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On some anti λ-Ideal convergent fuzzy double sequence spaces which contains ${}_2l_\infty$, ${}_2c$ and ${}_2c_0$

Mohammad Imran Idrisi[a,*], Kavita Saini[a], Nadeem Rao[b]

[a] *Department of Mathematics, Chandigarh University, Mohali Punjab, India*
[b] *Department of Mathematics, University Centre for Research and Development, Chandigarh University, Mohali Punjab-140413, India*

Abstract. The concept of fuzzy sets was introduced by Zadeh as a means of representing data that was not precise but rather fuzzy. Recently, Kočinac [19] studied some topological properties of fuzzy antinormed linear spaces. This has motivated us to introduce and study the fuzzy antinormed double sequence spaces with respect to ideal by using a bounded linear operator and prove some theorems, in particular convergence and completeness theorems on these new double sequence spaces.

1. Introduction and Preliminaries

Fuzzy set theory was formalised by Professor Lofti Zadeh [28] at the University of California in 1965. Thereafter, fuzzy set theory found applications in different areas of mathematics and in other fields. The concept of fuzzy norm was introduced by Katsaras [12] in 1984. In 1992, by using fuzzy numbers, Felbin [8] introduced the fuzzy norm on a linear space. Cheng and Mordeson [4] introduced another idea of fuzzy norm on a linear space, and in 2003 Bag and Samanta [2] modified the defnition of fuzzy norm of Cheng-Mordeson [4]. In [3] a comparative study of the fuzzy norms defined by Katsaras [12], Felbin [8] and Bag and Samanta [2] was given.

Later on, Jebril and Samanta [11] introduced the concept of fuzzy anti-norm on a linear space depending on the idea of fuzzy anti norm, introduced by Bag and Samanta [3]. The motivation of introducing fuzzy anti-norm is to study fuzzy set theory with respect to the non-membership function. It is useful in the process of decision making.

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast[7] and Schoenberg[27]. There has been an effort to introduce several generalizations and variants of statistical convergence in different spaces (see, for example, [6] and references therein). One such very important generalization of this notion was introduced by Kostyrko et al. [21] by using an ideal I of subsets of the set of natural numbers, which they called I-convergence. After that the idea of I-convergence for double sequence was introduced by Das et al. [5] in 2008 (see also [25], [20] for ideal convergence in fuzzy context).

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* Corresponding author: Mohammad Imran Idrisi
Email addresses: mhdimranidrisi@gmail.com (Mohammad Imran Idrisi), kavitasainitg3@gmail.com (Kavita Saini), nadeemrao1990@gmail.com (Nadeem Rao)

Now, we recall some terms and definitions which will be used throughout the article.

Let X be a non empty set then a family $I \subset 2^X$ is said to be an **ideal** in X if $\emptyset \in I$, I is additive i.e for all $A, B \in I \Rightarrow A \cup B \in I$ and I is hereditary i.e for all $A \in I, B \subseteq A \Rightarrow B \in I$ [13, 14]. A non empty family of sets $\mathcal{F} \subset 2^X$ is said to be a **filter** on X if for all $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$ and for all $A \in \mathcal{F}$ with $A \subseteq B$ implies $B \in \mathcal{F}$. An ideal $I \subset 2^X$ is said to be **non trivial** if $I \neq 2^X$; a non trivial ideal is said to be admissible if $I \supseteq \{\{x\} : x \in X\}$ and is said to be **maximal** if there cannot exist any non trivial ideal $J \neq I$ containing I as a subset. One may refer to [1, 9, 10, 15–18] to know more about this concept. For each ideal I there is a filter $\mathcal{F}(I)$ called the filter associate with ideal I, that is

$$\mathcal{F}(I) = \{K \subseteq X : K^c \in I\}, \text{ where } K^c = X \backslash K.$$

Example 1.1. *Consider:*

$$X = \{1, 2, 3\} \text{ and } 2^X = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

then the ideal $I \subset 2^X$ *such that* $I = \{\phi,\{1\},\{2\},\{1,2\}\}$
and the filter associated with ideal I
$\mathcal{F}(I) = \{\{3\},\{1,3\},\{2,3\},\{1,2,3\}\}$

Throughout the article, I is an admissible ideal on $\mathbb{N} \times \mathbb{N}$, and $_2\omega$ denotes the class of all double real sequences. The spaces $_2l_\infty$, $_2c$ and $_2c_0$ are the Banach spaces of bounded, convergent, and null double sequences of reals respectively with the norm

$$\|x\| = \sup_{i,j \in \mathbb{N}} |x_{ij}|. \tag{1}$$

Definition 1.2. *[23, 24] A double sequence* $x = (x_{ij}) \in {_2\omega}$ *is said to be I-convergent to a number L if for every* $\epsilon > 0$ *there are* $m, n \in \mathbb{N}$, *such that*

$$\{(i, j) : |x_{ij} - L| \geq \epsilon, i \leq m, j \leq n\} \in I. \tag{2}$$

In this case, we write $I - \lim x_{ij} = L$.

Definition 1.3. *[23, 24] A double sequence* $(x_{ij}) \in {_2\omega}$ *is said to be I-Cauchy if for every* $\epsilon > 0$ *there exist numbers* $m = m(\epsilon), n = n(\epsilon) \in \mathbb{N}$ *such that for all* $i, p \geq m$ *and* $j, q \geq n$

$$\{(i, j) : |x_{ij} - x_{pq}| \geq \epsilon, i \leq m, j \leq n\} \in I. \tag{3}$$

Definition 1.4. *[23, 24] A double sequence* $(x_{ij}) \in {_2\omega}$ *is said to be I-bounded if there exists* $M > 0$ *such that*

$$\{(i, j) : |x_{ij}| > M\} \in I. \tag{4}$$

Definition 1.5. *[22, 26] A binary operation* $\diamond : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ *is said to be a continuous t-conorm if it satisfies the following conditions:*
(a) $\diamond$ *is associative and commutative,*
(b) $\diamond$ *is continuous,*
(c) $a \diamond 0 = a$ *for all* $a \in [0, 1]$,
(d) $a \diamond b \leq c \diamond d$ *whenever* $a \leq c$ *and* $b \leq d$ *for each* $a, b, c, d \in [0, 1]$.

Some example of continuous t-conorm are $a \diamond b = a + b - ab, a \diamond b = \max a, b, a \diamond b = \min a + b, 1$.

Remark 1.6. *(a) For any* $r_1, r_2 \in (0, 1)$ *with* $r_1 > r_2$ *, there exist* $r_3 \in (0, 1)$ *such that* $r_1 > r_4 \diamond r_2$.
(b) For any $r_4 \in (0, 1)$, *there exist* $r_5 \in (0, 1)$ *such that and* $r_5 \diamond r_5 \leq r4$.

Recall now the notion of fuzzy antinorm in a linear space with respect to a continuous t-conorm following [9].

Definition 1.7. *[19] Let X be a real linear space and $\diamond$ a t-conorm. A fuzzy subset $\nu : X \times \mathbb{R} \to \mathbb{R}$ of $X \times \mathbb{R}$ is called a fuzzy antinorm on X with respect to the t-conorm if, for all $x, y \in X$*

(FaN1) for each $t \in (-\infty, 0]$, $\nu(x,t) = 1$;
(FaN2) for each $t \in (0,\infty)$, $\nu(x,t) = 0$ if and only if $x = \theta$;
(FaN3) for each $t \in (0,\infty)$, $\nu(\alpha x,t) = \nu(x, t|\alpha|)$ if $\alpha \neq 0$;
(FaN4) for all $s, t \in \mathbb{R}$, $\nu(x+y, s+t) \leq \nu(x,s) \diamond \nu(y,t)$;
(FaN5) $\lim_{t\to\infty} \nu(x,t) = 0$.

Note that if ν is the antinorm in the definition above, then $\nu(x,t)$ is nonincreasing with respect to t for each $x \in X$. The following are example of fuzzy antinorms with respect to a corresponding t-conorm and show how a fuzzy antinorm can be obtained from a norm.

Example 1.8. *Let $(X, ||.||)$ be a normed linear space and let the t-conorm $\diamond$ be given by $a \diamond b = a + b - ab$. Define $\nu : X \times \mathbb{R} \to [0,1]$ by*

$$\nu(x,t) = \begin{cases} 0, & \text{if } t > ||x|| \\ 1, & \text{if } t \leq ||x||. \end{cases}$$

Then ν is a fuzzy antinormon X with respect to the t-conorm $\diamond$. This antinorm ν satisfies also the following:

(FaN6) *$\nu(x,t) < 1$ for each $t > 0$ implies $x = \theta$.*

Example 1.9. *Let $(X, ||.||)$ be a normed linear space and consider the t-conorm $\diamond$ defined by $a \diamond b = \min\{a+b, 1\}$. Define $\nu : X \times \mathbb{R} \to [0,1]$ by*

$$\nu(x,t) = \begin{cases} \frac{||x||}{2t-||x||}, & \text{if } t > ||x|| \\ 1, & \text{if } t \leq ||x||. \end{cases}$$

Then ν is a fuzzy antinorm on X with respect to the t-norm $\diamond$. Note that this ν satisfies the condition (FaN6) and also the following:

(FaN7) $\nu(x, .)$ is a continuous function on $\mathbb{R}$ and strictly decreasing on the subset $\{t : 0 < \nu(x,t) < 1\}$ of $\mathbb{R}$.

Definition 1.10. *[19] A sequence $(x_n)_{n\in\mathbb{N}}$ in a fuzzy antinormed linear space $(X, \nu, \diamond)$ is said to be ν-convergent to a point $x \in X$ if for each $\epsilon > 0$ and each $t > 0$ there is $n_0 \in \mathbb{N}$ such that*

$$\nu(x_n - x, t) < \epsilon \text{ for each } n \geq n_0 \tag{5}$$

Let $(X, \nu, \diamond)$be a fuzzy antinormed linear space with respect to an idempotent t-conorm $\diamond$, and let ν satisfy (FaN6). Then for each $\lambda \in (0,1)$ the function $||x||_\lambda : X \to [0,\infty)$ defined by

$$||x||_\lambda = \{t > 0 : \nu(x,t) \leq 1 - \lambda\} \tag{6}$$

is a norm on X and $v = \{||x||_\lambda : \lambda \in (0,1)\}$ is an asscending family of norms on X. In this paper we generalize the definition of fuzzy anti-norm on a linear space. Later on we study some relations and results on them.

2. Fuzzy(anti) I_λ- convergence

Now, in this section we define fuzzy I_λ-convergence, fuzzy I_λ- anti-convergence, fuzzy I_λ- anti-Cauchy and fuzzy I_λ- compactness for double sequences with respect to an ideal I on $\mathbb{N} \times \mathbb{N}$.

Definition 2.1. *A double sequence (x_{ij}) in a fuzzy antinormed linear space X is said to be fuzzy I_ν-convergent to a point $x \in X$ if for each $\epsilon > 0$ and each $t > 0$ the set*

$$\{(i,j) : \nu(x_{ij} - x, t) < \epsilon\} \in I. \tag{7}$$

In this case, we write fuzzy $I_\nu - \lim x_{ij} = x$ and x is called a fuzzy I_ν-limit of (x_{ij}).

Definition 2.2. *Let X be a fuzzy antinormed double sequence space and $\lambda \in (0,1)$. A sequence $(x_{ij}) \in X$ is said to be fuzzy I_λ-convergent to $x \in X$ if for all $t > 0$, the set*

$$\{(i,j) : \nu(x_{ij} - x, t) < 1 - \lambda\} \in I. \tag{8}$$

In this case we write fuzzy $I_\lambda - \lim \nu(x_{ij} - x, t) = 0$ and x is called a fuzzy I_λ-limit of (x_{ij}).

Definition 2.3. *Let X be a fuzzy antinormed double sequence space and $\lambda \in (0,1)$. A sequence $(x_{ij}) \in X$ is said to be fuzzy I_λ-anti-convergent in X if there exist $x \in X$ and $M \in \mathcal{F}(I)$ such that for all $t > 0$,*

$$M = \{(i,j) : \nu(x_{ij} - x, t) < 1 - \lambda\}. \tag{9}$$

In this case, we write $(x_{ij}) \to x$ and x is called a fuzzy I_λ-anti-limit of (x_{ij}).

Definition 2.4. *Let $\lambda \in (0,1)$. A sequence (x_{ij}) in a fuzzy antinormed double sequence space X is said to be fuzzy I_λ-anti-Cauchy if there exist numbers $m, n \in \mathbb{N}$ and $M \in \mathcal{F}(I)$ such that for all $i, p \geq m$, $j, q \geq n$ and all $t > 0$,*

$$M = \{(i,j) : \nu(x_{ij} - x_{pq}, t) < 1 - \lambda\}. \tag{10}$$

Definition 2.5. *A fuzzy antinormed double sequence space X is said to be fuzzy I_λ-anti-complete, $\lambda \in (0,1)$, if for every fuzzy I_λ-anti-Cauchy sequence in X is fuzzy I_λ- anti-convergent in X.*

Now, we define two fuzzy antinormed double sequence spaces with the help of a bounded linear operator T:

$$_2\mathcal{X}^I_\nu(T) = \{(x_{ij}) \in {}_2\ell_\infty : (i,j) : \nu(T(x_{ij} - x), t) < \epsilon\}; \tag{11}$$

$$_2\mathcal{X}^I_{0\nu}(T) = \{(x_{ij}) \in {}_2\ell_\infty : (i,j) : \nu(T(x_{ij}), t) < \epsilon\}. \tag{12}$$

We also define an open ball with centre x and radius r with respect to t as follows:

$$_2\mathcal{B}_x(r,t)(T) = \{(y_{ij}) \in {}_2\ell_\infty : (i,j) : \nu(T(x_{ij}) - T(y_{ij}), t) < r\}. \tag{13}$$

Example 2.6. *Let (x_{ij}) be a double sequence in a fuzzy antinormed linear double sequence space $_2\mathcal{X}^I_\nu(T)$ with $\diamond$ as idempotent t-conorm and $\nu(x,t) = \frac{\|x\|}{2t + \|x\|}$, we define*

$$\left(x_{ij}\right) = \begin{cases} (\frac{1}{i} \cdot \frac{1}{j}), & \text{if } i, j \text{ are cubes}, \\ (-1, 1), & \text{otherwise}. \end{cases}$$

Then by using (FaN6) and (FaN7) we get,

$$fuzzy - I_\nu - \lim(x_{ij}) = (0,0) \;\; \text{and} \;\; fuzzy - I_\lambda anti - \lim(x_{ij}) = (-1, 1). \tag{14}$$

Theorem 2.7. *In the fuzzy antinormed linear double sequence space $_2\mathcal{X}^I_\nu(T)$ with respect to an idempotent t-conorm $\diamond$ satisfying (FaN6) and (FaN7) a sequence is I_ν-convergent if and only if it is I_λ-convergent for each $\lambda \in (0,1)$.*

Proof. Let (x_{ij}) be a sequence in ${}_2\mathcal{X}^I_\nu(T)$ such that (x_{ij}) is I_ν- convergent to x, i.e., for each $t > 0$

$$I_\nu - \lim_{i,j\to\infty} \nu(T(x_{ij} - x), t) = 0. \tag{15}$$

Fix $\lambda \in (0, 1)$. So, $I_\nu - \lim_{i,j\to\infty} \nu(T(x_{ij} - x), t) = 0 < 1 - \lambda$. There exists a set $P \in I$ such that for each $(m, n) \in P$,

$$\nu(T(x_{mn} - x), t) < 1 - \lambda \tag{16}$$

Since $\|T(x_{mn}) - T(x)\|_\lambda = v\{t > 0 : \nu(T(x_{mn} - T(x), t) < 1 - \lambda\}$, we have
$\|T(x_{mn}) - T(x)\|_\lambda \leq t$ for all $(m, n) \in P$. As $t > 0$ was arbitrary and T is a compact linear operator, for each $\lambda \in (0, 1)$, by (FaN6), we have $\|x_{mn} - x\|_\lambda$ I- converges to 0.

Conversely, suppose now that for each $\lambda \in (0, 1)$, $\|x_{ij} - x\|_\lambda$ I-converges to 0. This means that for each $\lambda \in (0, 1)$ and each $\epsilon > 0$ there is a set $P_\lambda \in I$ such that, for each $(i, j) \in P$

$$\|T(x_{ij} - x)\|_\lambda \leq \epsilon. \tag{17}$$

Therefore,

$$\nu(T(x_{ij} - x), \epsilon) = v\{1 - \lambda : \|T(x_{ij} - x)\|_\lambda \leq \epsilon\} \tag{18}$$

implies $\nu(T(x_{ij} - x), \epsilon) \leq 1 - \lambda$ for each $\lambda \in (0, 1)$ and each $(i, j) \in P$, which means

$$I_\nu - \lim \nu(T(x_{ij} - x), \epsilon) \to 0 \tag{19}$$

that is, (x_{ij}) is I_ν-convergent to x as $i, j \to \infty$. □

Theorem 2.8. *Let ${}_2\mathcal{X}^I_\nu(T)$ be a fuzzy antinormed double sequence space with respect to an idempotent t-conorm $\diamond$ satisfying (FaN6). Then fuzzy I_λ-anti-limit of a fuzzy I_λ-anti-convergent sequence is unique.*

Proof. Let $(x_{ij}) \in {}_2\mathcal{X}^I_\nu(T)$ be fuzzy I_λ- convergent double sequence converging two distinct points x and y in ${}_2\mathcal{X}^I_\nu(T)$. This means that for each $t > 0$, there exist $x, y \in X$ and $A_1, A_2 \in \mathcal{F}(I)$ such that

$$A_1 = \{(i, j) : \nu(T(x_{ij} - x), t) < 1 - \lambda\} \in \mathcal{F}(I); \tag{20}$$

$$A_2 = \{(i, j) : \nu(T(x_{ij} - y), t) < 1 - \lambda\} \in \mathcal{F}(I). \tag{21}$$

The set $A = A_1 \cap A_2 \in \mathcal{F}(I)$ and by the assumption on $\diamond$ for each $(i, j) \in A$, we have

$$\nu(x - y, t) \leq \nu(T(x_{ij} - x), t) \diamond \nu(T(x_{ij} - y), t)$$
$$< (1 - \lambda) \diamond (1 - \lambda) = (1 - \lambda).$$

So we have
$$\{(i, j) : \nu(T(x - y), t) < 1 - \lambda\} \supseteq \{(i, j) : \nu(T(x_{ij} - x), t) < 1 - \lambda\}$$
$$\cap\{(i, j) : \nu(T(x_{ij} - y), t) < 1 - \lambda\} \tag{22}$$

Thus, the sets on right hand side of the above equation (21) belong to $\mathcal{F}(I)$. Therefore,
$\nu(T(x - y), t) < 1 - \lambda$ for each $t > 0$. Since T is a bounded linear operator, by (*FaN*6) one obtains $x - y = \theta$ i.e., $x = y$. □

Theorem 2.9. *Let ${}_2\mathcal{X}^I_\nu(T)$ and ${}_2\mathcal{X}^I_{0\nu}(T)$ be a fuzzy antinormed double sequence spaces with respect to an idempotent t-conorm $\diamond$ satisfying (FaN6). Then*

1. *if $I_\lambda - anti - \lim x_{ij} = x$ and $I_\lambda - anti - \lim y_{ij} = y$, then*
   $I_\lambda - anti - \lim(x_{ij} + y_{ij}) = x + y$

2. *if* $I_\lambda - anti - \lim x_{ij} = x$ *and* $r \in \mathbb{R}$, *then* $I_\lambda - anti - \lim rx_{ij} = rx$.

Proof. Since $I_\lambda - anti - \lim x_{ij} = x$ and $I_\lambda - anti - \lim y_{ij} = y$, there exist $M_1, M_2 \in \mathcal{F}(I)$ such that for all $t > 0$ we have

$$M_1 = \{(i,j) : \nu(T(x_{ij} - x), \frac{t}{2}) < 1 - \lambda\} \in \mathcal{F}(I); \tag{23}$$

$$M_2 = \{(i,j) : \nu(T(y_{ij} - y), \frac{t}{2}) < 1 - \lambda\} \in \mathcal{F}(I). \tag{24}$$

The set $M = M_1 \cap M_2 \in \mathcal{F}(I)$ and by the assumption on $\diamond$ for each $(i,j) \in M$, we have

$$\nu(T(x_{ij} + y_{ij}) - T(x+y), t) \leq \nu(T(x_{ij} - x), \frac{t}{2}) \diamond \nu(T(y_{ij} - y), \frac{t}{2})$$
$$< (1-\lambda) \diamond (1-\lambda) = (1-\lambda).$$

So we have
$\{(i,j) : \nu(T(x_{ij} + y_{ij}) - (x+y), t) < 1 - \lambda\} \supseteq \{(i,j) : \nu(T(x_{ij} - x), \frac{t}{2}) < 1 - \lambda\}$

$$\cap \{(i,j) : \nu(T(x_{ij} - y), \frac{t}{2}) < 1 - \lambda\} \tag{25}$$

Thus, the sets on right hand side of the above equation (24) belong $\mathcal{F}(I)$. So we have $M = \{(i,j) : \nu(T(x_{ij} + y_{ij}) - (x+y), t) < 1 - \lambda\} \notin I$ which means that $I_\lambda - anti - \lim(x_{ij} + y_{ij}) = x + y$.

(2) The fact $I_\lambda - anti - \lim x_{ij} = x$ implies that there exists $M \in \mathcal{F}(I)$ such that for all $t > 0$ we have

$$M = \{(i,j) : \nu(T(x_{ij} - x), t) < 1 - \lambda\} \in \mathcal{F}(I). \tag{26}$$

Therefore, for each $(i,j) \in M$, we have

$$\nu(rT(x_{ij}) - rT(x), t) = \nu(T(x_{ij} - x), \frac{t}{|r|}) < 1 - \lambda.$$

We have

$$\{(i,j) : \nu(rT(x_{ij}) - rT(x), t) < 1 - \lambda\} \supseteq \{(i,j) : \nu(T(x_{ij}) - T(x), t) < 1 - \lambda\} \tag{27}$$

From this we get,$\{(i,j) : \nu(rT(x_{ij}) - rT(x), t) \geq 1 - \lambda\} \notin I$ which shows that $I_\lambda - anti - \lim \; rx_{ij} = rx$.
□

Theorem 2.10. *Let* ${}_2\mathcal{X}^I_\nu(T)$ *be a fuzzy antinormed double sequence space with respect to an idempotent t-conorm* $\diamond$. *If* $(x_{ij}) \in {}_2\mathcal{X}^I_\nu(T)$ *is* I_λ*-anticonvergent to* $x \in {}_2\mathcal{X}^I_\nu(T)$, *then* $||x_{ij} - x||_\lambda$ *is I-convergent to 0.*

Proof. Let $(x_{ij}) \in {}_2\mathcal{X}^I_\nu(T)$ be I_λ-anticonvergent double sequence converges to $x \in {}_2\mathcal{X}^I_\nu(T)$ then for $N \in \mathcal{F}(I)$ and for all $t > 0$, we have

$$N = \{(i,j) : \nu(T(x_{ij} - x), t) < 1 - \lambda\}. \tag{28}$$

Since for a compact linear operator T, $||T(x_{ij}) - T(x)||_\lambda = v\{t > 0 : \nu(T(x_{mn} - T(x), t) < 1 - \lambda\}$, we have
$||T(x_{ij}) - T(x)||_\lambda \leq t$ for all $(i,j) \in M \subseteq I$. As $t > 0$ was arbitrary then for each $\lambda \in (0,1)$, by (FaN6), we have $||x_{ij} - x||_\lambda$ *I*- converges to 0.
□

Theorem 2.11. *Let ${}_2\mathcal{X}^I_\nu(T)$ be a fuzzy antinormed double sequence space with respect to an idempotent t-conorm $\diamond$ satisfying (FaN6) and $\lambda \in (0,1)$. Then every I_λ-anti-convergent double sequence $(x_{ij}) \in {}_2\mathcal{X}^I_\nu(T)$ is fuzzy I_λ-anti-Cauchy.*

Proof. Let $(x_{ij}) \in {}_2\mathcal{X}^I_\nu(T)$ be fuzzy I_λ- anti-convergent double sequence. This shows that there exists $M \in \mathcal{F}(I)$ such that for all $t > 0$ we have

$$\{(i,j) : \nu(T(x_{ij} - x), \frac{t}{2}) < 1 - \lambda\} \in \mathcal{F}(I). \tag{29}$$

Therefore for each $(i,j),(m,n) \in M$, we have

$$\nu(T(x_{ij}) - T(x_{mn}), t) \leq \nu(T(x_{ij} - x), \frac{t}{2}) \diamond \nu(T(x_{ij} - x), \frac{t}{2}).$$

$$< (1-\lambda) \diamond (1-\lambda) = (1-\lambda)$$

which means that (x_{ij}) is fuzzy I_λ-anti-quasi- Cauchy in ${}_2\mathcal{X}^I_\nu(T)$.
□

Theorem 2.12. *Let $({}_2\mathcal{X}^I_\nu(T), v)$ be a fuzzy antinormed double sequence space with respect to an idempotent t-conorm $\diamond$. Then I-Cauchy sequence (x_{ij}) in $({}_2\mathcal{X}^I_\nu(T), \|.\|_\lambda)$, $\lambda \in (0,1)$, is fuzzy I_λ-anti-quasi-Cauchy in $({}_2\mathcal{X}^I_\nu(T), v)$.*

Proof. Let $(x_{ij}) \in {}_2\mathcal{X}^I_\nu(T)$ be a double I-Cauchy sequence with respect to the norm $\|.\|_\lambda$ and $\lambda \in (0,1)$ then there exist numbers $r, s \in \mathbb{N}$ and $N \in \mathcal{F}(I)$ such that for all $i, p \geq r$, $j, q \geq s$ and for all $t > 0$,

$$N = \{(i,j) : \nu(T(x_{ij} - x_{pq}), t) < 1 - \lambda\}. \tag{30}$$

Therefore we can write as,

$$\nu(T(x_{ij}) - T(x_{pq}), t) \leq \nu(T(x_{ij} - x), \frac{t}{2}) \diamond \nu(T(x_{ij} - x), \frac{t}{2}).$$

$$< (1-\lambda) \diamond (1-\lambda) = (1-\lambda)$$

which implies that (x_{ij}) is fuzzy I_λ-anti-quasi- Cauchy in ${}_2\mathcal{X}^I_\nu(T)$.
□

Theorem 2.13. *Let ${}_2\mathcal{X}^I_\nu(T)$ be a fuzzy antinormed double sequence space with respect to an idempotent t-conorm $\diamond$. If ${}_2\mathcal{X}^I_\nu(T)$ is fuzzy I_λ-anti-complete, then ${}_2\mathcal{X}^I_\nu(T)$ is I- complete with respect to $\|.\|_\lambda, \lambda \in (0,1)$.*

Proof. Let (x_{ij}) be fuzzy I_λ- anti-Cauchy sequence in ${}_2\mathcal{X}^I_\nu(T)$. As ${}_2\mathcal{X}^I_\nu(T)$ is fuzzy I_λ-anti-complete then fuzzy I_λ- anti-Cauchy sequence (x_{ij}) is fuzzy I_λ- anti-convergent to x (say). By Theorem (2.4), this means that $\|T(x_{ij} - x)\|_\lambda$ is convergent to 0; i.e., as T is a compact operator, (x_{ij}) is I_λ- is convergent to 0. Hence ${}_2\mathcal{X}^I_\nu(T)$ is I_λ-complete with respect to $\|.\|_\lambda$, $\lambda \in (0,1)$. Therefore $({}_2\mathcal{X}^I_\nu(T), \|.\|_\lambda)$ is I- complete.
□

Conclusion

The concept of fuzzy sets is well established as an important and practical construct for modeling. In the present paper we have studied the concept of fuzzy antinormed double sequence spaces with the help of an ideal and defined by a bounded linear operator and studied the fuzzy topology on the said spaces.

Conflict of interest

The authors declare that they have no competing interests.

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