Filomat 38:25 (2024), 8869–8876
https://doi.org/10.2298/FIL2425869K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Faber-hypercyclic semigroups of linear operators

Noureddine Karim[a], Otmane Benchiheb[a,*], Mohamed Amouch[a]

[a]*Chouaib Doukkali University. Department of Mathematics, Faculty of science El Jadida, Morocco*

Abstract. In this research, the Ω-hypercyclic and Ω-transitive behavior are studied within the framework of linear strongly continuous semigroups. We give sufficient constraints on the spectrum of an operator to yield a Ω-hypercyclic semigroup. Also, we establish necessary and sufficient conditions on the semigroup to be Ω-transitive.

1. Introduction and preliminary

In this study, X will be used to represent a complex topological vector space over the field $\mathbb{K}$, and the set of all continuous linear operators on X will be denoted by $\mathcal{B}(X)$. Operator refers to a continuous linear operator acting on X in the following. A pair (X, T) made up of an operator T on X and a complex topological vector space X is known as a linear dynamical system. The most studied property in a linear dynamical system (X, T) is its hypercyclic nature.

An operator $T \in \mathcal{B}(X)$ is said to be *hypercyclic* if there is a vector $x \in X$, called a hypercyclic vector for T, whose orbit beneath T;

$$\mathcal{O}(T, x) := \{T^n x : \ n \in \mathbb{N}\},$$

is dense in X. The operator T is said to be *supercyclic* if there is a vector $x \in X$, called supercyclic, whose projective orbit under T;

$$\mathbb{K}.\mathcal{O}(T, x) := \{\lambda T^n x : \ \lambda \in \mathbb{K}, n \in \mathbb{N}\},$$

is dense in X. These definitions were extended to C_0-semigroups of bounded linear operators, see [17, 22]

Recent years have seen a significant amount of research focused on these aspects of the dynamical system (X, T). For a discussion of the study findings in this area, we direct the reader to two books [4, 13] and the paper [16]. An operator T on a separable Banach space is hypercyclic if and only if it is topologically transitive, and this means that for every two nonempty and open subsets U, V of X, there is $n \in \mathbb{N}$ such that

$$T^n U \cap V \neq \emptyset.$$

This can be demonstrated using Baire's theorem [12, Theorem 1.2]. T is said to be recurrent if, for each nonempty open subset U of X, there exists $n \in \mathbb{N}$, such that

$$T^n U \cap U \neq \emptyset.$$

2020 *Mathematics Subject Classification*. Primary 47A16 mandatory; Secondary 47D03.
Keywords. C_0-semigroups, Ω-hypercyclic operators, Ω-transitive operators, C_0-semigroup of operators.
Received: 20 January 2024; Revised: 10 April 2024; Accepted: 15 May 2024
Communicated by Snežana Č. Živković-Zlatanović
* Corresponding author: Otmane Benchiheb
Email addresses: noureddinekarim1894@gmail.com (Noureddine Karim), otmane.benchiheb@gmail.com, benchiheb.o@ucd.ac.ma (Otmane Benchiheb), amouch.m@ucd.ac.ma (Mohamed Amouch)

This definition was extended to C_0-semigroups by M. Moosapoor in her paper [22].

In recent times, many researchers have delved into innovative ideas within the realm of linear dynamics. For instance, Moosapoor [21] introduced the concept of subspace-recurrent operators, while in another work [23], she proposed significant criteria related to subspace supercyclicity. Furthermore, her exploration in [20] scrutinized the subspace diskcyclic behavior within the framework of C_0-semigroups. The secend and the third authors of this paper study the super-recurrence of operators in [1], the subspece-super recurrence of operators in [6], and the supercyclicity of a set of operators in [2]. is a source of dynamics examples for classical operators.

The purpose of this paper is to study strongly continuous semigroups of operators in Banach spaces and the concept of Faber-hypercyclicity.

Recalling that a strongly continuous semigroup (also known as a C_0-semigroup) of operators in $\mathcal{B}(X)$ is a one-parameter family $\mathcal{T} = \{T_t\}_{t\geq 0}$ of continuous linear operators in $\mathcal{B}(X)$ such that

- $T_0 = I$;
- $T_t T_s = T_{t+s}$, for all $t, s \geq 0$,
- $\lim_{t\to s} T_t x = T_s x$, for all $s \geq 0$, $x \in X$.

A C_0-semigroup $\mathcal{T} = \{T_t\}_{t\geq 0}$ is considered hypercyclic if

$$\mathcal{O}(\mathcal{T}, x) := \{T_t x : \ t \geq 0\}$$

is dense in X for some $x \in X$.

In this case, x is called a hypercyclic vector for the semigroup $\mathcal{T}$.

Hypercyclic semigroups can only exist in separable Banach spaces because, for any hypercyclic vector x, $\{T_t x : \ t \in \mathbb{Q}, t \geq 0\}$ is a dense subset of X due to the strong continuity of the semigroup, see [15].

Desch, Schappacher, and Webb initiated the study of hypercyclic semigroups in [10].

Based on a study of the point spectra of the semigroup generator, they provided a sufficient criterion for a semigroup's hypercyclicity. Additionally, they described hypercyclic translation semigroups defined on weighted spaces of continuous or integrable functions on the real line. For more examples and properties of hypercyclic strongly continuous semigroups, see [7, 9, 11, 15, 25]. The importance of studying the dynamics of C_0-semigroups is their connection with the invariant subspace problem of Hilbert spaces see [5, 12, 14, 26] and also their relationship with partial differential equations, see [18, p. 297] and [19, p. 339].

To begin with, note that the unit disk $\mathbb{D}$ or the unit circle $\mathbb{T}$ are involved in numerous results in the spectral theory of hypercyclic operators. Since the iterates T^n coincide with $F_n^{\mathbb{D}}(T)$, where $F_n^{\mathbb{D}}(z) = z^n$ represents the basic Taylor polynomials associated to $\mathbb{D}$, Badea and Grivaux in [3] observed that the explanation for this frequent occurrence is that the unit disk is hidden in the definition of a hypercyclic operator. They then introduced the concept of Ω-hypercyclicity.

Let Ω be an open, non-empty connected subset of $\mathbb{C}$, having a rectifiable boundary $\partial\Omega$ and a compact closure symbolized by $\overline{\Omega}$. The Taylor polynomials of the disk naturally generalize to the Faber polynomials F_n^{Ω} associated with the domain Ω.

Faber polynomials are essential in many complicated approximation problems, as well as the theory of univalent functions in complex analysis. The Faber polynomials are the subject of extensive literature.

In the following, Ω will be considered a bounded domain of the complex plane, with a closed Jordan curve serving as its boundary $\partial\Omega = C$ and its complement $\overline{\Omega}^c$ being simply connected in the extended complex plane $\mathbb{C} \cup \{\infty\}$. According to the Riemann mapping theorem, there is a unique function

$$\psi : \overline{\mathbb{D}}^c \to \overline{\Omega}^c$$

that is meromorphic outside $\mathbb{D}$ and that maps $\overline{\mathbb{D}}^c$ conformally and univalently onto $\overline{\Omega}^c$. It is such that $\psi(\infty) = \infty$ and $\psi'(\infty) > 0$. For $|w| > 1$, the Laurent expansion of ψ takes the following form:

$$\psi(w) = aw + d_0 + d_1/w + d_2/w^2 + \cdots$$

in which $a > 0$. For any ψ, the inverse function ϕ maps $\overline{\Omega}^c$ conformally and univalently on $\overline{\mathbb{D}}^c$. ϕ has a Laurent expansion of the form

$$\phi(z) = (1/a)z + b_0 + b_1/z + b_2/z^2 + \cdots$$

in a neighborhood of ∞.

The polynomial component of the Laurent expansion of $\phi(z)^n$ at infinity for $n \geq 1$ is the n-th Faber polynomial F_n^Ω of the domain Ω. F_0^Ω is identically equal to 1. Instead of writing F_n^Ω, we typically write F_n when there is no chance of misunderstanding.

For examples of Faber polynomials, see [8].

When a bounded operator T on X has a vector x such that

$$\{F_n^\Omega(T)x : \ n \geq 0\}$$

is dense in X, it is called Ω-hypercyclic. Such a vector is a Ω-hypercyclic vector for T, while the set of such vectors is denoted by $HC_\Omega(T)$. When for every two non-empty open subsets $U, V \subset X$ we can find an integer n such that

$$F_n^\Omega(T)U \cap V \neq \emptyset,$$

then T is said to be Ω-transitive, for a general overview about these two notions see [3].

The majority of findings in the spectral theory of hypercyclic operators pertaining to the unit disk or circle, as it turns out, have analogs for Ω-hypercyclic operators that deal with the corresponding open domain Ω or its boundary.

For instance, in [3], Badea and Grivaux demonstrated that in the case of a Ω-hypercyclic operator $T \in \mathcal{B}(X)$, where Ω is a UB-domain, the point spectrum $\sigma_p(T^*)$ of its adjoint is empty. Also it was shown that if the boundary of Ω is a rectifiable Jordan curve, then every connected component of the spectrum $\sigma(T)$ of T meets the boundary of Ω.

Recall that a domain Ω is considered to be UB-domain if the Faber polynomials of Ω are uniformly bounded on $\overline{\Omega}$.

They also adapted the Godefroy-Shapiro Criterion to provide the following results: If Ω is a bounded domain with a rectifiable Jordan curve as its boundary and $T \in \mathcal{B}(X)$ a bounded operator on X, then T is Ω-hypercyclic if the two vector spaces that follow are dense in X:

$$X_1 = \text{span}\{\ker(T - zI) : \ z \in \overline{\Omega}^c\} \text{ and } X_2 = \text{span}\{\ker(T - zI) : z \in \Omega\}.$$

They then produced a number of instances of Ω-hypercyclic operators using this criterion:

- For the adjoint of the multiplication operator M_ϕ, induced by $\phi \in H^\infty(\mathbb{D})$ acting on Hardy space, they showed that for M_ϕ^* to be Ω-hypercyclic it must and is enough that $\overline{\phi(\mathbb{D})}$ reach the boundary of Ω.
- For the backward shift B on ℓ_p, $1 \leq p < +\infty$, or c_0, they proved that wB is Ω-hypercyclic for every complex number w with $|w| > d(0, \partial\Omega)$.

Here, we study Ω-transitivity and Ω-hypercyclicity in the framework of strongly continuous semigroups of bounded linear operators in Banach spaces. One can consider Ω-hypercyclicity and Ω-transitivity behavior in strongly continuous linear semigroups as the continuous-time counterpart of the previously discussed discrete-time case.

2. Ω-Hypercyclic semigroups

In the rest of the paper, let $\mathcal{T} = (T_t)_{t \geq 0}$ be a strongly continuous linear semigroup on a separable Banach space X.

Definition 2.1. *We say that a semigroup $\mathcal{T} \subset \mathcal{B}(X)$ is Ω-hypercyclic if there exists some $x \in X$ for which the set*

$$\{F_n^{\Omega}(T_t)x : \ n \in \mathbb{N}, t \geq 0\},$$

is dense in X. Such vector x is called a Ω-hypercyclic vector for $\mathcal{T}$ or a just Ω-hypercyclic vector. The set of all Ω-hypercyclic vectors for $\mathcal{T}$ will be denoted by $HC_{\Omega}(\mathcal{T})$.

Example 2.2. *Take $\Omega = \mathbb{D}$. Then each hypercyclic semigroup is Ω-hypercyclic.*

We denote by $\mathcal{T}'$ the set of all elements of $\mathcal{B}(X)$ that commute with every element of $\mathcal{T}$.

Proposition 2.3. *Let $\mathcal{T} \subset \mathcal{B}(X)$ be a Ω-hypercyclic strongly continuous linear semigroup on X and $T \in \mathcal{B}(X)$ be an operator with dense range. If $T \in \mathcal{T}'$, then $Tx \in HC_{\Omega}(\mathcal{T})$ for all $x \in HC_{\Omega}(\mathcal{T})$ with $Tx \neq 0$.*

Proof. Since $x \in HC_{\Omega}(\mathcal{T})$, we have that

$$\overline{\{F_n^{\Omega}(T_t)x : \ n \in \mathbb{N}, t \geq 0\}} = X.$$

Then,

$$\begin{aligned} X &= \overline{T(\overline{\{F_n^{\Omega}(T_t)x : \ n \in \mathbb{N}, t \geq 0\}})} \\ &\subset \overline{\overline{T(\{F_n^{\Omega}(T_t)x : \ n \in \mathbb{N}, t \geq 0\})}} \\ &= \overline{T(\{F_n^{\Omega}(T_t)x : \ n \in \mathbb{N}, t \geq 0\})} \\ &= \overline{\{F_n^{\Omega}(T_t)Tx : \ n \in \mathbb{N}, t \geq 0\}}, \end{aligned}$$

since $T \in \mathcal{T}'$. Hence, Tx is Ω-hypercyclic for $\mathcal{T}$. □

Corollary 2.4. *Let $\mathcal{T} \subset \mathcal{B}(X)$ be a Ω-hypercyclic strongly continuous linear semigroup on X. If $x \in HC_{\Omega}(\mathcal{T})$, then $ax \in HC_{\Omega}(\mathcal{T})$ for all $a \in \mathbb{C}\backslash\{0\}$.*

Proof. Let $a \in \mathbb{C}\backslash\{0\}$ and take $T = aI$, where I is identity operator of X. We have that $T \in \mathcal{T}'$. Then we apply Proposition 2.3 to conclude. □

Definition 2.5. *[24] Let $\mathcal{T} = (T_t)_{t\geq 0}$ and $\mathcal{S} = (S_t)_{t\geq 0}$ be two strongly continuous linear semigroup on X. $\mathcal{S}$ and $\mathcal{T}$ are called similar if there exists an isomorphism P on X such that*

$$T_t = P^{-1}S_tP, \text{ for all } t \geq 0.$$

Proposition 2.6. *Let $\mathcal{S}$ and $\mathcal{T}$ be two similar strongly continuous linear semigroups on X. If $\mathcal{S}$ is Ω-hypercyclic, then $\mathcal{T}$ is also Ω-hypercyclic.*

Proof. Let $y \in X$ be an arbitrary element. For $x = Py \in X$ and $\varepsilon > 0$, there exist $x_0 \in X$, $n \in \mathbb{N}$ and $t \geq 0$ such that

$$\|F_n^{\Omega}(S_t)x_0 - x\| \leq \varepsilon M^{-1},$$

where $M = \|P^{-1}\|$. Then, if $y_0 = P^{-1}x_0$ we have

$$\|F_n^{\Omega}(T_t)y_0 - y\| \leq M\|F_n^{\Omega}(S_t)x_0 - x\| < \varepsilon.$$

□

By A, we denote the infinitesimal generator of $\mathcal{T}$. We will consider the following subsets of X:
$X_0 := \{x \in X : \ \lim_{n\to\infty} \lim_{t\to\infty} F_n^{\Omega}(T_t)x = 0\}$.
$X_\infty := \{x \in X : \ \text{for each } \varepsilon > 0 \text{ there exist some } w \in X, n \in \mathbb{N} \text{ and some } t > 0 \text{ with } \|w\| < \varepsilon \text{ and } \|F_n^{\Omega}(T_t)w - x\| < \varepsilon\}$.

Proposition 2.7. *Assume that $\mathcal{T}$ and $\mathcal{S}$ are two C_0-semigroups acting on X and Y respectively. If the direct sum of $\mathcal{T}$ and $\mathcal{S}$ is Ω-hypercyclic on $X \oplus Y$, then $\mathcal{T}$ and $\mathcal{S}$ are Ω-hypercyclic on X and Y respectively.*

Proof. Let $x \oplus y$ be an Ω-hypercyclic vector for $\mathcal{T} \oplus \mathcal{S}$. Note that

$$\{F_n^{\Omega}(T_t \oplus S_t)(x \oplus y) : \ n \in \mathbb{N},\ t \geq 0\} \subseteq \{F_n^{\Omega}(T_t)x : \ n \in \mathbb{N},\ t \geq 0\} \oplus \{F_n^{\Omega}(S_t)y : \ n \in \mathbb{N},\ t \geq 0\}.$$

This implies that x and y are Ω-hypercyclic vectors for $\mathcal{T}$ and $\mathcal{S}$ respectively. □

Definition 2.8. *A semigroup $\mathcal{T} \subset \mathcal{B}(X)$ is Ω-transitive, if for every two non empty open subsets U, V of X, there exists an integer n and a nonnegative real number t such that*

$$F_n^{\Omega}(T_t)U \cap V \neq \emptyset.$$

Theorem 2.9. *The following assertions are equivalent:*

1. *$HC_{\Omega}(\mathcal{T})$ dense in X;*
2. *$\mathcal{T}$ is Ω-transitive.*

Proof. (1) ⇒ (2) : Let U, V be two nonempty open subsets of X, since $HC_{\Omega}(\mathcal{T})$ dense, we have that

$$HC_{\Omega}(\mathcal{T}) \cap U \neq \emptyset.$$

Let $x \in U$ such that $\{F_n^{\Omega}(T_t)x : \ n \in \mathbb{N}, t \geq 0\}$ is dense in X. Thus, $\{F_n^{\Omega}(T_t)x : \ n \in \mathbb{N}, t \geq 0\}$ intersects V, then there are $n \in \mathbb{N}$ and a $t \geq 0$ such that $F_n^{\Omega}(T_t)x \in V$, but $x \in U$, which implies that $F_n^{\Omega}(T_t)x \in F_n(T_t)U \cap V$. Hence, $\mathcal{T}$ is really Ω-transitive.
(2) ⇒ (1) : Let $(U_n)_{n\in\mathbb{N}}$ be a basis of open subsets of X. Then,

$$HC_{\Omega}(\mathcal{T}) = \cap_{i\in\mathbb{N}} \cup_{n\in\mathbb{N}} \cup_{t\geq 0} F_n^{\Omega}(T_t)^{-1}(U_i).$$

Since $\mathcal{T}$ is supposed to be Ω-transitive, for every V a nonempty open set, there exist an integer n and a positive number t such that $F_n^{\Omega}(T_t)V \cap U_i \neq \emptyset$, for every $i \in \mathbb{N}$. Thus each open set $W_i := \cup_{n\in\mathbb{N}} \cup_{t\geq 0} F_n^{\Omega}(T_t)^{-1}(U_i)$ is dense in X, it follows by the Baire category theorem that $\cap_{i\in\mathbb{N}} W_i = HC_{\Omega}(\mathcal{T})$ is dense in X. □

Theorem 2.10. *The following assertions are equivalent:*

1. *$\mathcal{T}$ is Ω-transitive;*
2. *For each $x, y \in X$, there exist sequences $\{x_k\}$ in X, $\{n_k\}$ in $\mathbb{N}$ and $\{t_k\}$ in $\mathbb{R}_+$ such that*

$$x_k \to x \text{ and } F_{n_k}^{\Omega}(T_{t_k})x_k \to y.$$

3. *For each $x, y \in X$, and for W a neighborhood of the origin, there exist sequences $z \in X$, $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$ such that*

$$x - z \in W \text{ and } F_n^{\Omega}(T_t)z - y \in W.$$

Proof. (1) ⇒ (2) : Let $x, y \in X$. For all $k \geq 1$, let $U_k = B(y, \frac{1}{k})$ and $V_k = B(z, \frac{1}{k})$. Then U_k and V_k are nonempty open subsets of X. Since is Ω-transitive, there exist $\{n_k\} \subset \mathbb{N}$ and $\{t_k\} \subset [0, +\infty)$ with $F_{n_k}^{\Omega}(T_{t_k})(U_k) \cap V_k \neq \emptyset$. Let $x_k \in U_k$ such that $F_{n_k}^{\Omega}(T_{t_k})x_k \in V_k$, then

$$\|x_k - x\| < \frac{1}{k} \text{ and } \|F_{n_k}(T_{t_k})x_k - y\| < \frac{1}{k},$$

which implies that

$$x_k \to x \text{ and } F_{n_k}^{\Omega}(T_{t_k})x_k \to y.$$

(2) → (3) : Let $x, y \in X$, then there exists sequences $\{x_k\}$ in X and $\{n_k\}$ in $\mathbb{N}$ and $\{t_k\}$ of $\mathbb{R}_+$ such that

$$x_k - x \to 0 \text{ and } F_{n_k}^{\Omega}(T_{t_k})(x_k) - y \to 0.$$

If W is a neighborhood of 0, then there exists $N \in \mathbb{N}$, such that $x_k - x \in W$ and $F^{\Omega}_{n_k}(T_{t_k})(x_k) - y \in W$ for all $k \geq N$.
(3) → (1) : Let U and V be two nonempty open subsets of X. Then there exist $x, y \in X$ such that $x \in U$ and $y \in V$. Since for all $k \geq 1$, $W_k = B(0, \frac{1}{k})$ is neighborhood of 0, there exist $z_k \in X$ and $\{n_k \in \mathbb{N}$ and $t_k \in \mathbb{R}_+$ such that

$$\|F^{\Omega}_{n_k}(T_{t_k})z_k - y\| < \frac{1}{k} \text{ and } \|z_k - x\| < \frac{1}{k}.$$

This implies that $z_k \to x$ and $F^{\Omega}_{n_k}(T_{t_k})z_k \to y$. Then there exists $N \in \mathbb{N}$ such that $z_k \in U \cap F^{\Omega}_{n_k}(T_{t_k})V$, for all $k \geq \mathbb{N}$. □

Theorem 2.11. *The following assertions are equivalent:*

1. $\mathcal{T}$ *is* Ω*-hypercyclic;*
2. *for all* $y \in X$, $z \in X$ *and all* $\varepsilon > 0$ *there exist some* $v \in X$, $n \in \mathbb{N}$ *and some* $t > 0$ *such that* $\|y - v\| < \varepsilon$ *and* $\|z - F^{\Omega}_n(T_t)v\| < \varepsilon$;
3. *for all* $\varepsilon > 0$ *there exists a dense subset* $D \subset X$ *such that for all* $z \in D$ *there exists a dense subset* $D' \subset X$ *such that for all* $y \in D'$ *there exist* $v \in X$, $n \in \mathbb{N}$ *and* $t > 0$ *such that* $\|y - v\| < \varepsilon$ *and* $\|z - F^{\Omega}_n(T_t)v\| < \varepsilon$.

Proof. (1) ⇒ (2) : Let $y, z \in X$, and $\varepsilon > 0$, then $U := B(x, \varepsilon)$ and $V := B(y, \varepsilon)$ are two nonempty open sets. Since T is Ω-transitive there exist $n \in \mathbb{N}$ and $t \geq 0$ such that $F^{\Omega}_n(T_t)U \cap V \neq \emptyset$. Thus, there is $v \in X$ such that $v \in U$ and $F^{\Omega}_n(T_t)v \in V$, which means that

$$\|y - v\| < \varepsilon \text{ and } \|z - F_n(T_t)v\| < \varepsilon.$$

(2) ⇒ (1) : Let $\{z_1, z_2, z_3, ...\}$ be a dense sequence in X. We shall construct sequences $\{y_1, y_2, y_3, ...\} \subset X$, $\{n_1, n_2, n_3, ...\} \subset \mathbb{N}$ and $\{t_1, t_2, t_3, ...\} \subset [0, +\infty)$ inductively:

- let $y_1 = z_1$, $n_1 = t_1 = 0$;
- for $i > 1$ we find y_i, n_i and t_i such that

$$\|y_i - y_{i-1}\| \leq \frac{2^{-i}}{\sup\{\|F^{\Omega}_{n_j}(T_{t_j})\| : \ j < i\}}, \tag{1}$$

$$\|z_i - F^{\Omega}_{n_i}(T_{t_i})y_i\| \leq 2^{-i}. \tag{2}$$

In particular, 1 implies $\|y_i - y_{i-1}\| \leq 2^{-i}$, so that the sequence y_i has a limit x. Applying 2 and once again 1, we infer that

$$\begin{aligned}\|z_i - F^{\Omega}_{n_i}(T_{t_i})x\| &\leq \|z_i - F_{n_i}(T_{t_i})y_i\| + \|F_{n_i}(T_{t_i})\|\|y_i - x\| \\ &\leq \|z_i - F^{\Omega}_{n_i}(T_{t_i})y_i\| + \sum_{j=i+1}^{\infty} \|F^{\Omega}_{n_i}(T_{t_i})\|\|y_j - y_{j-1}\| \\ &\leq 2^{-i} + \sum_{j=i+1}^{\infty} 2^{-j} = 2^{-i+1}.\end{aligned}$$

Given $z \in X$ and $\varepsilon > 0$ there are arbitrarily large n such that $\|z_i - z\| < \varepsilon/2$. Choosing n large enough such that $2^{-i+1} < \varepsilon/2$, we obtain

$$\|F^{\Omega}_{n_i}(T_{t_i})x - z\| \leq \|z - z_i\| + \|z_i - F^{\Omega}_{n_i}(T_{t_i})x\| < \varepsilon.$$

Therefore, $\{F^{\Omega}_n(T_t)x : \ n \in \mathbb{N}, t \geq 0\}$ is dense.
(2) ⇒ (3) : Put $D = D' = X$.
(3) ⇒ (2) : Let $\varepsilon > 0$ and $z \in X$. Pick $\tilde{z} \in D$ such that $\|z - \tilde{z}\| < \varepsilon/2$. Then specify D' according to (3) with $\tilde{z}$ instead of z and $\varepsilon/2$ instead of ε. For $y \in X$ pick $\tilde{y} \in D'$ with $\|\tilde{y} - y\| < \varepsilon/2$. Finally, we choose n, t and v according to (3) with $\varepsilon/2$, $\tilde{y}, \tilde{z}$ instead of ε, y, z and obtain

$$\|F^{\Omega}_n(T_t)v - z\| \leq \|F^{\Omega}_n(T_t)v - \tilde{z}\| + \|\tilde{z} - z\| < \varepsilon,$$

$$\|v - y\| \leq \|v - \tilde{y}\| + \|\tilde{y} - y\| < \varepsilon.$$

□

Theorem 2.12. *If both X_∞ and X_0 are both dense in X, then $\mathcal{T}$ is Ω-hypercyclic.*

Proof. We use Theorem 2.11 (3) with $D = X_\infty$ and $D' = X_0$. Let $z \in X_\infty$ and $y \in X_0$. Then for each $\varepsilon > 0$ there are arbitrarily large $n \in \mathbb{N}$, $t > 0$ and $w \in X$ such that

$$||F_n^{\Omega}(T_t)w - z|| < \frac{\varepsilon}{2} \text{ and } ||w|| < \varepsilon.$$

Since $y \in X_0$, for sufficiently large n and t we have $||F_n^{\Omega}(T_t)y|| < \varepsilon/2$. We put $v = y + w$, then

$$||z - F_n^{\Omega}(T_t)v|| \leq ||z - F_n^{\Omega}(T_t)w|| + ||F_n^{\Omega}(T_t)y|| < \varepsilon,$$

$$||y - v|| = ||w|| < \varepsilon.$$

□

The following necessary condition on the spectrum of the generator $A : D(A) \to X$ was provided by Desch, Schappacher, and Webb [10] for the semigroup to be hypercyclic:

(DSW) For some open subset U of the point spectrum $\sigma_p(A)$ of A intersecting the imaginary axis, there exist eigenvectors x_λ corresponding to $\lambda \in U$ such that for each $\phi \in X\backslash\{0\}$ the mapping $F_\phi(\lambda) = \phi(x_\lambda)$ is holomorphic on U and does not vanish identically.

Theorem 2.13. *A strongly continuous semigroup $\mathcal{T}$ on a separable Banach space is hypercyclic whenever its generator A satisfies (DSW).*

Kalmes [15] showed that if the (DSW) is satisfied, then every operator $T_t \in \mathcal{T}$ is hypercyclic. In the following, we extend this result and give an explanation to why exactly the imaginary axis appeared in [10, Theorem 3.1].

Theorem 2.14. *Let A be the infinitesimal generator of $\mathcal{T}$. Let U be an open subset of $\sigma_p(A)$, the point spectrum of A, such that $U \cap \{\lambda \in \mathbb{C} : \ e^{\lambda t_0} \in \partial\Omega\} \neq \emptyset$ for some $t_0 > 0$, and for each $\lambda \in U$ let x_λ be a nonzero eigenvector. For each $\phi \in X^*$ we define a function $F_\phi : U \to \mathbb{C}$ by $F_\phi(\lambda) = \langle \phi, x_\lambda \rangle$. Assume that for each $\phi \in X^*$ the function F_ϕ is analytic and that F_ϕ does not vanish identically on U unless $\phi = 0$. Then $\mathcal{T}$ is Ω-hypercyclic.*

Proof. We start with the statement, if $V \subset U$ is a non-empty open subset admitting a cluster point in U, then the set

$$Y_V := \{x_\lambda : \ \lambda \in V\}$$

is dense in X. Indeed, if $\phi \in X'$ is such that $\phi(x_\lambda) = 0$ for every $\lambda \in V$, then $F_\phi(\lambda) = 0$, it follows that the holomorphic function F_ϕ vanishes identically on U, since V has accumulation points in U. By hypothesis, this implies ϕ is null so that from the Hahn-Banach theorem, we obtain the density of Y_V in X.

Now we shall show that $X_1 := sev\{ker(T - zI) : \ z \in \Omega\}$ and $X_2 := sev\{ker(T - zI) : \ z \in \overline{\Omega}^c\}$ are dense. Put $\phi(z) = e^{z.t_0}$. Let V_1 be an open subset of $\{\lambda \in U : \ \phi(\lambda) \in \Omega\}$ which admits a cluster point in U, and V_2 be an open subset of $\{\lambda \in U : \ \phi(\lambda) \in \overline{\Omega}^c\}$ which admits a cluster point in U, this is possible by the open mapping theorem for the analytic function ϕ. The set Y_{V_1} is a subset of X_1 and Y_{V_2} is a subset of X_2. As Y_{V_1} and Y_{V_2} are dense, so are X_1 and X_2. Then the operator T_{t_0} is Ω-hypercyclic, which implies the Ω-hypercyclicity of $\mathcal{T}$ as a C_0-semigroup. □

3. Questions

1. Does the converse of Proposition 2.7 hold? In other words, does the direct sum of two Ω-hypercyclic strongly continuous semigroups remain Ω-hypercyclic?
2. Is Ω-hypercyclicity of a strongly continuous semigroup, implies that any of its operators is Ω-hypercyclic?

Acknowledgment. The authors are sincerely grateful to the handling editor and anonymous referee for their careful reading, critical comments, and valuable suggestions that significantly contributed to improving the manuscript during the revision.

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