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On strongly quasipolar rings

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Abstract. Idempotent elements, invertible elements and quasinilpotents are some important tools to study structures of rings. By using these kinds of elements, we study a class of rings, called strongly quasipolar rings, which is a subclass of that of quasipolar rings. Let R be a ring with identity. An element $a \in R$ is said to be strongly quasipolar if there exists $p^2 = p \in comm^2(a)$ such that $a + p$ is invertible and a^2p is quasinilpotent. The ring R is called strongly quasipolar in case each of its elements is strongly quasipolar. Some basic properties of the strongly quasipolar rings are obtained. The class of strongly quasipolar rings lies properly between the classes of pseudopolar rings and quasipolar rings. We determine the conditions under which a quasipolar ring is strongly quasipolar. We also show that strongly quasipolarity is a generalization of uniquely cleanness. When we consider this concept in terms of generalized inverses, we get that every pseudo Drazin invertible element is strongly quasipolar, and every strongly quasipolar element is generalized Drazin invertible.

1. Introduction

Throughout this paper, all rings are associative with identity. For a ring R, $\mathrm{Id}(R)$, $U(R)$, $\mathrm{nil}(R)$, $C(R)$ and $J(R)$, denote the set of all idempotents, the set of all invertible elements, the set of all nilpotent elements, the center of R and the Jacobson radical of R, respectively. Let R be a ring and $a \in R$. The *commutant* of a is defined by $comm(a) = \{x \in R \mid ax = xa\}$ and the *double commutant* of a is defined by $comm^2(a) = \{x \in R \mid xy = yx \text{ for all } y \in comm(a)\}$. Due to Harte [26], an element $a \in R$ is called quasinilpotent if $1 - ax \in U(R)$ for every $x \in comm(a)$ and the set of all quasinilpotents of R is denoted by R^{qnil}. It is clear that $\mathrm{nil}(R)$ and $J(R)$ are contained in R^{qnil}.

Koliha and Patricio in [28] introduced quasipolarity of rings. An element a of a ring R is called *quasipolar* provided that there exists $p \in \mathrm{Id}(R)$ such that $p \in comm^2(a)$, $a + p \in U(R)$ and $ap \in R^{qnil}$ where p is said to be *spectral idempotent of a* and denoted by a^π. A ring R is said to be *quasipolar* in case every element in R is quasipolar. As is well-known, a ring R is quasipolar if and only if for any $a \in R$, there exists $x \in comm^2(a)$ such that $x = xax$ and $a - a^2x \in R^{qnil}$ (see [28, Theorem 4.2]), that is, a is generalized Drazin invertible.

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In [35], pseudo Drazin invertible elements and pseudopolar rings are defined and investigated. Let R be a ring. Then $a \in R$ is called *pseudo Drazin invertible* if there exists $x \in comm^2(a)$ such that $xax = x$, $a^k - a^{k+1}x \in J(R)$ for some $k \geq 1$, and $a \in R$ is said to be *pseudopolar* if there exists $p \in R$ such that $p^2 = p \in comm^2(a)$, $a + p \in U(R)$ and $a^k p \in J(R)$ for some $k \geq 1$. It is proved in [35, Theorem 3.2] that a is pseudo Drazin invertible if and only if a is pseudopolar. Also, every pseudopolar element is quasipolar. In [32], Mosic introduced the concept of the pseudo n-strong Drazin inverse which generalizes the concept of pseudo Drazin inverses, and Cui, Danchev and Zeng continued to study this concept in [15].

The notion of uniquely clean ring was defined in [1] for the commutative case, and in [33] for the noncommutative case, also studied in [6, 11]. An element a in a ring R is called *uniquely clean* provided that there exists a unique idempotent $e \in R$ such that $a - e \in U(R)$. A ring R is said to be *uniquely clean* in case every element in R is uniquely clean. It is known by [36, Corollary 3.3] that every uniquely clean ring is quasipolar. Furthermore, a ring is said to be *uniquely strongly clean* if every element is uniquely the sum of an idempotent and a unit that commute. By [11, Example 4], uniquely clean rings are uniquely strongly clean. Also by [35, Corollary 2.8], uniquely strongly clean rings are pseudopolar.

It is seen that units, idempotents and quasinilpotents in rings are key elements determining the structure of the rings. Motivated by the aforementioned notions in ring theory, in this paper, we introduce the notion of strongly quasipolar rings and connect it to these notions. We observe that the class of strongly quasipolar rings is located between the classes of pseudopolar rings and quasipolar rings. By the perspective of generalized inverses, we obtain that pseudo Drazin invertible elements are strongly quasipolar and strongly quasipolar elements are generalized Drazin invertible. We also introduce *rings with qnil-units*, shortly, *qnilU* rings, that is, a ring R is said to be *qnilU* if $U(R) = 1 + R^{qnil}$, equivalently, $R^{qnil} = \{r \in R \mid 1 + r \in U(R)\}$. This notion is also introduced and studied in [18] independently, under the name UQ rings. We prove that quasipolarity and strongly quasipolarity coincide for *qnilU* rings. Moreover, for a ring R, we prove that if R^{qnil} is contained in $\mathrm{nil}(R)$ or $J(R)$ or $C(R)$, then quasipolarity and strongly quasipolarity coincide. Some characterizations of strongly quasipolar elements are given. Various basic properties of strongly quasipolar elements are proved and some examples are given to illustrate the obtained results.

Throughout this paper, $M_n(R)$ and $U_n(R)$ denote the ring of all $n \times n$ matrices and the ring of all $n \times n$ upper triangular matrices over R, respectively. Also, $D_n(R)$ stands for the subring of $U_n(R)$ having all diagonal entries are equal and $V_n(R) = \{(a_{ij}) \in D_n(R) \mid a_{ij} = a_{(i+1)(j+1)}$ for $i = 1, \ldots, n-2$ and $j = 2, \ldots, n-1\}$ is a subring of $D_n(R)$. In what follows, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ denote the ring of integers, the ring of rational numbers and the ring of real numbers and for a positive integer n, $\mathbb{Z}_n$ is the ring of integers modulo n, and $\mathbb{Z}_{(2)}$ is the ring of the localization of $\mathbb{Z}$ at 2.

2. Quasinilpotent elements in rings

In this section we summarize some properties of quasinilpotent elements in rings, so that they may be easily referenced in the paper. It is noted in [9, 24, 26] that quasinilpotents play an important role in Banach algebras and ring theory. Quasinilpotents ensued from Banach algebras. Indeed, for a Banach algebra $\mathcal{B}$ (see [25, Page 251]), $a \in \mathcal{B}^{qnil}$ if and only if $\lim_{n\to\infty} \| a^n \|^{1/n} = 0$ if and only if $a - x \in U(\mathcal{B})$ for all complex number x.

Proposition 2.1. *Let R be a ring. Then the following hold.*

(1) (i) *For any positive integer n, if* $a^n \in R^{qnil}$, *then* $a \in R^{qnil}$.
(ii) *If* $a \in R^{qnil}$ *and* $c \in C(R)$, *then* $ac \in R^{qnil}$ ([12, Proposition 2.7]).
(2) *R is a local ring if and only if* $R = U(R) \cup R^{qnil}$ ([12, Theorem 3.2]).
(3) *Let* $a, b \in R$. *Then* $ab \in R^{qnil}$ *if and only if* $ba \in R^{qnil}$ ([31, Lemma 2.2]).
(4) *Let* $a \in R^{qnil}$ *and* $r \in U(R)$. *Then* $r^{-1}ar \in R^{qnil}$ ([12, Lemma 2.3]).
(5) *Let D be a division ring and* $R = M_n(D)$. *Then* $R^{qnil} = nil(R)$ ([12, Example 2.2]).

Corollary 2.2. [30, Corollary 2.2] *Let R be a ring and* $e \in Id(R)$. *Then* $xe - exe, ex - exe \in R^{qnil}$ *for each* $x \in R$.

Proposition 2.3. [36, Lemma 3.5] *Let e be an idempotent in a ring R. Then* $(eRe)^{qnil} = (eRe) \cap R^{qnil} \subseteq eR^{qnil}e$.

Proposition 2.4. *Let $\{R_i\}_{i\in I}$ be a class of rings with index set I. Then the following hold.*

(1) $\prod_{i\in I} U(R_i) = U\big(\prod_{i\in I} R_i\big)$.

(2) $\prod_{i\in I} R_i^{qnil} = \big(\prod_{i\in I} R_i\big)^{qnil}$.

Proof. (1) It is well known result.
(2) It is proved in [30, Proposition 2.4]. □

For $a \in R$, l_a and r_a will denote the abelian group endomorphisms of R given by left and right multiplication by a, respectively. So they are defined by $l_a(r) = ar$ and $r_a(r) = ra$ where $r \in R$. According to [3], a ring R is called *uniquely bleached* if for $j \in J(R)$ and $u \in U(R)$, the abelian group endomorphisms $l_u - r_j$ and $l_j - r_u$ of R are isomorphisms.

Remark 2.5. [3, Remark 11] Over a commutative local ring R, for $u \in U(R)$, $j \in J(R)$, the maps $l_u - r_j$ and $l_j - r_u$ are injective as well as surjective, i.e., commutative local rings are uniquely bleached.

Theorem 2.6. [30, Theorem 2.5] *Let R be a ring. Then the following hold.*

(1) (i) $T = \left\{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, c \in R^{qnil}, b \in R\right\} \subseteq U_2(R)^{qnil}$. *The reverse inclusion holds if for any* $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in U_2(R)^{qnil}$ *and any* $x \in comm(a)$ *and* $y \in comm(c)$, *we have* $b \in Ker(l_x - r_y)$.
(ii) *If R is a division ring, then* $U_2(R)^{qnil} = nil(U_2(R))$.

(2) (i) $K = \left\{\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a \in R^{qnil}, b \in R\right\} \subseteq D_2(R)^{qnil}$. *The reverse inclusion holds if for any* $A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \in D_2(R)^{qnil}$ *and any* $x \in comm(a)$, *we have* $b \in Ker(l_x - r_x)$.
(ii) *If R is a division ring, then* $D_2(R)^{qnil} = nil(D_2(R))$.

(3) *Let* $S = \left\{\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \mid a \in R\right\}$ *and consider the isomorphism* $f : R \to S$ *defined by* $f(a) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$. *Then* $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \in D_2(R)^{qnil}$ *if and only if* $a \in R^{qnil}$.

Let R be a ring and S a subring of R with the same identity as that of R and

$$T[R, S] = \{(r_1, r_2, r_3, \dots, r_n, s, s, s, \dots) : r_i \in R, s \in S, n \geq 1, 1 \leq i \leq n\}.$$

Then $T[R, S]$ is a ring under the componentwise addition and multiplication. Note that $\mathrm{nil}(T[R, S]) = T[\mathrm{nil}(R), \mathrm{nil}(S)]$, $J(T[R, S]) = T[J(R), J(R) \cap J(S)]$ and $C(T[R, S]) = T[C(R), C(R) \cap C(S)]$.

Proposition 2.7. [30, Proposition 2.8] *For the ring* $T[R, S]$, *the following hold.*

(1) *If* $A = (a_1, a_2, a_3, \dots, a_n, s, s, s, \dots) \in T[R, S]^{qnil}$, *then* $a_i \in R^{qnil}$ *for* $i = 1, 2, 3, \dots, n$ *and* $s \in S^{qnil}$.
(2) *If* $S^{qnil} \subseteq R^{qnil}$ *and* $a \in R^{qnil}$ *and* $s \in S^{qnil}$, *then* $A = (a, s, s, s, \dots) \in T[R, S]^{qnil}$.

Let R be an algebra over a commutative ring S. Following Dorroh [21], the *Dorroh extension* (or *ideal extension*) of R by S denoted by $I(R, S)$ is the direct product $R \times S$ with the usual addition and multiplication defined by $(a_1, b_1)(a_2, b_2) = (a_1a_2 + b_1a_2 + a_1b_2, b_1b_2)$ for $a_1, a_2 \in R$ and $b_1, b_2 \in S$.

Lemma 2.8. [30, Lemma 2.6] *Let* $I(R, S)$ *be an ideal extension of R by S. Then the following hold.*

(1) $(r, s) \in I(R, S)$ *is invertible with inverse* (u, v) *if and only if* $r + s$ *and* s *are invertible with inverse* $u + v$ *and* v, *respectively.*
(2) $(R, 0)^{qnil} = (R, 0) \cap I(R, S)^{qnil}$.
(3) $(0, S) \cap I(R, S)^{qnil} \subseteq (0, S)^{qnil}$.
(4) $(e, f) \in I(R, S)$ *is idempotent in* $I(R, S)$ *if and only if* $e + f$ *and* f *are idempotent in R.*
(5) *If* $U(R) = 1 + R^{qnil}$ *and* $S^{qnil} \subseteq R^{qnil}$, *then* $(a, b) \in I(R, S)^{qnil}$ *implies* $a \in R^{qnil}$ *and* $b \in S^{qnil}$.

3. Strongly quasipolar rings

In this section, we introduce strongly quasipolarity which is a stronger concept than the quasipolarity for rings. The aim of this section is to provide some properties of strongly quasipolar rings. We also investigate the conditions under which quasipolarity and strongly quasipolarity coincide. We begin with the main definition.

Definition 3.1. Let R be a ring and $a \in R$. Then a is called *strongly quasipolar* if there exists $p^2 = p \in comm^2(a)$ such that $a + p \in U(R)$, $a^2p \in R^{qnil}$, and p is called *strongly spectral idempotent of a* which is denoted by $a^{s\pi}$. A ring R is said to be *strongly quasipolar* in case every element in R is strongly quasipolar.

In the next result which is an analog of a well-known result [28, Proposition 2.3], we show the uniqueness of strongly spectral idempotent.

Proposition 3.2. *Let R be a ring and $a \in R$ strongly quasipolar. Then a has a unique strongly spectral idempotent.*

Proof. Let p and q be strongly spectral idempotents of a. Then $(a + p)^2, (a + q)^2 \in U(R)$. On the one hand, we have

$$1 - (1 - p)q = 1 - (1 - p)(a + p)^{-2}(a + p)^2q = 1 - (1 - p)(a + p)^{-2}a^2q.$$

Having $(1 - p)(a + p)^{-2} \in comm(a^2q)$ and $a^2q \in R^{qnil}$ imply $1 - (1 - p)q \in U(R)$. Note that $1 - (1 - p)q = (1 - (1 - p)q)(1 + (1 - p)q)$. This yields $1 = 1 + (1 - p)q$, and so $q = pq$. On the other hand, we obtain $p = qp$ in a similar way. Since $p, q \in comm^2(a)$, we have $pq = qp$. Therefore $p = q$. □

Proposition 3.3. *The elements of the following subsets of a ring R are strongly quasipolar:*

$$(1)\ \mathrm{Id}(R),\ (2)\ U(R),\ (3)\ \mathrm{nil}(R),\ (4)\ J(R).$$

Proof. (1) Let $e \in \mathrm{Id}(R)$ and $p^2 = p = 1 - e \in comm^2(e)$. Then $e + p \in U(R)$ and $e^2p \in R^{qnil}$.
(2) Let $u \in U(R)$ and $p^2 = p = 0 \in comm^2(u)$. Then $u + p \in U(R)$ and $u^2p \in R^{qnil}$.
(3) Let $n \in \mathrm{nil}(R)$, $p^2 = p = 1$ and $n^t = 0$ for some $t \geq 2$. We may assume that $t = 2k$ for some $k \in \mathbb{Z}^+$. Then $n + p \in U(R)$ and $((np)^2)^k = 0$. By Proposition 2.1(1)(i), $n^2p \in R^{qnil}$.
(4) Let $a \in J(R)$. Then $a^2 \in J(R)$. Since $J(R) \subseteq R^{qnil}$, $a^2 \in R^{qnil}$. The rest is clear. □

Example 3.4. The ring $M_2(\mathbb{Z}_2)$ is strongly quasipolar since

$$M_2(\mathbb{Z}_2) = U(M_2(\mathbb{Z}_2)) \cup \ \mathrm{nil}(M_2(\mathbb{Z}_2)) \cup \ \mathrm{Id}(M_2(\mathbb{Z}_2)).$$

Corollary 3.5. *Let R be a ring, $a \in R$ and $e^2 = e \in Id(R)$. Then $ea - eae, ae - eae, e + ea - eae, e + ae - eae, 1 + ea - eae, 1 + ae - eae$ are strongly quasipolar elements of R.*

Proof. Let $e^2 = e,\ a \in R$. Then $ea - eae, ae - eae$ are nilpotents and $1 + ea - eae$ and $1 + ae - eae$ are units, and so by Proposition 3.3, they are strongly quasipolar. Also $e + ea - eae,\ e + ae - eae$ are idempotents so by Proposition 3.3 (1), they are strongly quasipolar. □

The following result is clear by definitions.

Lemma 3.6. *Let R be a ring and $a \in R$. Then the following hold.*

(1) *$a \in R^{qnil}$ if and only if a is quasipolar and $a^{\pi} = 1$.*
(2) *$a^2 \in R^{qnil}$ if and only if a is strongly quasipolar and $a^{s\pi} = 1$.*

The next result provides another source of examples for strongly quasipolar rings.

Examples 3.7. (1) Every Boolean ring is strongly quasipolar.
(2) Every local ring is strongly quasipolar.

Proof. (1) Clear by Proposition 3.3.
(2) Let R be a local ring. It is well-known that $R = U(R) \cup J(R)$. The rest is clear by Proposition 3.3. □

We now present that the strongly quasipolarity is a stronger concept than the quasipolarity.

Theorem 3.8. *Every strongly quasipolar element is quasipolar.*

Proof. Let $a \in R$ be strongly quasipolar. Then there exists $p^2 = p \in comm^2(a)$ such that $a + p \in U(R)$ and $a^2p \in R^{qnil}$. Then we have $a^2p = (ap)^2 \in R^{qnil}$, so by Proposition 2.1(1)(i), $ap \in R^{qnil}$. □

The next results are immediate consequences of Theorem 3.8.

Corollary 3.9. *Let R be a ring and $a \in R$ strongly quasipolar. Then the strongly spectral idempotent of a is also spectral idempotent.*

Corollary 3.10. *Every strongly quasipolar ring is quasipolar.*

Despite all our efforts, we have not succeeded in giving an example to show that quasipolar rings are not strongly quasipolar, but we determine under what conditions the strongly quasipolarity and the quasipolarity are the same, as follows.

Theorem 3.11. *Let R be a quasipolar ring. Then the following hold.*

(1) *If $R^{qnil} \subseteq nil(R)$, then R is strongly quasipolar.*
(2) *If $R^{qnil} \subseteq J(R)$, then R is strongly quasipolar.*
(3) *If $R^{qnil} \subseteq C(R)$, then R is strongly quasipolar.*

Proof. Assume that R is quasipolar and let $a \in R$. Then there exists $p^2 = p \in comm^2(a)$ such that $a + p \in U(R)$ and $ap \in R^{qnil}$.
(1) By hypothesis, $(ap)^k = a^kp = 0$ for some $k \geq 1$. Note that $ap = pa$. We have $a^2p \in \mathrm{nil}(R)$ and so $a^2p \in R^{qnil}$ since $\mathrm{nil}(R) \subseteq R^{qnil}$. Therefore, R is strongly quasipolar.
(2) We have $a^2p = (ap)(ap) \in J(R) = R^{qnil}$. Hence R is strongly quasipolar.
(3) By (2) and [12, Proposition 2.6]. □

To illustrate Theorem 3.11, we give an example of strongly quasipolar matrix rings.

Example 3.12. Let R be a commutative local ring and consider the ring

$$\mathcal{S} = \left\{ \begin{bmatrix} x & y \\ y & x \end{bmatrix} \in M_2(R) : x, y \in R \right\}.$$

Then S is a commutative subring of $M_2(R)$ and $J(\mathcal{S}) = \mathcal{S}^{qnil}$. On the one hand, in [13, Theorem 3.5], it is proved that $\mathcal{S}$ is quasipolar. On the other hand, Theorem 3.11(2) yields $\mathcal{S}$ is strongly quasipolar.

In the following, we deal with some properties of strongly quasipolar elements.

Proposition 3.13. *Let R be a ring and let $a \in R$, $u \in U(R)$. Then a is strongly quasipolar if and only if $u^{-1}au$ is strongly quasipolar.*

Proof. Assume that $a \in R$ is strongly quasipolar. Then there exists $p^2 = p \in comm^2(a)$ such that $a + p \in U(R)$ and $a^2p \in R^{qnil}$. Say $e = u^{-1}pu$. Then we have $e^2 = (u^{-1}pu)(u^{-1}pu) = u^{-1}p(uu^{-1})pu = u^{-1}p^2u = u^{-1}pu = e$. Moreover for any $x \in comm(u^{-1}au)$, we get $uxu^{-1} \in comm(a)$ and so $p(uxu^{-1}) = (uxu^{-1})p$. Then $ex = (u^{-1}pu)x = x(u^{-1}pu) = xe$. Since $a + p \in U(R)$, we have $u^{-1}au + e \in U(R)$. Since $a^2p \in R^{qnil}$, we get $(u^{-1}au)^2e = u^{-1}(a^2p)u \in R^{qnil}$ by Proposition 2.1(4). Hence $u^{-1}au$ is strongly quasipolar. The converse is clear. □

The following example shows that strongly quasipolar rings need not be closed under subrings.

Example 3.14. Consider the ring of real numbers. Since $\mathbb{R} = \{0\} \cup U(\mathbb{R})$, $\mathbb{R}$ is a strongly quasipolar ring. But the subring $\mathbb{Z}$ of $\mathbb{R}$ is not strongly quasipolar.

Theorem 3.15. *Let R be a ring. Then $a \in R$ is strongly quasipolar if and only if $-a$ is strongly quasipolar.*

Proof. Assume that $-a \in R$ is strongly quasipolar. Then there exists $p^2 = p \in comm^2(-a)$ such that $-a + p \in U(R)$, $a^2p \in R^{qnil}$. So $1 + ap \in U(R)$. Set $u = (1 + ap)^{-1}p - (-a + p)^{-1}(1 - p)$. Then

$$\begin{aligned}(a + p)u &= (a + p)[(1 + ap)^{-1}p - (-a + p)^{-1}(1 - p)]\\ &= (1 + ap)^{-1}ap + (1 + ap)^{-1}p - (-a + p)^{-1}a(1 - p)\\ &= (1 + ap)^{-1}(1 + ap)p + (-a + p)^{-1}(-a + p)(1 - p) = p + (1 - p) = 1.\end{aligned}$$

Thus $a + p \in U(R)$. Hence a is strongly quasipolar. □

We close this section by obtaining a characterization of strongly quasipolarity.

Theorem 3.16. *Let $\{R_i\}_{i\in I}$ be a family of rings for some index set I. Then R_i is strongly quasipolar for each $i \in I$ if and only if $\prod_{i\in I} R_i$ is strongly quasipolar.*

Proof. Assume that R_i is strongly quasipolar for each $i \in I$. Let $a = (a_i) \in \prod_{i\in I} R_i$. Then by assumption there exists $p_i^2 = p_i \in comm^2(a_i)$ such that $a_i + p_i \in U(R_i)$ and $a_i^2p_i \in R_i^{qnil}$. For $p = (p_i) \in \prod_{i\in I} R_i$, we have $p^2 = p \in comm^2(a)$, $a + p \in \prod_{i\in I} U(R_i)$ and $a^2p \in \prod_{i\in I} R_i^{qnil}$. By Proposition 2.4 the result is clear. Conversely, suppose that $\prod_{i\in I} R_i$ is strongly quasipolar. Let $a_i \in R_i$ for any $i \in I$. Consider $a = (\dots, a_i, \dots) \in \prod_{i\in I} R_i$ where i^{th} entry is a_i and other entries are zero. Then by supposition, there exists $p^2 = p \in comm^2(a)$ such that $a + p \in U\Big(\prod_{i\in I} R_i\Big)$ and $a^2p \in \Big(\prod_{i\in I} R_i\Big)^{qnil}$. So we have an idempotent $p_i \in comm^2(a_i)$ with $a_i + p_i \in U(R_i)$ and $a_i^2p_i \in R_i^{qnil}$ by Proposition 2.4. □

Corner rings of a strongly quasipolar ring are strongly quasipolar as shown below.

Theorem 3.17. *Let R be a ring and $e \in Id(R)$. If R is strongly quasipolar, then eRe is strongly quasipolar.*

Proof. Let $a \in eRe$. By hypothesis for $a \in R$ there exists $p^2 = p \in comm^2(a)$ such that $a + p \in U(R)$ and $a^2p \in R^{qnil}$. Since e is central eRe, we have $ae = ea$ and so $ep = pe$. Thus we get $x(epe) = (epe)x$ for all $x \in comm_{eRe}(a)$. Moreover we have $a + epe = e(a + p)e \in U(eRe)$. Hence by [36, Lemma 3.5], we have $a^2(epe) = ea^2ep = a^2p \in eRe \cap R^{qnil} = (eRe)^{qnil}$. Therefore eRe is strongly quasipolar. □

Theorem 3.18. *Let R be a ring. Then $a \in R$ is strongly quasipolar if and only if there exists some $x \in R$ such that $x \in comm^2(a)$, $x = xax$ and $a^2 - a^3x \in R^{qnil}$.*

Proof. Assume that a is strongly quasipolar. Then there exists $p^2 = p \in comm^2(a)$ such that $a + p \in U(R)$ and $a^2p \in R^{qnil}$. Write $x = (a + p)^{-1}(1 - p)$. Since $p \in comm^2(a)$, we have $xa = ax$ and $ax = a(a + p)^{-1}(1 - p) = (a + p)^{-1}(a + p)(1 - p) = 1 - p$. Moreover $xax = x(1 - p) = x$. Let $y \in comm(a)$. So we get $py = yp$ and $xy = yx$. Also note that $a^2 - a^3x = a^2(1 - ax) = a^2(1 - (1 - p)) = a^2p \in R^{qnil}$. Conversely, let $a \in R$. By supposition, there exists $x \in comm^2(a)$ such that $x = xax$ and $a^2 - a^3x \in R^{qnil}$. Write $p = 1 - ax$. Since $a, x \in comm^2(a)$, we have $p^2 = p \in comm^2(a)$ and $px = xp = 0$. Being $a^2 - a^3x = a^2(1 - ax) \in R^{qnil}$, we get $a^2p \in R^{qnil}$ and so $ap \in R^{qnil}$. Also we have $(a + p)(x + p) = ax + ap + p = 1 + ap \in U(R)$. Thus $a + p \in U(R)$. Hence a is strongly quasipolar. □

Recall that in [27], a ring R is called *unit-central* if all unit elements are central in R.

Proposition 3.19. *Let R be a ring with unit-central element $c \in R$. Then a is strongly quasipolar if and only if ac is strongly quasipolar.*

Proof. Assume that $a \in R$ is strongly quasipolar. Then there exists some $x \in R$ such that $x \in comm^2(a)$, $x = xax$ and $a^2 - a^3x \in R^{qnil}$. Write $y = c^{-1}x$. Since $x \in comm^2(a)$, we have $y \in comm^2(ac)$. Also we get $y(ac)y = (c^{-1}x)(ac)(c^{-1}x) = xac^{-1}x = c^{-1}(xax) = y$. Moreover $(ac)^2 - (ac)^3y = a^2c^2 - a^3c^3y = c^2(a^2 - a^3cc^{-1}x) = c^2(a^2 - a^3x) \in R^{qnil}$ by Proposition 2.1(1). □

4. Some examples of strongly quasipolar rings

In this section we focus on the examples of strongly quasipolar rings. Thus, we determine the position of the class of strongly quasipolar rings among classes of some known rings.

Recall that a ring is called *abelian* if every idempotent is central, and a ring R is called *strongly J-clean* provided that for any $a \in R$, there exists an $e \in \mathrm{Id}(R)$ such that $a - e \in J(R)$ and $ae = ea$ (cf. [5] and [7]). It is known by [5, Corollary 2.4] that a ring R is uniquely clean if and only if it is strongly J-clean and abelian. We use this fact to prove that every clean ring is strongly quasipolar.

Theorem 4.1. *If R is a strongly J-clean ring, then it is strongly quasipolar.*

Proof. Let $a \in R$. By hypothesis there exist $e^2 = e \in R$ and $w \in J(R)$ such that $a = e + w$, $ew = we$. Since $w \in J(R)$, we have $a + (1 - e) = 1 + w \in U(R)$. Moreover since $J(R)$ is an ideal of R we get $a^2(1 - e) = (e + w)^2(1 - e) = w^2(1 - e) \in J(R) \subseteq R^{qnil}$. Also $1 - e \in comm^2(a)$. Thus $a \in R$ is strongly quasipolar. Hence R is strongly quasipolar. □

The following example shows that the converse statement of Theorem 4.1 is not true in general.

Example 4.2. Consider the ring $\mathbb{Z}_3$. It is clear that the ring $\mathbb{Z}_3$ is strongly quasipolar. But it is not strongly J-clean by [5, Proposition 3.1], and so it is not uniquely clean.

Recall that an element a of a ring R is said to be *strongly nil clean* in case there is an idempotent $e \in R$ such that $e \in comm(a)$ and $a - e \in \mathrm{nil}(R)$. A ring R is called *strongly nil clean* in case every element in R is strongly nil clean [20].

Theorem 4.3. *Every strongly nil clean ring is strongly quasipolar.*

Proof. Let $a \in R$. By hypothesis there exist $e^2 = e \in R$ and $b \in \mathrm{nil}(R)$ such that $a = e + b$, $eb = be$. Then $a + (1 - e) = 1 + b \in U(R)$. Also we have $a^2(1 - e) = b^2(1 - e) \in \mathrm{nil}(R) \subseteq R^{qnil}$. On the other hand $1 - e \in comm^2(a)$ by the proof of [23, Theorem 2.4]. Thus $a \in R$ is strongly quasipolar. Hence R is strongly quasipolar. □

There exist strongly quasipolar rings which are not strongly nil-clean.

Example 4.4. (1) Let $R_i = \mathbb{Z}_{2^i}$ where $(i = 1, 2, 3, \dots)$ and consider the ring $R = \prod_{i=1}^{\infty} R_i$. Each ring R_i is commutative local. By Examples 3.7(2), each R_i is strongly quasipolar. By Theorem 3.16, R is strongly quasipolar. Let $a = (0, 2, 2, 2, \dots) \in R$. By [20, Remark, page 204], a is not nil clean. So R is not strongly nil clean.

(2) Let R be a commutative local ring. Then $M_2(R)$ is not strongly nil clean which is clear from [8, Corollary 2.3]. On the other hand, $M_2(\mathbb{Z}_2)$ is strongly quasipolar by Example 3.4.

Nil-quasipolar rings were introduced in [23]. For a ring R an element $a \in R$ is called *nil-quasipolar* if there exists $p^2 = p \in comm^2(a)$ such that $a + p \in \mathrm{nil}(R)$. A ring R is called *nil-quasipolar* in case every element in R is nil-quasipolar.

Corollary 4.5. *If R is a nil-quasipolar ring, then it is strongly quasipolar.*

Proof. Assume that R is nil-quasipolar. Then by [23, Theorem 2.4] R is strongly nil-clean. The rest is clear from Theorem 4.3. □

Example 4.6. Let $R = T[\mathbb{Q}, \mathbb{Z}_{(2)}] = \{(q_1, q_2, \dots, q_n, a, a, \dots) \mid n \geq 1, q_i \in \mathbb{Q}, a \in \mathbb{Z}_{(2)}\}$. Then R is strongly clean but not quasipolar by [36, Example 3.4 (iii)]. So R is not strongly quasipolar by Corollary 3.10.

In [17], a ring R is called a *JU ring* or a *ring with Jacobson units* if $U(R) = 1 + J(R)$. It is named *UJ ring* in [29]. In [4], a ring R is said to be a *UU ring* if $U(R) = 1+ \text{nil}(R)$. It is well known that $1 + R^{qnil} \subseteq U(R)$, that is, $R^{qnil} \subseteq 1 + U(R)$. There are rings these inclusions could be strict, so that it is rather natural to consider the following condition.

Definition 4.7. A ring R is called a *qnilU ring* or *a ring with qnil-units* if

$$U(R) = 1 + R^{qnil}, \text{ equivalently } R^{qnil} = \{r \in R \mid 1 + r \in U(R)\}.$$

We note the following relations between JU rings and UU rings.

Proposition 4.8. *A ring R is a JU ring with nil Jacobson radical if and only if R is a UU ring and nil(R) is a one sided ideal of R.*

Proof. It is routine. □

Lemma 4.9. *We have the following statements.*

(1) *Every JU ring is qnilU.*
(2) *Every UU ring is qnilU.*

Proof. Clear by $J(R) \subseteq R^{qnil}$ and $\text{nil}(R) \subseteq R^{qnil}$. □

We now consider converse relations between JU rings, UU rings and qnilU rings.

Examples 4.10. (1) There are qnilU rings that are not UU rings.
(2) There are qnilU rings and UU rings that are not JU rings.
(3) There are qnilU rings and JU rings that are not UU rings.

Proof. (1) Let R be a commutative local ring. Then $J(R[[x]]) = J(R) + xR[[x]] = (R[[x]])^{qnil}$ and $U(R[[x]]) = 1+(R[[x]])^{qnil}$. Hence $R[[x]]$ is a qnilU ring. Since x is not nilpotent in $R[[x]]$ and $1+ \text{nil}(R[[x]]) \subseteq 1+(R[[x]])^{qnil}$, $R[[x]]$ is not UU.
(2) Let R denote the Bergman's ring in [2]. Let R be the $\mathbb{Z}_2$-algebra generated by x, y with $x^2 = 0$. It is noted in [19, Example 2.5] that Bergman showed in [2, Corollary 2.16] that $\text{nil}(R) = \mathbb{Z}_2x + xRx$ and $U(R) = 1 + \mathbb{Z}_2x + xRx$. So R is a UU ring. However, $(\mathbb{Z}_2x + xRx)^2 = 0$ and $J(R) = (0)$ entails that R is not a JU-ring.
Since $\mathbb{Z}_2x + xRx \subseteq R^{qnil}$, $U(R) = 1 + \mathbb{Z}_2x + xRx \subseteq 1 + R^{qnil}$. The reverse inclusion always holds. It follows that $U(R) = 1 + \mathbb{Z}_2x + xRx = 1 + R^{qnil}$ or $R^{qnil} = 1 + U(R)$. It follows that R is a qnilU ring.
(3) Let R be a local ring with $R/J(R) \cong \mathbb{Z}_2$. Then $R \setminus J(R) = U(R)$ and $U(R) = 1 + J(R)$. Since $1 + R^{qnil} \subseteq U(R) = 1 + J(R)$, $U(R) = 1 + J(R) = 1 + R^{qnil}$. So R is both a JU ring and a qnilU ring. Assume that $J(R) \neq \text{nil}(R)$ (For instance, $R = \mathbb{Z}_{(2)}$). Then $1+ \text{nil}(R) \neq U(R)$. Hence R is not a UU ring. □

Proposition 4.11. [12, Proposition 2.10] *Let R be a qnilU ring. For $a, b \in R$ with $a \in R^{qnil}$ and $b \in comm(a)$, we have $ab \in R^{qnil}$.*

Proof. Having $a \in R^{qnil}$ and $b \in comm(a)$ imply $1 + ab \in U(R)$ by definition of R^{qnil}. Since R is qnilU, that is, $R^{qnil} = 1 + U(R)$, $ab \in R^{qnil}$. □

We have discussed the conditions under which any quasipolar ring is strongly quasipolar as in Theorem 3.11. We also give another condition such as being a qnilU ring under which any quasipolar ring is strongly quasipolar.

Theorem 4.12. *Let R be a qnilU ring. Then R is strongly quasipolar if and only if it is quasipolar.*

Proof. One way is known. For the other way, suppose that R is a quasipolar ring. Let $a \in R$ and $p^2 = p \in \mathrm{comm}^2(a)$ with $a + p \in U(R)$, $ap \in R^{qnil}$. Since $a \in \mathrm{comm}(ap)$, $a^2p \in R^{qnil}$ by Proposition 4.11. □

Theorem 4.13. *Let R be a ring, $a \in R$ and consider the following conditions.*

(1) *a is strongly quasipolar.*
(2) *There exists $x \in comm^2(a)$ such that $xax = x$ and $a^2 - a^3x \in R^{qnil}$.*
(3) *There exists $x \in comm^2(a)$ such that $xax = x$ and $a^k - a^{k+1}x \in R^{qnil}$ for each $k \geq 2$.*
(4) *There exists $p \in R$ such that $p^2 = p \in comm^2(a)$, $a^k + p \in U(R)$ and $a^kp \in R^{qnil}$ for some $k \geq 2$.*

Then (1) $\Leftrightarrow$ (2), (3) $\Rightarrow$ (4) *and* (3) $\Rightarrow$ (2). *If R is a qnilU ring, then* (2) $\Rightarrow$ (3).

Proof. (1) $\Leftrightarrow$ (2) is proved in Theorem 3.18.
(2) $\Rightarrow$ (3) Suppose that R is a qnilU ring. For $a \in R$, assume that there exists $x \in comm^2(a)$ such that $xax = x$ and $a^2 - a^3x \in R^{qnil}$. Consider $a^3 - a^4x = a(a^2 - a^3x)$. Since $x \in comm^2(a)$, we have $a(a^2 - a^3x) = (a^2 - a^3x)a = a^3 - a^4x$. Having $a^2 - a^3x \in R^{qnil}$ and Proposition 4.11 imply $a^3 - a^4x \in R^{qnil}$. Continuing in this way, $a^k - a^{k+1}x \in R^{qnil}$ for each $k \geq 2$.
(3) $\Rightarrow$ (2) Clear.
(3) $\Rightarrow$ (4) For $a \in R$, assume that there exists $x \in comm^2(a)$ such that $xax = x$ and $a^k - a^{k+1}x \in R^{qnil}$ for each $k \geq 2$. Let $p = 1 - ax$. Then $p^2 = p$ and $p \in comm^2(a)$ since $x \in comm^2(a)$. It follows that $a^k - a^{k+1}x \in R^{qnil}$ implies $a^kp \in R^{qnil}$. Since $ax = 1 - p$ and $px = 0$ and $1 + a^kp \in U(R)$, $(a^k + p)(x^k + p) = a^kx^k + px^k + a^kp + p = (ax)^k + a^kp + p = 1 - p + a^kp + p = 1 + a^kp$ is invertible. Thus $a^k + p$ is invertible. □

In the light of the next result, every pseudo Drazin invertible element is strongly quasipolar.

Theorem 4.14. *Every pseudopolar element is strongly quasipolar.*

Proof. Let R be a ring and $a \in R$ pseudopolar. Then there exists $p^2 = p \in comm^2(a)$ such that $a + p \in U(R)$ and $a^kp \in J(R)$ for some positive integer k. If $k = 1$, then $ap \in J(R)$, and so $a^2p \in J(R)$. This yields $a^2p \in R^{qnil}$. If $k = 2$, then there is nothing to show. Assume that $k \geq 3$. Since $p^2 = p$ and $pa = ap$, we have $a^kp = (ap)^k \in J(R)$. It follows that $(ap)^{2k} \in R^{qnil}$. By Proposition 2.1(1), $(ap)^2 \in R^{qnil}$. This entails $a^2p \in R^{qnil}$. Therefore a is strongly quasipolar. □

Corollary 4.15. *Every pseudopolar ring is strongly quasipolar.*

As a consequence of Theorem 4.14, we have the following hierarchy:

{Drazin invertible} ⊆ {pseudo Drazin invertible} ⊆ {strongly quasipolar} ⊆ {generalized Drazin invertible}

There are strongly quasipolar elements which are not pseudo Drazin invertible as shown below.

Example 4.16. Let R denote the Banach algebra of all bounded linear operators on l^1 and consider the infinite matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0\dots \\ 1 & 0 & 0 & 0 & 0 & 0\dots \\ 0 & 1/2 & 0 & 0 & 0 & 0\dots \\ 0 & 0 & 1/3 & 0 & 0 & 0\dots \\ 0 & 0 & 0 & 1/4 & 0 & 0\dots \\ 0 & 0 & 0 & 0 & 1/5 & 0\dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \in R.$$

On the one hand, it is known by [35, Example 4.2] that $A \in R^{qnil}$, so it is generalized Drazin invertible, but not pseudo Drazin invertible. On the other hand, we claim that A is strongly quasipolar. Note that

$$A^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0\dots \\ 0 & 0 & 0 & 0 & 0 & 0\dots \\ 1/2 & 0 & 0 & 0 & 0 & 0\dots \\ 0 & 1/6 & 0 & 0 & 0 & 0\dots \\ 0 & 0 & 1/12 & 0 & 0 & 0\dots \\ 0 & 0 & 0 & 1/20 & 0 & 0\dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

i.e., for $n \geq 2$, the entry of A^2 in the $n+1$-row and $n-1$-column is $1/n(n-1)$, and the other entries are zero. Let $X \in comm(A^2)$. Then X has the form

$$X = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 & 0 & 0\dots \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 & 0 & 0\dots \\ a_{31} & a_{32} & a_{33} & a_{34} & 0 & 0 & 0 & 0\dots \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & 0 & 0 & 0\dots \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & 0 & 0\dots \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & 0\dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

and so A^2X has the form

$$A^2X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0\dots \\ 0 & 0 & 0 & 0 & 0 & 0\dots \\ x_{31} & x_{32} & 0 & 0 & 0 & 0\dots \\ x_{41} & x_{42} & 0 & 0 & 0 & 0\dots \\ x_{51} & x_{52} & x_{53} & x_{54} & 0 & 0\dots \\ x_{61} & x_{62} & x_{63} & x_{64} & x_{65} & 0\dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

It follows that $I + A^2X$ is invertible. Hence $A^2 \in R^{qnil}$. Lemma 3.6(2) yields that A is strongly quasipolar as claimed.

A ring R is called *von Neumann regular* (simply *regular*) if for each $a \in R$, there exists $x \in R$ such that $a = axa$. A ring R is called *π-regular* if for each $a \in R$ there exist a positive integer n and $x \in R$ such that $a^n = a^nxa^n$. Regular rings are clearly π-regular. A ring R is called *strongly regular* if for each $x \in R$, there exists $y \in R$ such that $x = x^2y$. A ring R is called *strongly π-regular* if for each $a \in R$, there exist a positive integer n and $x \in R$ such that $a^n = a^{n+1}x$. Strongly regular rings are strongly π-regular. Note that a ring R is strongly regular if and only if R is abelian regular by [22, Theorem 3.2].

Theorem 4.17. *Let R be a ring. Then the following are equivalent.*

(1) *R is strongly π-regular.*
(2) *R is quasipolar and* $R^{qnil} = nil(R)$.
(3) *R is strongly quasipolar and* $R^{qnil} = nil(R)$.

Proof. (1) ⇔ (2) By [14, Theorem 2.6].
(2) ⇒ (3) Let $a \in R$. There exists $p^2 = p \in comm^2(a)$ such that $a + p \in U(R)$, $ap \in R^{qnil}$. Since $a(ap) = (ap)a$ and $R^{qnil} = \text{nil}(R)$, $a^2p \in R^{qnil}$. So (3) holds.
(3) ⇒ (2) By Theorem 3.8, every strongly quasipolar ring is quasipolar. Hence (2) holds. This completes the proof. □

The assumption $R^{qnil} = \text{nil}(R)$ in Theorem 4.17 is not superfluous by the following example.

Example 4.18. Let S be a finite ring and $R = \mathbb{Z}_{(2)} \oplus M_2(S)$. Both $\mathbb{Z}_{(2)}$ and $M_2(S)$ are strongly quasipolar and R is strongly quasipolar by Theorem 3.16. Since $J(R) = R^{qnil} \neq \text{nil}(R)$, by Theorem 4.17, R is not strongly π-regular.

We have the following corollaries from Theorem 4.17 and Theorem 3.11.

Corollary 4.19. *Let R be a strongly π-regular ring. Then R is strongly quasipolar.*

Corollary 4.20. *Let R be a strongly regular ring. Then R is strongly quasipolar.*

The following examples illustrate that the converse statements of Corollary 4.19 and Corollary 4.20 need not hold in general.

Examples 4.21. (1) There is a strongly quasipolar ring that is not strongly π-regular.
(2) There is a strongly quasipolar ring that is not strongly regular.

Proof. (1) Consider the ring $R = \mathbb{Z}_{(2)}$. Note that R is a commutative local ring. Then it is quasipolar. By Examples 3.7, R is strongly quasipolar. But R is not strongly π-regular because it is a domain.
(2) Let $R = M_2(\mathbb{Z}_2)$. Then R is strongly quasipolar by Example 3.4. However, R is not strongly regular because it is not abelian. □

Recall that if a is quasipolar with spectral idempotent p in a ring R and $ap \in \text{nil}(R)$ with the nilpotency index k, it is said that a is *polar* of order k. A ring R is called *a polar ring* if every element of R is polar (see [28] for detail). We mention the following result to close relations.

Proposition 4.22. [23, Proposition 2.11] *A ring R is strongly π-regular if and only if R is polar.*

In [10], an element a of a ring R is called *perfectly clean* if there exists an idempotent $e \in comm^2(a)$ such that $a - e \in U(R)$. A ring R is *perfectly clean* in case every element in R is perfectly clean. It is known that every quasipolar ring is perfectly clean.

Theorem 4.23. [10, Theorem 2.2] *Let R be a ring. Then R is strongly nil clean if and only if*

(1) *R is perfectly clean.*
(2) *R is a UU ring.*

Proof. We summarize the proof. Forward direction. Let $a \in R$, we see that $a - a^2 \in \text{nil}(R)$. There exists a positive integer n such that $(a - a^2)^n = 0$. Let $f(t) = \sum_{i=0}^{n} \binom{2n}{i} t^{2n-i}(1-t)^i \in \mathbb{Z}[t]$. Let $e = f(a)$. Then $e \in \text{Id}(R)$. For any $x \in comm(a)$, we see that $xe = xf(a) = f(a)x = ex$. Thus, $x \in comm^2(a)$ and $a - e \in \text{nil}(R)$. Then $a = (1-e) + (2e - 1 + a - e)$ with $1 - e \in comm^2(a)$ and $(2e-1) + (a-e) \in U(R)$. Therefore, R is perfectly clean. Clearly, $\text{nil}(R) \subseteq \{x \in R \mid 1 - x \in U(R)\}$. Let $1 - x \in U(R)$, then $x = e + n$ where $e \in \text{Id}(R)$ and $e \in comm(x)$ and $n \in \text{nil}(R)$. Hence $1 - e = (1 - x) + n \in U(R)$. So $1 - e = 1$ and $e = 0$. Thus $x = n \in \text{nil}(R)$. It entails that $\text{nil}(R) = \{x \in R \mid 1 - x \in U(R)\}$ and $U(R) = 1 + \text{nil}(R)$.
Backward direction. Assume that (1) and (2) hold. Let $a \in R$, there exist $e \in \text{Id}(R)$ and $u \in U(R)$ such that $e \in comm^2(a)$ and $-a = e - u$. We have $a = (1-e) + (-1+u)$. By (2), $-1 + u \in \text{nil}(R)$. So R is strongly nil clean. □

Theorem 4.24. *Let R be a ring. Then R is strongly nil clean if and only if*

(1) *R is quasipolar.*
(2) *$nil(R) = \{x \in R \mid 1 - x \in U(R)\}$, that is, R is a UU ring, that is, $U(R) = 1 + nil(R)$.*

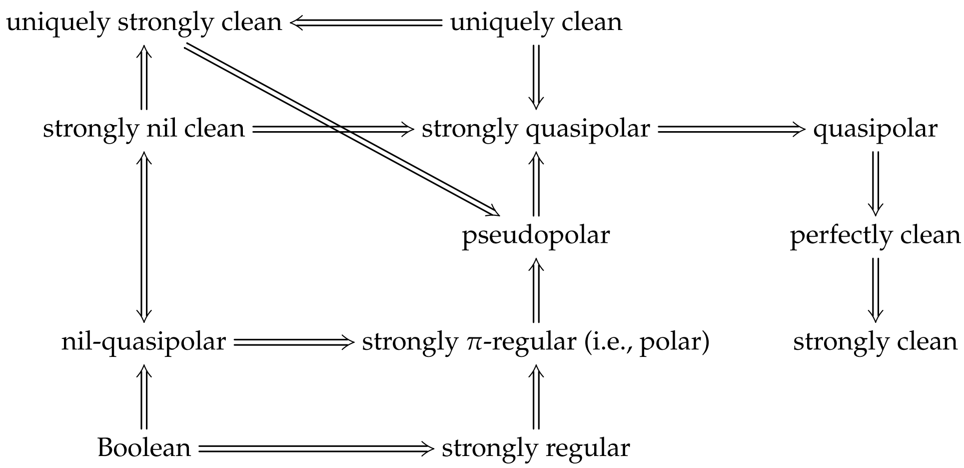
Proof. Forward direction. By Theorem 4.23, R is a UU ring. Let $a \in R$, as in the proof of Theorem 4.23, there exist $e^2 = e \in comm^2(a)$, $n \in \text{nil}(R)$ such that $-a = e + n$. Then $-a = (1-e) + (2e - 1 + n)$ and $(2e-1) + n \in U(R)$ since $2e - 1 \in U(R)$. Furthermore, $-a(1-e) = n(1-e) \in \text{nil}(R)$ entails that R is quasipolar.
Backward direction. Assume that (1) and (2) hold. Then R is perfectly clean. Accordingly, R is strongly nil clean by Theorem 4.23. □

The following result is proved by using definitions in Theorem 4.3. We now give an alternative proof.

Corollary 4.25. *Every strongly nil clean ring is strongly quasipolar.*

Proof. Let R be a strongly nil clean ring. By Theorem 4.24, R is quasipolar and UU. Every UU ring is qnilU by Lemma 4.9. Hence R is strongly quasipolar by Theorem 4.12. □

In the following diagram we summarize visually the various containment relationships between the rings mentioned in this section. An arrow signifies containment of the class of rings at the start of the arrow into the class of rings to which the arrow points.



5. Strongly quasipolarity of matrix rings and some ring extensions

Being strongly quasipolar is not a Morita invariant property. There are some examples to show that $M_n(R)$ need not be strongly quasipolar for a commutative local ring R. The first such an example was given in [34, Example 1].

Example 5.1. Consider $\mathbb{Z}_{(2)} = \{a/b \in \mathbb{Q} \mid 2 \nmid b\}$. Let $A = \begin{bmatrix} 8 & 6 \\ 3 & 7 \end{bmatrix} \in M_2(\mathbb{Z}_{(2)})$. Then A has no strongly clean decomposition by [34, Example 1]. It follows that A has no strongly quasipolar decomposition either. Hence $M_2(\mathbb{Z}_{(2)})$ is not strongly quasipolar.

We state the following lemma before mentioning one of the main theorems.

Lemma 5.2. *Let R be a local ring, $u \in U(R)$ and $j \in J(R)$. The following are equivalent.*

(1) *The matrix* $\begin{bmatrix} j & 0 \\ 0 & u \end{bmatrix} \in M_2(R)$ *is quasipolar.*

(2) *The matrix* $\begin{bmatrix} j & 0 \\ 0 & u \end{bmatrix} \in M_2(R)$ *is strongly quasipolar.*

(3) *The matrix* $\begin{bmatrix} u & 0 \\ 0 & j \end{bmatrix} \in M_2(R)$ *is strongly quasipolar.*

(4) *The endomorphisms $l_u - r_j$ and $l_j - r_u$ are injective.*

Proof. It is enough to prove (1) ⇒ (2). Let $A = \begin{bmatrix} j & 0 \\ 0 & u \end{bmatrix} \in M_2(R)$ and $E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathrm{Id}(M_2(R))$. Then $AE \in J(M_2(R))$ implies $A^2E \in M_2(R)^{qnil}$. The rest follows by [16, Corollary 3.3]. □

Theorem 5.3. *Let R be a commutative local ring and $A \in M_2(R)$. Then A is strongly quasipolar if and only if it is quasipolar.*

Proof. Assume that A is quasipolar. In [16, Theorem 3.4], it is proved that over a local ring R, $A \in M_2(R)$ is quasipolar if and only if either A is invertible or $A \in M_2(R)^{qnil}$ or A is similar to a diagonal matrix of the form $\begin{bmatrix} u & 0 \\ 0 & j \end{bmatrix}$ where $u \in U(R)$, $j \in J(R)$ and $l_u - r_j$, $l_j - r_u$ are injective.
Case I. Let $A \in U(M_2(R))$, by Proposition 3.3(3), A is strongly quasipolar.
Case II. Let $A \in M_2(R)^{qnil}$. By [16, Lemma 4.1], we have $A^2 \in J(M_2(R))$. Since $J(M_2(R)) \subseteq M_2(R)^{qnil}$, $A^2 \in M_2(R)^{qnil}$. Write $P = I_2 \in comm^2(A)$. Then $A + P \in U(M_2(R))$ and $A^2P \in M_2(R)^{qnil}$. Thus A is strongly quasipolar.
Case III. A is similar to a diagonal matrix of the form $B = \begin{bmatrix} u & 0 \\ 0 & j \end{bmatrix}$ where $u \in U(R)$, $j \in J(R)$ and $l_u - r_j$, $l_j - r_u$ are injective. By Lemma 5.2, B is strongly quasipolar. By Proposition 3.13, we have A is strongly quasipolar.
The converse is clear by Theorem 3.8. □

Theorem 5.4. *Let R be a ring. Then $U_n(R)$ is strongly quasipolar for any integer $n \geq 1$ if one of the following holds:*

(1) *R is commutative uniquely clean,*
(2) *R is commutative local.*

Proof. Let $A \in U_n(R)$. In [36, Theorem 4.1 and Theorem 4.3], it is proved that A is quasipolar. In either case, to prove the quasipolarity of A it is shown that there exists $E^2 = E \in comm^2(A)$ such that $A - E \in U(U_n(R))$ and $AE \in J(U_n(R))$. Since $AE \in J(U_n(R)) \subseteq U_n(R)^{qnil}$ and $A^2E \in J(U_n(R))$, $A^2E \in U_n(R)^{qnil}$. Thus A is strongly quasipolar. □

We end this paper by studying some ring extensions in terms of strongly quasipolarity.

Theorem 5.5. *Let S be a subring of a ring R and $T = T[R, S]$. Then T is strongly quasipolar if and only if R and S are strongly quasipolar.*

Proof. Necessity. Assume that T is strongly quasipolar. Let $a \in R$ and $x = (a, 0, 0, \dots) \in T[R, S]$. There exists $y = (y_1, y_2, \dots, y_n, t, t, \dots) \in T$ such that $y \in comm^2(x)$ and $yxy = y$ and $x^2 - x^3y \in T^{qnil}$. Obviously, $y_1 \in comm^2(a)$, $y_1ay_1 = y_1$. By Proposition 2.7(1), $a^2 - a^3y_1 \in R^{qnil}$. Let $s \in S$ and $b = (0, s, s, s, \dots) \in T$. By hypothesis, there exists $z = (z_1, z_2, \dots, z_n, u, u, \dots) \in T$ such that $z \in comm^2(b)$ and $zbz = z$ and $b^2 - b^3z \in T^{qnil}$. Then $u \in comm^2(s)$ and $usu = u$. By Proposition 2.7(1), $s^2 - s^3u \in S^{qnil}$. Therefore R and S are strongly quasipolar.
Sufficiency. Let $x = (r_1, r_2, r_3, \dots, r_n, s, s, s, \dots) \in T$. Since R is strongly quasipolar, there exist $x_i \in R$ with $x_i \in comm^2(r_i)$ such that $x_ir_ix_i = x_i$ and $r_i^2 - r_i^3x_i \in R^{qnil}$ where $i = 1, 2, 3, \dots, n$ and $t \in S$ with $t \in comm^2(s)$ such that $tst = t$ and $s^2 - s^3t \in S^{qnil}$. Let $y = (x_1, x_2, x_3, \dots, x_n, t, t, t, \dots) \in T$. Then $yxy = y$, $y \in comm^2(x)$ and $x^2 - x^3y \in T^{qnil}$. Therefore T is strongly quasipolar. □

To illustrate Theorem 5.5 we give some examples.

Example 5.6. Let D be a division ring, $R = M_2(D)$ and $S = D_2(D)$. Then R is strongly π-regular by [22]. In view of Theorem 4.17, R is strongly quasipolar. Similarly, S is also strongly quasipolar. Then $T = T[R, S]$ is strongly quasipolar.

Theorem 5.7. *Let $D = I(R, S)$ be an ideal extension described in Lemma 2.8. Consider the following conditions.*

(1) *D is strongly quasipolar.*
(2) *R is strongly quasipolar.*
(3) *S is strongly quasipolar.*

Then (1) ⇒ (2). If R is qnilU and $S^{qnil} \subseteq R^{qnil}$, then (1) ⇒ (3).

Proof. (1) ⇒ (2) Let $a \in R$. There exists $(e,f)^2 = (e,f) \in comm^2(a,0)$ such that $(a,0)+(e,f) \in U(D)$ and $(a,0)^2(e,f) \in D^{qnil}$. By Lemma 2.8, $e+f$, $f \in \mathrm{Id}(R)$, $a+e+f$, $f \in U(R)$. Then $(a,0)^2(e,f) \in D^{qnil}$ implies $a^2(e+f) \in R^{qnil}$. We claim $e+f \in comm^2(a)$. In fact, let $r \in comm(a)$. Then $(r,0) \in comm(a,0)$. Hence $(e,f)(r,0) = (r,0)(e,f)$ implies $r(e+f) = (e+f)r$. Thus $e+f \in comm^2(a)$. It entails that R is strongly quasipolar.
(1) ⇒ (3) Assume that R is qnilU and $S^{qnil} \subseteq R^{qnil}$. Let $s \in S$. For $(0,s) \in D$, there exists an idempotent $(e,f) \in comm^2(0,s)$ such that $(0,s)+(e,f) \in U(D)$ and $(0,s)^2(e,f) \in D^{qnil}$. Then $(0,s)+(e,f) \in U(D)$ implies $(e, s+f) \in U(D)$. By Lemma 2.8 $e+f+s \in U(R)$ and $s+f \in U(S)$. And $(0,s)^2(e,f) \in D^{qnil}$ implies $(s^2e, s^2f) \in D^{qnil}$. By Lemma 2.8 $s^2e \in R^{qnil}$ and $s^2f \in S^{qnil}$. □

For a further research study on the subject, one can consider the following questions:

1. Is there a generalized inverse corresponding to the concept of strongly quasipolarity?
2. Is there a new class of rings between the classes of strongly quasipolar rings and quasipolar rings?
3. What kind of properties does a ring gain when we consider an integer n greater than 2 in the definition of strongly quasipolar rings?

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