)W}(T)$.

Proof. Suppose that T has SVEP at every $\lambda \notin \sigma_{gD(gM)W}(T)$. It is sufficient to prove that $\sigma_{gD(gM)}(T) \subset \sigma_{gD(gM)W}(T)$. Let $\lambda \notin \sigma_{gD(gM)W}(T)$ then $\lambda I - T$ admits a $GK(gM)D$ and $\lambda \notin \mathrm{acc}\sigma_{gDW}(T)$. Since $\sigma_{gD(gM)W}(T) \subset \sigma_w(T)$, T has SVEP at every $\lambda \notin \sigma_w(T)$. Therefore, by [1, Theorem 5.4] we have $\sigma_w(T) = \sigma_b(T)$. Now we prove $\sigma_{gDW}(T) = \sigma_{gD}(T)$. Clearly, $\sigma_{gDW}(T) \subset \sigma_{gD}(T)$. Let $\mu \notin \sigma_{gDW}(T)$, then by Theorem 2.2, we have $\mu I - T$ has GKD and $\mu \notin acc\sigma_w(T) = acc\sigma_b(T)$. Therefore, by [11, Theorem 3.9] $\mu \notin \sigma_{gD}(T)$. Thus, $\sigma_{gDW}(T) = \sigma_{gD}(T)$. This implies that $\lambda \notin \mathrm{acc}\sigma_{gD}(T)$. Therefore, by [19, Theorem 3.10] $\lambda I - T$ is generalized Drazin-g-meromorphic invertible.

Now suppose that T^* has SVEP at every $\lambda \notin \sigma_{gD(gM)W}(T)$. Since $\sigma_{gD}(T) = \sigma_{gD}(T^*)$ and $\sigma_{gDW}(T) = \sigma_{gDW}(T^*)$ we have $\sigma_{gD(gM)}(T) = \sigma_{gD(gM)W}(T)$. The converse is an immediate consequence of Corollary 2.6. □

Recall that an operator $T \in B(X)$ is said to satisfy generalized a-Browder's theorem if $\sigma_{usbb}(T) = \sigma_{usbw}(T)$. An operator $T \in B(X)$ satisfies a-Browder's theorem if $\sigma_{ub}(T) = \sigma_{uw}(T)$. By [4, Theorem 2.2] we know that a-Browder's theorem is equivalent to generalized a-Browder's theorem.

Theorem 2.8. *Let $T \in B(X)$, then the following holds:*
(i) a-Browder's theorem holds for T if and only if $\sigma_{gD(gM)\mathcal{J}}(T) = \sigma_{gD(gM)W_+}(T)$,
(ii) a-Browder's theorem holds for T^* if and only if $\sigma_{gD(gM)Q}(T) = \sigma_{gD(gM)W_-}(T)$,
(iii) Browder's theorem holds for T if and only if $\sigma_{gD(gM)}(T) = \sigma_{gD(gM)W}(T)$.

Proof. (i) Suppose that a-Browder's theorem holds for T which implies that $\sigma_{uw}(T) = \sigma_{ub}(T)$. Then by proof of Proposition 2.5, we have $\sigma_{gD\mathcal{J}}(T) = \sigma_{gDW_+}(T)$. It is sufficient to prove that $\sigma_{gD(gM)\mathcal{J}}(T) \subset \sigma_{gD(gM)W_+}(T)$. Let $\lambda \notin \sigma_{gD(gM)W_+}(T)$, then $\lambda I - T$ is generalized Drazin-g-meromorphic upper semi-Weyl. By Theorem 2.1 it follows that $\lambda I - T$ admits a $GK(gM)D$ and $\lambda \notin \mathrm{acc}\sigma_{gDW_+}(T)$. This gives $\lambda \notin \mathrm{acc}\sigma_{gD\mathcal{J}}(T)$. Therefore, by [19, Theorem 3.13] $\lambda I - T$ is generalized Drazin-g-meromorphic bounded below which gives $\lambda \notin \sigma_{gD(gM)\mathcal{J}}(T)$. Conversely, suppose that $\sigma_{gD(gM)\mathcal{J}}(T) = \sigma_{gD(gM)W_+}(T)$. Using Proposition 2.4 we deduce that T has SVEP at every $\lambda \notin \sigma_{gD(gM)W_+}(T)$. Since $\sigma_{gD(gM)W_+}(T) \subset \sigma_{uw}(T)$, T has SVEP at every $\lambda \notin \sigma_{uw}(T)$. By [1, Theorem 5.27] T satisfies a-Browder's theorem.
(ii) Suppose that a-Browder's theorem holds for T^* which implies that $\sigma_{lb}(T) = \sigma_{lw}(T)$. By proof of Proposition 2.5, we have $\sigma_{gDQ}(T) = \sigma_{gDW_-}(T)$. It is sufficient to prove that $\sigma_{gD(gM)Q}(T) \subset \sigma_{gD(gM)W_-}(T)$. Let $\lambda \notin \sigma_{gD(gM)W_-}(T)$, then $\lambda I - T$ is generalized Drazin-g-meromorphic lower semi-Weyl. By Theorem 2.1 it follows that $\lambda I - T$ admits a $GK(gM)D$ and $\lambda \notin \mathrm{acc}\sigma_{gDW_-}(T)$. This gives $\lambda \notin \mathrm{acc}\sigma_{gDQ}(T)$. Therefore, by [19, Theorem 3.14] $\lambda I - T$ is generalized Drazin-g-meromorphic surjective which gives $\lambda \notin \sigma_{gD(gM)Q}(T)$. Conversely, suppose that $\sigma_{gD(gM)Q}(T) = \sigma_{gD(gM)W_-}(T)$. Using Proposition 2.5 we deduce that T^* has SVEP at every $\lambda \notin \sigma_{gD(gM)W_-}(T)$. Since $\sigma_{gD(gM)W_-}(T) \subset \sigma_{lw}(T)$, T^* has SVEP at every $\lambda \notin \sigma_{lw}(T) = \sigma_{uw}(T^*)$. Therefore,

a-Browder's theorem holds for T^*.
(iii) Suppose that Browder's theorem holds for T which implies that $\sigma_b(T) = \sigma_w(T)$. Then by proof of Theorem 2.7, we have $\sigma_{gD}(T) = \sigma_{gDW}(T)$. It is sufficient to prove that $\sigma_{gD(gM)}(T) \subset \sigma_{gD(gM)W}(T)$. Let $\lambda \notin \sigma_{gD(gM)W}(T)$, then $\lambda I - T$ is generalized Drazin-g-meromorphic Weyl. By Theorem 2.1 it follows that $\lambda I - T$ admits a $GK(gM)D$ and $\lambda \notin \text{acc}\sigma_{gDW}(T)$. This gives $\lambda \notin \text{acc}\sigma_{gD}(T)$. Therefore, by [19, Theorem 3.10] $\lambda I - T$ is generalized Drazin-g-meromorphic invertible which gives $\lambda \notin \sigma_{gD(gM)}(T)$. Conversely, suppose that $\sigma_{gD(gM)}(T) = \sigma_{gD(gM)W}(T)$. Using Corollary 2.6 we deduce that T and T^* have SVEP at every $\lambda \notin \sigma_{gD(gM)W}(T)$. Since $\sigma_{gD(gM)W}(T) \subset \sigma_w(T)$, T and T^* have SVEP at every $\lambda \notin \sigma_w(T)$. Therefore, by [1, Theorem 5.4] Browder's theorem holds for T. □

Using Theorem 2.8, [2, Theorem 2.3], [4, Theorem 2.1], [5, Proposition 2.2], [16, Theorem 2.6] and [14, Theorem 2.8] we have the following theorem:

Theorem 2.9. *Let $T \in B(X)$, then the following statements are equivalent:*
(i) Browder's theorem holds for T,
(ii) Browder's theorem holds for T^*,
(iii) T has SVEP at every $\lambda \notin \sigma_w(T)$,
(iv) T^* has SVEP at every $\lambda \notin \sigma_w(T)$,
(v) T has SVEP at every $\lambda \notin \sigma_{bw}(T)$,
(vi) generalized Browder's theorem holds for T,
(vii) T or T^* has SVEP at every $\lambda \notin \sigma_{gDRW}(T)$,
(viii) $\sigma_{gDR}(T) = \sigma_{gDRW}(T)$,
(ix) T or T^* has SVEP at every $\lambda \notin \sigma_{gDMW}(T)$,
(x) T or T^* has SVEP at every $\lambda \notin \sigma_{gD(gM)W}(T)$,
(xi) $\sigma_{gDM}(T) = \sigma_{gDMW}(T)$,
(xii) $\sigma_{gD}(T) = \sigma_{gDW}(T)$,
(xiii) $\sigma_{gD(gM)}(T) = \sigma_{gD(gM)W}(T)$.

Using [4, Theorem 2.2], [16, Theorem 2.7] and [14, Theorem 2.9] a similar result for a-Browder's theorem can be stated as follows:

Theorem 2.10. *Let $T \in B(X)$, then the following statements are equivalent:*
(i) a-Browder's theorem holds for T,
(ii) generalized a-Browder's theorem holds for T,
(iii) T has SVEP at every $\lambda \notin \sigma_{gDRW_+}(T)$,
(iv) $\sigma_{gDR\mathcal{J}}(T) = \sigma_{gDRW_+}(T)$,
(v) T has SVEP at every $\lambda \notin \sigma_{gDMW_+}(T)$,
(vi) T has SVEP at every $\lambda \notin \sigma_{gD(gM)W_+}(T)$,
(vii) $\sigma_{gDM\mathcal{J}}(T) = \sigma_{gDMW_+}(T)$,
(viii) $\sigma_{gD(gM)\mathcal{J}}(T) = \sigma_{gD(gM)W_+}(T)$.

Lemma 2.11. *Let $T \in B(X)$, then*
(i) $\sigma_{uf}(T) = \sigma_{ub}(T) \Leftrightarrow \sigma_{gD\phi_+}(T) = \sigma_{gD\mathcal{J}}(T)$,
(ii) $\sigma_{lf}(T) = \sigma_{lb}(T) \Leftrightarrow \sigma_{gD\phi_-}(T) = \sigma_{gD\mathcal{Q}}(T)$.

Proof. (i) Let $\sigma_{ub}(T) = \sigma_{uf}(T)$. It is sufficient to show that $\sigma_{gD\mathcal{J}}(T) \subset \sigma_{gD\phi_+}(T)$. Let $\lambda \notin \sigma_{gD\phi_+}(T)$. Then $\lambda I - T$ is generalized Drazin upper semi-Fredholm. Then by Theorem 2.2, $\lambda I - T$ admits a GKD and $\lambda \notin \text{acc}\sigma_{uf}(T)$ which implies that $\lambda \notin \text{acc}\sigma_{ub}(T)$. Then by Theorem [11, Theorem 3.6], we have $\lambda \notin \sigma_{gD\mathcal{J}}(T)$. Coversely, let $\sigma_{gD\phi_+}(T) = \sigma_{gD\mathcal{J}}(T)$. It is sufficient to show that $\sigma_{ub}(T) \subset \sigma_{uf}(T)$. Let $\lambda \notin \sigma_{uf}(T)$. Then $\lambda \notin \sigma_{gD\phi_+}(T) = \sigma_{gD\mathcal{J}}(T)$. This implies that $\lambda \notin acc\sigma_{ap}(T)$. Then by [1, Remark 2.11], we have T has SVEP at λ. This gives $p(\lambda I - T) < \infty$. Thus, $\lambda \notin \sigma_{ub}(T)$.

(ii) Using a similar argument as above we can get the desired result. □

The following example demonstrates that the inclusions $\sigma_{gD(gM)\phi_+}(T) \subset \sigma_{gD(gM)\mathcal{J}}(T)$, $\sigma_{gD(gM)\phi_-}(T) \subset \sigma_{gD(gM)\mathcal{Q}}(T)$ and $\sigma_{gD(gM)\phi}(T) \subset \sigma_{gD(gM)}(T)$ can be proper:

Example 2.12. Let $X = c(\mathbb{N})$, $c_0(\mathbb{N})$, $l^p(\mathbb{N})$ $(p \geq 1)$ or $l^\infty(\mathbb{N})$. Let U and V be the forward and the backward unilateral shifts on X, respectively. Then $\sigma(U) = \sigma(V) = \mathbb{D}$, where $\mathbb{D}$ denotes the closed unit disc, $\sigma_a(U) = \sigma_s(V) = \partial\mathbb{D}$ and by [21, Theorem 4.2], we have $\sigma_f(U) = \sigma_f(V) = \partial\mathbb{D}$. Therefore, by [19, Theorem 4.13], $\sigma_{gK(gM)}(U) = \sigma_{gD(gM)\phi_+}(U) = \sigma_{gD(gM)\mathcal{J}}(U) = \partial\mathbb{D}$ which gives $\sigma_{gD(gM)\phi_-}(U)=\sigma_{gD(gM)\phi}(U)=\partial\mathbb{D}$. Also, by [19, Corollary 4.1], we have $\sigma_{gD(gM)\mathcal{Q}}(U) = \sigma_{gD(gM)}(U) = \mathbb{D}$. Hence, the inclusions $\sigma_{gD(gM)\phi_-}(U) \subset \sigma_{gD(gM)\mathcal{Q}}(U)$ and $\sigma_{gD(gM)\phi}(U) \subset \sigma_{gD(gM)}(U)$ are proper. Also, by [19, Theorem 4.14], $\sigma_{gK(gM)}(V) = \sigma_{gD(gM)\phi_-}(V) = \sigma_{gD(gM)\mathcal{Q}}(V) = \partial\mathbb{D}$ which gives $\sigma_{gD(gM)\phi_+}(V) = \sigma_{gD(gM)\phi}(V) = \partial\mathbb{D}$. By [19, Corollary 4.1], we have $\sigma_{gD(gM)\mathcal{J}}(V) = \sigma_{gD(gM)}(V) = \mathbb{D}$. Hence, the inclusion $\sigma_{gD(gM)\phi_+}(V) \subset \sigma_{gD(gM)\mathcal{J}}(V)$ is proper.

In the following results we obtain necessary and sufficient conditions to get equality.

Theorem 2.13. *Let $T \in B(X)$, then the following statements are equivalent:*
(i) $\sigma_{gD\phi_+}(T) = \sigma_{gD\mathcal{J}}(T)$,
(ii) T has SVEP at every $\lambda \notin \sigma_{gD\phi_+}(T)$,
(iii) T has SVEP at every $\lambda \notin \sigma_{gD(gM)\phi_+}(T)$,
(iv) $\sigma_{gD(gM)\mathcal{J}}(T) = \sigma_{gD(gM)\phi_+}(T)$.

Proof. (i) ⇔ (ii) Suppose that $\sigma_{gD\phi_+}(T) = \sigma_{gD\mathcal{J}}(T)$. Let $\lambda \notin \sigma_{gD\phi_+}(T)$, then $\lambda \notin \sigma_{gD\mathcal{J}}(T)$ which gives T has SVEP at λ. Now suppose that T has SVEP at every $\lambda \notin \sigma_{gD\phi_+}(T)$ which gives T has SVEP at every $\lambda \notin \sigma_{uf}(T)$. This implies that $\sigma_{uf}(T) = \sigma_{ub}(T)$. Thus by Lemma 2.11, we have $\sigma_{gD\phi_+}(T) = \sigma_{gD\mathcal{J}}(T)$.
(iii) ⇔ (iv) Suppose that T has SVEP at every $\lambda \notin \sigma_{gD(gM)\phi_+}(T)$ which implies that $\lambda I - T$ is generalized Drazin-g-meromorphic upper semi-Fredholm. It is sufficient to show that $\sigma_{gD(gM)\mathcal{J}}(T) \subset \sigma_{gD(gM)\phi_+}(T)$. Let $\lambda \notin \sigma_{gD(gM)\phi_+}(T)$, then by Theorem 2.1 there exists $(M,N) \in Red(\lambda I - T)$ such that $(\lambda I - T)_M$ is semi-regular and $(\lambda I - T)_N$ is g-meromorphic. Since T has SVEP at λ, $(\lambda I - T)_M$ has SVEP at 0. Therefore, by [1, Theorem 2.91] $(\lambda I - T)_M$ is bounded below. Thus, $\lambda \notin \sigma_{gD(gM)\mathcal{J}}(T)$. Conversely, suppose that $\sigma_{gD(gM)\mathcal{J}}(T) = \sigma_{gD(gM)\phi_+}(T)$. Let $\lambda \notin \sigma_{gD(gM)\phi_+}(T)$, then $\lambda I - T$ is generalized Drazin-g-meromorphic bounded below. Therefore, by [19, Theorem 3.13] it follows that T has SVEP at λ.
(i) ⇔ (iv) Suppose that $\sigma_{gD\phi_+}(T) = \sigma_{gD\mathcal{J}}(T)$. It is sufficient to prove that $\sigma_{gD(gM)\mathcal{J}}(T) \subset \sigma_{gD(gM)\phi_+}(T)$. Let $\lambda \notin \sigma_{gD(gM)\phi_+}(T)$, then $\lambda I - T$ is generalized Drazin-g-meromorphic upper semi-Fredholm. By Theorem 2.1 it follows that $\lambda I - T$ admits a $GK(gM)D$ and $\lambda \notin \mathrm{acc}\,\sigma_{gD\phi_+}(T)$. This gives $\lambda \notin \mathrm{acc}\,\sigma_{gD\mathcal{J}}(T)$. Therefore, by [19, Theorem 3.13] $\lambda I - T$ is generalized Drazin-g-meromorphic bounded below which gives $\lambda \notin \sigma_{gD(gM)\mathcal{J}}(T)$. Conversely, suppose that $\sigma_{gD(gM)\mathcal{J}}(T) = \sigma_{gD(gM)\phi_+}(T)$. Then by (iv) ⇒ (iii) T has SVEP at every $\lambda \notin \sigma_{gD(gM)\phi_+}(T)$. Since $\sigma_{gD(gM)\phi_+}(T) \subset \sigma_{uf}(T)$, T has SVEP at every $\lambda \notin \sigma_{uf}(T)$. Therefore, $\sigma_{uf}(T) = \sigma_{ub}(T)$. Thus, by Lemma 2.11 $\sigma_{gD\phi_+}(T) = \sigma_{gD\mathcal{J}}(T)$. □

Theorem 2.14. *Let $T \in B(X)$, then the following statements are equivalent:*
(i) $\sigma_{gD\phi_-}(T) = \sigma_{gD\mathcal{Q}}(T)$,
(ii) T^* has SVEP at every $\lambda \notin \sigma_{gD\phi_-}(T)$,
(iii) T^* has SVEP at every $\lambda \notin \sigma_{gD(gM)\phi_-}(T)$,
(iv) $\sigma_{gD(gM)\mathcal{Q}}(T) = \sigma_{gD(gM)\phi_-}(T)$.

Proof. (i) ⇔ (ii) Suppose that $\sigma_{gD\phi_-}(T) = \sigma_{gD\mathcal{Q}}(T)$. Let $\lambda \notin \sigma_{gD\phi_-}(T)$, then $\lambda \notin \sigma_{gD\mathcal{Q}}(T)$ which gives T^* has SVEP at every $\lambda \notin \sigma_{gD\phi_-}(T)$. Now suppose that T^* has SVEP at every $\lambda \notin \sigma_{gD\phi_-}(T)$ which gives T^* has SVEP at every $\lambda \notin \sigma_{lf}(T)$. This implies that $\sigma_{lf}(T) = \sigma_{lb}(T)$. Thus by Lemma 2.11, we have $\sigma_{gD\phi_-}(T) = \sigma_{gD\mathcal{Q}}(T)$.
(iii) ⇔ (iv) Suppose that T^* has SVEP at every $\lambda \notin \sigma_{gD(gM)\phi_-}(T)$ which implies that $\lambda I - T$ is generalized Drazin-g-meromorphic lower semi-Fredholm. It is sufficient to show that $\sigma_{gD(gM)\mathcal{Q}}(T) \subset \sigma_{gD(gM)\phi_-}(T)$. Let $\lambda \notin \sigma_{gD(gM)\phi_-}(T)$. By Theorem 2.1 it follows that $\lambda I - T$ admits a $GK(gM)D$ and $\lambda \notin \mathrm{acc}\,\sigma_{gD\phi_-}(T)$. Since $\sigma_{gD(gM)\phi_-}(T) \subset \sigma_{lf}(T)$, T^* has SVEP at every $\lambda \notin \sigma_{lf}(T)$. Therefore, we have $\sigma_{lf}(T) = \sigma_{lb}(T)$. Thus, by Lemma 2.11 $\sigma_{gD\phi_-}(T) = \sigma_{gD\mathcal{Q}}(T)$ which implies that $\lambda \notin \mathrm{acc}\,\sigma_{gD\mathcal{Q}}(T)$. Hence, $\lambda \notin \sigma_{gD(gM)\mathcal{Q}}(T)$. Conversely, suppose that $\sigma_{gD(gM)\mathcal{Q}}(T) = \sigma_{gD(gM)\phi_-}(T)$. Let $\lambda \notin \sigma_{gD(gM)\phi_-}(T)$, then $\lambda I - T$ is generalized Drazin-g-meromorphic surjective. Therefore, by [19, Theorem 3.14] it follows that T^* has SVEP at λ.

(i) ⇔ (iv) Suppose that $\sigma_{gD\phi_-}(T) = \sigma_{gDQ}(T)$. It is sufficient to prove that $\sigma_{gD(gM)Q}(T) \subset \sigma_{gD(gM)\phi_-}(T)$. Let $\lambda \notin \sigma_{gD(gM)\phi_-}(T)$, then $\lambda I - T$ is generalized Drazin-g-meromorphic lower semi-Fredholm. By Theorem 2.1 it follows that $\lambda I - T$ admits a $GK(gM)D$ and $\lambda \notin \mathrm{acc}\sigma_{gD\phi_-}(T)$. This gives $\lambda \notin \mathrm{acc}\sigma_{gDQ}(T)$. Therefore, by [19, Theorem 3.14] $\lambda I - T$ is generalized Drazin-g-meromorphic surjective which gives $\lambda \notin \sigma_{gD(gM)Q}(T)$. Conversely, suppose that $\sigma_{gD(gM)Q}(T) = \sigma_{gD(gM)\phi_-}(T)$. Then by (iv) ⇒ (iii) T^* has SVEP at every $\lambda \notin \sigma_{gD(gM)\phi_-}(T)$. Since $\sigma_{gD(gM)\phi_-}(T) \subset \sigma_{lf}(T)$, T^* has SVEP at every $\lambda \notin \sigma_{lf}(T)$. Therefore, $\sigma_{lf}(T) = \sigma_{lb}(T)$. Thus, by Lemma 2.11 $\sigma_{gD\phi_-}(T) = \sigma_{gDQ}(T)$. □

Using [16, Corollary 2.10], [14, Corollary 2.14] and Theorems 2.13, 2.14 we have the following result:

Corollary 2.15. *Let $T \in B(X)$, then the following statements are equivalent:*
(i) $\sigma_f(T) = \sigma_b(T)$,
(ii) T and T^* have SVEP at every $\lambda \notin \sigma_f(T)$,
(iii) $\sigma_{bf}(T) = \sigma_{bb}(T)$,
(iv) T and T^* have SVEP at every $\lambda \notin \sigma_{bf}(T)$,
(v) $\sigma_{gD}(T) = \sigma_{gD\phi}(T)$,
(vi) T and T^* have SVEP at every $\lambda \notin \sigma_{gD\phi}(T)$,
(viii) $\sigma_{gDR}(T) = \sigma_{gDR\phi}(T)$,
(viii) T and T^* have SVEP at every $\lambda \notin \sigma_{gDR\phi}(T)$,
(ix) $\sigma_{gDM}(T) = \sigma_{gDM\phi}(T)$,
(x) T and T^* have SVEP at every $\lambda \notin \sigma_{gDM\phi}(T)$,
(xi) $\sigma_{gD(gM)}(T) = \sigma_{gD(gM)\phi}(T)$,
(xii) T and T^* have SVEP at every $\lambda \notin \sigma_{gD(gM)\phi}(T)$.

3. Cline's Formula for the generalized Drazin-g-meromorphic invertibility

For a ring R with identity, Drazin[12] introduced the concept of Drazin inverses in a ring. An element $a \in R$ is said to be *Drazin invertible* if there exist an element $b \in R$ and $r \in \mathbb{N}$ such that

$$ab = ba,\ bab = b,\ a^{r+1}b = a^r.$$

If such b exists then it is unique and is called *Drazin inverse* of a and denoted by a^D. For $a, b \in R$, Cline [10] proved that if ab is Drazin invertible, then ba is Drazin invertible and $(ba)^D = b((ab)^D)^2a$. Recently, Gupta and Kumar [13] generalized Cline's formula for Drazin inverses in a ring with identity to the case when $a^kb^ka^k = a^{k+1}$ for some $k \in \mathbb{N}$ and obtained the following result:

Theorem 3.1. ([13, Theorem 2.10]) *Let R be a ring with identity and suppose that $a^kb^ka^k = a^{k+1}$ for some $k \in \mathbb{N}$. Then a is Drazin invertible if and only if b^ka^k is Drazin invertible. Moreover, $(b^ka^k)^D = b^k(a^D)^2a^k$ and $a^D = a^k((b^ka^k)^D)^{k+1}$.*

Recently, Karmouni and Tajmouati [15] investigated for bounded linear operators A, B, C satisfying the operator equation $ABA = ACA$ and obtained that AC is generalized Drazin-Riesz invertible if and only if BA is generalized Drazin-Riesz invertible. Also, they generalized Cline's formula to the case of generalized Drazin-Riesz invertibility. Gupta and Kumar [14] established Cline's formula for the generalized Drazin-meromorphic invertibility for bounded linear operators A and B under the condition $A^kB^kA^k = A^{k+1}$. In this section, we establish Cline's formula for the generalized Drazin-g-meromorphic invertibility for bounded linear operators A and B under the condition $A^kB^kA^k = A^{k+1}$. By the proofs of [13, Proposition 2.1, Theorems 2.4, 2.5 and 2.8] and [7, Theorem 3] we can deduce the following result:

Proposition 3.2. *Let $A, B \in B(X)$ satisfies $A^kB^kA^k = A^{k+1}$ for some $k \in \mathbb{N}$, then A is g-meromorphic if and only if B^kA^k is g-meromorphic.*

Theorem 3.3. *Suppose that $A, B \in B(X)$ and $A^kB^kA^k = A^{k+1}$ for some $k \in \mathbb{N}$. Then A is generalized Drazin-g-meromorphic invertible if and only if B^kA^k is generalized Drazin-g-meromorphic invertible.*

Proof. Let A be generalized Drazin-g-meromorphic invertible, then there exists $T \in B(X)$ such that

$$TA = AT, \quad TAT = T \quad \text{and} \quad ATA - A \text{ is } g\text{-meromorphic.}$$

Let $S = B^kT^2A^k$. Then

$$(B^kA^k)S = (B^kA^k)(B^kT^2A^k) = B^k(A^kB^kA^k)T^2 = B^kA^{k+1}T^2 = B^kA^kT$$

and

$$S(B^kA^k) = (B^kT^2A^k)(B^kA^k) = B^kT^2A^{k+1} = B^kA^kT.$$

Therefore, $S(B^kA^k) = (B^kA^k)S$. Now

$$S(B^kA^k)S = B^kT^2A^k(B^kA^k)B^kT^2A^k = (B^kT^2A^k)(B^kA^kT) = B^kT^2A^{k+1}T = B^kT^2A^k = S.$$

Let $Q = I - AT$, then Q is a bounded projection commuting with A which gives $Q^n = Q$ for all $n \in \mathbb{N}$. Also, observe that

$$(QA)^kB^k(QA)^k = Q^kA^kB^kQ^kA^k = Q^kA^{k+1}Q^k = Q^{k+1}A^{k+1} = (QA)^{k+1}$$

and

$$\begin{aligned} B^kA^k - (B^kA^k)^2S &= B^kA^k - (B^kA^k)^2B^kT^2A^K = B^kA^k - B^k(A^kB^kA^k)B^kT^2A^k \\ &= B^kA^k - B^kA^{k+2}T^2 = B^k(I - A^2T^2)A^k = B^k(I - AT)A^k \\ &= B^kQA^k = B^kQ^kA^k = B^k(QA)^k. \end{aligned}$$

Since QA is g-meromorphic and $(QA)^kB^k(QA)^k = (QA)^{k+1}$, by Proposition 3.2 $B^kA^k - (B^kA^k)^2S$ is g-meromorphic.

Conversely, let B^kA^k be generalized Drazin-g-meromorphic invertible. Then there exists $T' \in B(X)$ such that

$$T'B^kA^k = B^kA^kT', \quad T'B^kA^kT' = T' \quad \text{and} \quad B^kA^kT'B^kA^k - B^kA^k \text{ is } g\text{-meromorphic.}$$

Let $S' = A^kT'^{k+1}$. Then

$$S'A = A^kT'^{k+1}A = A^kT'^{k+2}B^kA^kA = A^kT'^{k+2}B^kA^{k+1} = A^kT'^{k+2}(B^kA^k)^2 = A^kT'^k$$

and

$$AS' = A^{k+1}T'^{k+1} = A^kT'^k.$$

Consider

$$\begin{aligned} AS' &= (A^kT'^{k+1}A)A^kT'^{k+1} = (A^kT'^k)A^kT'^{k+1} = A^kv^{k+1}B^kA^{2k}T'^{k+1} = A^kT'^{k+1}(B^kA^k)^{k+1} \\ &= S^{k+1} = A^kT'^{k+1} = S'. \end{aligned}$$

We assert that

$$(A - A^2S')^n = (A^n - A^{n+1}S') \text{ for all } n \in \mathbb{N}.$$

We prove it by induction. Clearly, the result holds for $n = 1$. Suppose that it is true for $n = m$. Consider

$$\begin{aligned} (A - A^2S')^{m+1} &= (A - A^2S')(A - A^2S')^m \\ &= (A - A^2S')(A^p - A^{m+1}S') \\ &= A^{m+1} - A^{m+2}S' - A^{m+2}S' + A^{m+3}S'^2 \\ &= A^{m+1} - A^{m+2}S'. \end{aligned}$$

Also,

$$\begin{aligned} B^k(A - A^2S')^k &= B^k(A^k - A^{k+1}S') = B^kA^k - B^kA^{k-1}A^2S' = B^kA^k - B^kA^{k-1}A^kT'^{k-1} \\ &= B^kA^k - B^kA^{2k-1}T'^{k-1} = B^kA^k - (B^kA^k)^kT'^{k-1} = B^kA^k - (B^kA^k)^2S'. \end{aligned}$$

Consider

$$(A - A^2S')^kB^k(A - A^2S')^k = (A^k - A^{k+1}S')B^k(A^k - A^{k+1}S')$$
$$= A^kB^kA^k - A^{k+1}S'B^kA^k - A^kB^kA^kB^kA^kS' + A^{k+1}(B^kA^k)^2S'^2$$
$$= A^{k+1} - A^{k+2}S' = (A - A^2S')^{k+1}.$$

Since $B^k(A - A^2S')^k = B^kA^k - (B^kA^k)^2T'$ is g-meromorphic, using Proposition 3.2 we deduce that $A - A^2S'$ is g-meromorphic. □

Acknowledgement

The authors are grateful to the referees for their valuable comments and suggestions.

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