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Generalized Cartesian symmetry classes

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Abstract. Let V be a unitary space. Suppose G is a subgroup of the full symmetric group S_m and $\mathfrak{X}$ is an irreducible unitary representation of G. In this paper, we introduce the generalized Cartesian symmetry class over V associated with G and $\mathfrak{X}$. Then we investigate some important properties of this vector space. Also, we study some basic properties of the induced linear operators on the generalized Cartesian symmetry classes. Some open problems are also given.

1. Introduction and Preliminaries

In recent years, the study of symmetry classes has played a fundamental role in various branches of mathematics (see [1, 2, 4, 5, 7–10]). In this paper, we focus on the generalized Cartesian symmetry class associated with an irreducible unitary representation of a subgroup of the full symmetric group. Our main goal is to establish important properties of this vector space.

Let S_m denote the full symmetric group of degree m, and let G be a subgroup of S_m. Let U be a unitary space, meaning a finite dimensional complex vector space equipped with an inner product. The set of all linear operators on U is denoted by End (U). Assume that $\mathfrak{X}$ is an irreducible unitary representation of G over U. The generalized trace function $Tr_{\mathfrak{X}} : \mathbb{C}_{m\times m} \to \text{End }(U)$ is defined by

$$Tr_{\mathfrak{X}}(A) = \sum_{\sigma\in G} \mathfrak{X}(\sigma) \sum_{i=1}^{m} a_{i\sigma(i)}$$

for $A = (a_{ij}) \in \mathbb{C}_{m\times m}$.
It is proved that $Tr_{\mathfrak{X}}(A^*) = Tr_{\mathfrak{X}}(A)^*$. In particular, if A is Hermitian, then $Tr_{\mathfrak{X}}(A)$ is Hermitian (see [8]).

Let V be a unitary space of dimension n and denote by $\times^m V$ be the Cartesian product of m-copies of V. Then $U \otimes \times^m V$ is a unitary space with an induced inner product given by

$$\langle u \otimes x^{\times}, v \otimes y^{\times}\rangle = \langle u, v\rangle \sum_{i=1}^{m} \langle x_i, y_i\rangle,$$

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where $u, v \in U$ and $x^{\times} = (x_1, \cdots, x_m)$, $y^{\times} = (y_1, \cdots, y_m) \in \times^m V$.

The *generalized Cartesian symmetrizer associated with G and* $\mathfrak{X}$ is defined by

$$C_{\mathfrak{X}} = \frac{1}{|G|} \sum_{\sigma \in G} \mathfrak{X}(\sigma) \otimes Q(\sigma),$$

where

$$Q(\sigma)(v_1, \cdots, v_m) = (v_{\sigma^{-1}(1)}, \cdots, v_{\sigma^{-1}(m)})$$

is Cartesian permutation operator with respect to $\sigma \in G$.

By using [8, Proposition 2.4], we immediately deduce that $C_{\mathfrak{X}}$ is an orthogonal projection on $U \otimes \times^m V$.

Definition 1.1. *The range of* $C_{\mathfrak{X}}$,

$$V^{\mathfrak{X}}(G) := C_{\mathfrak{X}}(U \otimes \times^m V),$$

is called the generalized Cartesian symmetry class *over V associated with G and* $\mathfrak{X}$.

If $\dim U = 1$, then $V^{\mathfrak{X}}(G)$ reduces to $V^{\chi}(G)$, which is the Cartesian symmetry class associated with G and the irreducible character χ of G corresponding to the representation $\mathfrak{X}$ (see [3, 7, 11]). The elements of $V^{\mathfrak{X}}(G)$ that have the form $C_{\mathfrak{X}}(u \otimes x^{\times})$ are called the generalized Cartesian symmetrized vectors. The equality of two generalized symmetrized vectors has been studied in [8]. We will need the following theorem (see [8, Corollary 5.9]).

Theorem 1.2. *Let* $\mathfrak{X}$ *be a unitary representation of G over unitary space U and* $x^{\times}, y^{\times} \in \times^m V$. *Let* $A = [a_{ij}], B = [b_{ij}] \in \mathbb{C}_{m\times n}$ *such that* $x_i = \sum_{j=1}^{n} a_{ij} e_j$, $y_i = \sum_{j=1}^{n} b_{ij} e_j$, $i = 1, \cdots, m$. *Then the following are equivalent:*

(a) $C_{\mathfrak{X}}(u \otimes x^{\times}) = C_{\mathfrak{X}}(u \otimes y^{\times})$ *for all* $u \in U$.

(b) $Tr_{\mathfrak{X}}(AA^*) = Tr_{\mathfrak{X}}(AB^*) = Tr_{\mathfrak{X}}(BB^*)$.

The following theorem states the inner product two generalized symmetrized vectors in terms of the generalized trace function.

Theorem 1.3. *[8, Proposition 5.1]*
For all $u, v \in U$ *and* $x^{\times}, y^{\times} \in \times^m V$ *we have*

$$\langle C_{\mathfrak{X}}(u \otimes x^{\times}), C_{\mathfrak{X}}(v \otimes y^{\times}) \rangle = \frac{1}{|G|} \langle Tr_{\mathfrak{X}}(A)u, v \rangle,$$

where $A = [a_{ij}] \in \mathbb{C}_{m\times m}$ *and* $a_{ij} = \langle x_i, y_j \rangle$.

In this paper, we will refer to the following lemma frequently.

Lemma 1.4. *Let* $\sigma \in G$, $u \in U$ *and* $x^{\times} \in \times^m V$. *Then*

$$C_{\mathfrak{X}}(u \otimes x^{\times}_{\sigma}) = C_{\mathfrak{X}}(\mathfrak{X}(\sigma)u \otimes x^{\times}).$$

Proof. From definition $C_{\mathfrak{X}}$, we have

$$\begin{aligned} C_{\mathfrak{X}}(u \otimes x^{\times}_{\sigma}) &= \frac{1}{|G|} \sum_{\tau \in G} (\mathfrak{X}(\tau) \otimes Q(\tau))(u \otimes x^{\times}_{\sigma}) \\ &= \frac{1}{|G|} \sum_{\tau \in G} \mathfrak{X}(\tau)u \otimes Q(\tau)x^{\times}_{\sigma} \end{aligned}$$

$$
\begin{aligned}
&=\frac{1}{|G|}\sum_{\tau\in G}\mathfrak{X}(\tau)u\otimes Q(\tau)Q(\sigma^{-1})x^{\times}\\
&=\frac{1}{|G|}\sum_{\tau\in G}\mathfrak{X}(\tau)u\otimes Q(\tau\sigma^{-1})x^{\times} \qquad (\tau\sigma^{-1}=\pi)\\
&=\frac{1}{|G|}\sum_{\pi\in G}\mathfrak{X}(\pi\sigma)u\otimes Q(\pi)x^{\times}\\
&=\frac{1}{|G|}\sum_{\pi\in G}\mathfrak{X}(\pi)\mathfrak{X}(\sigma)u\otimes Q(\pi)x^{\times}\\
&=\left(\frac{1}{|G|}\sum_{\pi\in G}\mathfrak{X}(\pi)\otimes Q(\pi)\right)(\mathfrak{X}(\sigma)u\otimes x^{\times})\\
&=C_{\mathfrak{X}}(\mathfrak{X}(\sigma)u\otimes x^{\times}).
\end{aligned}
$$

□

Definition 1.5. *Suppose G_p is the stabilizer subgroup of p where $p = 1, 2, \cdots, m$. The linear map $T_p : U \to U$ defined by*

$$T_p = \frac{1}{|G_p|}\sum_{\sigma\in G_p}\mathfrak{X}(\sigma)$$

is called the linear map corresponding to p.

Theorem 1.6. *(a) The linear map T_p is an orthogonal projection on U.*

(b) rank $T_p = \frac{1}{|G_p|}\sum_{\sigma\in G_p}\chi(\sigma)$, where χ is the irreducible character of G corresponding to the representation $\mathfrak{X}$.

Proof. (a) We first prove that T_p is Hermitian. We have

$$T_p^* = \left(\frac{1}{|G_p|}\sum_{\sigma\in G_p}\mathfrak{X}(\sigma)\right)^* = \frac{1}{|G_p|}\sum_{\sigma\in G_p}\mathfrak{X}(\sigma)^* = \frac{1}{|G_p|}\sum_{\sigma\in G_p}\mathfrak{X}(\sigma^{-1}) = T_p\ .$$

Now we show that T_p is idempotent. We have

$$
\begin{aligned}
T_p^2 &=\left(\frac{1}{|G_p|}\sum_{\sigma\in G_p}\mathfrak{X}(\sigma)\right)\left(\frac{1}{|G_p|}\sum_{\pi\in G_p}\mathfrak{X}(\pi)\right)\\
&=\frac{1}{|G_p|^2}\sum_{\sigma\in G_p}\sum_{\pi\in G_p}\mathfrak{X}(\sigma)\mathfrak{X}(\pi)\\
&=\frac{1}{|G_p|^2}\sum_{\sigma\in G_p}\sum_{\pi\in G_p}\mathfrak{X}(\sigma\pi)\\
&=\frac{1}{|G_p|^2}\sum_{\sigma\in G_p}\sum_{\tau\in G_p}\mathfrak{X}(\tau) \qquad (\sigma\pi=\tau)\\
&=\frac{1}{|G_p|^2}\sum_{\sigma\in G_p}|G_p|T_p\\
&=T_p\ .
\end{aligned}
$$

(b)

$$\text{rank } T_p = tr\,(T_p) = tr\left[\frac{1}{|G_p|}\sum_{\sigma\in G_p}\mathfrak{X}(\sigma)\right] = \frac{1}{|G_p|}\sum_{\sigma\in G_p}\chi(\sigma).$$

□

In this paper, we study some important properties of the vector space $V^{\mathfrak{X}}(G)$.

2. The generalized Cartesian Symmetry Classes

Suppose $\mathbb{F} = \{u_1, \cdots, u_r\}$ and $\mathbb{E} = \{e_1, \cdots, e_n\}$ are orthonormal bases for unitary spaces U and V, respectively. Assume $[\mathfrak{X}(\sigma)]_{\mathbb{F}} = [m_{ij}(\sigma)]$ for any $\sigma \in G$. For $1 \leq i \leq n$ and $1 \leq j \leq m$, let

$$e_{ij} = (\delta_{1j}e_i, \delta_{2j}e_i, \cdots, \delta_{mj}e_i) \in \times^m V.$$

Then the set

$$\mathbb{B} = \{u_k \otimes e_{ij} \mid 1 \leq k \leq r, 1 \leq i \leq n, 1 \leq j \leq m\}$$

is an orthonormal basis of $U \otimes \times^m V$. Therefore,

$$V^{\mathfrak{X}}(G) = \langle C_{\mathfrak{X}}(u_k \otimes e_{ij}) \mid 1 \leq k \leq r, 1 \leq i \leq n, 1 \leq j \leq m\rangle.$$

The elements

$$C_{\mathfrak{X}}(u_k \otimes e_{ij}),\ 1 \leq k \leq r,\ 1 \leq i \leq n,\ 1 \leq j \leq m$$

of $V^{\mathfrak{X}}(G)$ are called *the generalized Cartesian standard symmetrized vectors*.

Definition 2.1. *For any $1 \leqslant j, s \leqslant m$, we define the linear map $T_{sj} : U \to U$ by*

$$T_{sj} = \frac{1}{|G_{sj}|}\sum_{\sigma\in G_{sj}}\mathfrak{X}(\sigma),$$

where

$$G_{sj} = \{\sigma \in G \mid \sigma(j) = s\}.$$

If G_{sj} is empty, then we define $T_{sj} = 0$. If $s = j$, then $G_{jj} = G_j$, the stabilizer of j in G and so $T_{jj} = T_j$, the linear map corresponding to j.

Theorem 2.2. *For any $1 \leqslant j, s \leqslant m, 1 \leqslant i, r \leqslant n, 1 \leqslant k, l \leqslant r$, we have*

$$\langle C_{\mathfrak{X}}(u_k \otimes e_{ij}), C_{\mathfrak{X}}(u_l \otimes e_{rs})\rangle = \begin{cases} 0 & s \nsim j \\ \delta_{ir}\dfrac{|G_{sj}|}{|G|}\langle T_{sj}u_k, u_l\rangle & s \sim j \end{cases}$$

In particular,

$$\| C_{\mathfrak{X}}(u_k \otimes e_{ij}) \|^2 = \frac{1}{[G : G_j]} \| T_j u_k \|^2 .$$

Proof. By using Theorem 1.3, we have

$$\begin{aligned}\langle C_{\mathfrak{X}}(u_k \otimes e_{ij}), C_{\mathfrak{X}}(u_l \otimes e_{rs})\rangle &= \langle C_{\mathfrak{X}}(u_k \otimes e_{ij}), u_l \otimes e_{rs}\rangle \\ &= \frac{1}{|G|}\langle Tr_{\mathfrak{X}}(A)u_k, u_l\rangle\end{aligned}$$

$$=\frac{1}{|G|}\langle\sum_{\sigma\in G}\mathfrak{X}(\sigma)\sum_{p=1}^{m}a_{p\sigma(p)}u_k,u_l\rangle,$$

where

$$a_{pq}=\langle\delta_{pj}e_i,\delta_{qs}e_r\rangle=\delta_{pj}\delta_{qs}\langle e_i,e_r\rangle=\delta_{pj}\delta_{qs}\delta_{ir}.$$

Therefore

$$\begin{aligned}\langle C_{\mathfrak{X}}(u_k\otimes e_{ij}),C_{\mathfrak{X}}(u_l\otimes e_{rs})\rangle &=\frac{1}{|G|}\langle\sum_{\sigma\in G}\mathfrak{X}(\sigma)\sum_{p=1}^{m}\delta_{pj}\delta_{\sigma(p)s}\delta_{ir}u_k,u_l\rangle\\ &=\delta_{ir}\frac{1}{|G|}\langle\sum_{\sigma\in G}\mathfrak{X}(\sigma)\delta_{\sigma(j)s}u_k,u_l\rangle\\ &=\begin{cases}0 & s\nsim j\\ \delta_{ir}\dfrac{1}{|G|}\langle\sum_{\sigma\in G_{sj}}\mathfrak{X}(\sigma)u_k,u_l\rangle & s\sim j\end{cases}\\ &=\begin{cases}0 & s\nsim j\\ \delta_{ir}\dfrac{|G_{sj}|}{|G|}\langle T_{sj}u_k,u_l\rangle & s\sim j\end{cases}\end{aligned}$$

In particular

$$\begin{aligned}\parallel C_{\mathfrak{X}}(u_k\otimes e_{ij})\parallel^2 &=\langle C_{\mathfrak{X}}(u_k\otimes e_{ij}),C_{\mathfrak{X}}(u_k\otimes e_{ij})\rangle\\ &=\frac{|G_j|}{|G|}\langle T_ju_k,u_k\rangle\\ &=\frac{1}{[G:G_j]}\langle T_ju_k,T_ju_k\rangle \qquad (T_j^2=T_j=T_j^*)\\ &=\frac{1}{[G:G_j]}\parallel T_ju_k\parallel^2 .\end{aligned}$$

□

From the above Theorem, we deduce that $C_{\mathfrak{X}}(u_k\otimes e_{ij})=0$ if and only if $T_ju_k=0$. For any $1\le k\le r$, let

$$\Omega_k=\{1\le j\le m\mid T_ju_k\neq 0\}.$$

Put $\Omega=\bigcup_{k=1}^{r}\Omega_k$. Then $\Omega=\{1\le j\le m\mid T_j\neq 0\}$. By Theorem 1.6, $T_j\neq 0$ if and only if $\sum_{\sigma\in G_j}\chi(\sigma)\neq 0$. Hence

$$\Omega=\{1\le j\le m\mid\sum_{\sigma\in G_j}\chi(\sigma)\neq 0\}=\{1\le j\le m\mid[\chi,1_{G_j}]\neq 0\},$$

where [,] is the inner product of characters (see [6]).

Let $\mathcal{D}$ be a set of representatives of orbits of the action of G on the set $\mathbf{I}_m=\{1,2,\cdots,m\}$. We put $\bar{\mathcal{D}}=\mathcal{D}\cap\Omega$. For each $1\le j\le m$ and $1\le i\le n$, the subspace

$$V_{ij}^{\mathfrak{X}}(G)=\langle C_{\mathfrak{X}}(u_k\otimes e_{ij})\mid 1\le k\le r\rangle$$

is called the *generalized cyclic subspace*. If dim $U=1$, then $V_{ij}^{\mathfrak{X}}(G)$ reduces to $V_{ij}^{\chi}(G)$, the cyclic subspace associated with G and the irreducible character χ of G (see [3, 11]).

Since $\langle \mathfrak{X}(\sigma)u_1 \mid \sigma \in G\rangle$ is a non-zero submodule of the irreducible $C[G]$-module U, so $\langle \mathfrak{X}(\sigma)u_1 \mid \sigma \in G\rangle = U$. Therefore it is to see that for every $1 \le j \le m$ and $1 \le i \le n$,

$$V^{\mathfrak{X}}_{ij}(G) = \langle C_{\mathfrak{X}}(u_1 \otimes e_{i\sigma(j)}) \mid \sigma \in G\rangle.$$

For each $1 \le i \le n$, we define

$$V^{\mathfrak{X}}_{i}(G) = \langle C_{\mathfrak{X}}(u_k \otimes e_{ij}) \mid 1 \le k \le r, 1 \le j \le m\rangle.$$

By Theorem 2.2, if $i \neq r$ then $V^{\mathfrak{X}}_{i}(G) \perp V^{\mathfrak{X}}_{r}(G)$. Thus

$$V^{\mathfrak{X}}(G) = \bigoplus_{i=1}^{n} V^{\mathfrak{X}}_{i}(G) \text{ (orthogonal)}.$$

For $1 \le j, s \le m$, if $j \sim s$ then by Lemma 1.4, $V^{\mathfrak{X}}_{ij}(G) = V^{\mathfrak{X}}_{is}(G)$, otherwise $V^{\mathfrak{X}}_{ij}(G) \perp V^{\mathfrak{X}}_{is}(G)$, by Theorem 2.2. Hence

$$V^{\mathfrak{X}}_{i}(G) = \bigoplus_{j\in\bar{\mathcal{D}}} V^{\mathfrak{X}}_{ij}(G) \text{ (orthogonal)}.$$

Therefore

$$V^{\mathfrak{X}}(G) = \bigoplus_{i=1}^{n} \bigoplus_{j\in\bar{\mathcal{D}}} V^{\mathfrak{X}}_{ij}(G) \text{ (orthogonal)}.$$

The following theorem provides a formula for computing the dimension of the generalized cyclic subspace.

Theorem 2.3. *Let $\mathfrak{X}$ be an irreducible unitary representation of G over a unitary space U. Suppose $\mathfrak{X}$ affords the irreducible character χ of G. If $j \in \bar{\mathcal{D}}$ then*

$$\dim\ V^{\mathfrak{X}}_{ij}(G) = [\chi, 1_{G_j}].$$

Proof. Let $j \in \bar{\mathcal{D}}$, $[G : G_j] = t$ and $G = \bigcup_{i=1}^{t} \sigma_i G_j$, be the left coset decomposition of G_j in G. Then $|Orb_G\,(j)| = t$. Suppose

$$Orb_G\,(j) = \{\sigma_1(j), \cdots, \sigma_t(j)\}.$$

Notice that

$$V^{\mathfrak{X}}_{ij}(G) = C_{\mathfrak{X}}(W_{ij}),$$

where

$$W_{ij} = \left\langle u_k \otimes e_{i\sigma(j)} \mid 1 \le k \le r,\ \sigma \in G\right\rangle.$$

Then

$$\mathbb{E}_{ij} = \{u_k \otimes e_{i\sigma_s(j)} \mid 1 \le k \le r,\ 1 \le s \le t\}$$

is a basis of W_{ij} but the set $C_{\mathfrak{X}}(\mathbb{E}_{ij})$ may not be a basis for $V^{\mathfrak{X}}_{ij}(G)$. Since W_{ij} is an invariant subspace of $C_{\mathfrak{X}}$, so the restriction $C_{\mathfrak{X}}\,|_{W_{ij}} = C^{ij}_{\mathfrak{X}} : W_{ij} \to W_{ij}$ is a linear operator. We put

$$[C^{ij}_{\mathfrak{X}}]_{\mathbb{E}_{ij}} = B = [b_{(k,l),(p,q)}].$$

Now we have

$$\begin{aligned} C^{ij}_{\mathfrak{X}}(u_p \otimes e_{i\sigma_q(j)}) &= C_{\mathfrak{X}}(u_p \otimes e_{i\sigma_q(j)}) \\ &= C_{\mathfrak{X}}(\mathfrak{X}(\sigma_q^{-1})u_p \otimes e_{ij}) \\ &= \frac{1}{|G|} \sum_{\sigma\in G} \mathfrak{X}(\sigma\sigma_q^{-1})u_p \otimes e_{i\sigma(j)} \end{aligned}$$

$$=\frac{1}{|G|}\sum_{l=1}^{t}\left(\sum_{\sigma\in\sigma_l G_j}\mathfrak{X}(\sigma\sigma_q^{-1})u_p\otimes e_{i\sigma(j)}\right)$$
$$=\frac{1}{|G|}\sum_{l=1}^{t}(\sum_{\tau\in G_j}\mathfrak{X}(\sigma_l\tau\sigma_q^{-1})u_p\otimes e_{i\sigma_l(j)})$$
$$=\frac{1}{|G|}\sum_{l=1}^{t}\sum_{\tau\in G_j}\sum_{k=1}^{r}m_{kp}(\sigma_l\tau\sigma_q^{-1})u_k\otimes e_{i\sigma_l(j)}$$
$$=\sum_{l=1}^{t}\sum_{k=1}^{r}\left[\frac{1}{|G|}\sum_{\tau\in G_j}m_{kp}(\sigma_l\tau\sigma_q^{-1})\right]u_k\otimes e_{i\sigma_l(j)}.$$

So

$$b_{(k,l),(p,q)}=\frac{1}{|G|}\sum_{\tau\in G_j}m_{kp}(\sigma_l\tau\sigma_q^{-1}).$$

We prove that B is an idempotent matrix. We have

$$\begin{aligned}(B^2)_{(k,l),(k',l')}&=\sum_{p=1}^{r}\sum_{q=1}^{t}b_{(k,l),(p,q)}b_{(p,q),(k',l')}\\
&=\sum_{p=1}^{r}\sum_{q=1}^{t}\left(\frac{1}{|G|}\sum_{\tau\in G_j}m_{kp}(\sigma_l\tau\sigma_q^{-1})\right)\left(\frac{1}{|G|}\sum_{\mu\in G_j}m_{pk'}(\sigma_q\mu\sigma_{l'}^{-1})\right)\\
&=\frac{1}{|G|^2}\sum_{p=1}^{r}\sum_{q=1}^{t}\sum_{\tau\in G_j}\sum_{\mu\in G_j}m_{kp}(\sigma_l\tau\sigma_q^{-1})m_{pk'}(\sigma_q\mu\sigma_{l'}^{-1})\\
&=\frac{1}{|G|^2}\sum_{\mu,\tau\in G_j}\sum_{q=1}^{t}m_{kk'}(\sigma_l\tau\mu\sigma_{l'}^{-1})\\
&=\frac{t|G_j|}{|G|^2}\sum_{g\in G_j}m_{kk'}(\sigma_l g\sigma_{l'}^{-1})\quad(g=\tau\mu)\\
&=\frac{1}{|G|}\sum_{g\in G_j}m_{kk'}(\sigma_l g\sigma_{l'}^{-1})\\
&=B_{(k,l),(k',l')}.\end{aligned}$$

Thus

$$\dim\ V_{ij}^{\mathfrak{X}}(G)=\text{rank}\ C_{\mathfrak{X}}^{ij}=\text{rank}\ B=\text{tr}\ B$$

Now we calculate tr B. We have

$$\begin{aligned}\text{tr}\ B&=\sum_{k=1}^{r}\sum_{l=1}^{t}b_{(k,l),(k,l)}\\
&=\sum_{k=1}^{r}\sum_{l=1}^{t}\left(\frac{1}{|G|}\sum_{\tau\in G_j}m_{kk}(\sigma_l\tau\sigma_l^{-1})\right)\\
&=\frac{1}{|G|}\sum_{\tau\in G_j}\sum_{l=1}^{t}\sum_{k=1}^{r}m_{kk}(\sigma_l\tau\sigma_l^{-1})\end{aligned}$$

$$=\frac{1}{|G|}\sum_{\tau\in G_j}\sum_{l=1}^{t}\chi(\sigma_l\tau\sigma_l^{-1})$$
$$=\frac{1}{|G|}\sum_{\tau\in G_j}\sum_{l=1}^{t}\chi(\tau)$$
$$=\frac{t}{|G|}\sum_{\tau\in G_j}\chi(\tau) \qquad ([G:G_j)]=t)$$
$$=\frac{1}{|G_j|}\sum_{\tau\in G_j}\chi(\tau)$$
$$=[\chi,1_{G_j}],$$

so the result holds. □

Now we construct a basis for the generalized Cartesian symmetry class $V^{\mathfrak{X}}(G)$. Since $V^{\mathfrak{X}}(G)=\bigoplus_{i=1}^{n}\bigoplus_{j\in\bar{D}}V_{ij}^{\mathfrak{X}}(G)$, in order to find a basis for $V^{\mathfrak{X}}(G)$, it suffices to find a basis for the generalized cyclic subspace $V_{ij}^{\mathfrak{X}}(G)$ for every $1\leq i\leq n$ and $j\in\bar{\mathcal{D}}$. Let $j\in\bar{\mathcal{D}}$ and $\dim V_{ij}^{\mathfrak{X}}(G)=s_j$. Since

$$V_{ij}^{\mathfrak{X}}(G)=\langle C_{\mathfrak{X}}(u_1\otimes e_{i\sigma(j)})\mid\sigma\in G\rangle,$$

so we can choose the ordered subset $\{j_1,\cdots,j_{s_j}\}$ from the orbit of j, such that the set

$$\{C_{\mathfrak{X}}(u_1\otimes e_{ij_1}),\cdots,C_{\mathfrak{X}}(u_1\otimes e_{ij_{s_j}})\}$$

is a basis for the generalized cyclic subspace $V_{ij}^{\mathfrak{X}}(G)$. Execute this procedure for each $k\in\bar{\mathcal{D}}$. If $\bar{\mathcal{D}}=\{j,k,l,\cdots\}$ $(j<k<l<\cdots)$, take

$$\hat{\mathcal{D}}=\{j_1,\cdots,j_{s_j};k_1,\cdots,k_{s_k};\cdots\}$$

to be ordered as indicated. Then

$$\{C_{\mathfrak{X}}(u_1\otimes e_{ij})\mid 1\leq i\leq n,\ j\in\hat{\mathcal{D}}\}$$

is a basis of $V^{\mathfrak{X}}(G)$. Hence

$$\dim V^{\mathfrak{X}}(G)=(\dim V)|\hat{\mathcal{D}}|=n\sum_{j\in\bar{\mathcal{D}}}s_j=n\sum_{j\in\bar{\mathcal{D}}}[\chi,1_{G_j}].$$

If $\mathfrak{X}$ is a linear representation of G, then $\dim V_{ij}^{\mathfrak{X}}(G)=1$ and the set

$$\{C_{\mathfrak{X}}(u_1\otimes e_{ij})\mid 1\leq i\leq n,\ j\in\bar{\mathcal{D}}\}$$

is an orthogonal basis of $V^{\mathfrak{X}}(G)$ (such representations of G are called *o.b.-representations*).

3. Induced Linear Operators on Generalized Cartesian Symmetry Classes

Let S_m be the full symmetric group of degree m, and let G be a subgroup of S_m. Let U be a unitary vector space. Given a linear operator $T:V\to V$, we can define the linear operator $\times^m T:\times^m V\to\times^m V$ by

$$(\times^m T)v^{\times}=(Tv_1,\cdots,Tv_m),$$

where $v^{\times}=(v_1,\cdots,v_m)\in\times^m V$. It is easy to see that $T\to\times^m T$ is an algebraic homomorphism. Moreover, $(\times^m T)Q(\sigma)=Q(\sigma)(\times^m T)$ for any $\sigma\in G$, which implies that

$$C_{\mathfrak{X}}(I\otimes\times^m T)=(I\otimes\times^m T)C_{\mathfrak{X}},$$

and hence, $V^{\mathfrak{X}}(G)$ is an invariant subspace of $U \otimes \times^m V$ under the mapping $C_{\mathfrak{X}}$. We denote the restriction of $I \otimes \times^m T$ to $V^{\mathfrak{X}}(G)$ by $K^{\mathfrak{X}}(T)$ and call it an induced operator. Note that $T \to K^{\mathfrak{X}}(T)$ is also an algebraic homomorphism.

Theorem 3.1. *Suppose $\mathfrak{X}$ is an irreducible unitary representation of G over unitary space U and let $S, T \in End\ (V)$ and $V^{\mathfrak{X}}(G) \neq 0$. Then*

(a) $K^{\mathfrak{X}}(T) = K^{\mathfrak{X}}(S) \iff T = S$,

(b) $K^{\mathfrak{X}}(T)$ *is invertible if and only if T is invertible.*

Proof. (a) Let $K^{\mathfrak{X}}(T) = K^{\mathfrak{X}}(S)$. Then for each $1 \leq i \leq n$, $1 \leq j \leq m$ and $u \in U$, we have

$$K^{\mathfrak{X}}(T)(C_{\mathfrak{X}}(u_k \otimes e_{ij})) = K^{\mathfrak{X}}(S)(C_{\mathfrak{X}}(u_k \otimes e_{ij})).$$

So

$$C_{\mathfrak{X}}(u_k \otimes (\delta_{1j}Te_i, \delta_{2j}Te_i, \cdots, \delta_{mj}Te_i)) = C_{\mathfrak{X}}(u_k \otimes (\delta_{1j}Se_i, \delta_{2j}Se_i, \cdots, \delta_{mj}Se_i)).$$

We put

$$x_\ell = \delta_{\ell j}Te_i = \delta_{\ell j}\sum_{p=1}^{n} a_{pi}e_p = \sum_{p=1}^{n} \delta_{\ell j}a_{pi}e_p,$$

$$y_\ell = \delta_{\ell j}Se_i = \delta_{\ell j}\sum_{p=1}^{n} b_{pi}e_p = \sum_{p=1}^{n} \delta_{\ell j}b_{pi}e_p.$$

Now we define two matrices C and D as follows:

$$C = (C_{\ell p}) = (\delta_{\ell j}a_{pi}), \quad D = (D_{\ell p}) = (\delta_{\ell j}b_{pi}).$$

Using Theorem 1.2, we get

$$Tr_{\mathfrak{X}}(CC^*) = Tr_{\mathfrak{X}}(CD^*) = Tr_{\mathfrak{X}}(DD^*). \tag{1}$$

We can easily see that

$$(CC^*)_{\ell p} = \delta_{\ell j}\delta_{pj}\sum_{q=1}^{n}|a_{qi}|^2 \tag{2}$$

$$(DD^*)_{\ell p} = \delta_{\ell j}\delta_{pj}\sum_{q=1}^{n}|b_{qi}|^2 \tag{3}$$

$$(CD^*)_{\ell p} = \delta_{\ell j}\delta_{pj}\sum_{q=1}^{n}a_{qi}\bar{b}_{qi} \tag{4}$$

$$Tr_{\mathfrak{X}}(CC^*) = \sum_{q=1}^{n}|a_{qi}|^2\sum_{\sigma\in G_j}\mathfrak{X}(\sigma) \tag{5}$$

$$Tr_{\mathfrak{X}}(DD^*) = \sum_{q=1}^{n}|b_{qi}|^2\sum_{\sigma\in G_j}\mathfrak{X}(\sigma) \tag{6}$$

$$Tr_{\mathfrak{X}}(CD^*) = \sum_{q=1}^{n}a_{qi}\bar{b}_{qi}\sum_{\sigma\in G_j}\mathfrak{X}(\sigma). \tag{7}$$

Applying the trace map on Equations (1), (5), (6), (7), we get

$$\sum_{q=1}^{n}|a_{qi}|^2\sum_{\sigma\in G_j}\chi(\sigma)=\sum_{q=1}^{n}|b_{qi}|^2\sum_{\sigma\in G_j}\chi(\sigma)=\sum_{q=1}^{mn}a_{qi}\bar{b}_{qi}\sum_{\sigma\in G_j}\chi(\sigma). \tag{8}$$

If we choose $j\in\bar{D}$, then $\sum_{\sigma\in G_j}\chi(\sigma)\neq 0$. Hence from Equation (8), we obtain

$$\sum_{q=1}^{n}|a_{qi}|^2=\sum_{q=1}^{n}|b_{qi}|^2=\sum_{q=1}^{n}a_{qi}\bar{b}_{qi}\ (1\le i\le n). \tag{9}$$

Thus

$$\sum_{i=1}^{n}\sum_{q=1}^{n}|a_{qi}|^2=\sum_{i=1}^{n}\sum_{q=1}^{n}|b_{qi}|^2=\sum_{i=1}^{n}\sum_{q=1}^{n}a_{qi}\bar{b}_{qi}, \tag{10}$$

which is equivalent to

$$tr(A^*A)=tr(B^*B)=tr(B^*A),$$

or

$$\|A\|^2=\|B\|^2=<A,B>, \tag{11}$$

where $\|.\|$ is the Frobenius norm $\mathbb{C}_{n\times n}$. From the equality condition in the Cauchy-Schwarz inequality, there exists a real number λ such that $A=\lambda B$. Now by substituting in Equation (11), we get $\lambda=1$ and then $A=B$. Therefore $T=S$. The converse is obvious.

(b) If T is invertible then $K^{\mathfrak{X}}(T)$ is invertible because $K^{\mathfrak{X}}$ is an algebraic homomorphism.
Conversely, if $K^{\mathfrak{X}}(T)$ is invertible then we prove that T is invertible. To show this, suppose that T is a singular operator. Then there exists a non-zero vector $e_1\in V$ such that $Te_1=0$. We can extend the set $\{e_1\}$ to an orthonormal basis $\{e_1,\cdots,e_n\}$ for V. Let $\{u_1,\cdots,u_r\}$ be an orthonormal basis for the unitary space U. Since $V^{\mathfrak{X}}(G)\neq 0$, we have $\bar{\mathcal{D}}\neq\emptyset$. Choose $j\in\bar{\mathcal{D}}$, then j belongs to $\Omega=\cup_1^r\Omega_k$. Therefore, there exists $1\le k\le r$ such that $j\in\Omega_k$. It follows that $C_{\mathfrak{X}}(u_k\otimes e_{ij})\neq 0$. Now we have

$$\begin{aligned}K^{\mathfrak{X}}(T)C_{\mathfrak{X}}(u_k\otimes e_{1j})=&(I\otimes\times^mT)C_{\mathfrak{X}}(u_k\otimes e_{1j})\\=&C_{\mathfrak{X}}(u_k\otimes\times^mT(\delta_{1j}e_1,\cdots,\delta_{mj}e_1))\\=&C_{\mathfrak{X}}(u_k\otimes(\delta_{1j}Te_1,\cdots,\delta_{mj}Te_1))\\=&C_{\mathfrak{X}}(u\otimes(0,\cdots,0))\\=&0,\end{aligned}$$

which is a contradiction. Therefore, T must be a non-singular operator. This completes the proof. □

Theorem 3.2. *Suppose $\mathfrak{X}$ is an irreducible unitary representation of G over unitary space U and let $S,T\in End\ (V)$. Then $K^{\mathfrak{X}}(T)^*=K^{\mathfrak{X}}(T^*)$ and $K^{\mathfrak{X}}(T)$ is (a) normal, (b) unitary, (c) Hermitian, (d) skew-Hermitian, (e) p.s.d, or (f) p.d if and only if T has the corresponding property.*

Proof. We know that $V^{\mathfrak{X}}(G)$ is an invariant subspace under of the both $I\otimes\times^mT$ and $(I\otimes\times^mT)^*=I\otimes\times^mT^*$. Thus

$$K^{\mathfrak{X}}(T)^*=((I\otimes\times^mT)\mid_{V^{\mathfrak{X}}(G)})^*=(I\otimes\times^mT)^*\mid_{V^{\mathfrak{X}}(G)}=K^{\mathfrak{X}}(T^*).$$

If T is (a) normal, (b) unitary, (c) Hermitian, (d) skew-Hermitian, (e) p.s.d, or (f) p.d, then $I\otimes\times^mT$ has the corresponding property and so, $K^{\mathfrak{X}}(T)=(I\otimes\times^mT)\mid_{V^{\mathfrak{X}}(G)}$ has also the corresponding property.
Conversely, if $K^{\mathfrak{X}}(T)$ is normal, then

$$K^{\mathfrak{X}}(T^*T)=K^{\mathfrak{X}}(T^*)K^{\mathfrak{X}}(T)=K^{\mathfrak{X}}(T)^*K^{\mathfrak{X}}(T)=K^{\mathfrak{X}}(T)K^{\mathfrak{X}}(T)^*=K^{\mathfrak{X}}(T)K^{\mathfrak{X}}(T^*)=K^{\mathfrak{X}}(TT^*).$$

By Theorem 3.1, $TT^* = T^*T$, i.e., T is normal. Similarly, if $K^{\mathfrak{X}}(T)$ is unitary or Hermitian, then so does T. If $K^{\mathfrak{X}}(T)$ is skew-Hermitian, then $K^{\mathfrak{X}}(T)^* = -K^{\mathfrak{X}}(T)$, so $K^{\mathfrak{X}}(T^*) = K^{\mathfrak{X}}(-T)$ because $K^{\mathfrak{X}}$ is an algebraic homomorphism. Then, by Theorem 3.1, $T^* = -T$, i.e; T is skew-Hermitian.
Now we consider $K^{\mathfrak{X}}(T)$ be a p.s.d operator. For every $v \in V$, define $v_j^* = (\delta_{1j}v, \cdots, \delta_{mj}v)$. Then we have

$$\begin{aligned}
&\langle K^{\mathfrak{X}}(T)C_{\mathfrak{X}}(u_1 \otimes v_j^*), C_{\mathfrak{X}}(u_1 \otimes v_j^*)\rangle \\
&=\langle C_{\mathfrak{X}}(u_1 \otimes \times^m T(\delta_{1j}v, \cdots, \delta_{mj}v)), C_{\mathfrak{X}}(u_1 \otimes (\delta_{1j}v, \cdots, \delta_{mj}v))\rangle \\
&=\langle C_{\mathfrak{X}}(u_1 \otimes (\delta_{1j}Tv, \cdots, \delta_{mj}Tv)), C_{\mathfrak{X}}(u_1 \otimes (\delta_{1j}v, \cdots, \delta_{mj}v))\rangle \\
&=\frac{1}{|G|}\langle Tr_{\mathfrak{X}}(A)u_1, u_1\rangle \\
&=\frac{1}{|G|}\langle \sum_{\sigma\in G} \mathfrak{X}(\sigma) \sum_{p=1}^{n} a_{p\sigma(p)}u_1, u_1\rangle \\
&=\frac{1}{|G|}\langle \sum_{\sigma\in G} \mathfrak{X}(\sigma) \sum_{p=1}^{n} \delta_{pj}\delta_{\sigma(p)j}\langle Tv, v\rangle u_1, u_1\rangle \\
&=\langle Tv, v\rangle \frac{1}{|G|}\langle \sum_{\sigma\in G} \mathfrak{X}(\sigma)\delta_{\sigma(j)j}u_1, u_1\rangle \\
&=\langle Tv, v\rangle\langle \frac{1}{|G|} \sum_{\sigma\in G_j} \mathfrak{X}(\sigma)u_1, u_1\rangle \\
&=\langle Tv, v\rangle \frac{|G_j|}{|G|}\langle T_j u_1, u_1\rangle \\
&=\langle Tv, v\rangle \frac{|G_j|}{|G|}\langle T_j u_1, T_j u_1\rangle \\
&=\langle Tv, v\rangle \frac{|G_j|}{|G|}\, \|T_j u_1\|^2 \geq 0,
\end{aligned}$$

where $a_{pq} = \langle \delta_{pj}Tv, \delta_{qj}v\rangle = \delta_{pj}\delta_{qj}\langle Tv, v\rangle$. Consequently, $\langle Tv, v\rangle \geq 0$. for all $v \in V$, i.e., T is p.s.d. If $K^{\mathfrak{X}}(T)$ is p.d, then $j \in \bar{\mathcal{D}}$. So $C_{\mathfrak{X}}(u_1 \otimes v_j^*) \neq 0$. Hence

$$\langle K^{\mathfrak{X}}(T)C_{\mathfrak{X}}(u_1 \otimes v_j^*), C_{\mathfrak{X}}(u_1 \otimes v_j^*)\rangle = \langle Tv, v\rangle \frac{|G_j|}{|G|}\, \|T_j u_1\|^2 > 0.$$

Consequently, $\langle Tv, v\rangle > 0$ for all $v \in V$, i.e., T is p.d. □

Corollary 3.3. *Suppose $\mathfrak{X}$ is an irreducible unitary representation of G over unitary space U and let $S, T \in End\ (V)$ such that $T \geq S$. Then $K^{\mathfrak{X}}(T) \geq K^{\mathfrak{X}}(S)$.*

Proof. Assume that $T \geq S$. Then $T - S \geq 0$. Hence $\times^m(T - S) \geq 0$. Therefore $\times^m T - \times^m S \geq 0$. Consequently $\times^m T \geq \times^m S$. This implies that

$$K^{\mathfrak{X}}(T) = (I \otimes \times^m T)|_{V^{\mathfrak{X}}(G)} \geq (I \otimes \times^m S)|_{V^{\mathfrak{X}}(G)} = K^{\mathfrak{X}}(S).$$

□

Theorem 3.4. *Suppose $\mathfrak{X}$ is an irreducible unitary representation of G over the unitary space U and let $T \in End\ (V)$. Then*

$$rank\ (K^{\mathfrak{X}}(T)) = rank\ (T)\ |\hat{\mathcal{D}}|.$$

Proof. Suppose $\mathbb{F} = \{u_1, \cdots, u_r\}$ and $\mathbb{E} = \{e_1, \cdots, e_n\}$ are orthonormal bases for unitary spaces U and V, respectively. We can assume that the set $\{e_1, \cdots, e_s\}$ is a basis for Ker T. Then the set

$$\{C_{\mathfrak{X}}(u_1 \otimes e_{ij}) \mid 1 \leq i \leq n, j \in \hat{\mathcal{D}}\}$$

is a basis of $V^{\mathfrak{X}}(G)$. For every $1 \leq i \leq s$, we have

$$\begin{aligned} K^{\mathfrak{X}}(T)C_{\mathfrak{X}}(u_1 \otimes e_{ij}) = &(I \otimes \times^m T)C_{\mathfrak{X}}(u_1 \otimes e_{ij}) \\ =&C_{\mathfrak{X}}(u_1 \otimes \times^m Te_{ij}) \\ =&C_{\mathfrak{X}}(u_1 \otimes (\delta_{1j}Te_i, \cdots, \delta_{mj}Te_i) \\ =&C_{\mathfrak{X}}(u_1 \otimes (0, \cdots, 0)) \\ =&0. \end{aligned}$$

Let

$$Te_i = \sum_{j=1}^{n} a_{ji}e_j \qquad (1 \leq i \leq n).$$

Let $s + 1 \leq i \leq n$ and $j \in \bar{\mathcal{D}}$. For every $1 \leq k \leq m$, we define

$$x_k = \delta_{kj}Te_i = \delta_{kj}\sum_{\ell=1}^{n} a_{\ell i}e_\ell = \sum_{\ell=1}^{n} \delta_{kj}a_{\ell i}e_\ell.$$

Let $B = [b_{k\ell}] = [\delta_{kj}a_{\ell i}]$. Then

$$\begin{aligned} Tr_{\mathfrak{X}}(BB^*) =& \sum_{\sigma\in G} \mathfrak{X}(\sigma) \sum_{k=1}^{n} (BB^*)_{k\sigma(k)} \\ =& \sum_{\sigma\in G} \mathfrak{X}(\sigma) \sum_{k=1}^{n}\sum_{\ell=1}^{n} b_{k\ell}\bar{b}_{\sigma(k)\ell} \\ =& \sum_{\sigma\in G} \mathfrak{X}(\sigma) \sum_{k,\ell=1}^{n} \delta_{kj}a_{\ell i}\delta_{\sigma(k)j}\bar{a}_{\ell i} \\ =& \sum_{\ell=1}^{n} |a_{\ell i}|^2 \sum_{\sigma\in G_j} \mathfrak{X}(\sigma) \\ =& \sum_{\ell=1}^{n} |a_{\ell i}|^2 |G_j| T_j \neq 0. \end{aligned}$$

Using Theorem 1.2, we deduce that $K^{\mathfrak{X}}(T)C_{\mathfrak{X}}(u_1 \otimes e_{ij}) = C_{\mathfrak{X}}(u_1 \otimes x^\times) \neq 0$. Also, for every $1 \leq i \leq n$ and $j \in \bar{\mathcal{D}}$, we have

$$\begin{aligned} K^{\mathfrak{X}}(T)C_{\mathfrak{X}}(u_1 \otimes e_{ij}) =&C_{\mathfrak{X}}\left(u_1 \otimes \left(\delta_{1j}Te_i, \cdots, \delta_{mj}Te_i\right)\right) \\ =&C_{\mathfrak{X}}\left(u_1 \otimes \left(\sum_{\ell=1}^{n} \delta_{1j}a_{\ell i}e_\ell, \cdots, \sum_{\ell=1}^{n} \delta_{mj}a_{\ell i}e_\ell\right)\right) \\ =&C_{\mathfrak{X}}\left(u_1 \otimes \sum_{\ell=1}^{n} a_{\ell i}(\delta_{1j}e_\ell, \cdots, \delta_{mj}e_\ell)\right) \\ =&C_{\mathfrak{X}}\left(u_1 \otimes \sum_{\ell=1}^{n} a_{\ell i}e_{\ell j}\right) \end{aligned}$$

$$= \sum_{\ell=1}^{n} a_{\ell i} C_{\mathfrak{X}}(u_1 \otimes e_{\ell j}).$$

This shows that the representation of $K^{\mathfrak{X}}(T)$ under the basis

$$\mathbb{S} = \bigcup_{j \in \hat{\mathcal{D}}} \left\{ C_{\mathfrak{X}}(u_1 \otimes e_{1j}), \cdots, C_{\mathfrak{X}}(u_1 \otimes e_{nj}) \right\}$$

is the following block matrix

$$\begin{bmatrix} \begin{bmatrix} T \end{bmatrix} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \begin{bmatrix} T \end{bmatrix} \end{bmatrix}_{|\hat{\mathcal{D}}| \times |\hat{\mathcal{D}}|}.$$

Therefore

$$\text{rank } K^{\mathfrak{X}}(T) = \text{rank } (T)\, |\hat{\mathcal{D}}|.$$

□

Using the above block matrix representation of $K^{\mathfrak{X}}(T)$, we obtain the following corollary.

Corollary 3.5. *Let $\mathfrak{X}$ be an irreducible unitary representation of G over unitary space U and let $T \in End\,(V)$. Then*

$$\det\ K^{\mathfrak{X}}(T) = (\det\ T)^{|\hat{\mathcal{D}}|}.$$

4. Open problems

Problem 4.1. *Characterize the subgroups of S_m whose irreducible representations are all o.b.-representations.*

Problem 4.2. *Let G be a subgroup of S_m and $\mathfrak{X}$ be an irreducible unitary representation of G. Determine the conditions on $\mathfrak{X}$ such that $V^{\mathfrak{X}}(G)$ has an orthogonal basis consisting the generalized Cartesian standard symmetrized vectors.*

Problem 4.3. *Determine the conditions on G and $\mathfrak{X}$ such that $V^{\mathfrak{X}}(G) \neq 0$.*

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Conflicts of Interest

The authors declare that they have no conflicts of interest.

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