Filomat 38:25 (2024), 8907–8916
https://doi.org/10.2298/FIL2425907K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some topological properties of intuitionistic fuzzy quasi normed space

Vakeel A. Khan[a,*], Mohd Faisal[a], Mohd Kamran

[a]*Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India*

Abstract. We establish the concept of an intuitionistic fuzzy quasi-normed space and provide an illustrative example. Through our exploration, we ascertain that an intuitionistic fuzzy quasi-norm can indeed transition into an intuitionistic fuzzy norm. However, it is worth noting that not every topology that supports intuitionistic fuzzy quasi-norms can be classified as metrizable. We establish the definitions of open balls and sequence convergence within intuitionistic fuzzy quasi-normed spaces. Additionally, we introduce the concepts of left/right $\mathcal{N}$-Cauchy sequences, both in the context of topology $\tau_{\mathcal{N}}$ and its inverse $\tau_{\mathcal{N}^{-1}}$, further elucidating the notion of $\mathcal{N}$-Cauchy sequences. We establish the proof of the open mapping theorem for intuitionistic fuzzy quasi-normed spaces.

1. Introduction

In 1965, L. A. Zadeh introduced a pioneering theory on fuzzy sets, extending the principles of crisp set theory [30]. The fuzzy norm concept for linear spaces was initially proposed by Katsaras [19], followed by Felbin's alternative definition in 1992, including a corresponding metric of the Kaleva and Seikkala type [14, 18]. Cheng and Mordeson further elaborated on this in 1994, and recent discussions by Xiao et al. [6] explored relationships between the axioms of KM fuzzy normed spaces and KM fuzzy metric spaces. Subsequently, Bag and Samanta introduced a modified fuzzy norm in their works (references [10, 11]), which may find application in specific scenarios. This concept has been utilized in the advancement of fuzzy functional analysis, as discussed in references ranging from [5] to [27].

In a different light, Atanassov put forward the concept of intuitionistic fuzzy set, which introduced a singular membership function delineating the degree of non-affiliation, as an extension of the fuzzy set theory. This proposal, distinct from conventional fuzzy sets, was advanced in Atanassov's work referenced as [7]. Following suit, Park introduced the notion of intuitionistic fuzzy metric space in 2004, as expounded upon in [24]. Further details on intuitionistic fuzzy metric space and associated findings can be found in references [2],[13],[25]. Additionally, in 2006, Saadati and Park introduced the concept of intuitionistic fuzzy normed space [26], with more comprehensive information available in references [31],[1] regarding intuitionistic fuzzy normed space and related developments.

Alegre and Romaguera's groundbreaking research [4] introduced the concept of a fuzzy quasi-norm, departing from the traditional symmetrical properties of fuzzy norms [10]. Expanding the application scope,

2020 *Mathematics Subject Classification*. Primary: 54A40; Secondary: 03B20, 46S40, 46A30.
Keywords. Fuzzy set, fuzzy quasi norm, intuitionistic fuzzy set, intuitionistic fuzzy quasi norm.
Received: 19 December 2023; Revised: 15 April 2024; Accepted: 30 April 2024
Communicated by Ljubiša D. R. Kočinac
The authors would like to thank the Joint CSIR-UGC, Government of India for JRF (NTA Ref. No. 191620117130).
* Corresponding author: Vakeel A. Khan
Email addresses: vakhanmaths@gmail.com (Vakeel A. Khan), faisalmohd114@gmail.com (Mohd Faisal), kamshafiq@gmail.com (Mohd Kamran)

they utilized fuzzy quasi-norms to model paratopological vector spaces [3], offering a novel perspective on the structure of such spaces. Subsequently, their work [6] explored fuzzy quasi-normed fields, uncovering crucial results such as the uniform boundedness theorem, which has significant implications in functional analysis and beyond.

Further advancing the field, Gao et al. [15] recently contributed the decomposition theorem for fuzzy quasi-norms, shedding light on the intricate inner workings of these mathematical structures. Meanwhile, Hussein and Al-Basri's investigation [17] into the completion of quasi-fuzzy normed algebras over fuzzy fields deepened our understanding of algebraic structures in fuzzy settings.

Highlighting the practical significance of fuzzy quasi-normed spaces, Alegre and Romaguera [4] emphasized their role in analyzing the complexities of exponential time algorithms, providing valuable insights into approximation theory and theoretical computer science.

In the domain of functional analysis, the open mapping theorem stands as a fundamental result, elucidating the behavior of mappings between topological spaces. Recent research by Jianrong Wu and Ruini Li [29] extended this theorem to fuzzy quasi-normed spaces, paving the way for new avenues of exploration in functional mappings within fuzzy environments and broadening the applicability of the theorem to diverse mathematical contexts. Through these combined efforts, the field of fuzzy mathematics continues to evolve, offering novel insights and practical solutions to complex problems.

Inspired by the preceding insights, this paper endeavors to introduce the concept of intuitionistic fuzzy quasi-normed spaces, thereby extending the framework initially proposed in [4] concerning fuzzy quasi-norm spaces. Additionally, we aim to contribute to the field by establishing the open mapping theorem within the context of intuitionistic fuzzy quasi-normed spaces, thus enriching the theoretical understanding and practical applications of this mathematical framework. .

2. Preliminaries

Definition 2.1. ([28]) A binary operation $\star : [0,1]\times[0,1] \to [0,1]$ qualifies as a continuous $t-norm$ if it adheres to the following properties:

(a) $s\star t = t\star s \quad \forall\, s,t\in[0,1]$;
(b) $s\star(t\star u) = (s\star t)\star u \quad \forall\, s,t,u\in[0,1]$;
(c) $s\star t \le u\star d$ whenever $s\le u$ and $t\le d \quad \forall\, s,t,u,d\in[0,1]$;
(d) $s\star 1 = s \quad \forall\, s\in[0,1]$;
(e) $\star$ is continuous.

Definition 2.2. ([28]) A binary operation $\circ : [0,1]\times[0,1] \to [0,1]$ qualifies as a continuous $t-conorm$ if it adheres to the following properties:

(a) $s\circ t = t\circ s \quad \forall\, s,t\in[0,1]$;
(b) $s\circ(t\circ u) = (s\circ t)\circ u \quad \forall\, s,t,u\in[0,1]$;
(c) $s\circ t \le u\circ d$ whenever $s\le u$ and $t\le d \quad \forall\, s,t,u,d\in[0,1]$;
(d) $s\circ 0 = s \quad \forall\, s\in[0,1]$;
(e) $\circ$ is continuous.

Example 2.3. Let $\star$ be a binary operation from $[0,1]\times[0,1]$ to $[0,1]$ and defined as $\star(s,u) = \min\{s,u\}$ for all $s,u\in[0,1]$. Then $\star$ is a continuous $t-norm$. Usually this $t-norm$ is denoted by $\wedge$.

Example 2.4. Let $\circ$ be a binary operation from $[0,1]\times[0,1]$ to $[0,1]$ and defined as $\circ(s,u) = \max\{s,u\}$ for all $s,u\in[0,1]$. Then $\circ$ is a continuous $t-conorm$. Usually this $t-conorm$ is denoted by $\vee$.

Proposition 2.5. ([16]) *Suppose $\star$ and $\circ$ function as continuous $t-norm$ and $t-conorm$, respectively. Then*

(i) *If $0<d_1<d_2<1$, there exists $d_3, d_4\in(0,1)$ such that $d_1\star d_3\ge d_2$ and $d_1\ge d_4\circ d_2$.*
(ii) *If $0<d_5<1$, then there exists $d_6, d_7\in(0,1)$ such that $d_6\star d_6\ge d_5$ and $d_7\circ d_7\le d_5$.*

Definition 2.6. Let X be real vector space and $Q : X \to [0, \infty)$ is functional then Q qualifies to be quasi-norm (also called asymmetric norm in [12]) if it satisfying the following properties:

(i) $Q(v) = Q(-v) = 0 \implies v = 0$
(ii) $Q(\alpha v) = \alpha Q(v) \quad \forall\, \alpha \geq 0$
(iii) $Q(v + u) \leq Q(v) + Q(u) \quad \forall\, v, u \in X$

Definition 2.7. ([29]) Let ℓ^p be the collection of all $p - summable$ sequences i.e.;

$$\ell^p = \Big\{x = (x_n) \in \omega : \sum_{n=1}^{\infty} |x_n|^p < \infty\Big\}.$$

For $0 < p < 1$, the space ℓ^p does not possess a norm; instead, it is equipped with a metric defined as: $d(\xi, \eta) = \sum_{n=1}^{\infty} |\xi_n - \eta_n|^p$. For $1 \leq p < \infty$, ℓ^p is a norm linear space with norm defined as follows

$$\|\xi\|_p = \Big(\sum_{n=1}^{\infty} |\xi_n|^p\Big)^{1/p}.$$

Definition 2.8. ([29]) A paratopological vector space is denoted by the 4-tuple $(\mathrm{V}, +, ., \tau)$, wherein (V, τ) constitutes a T_0 topology on V and the addition operation + is continuous. For any neighborhood B of $r\xi$, where $\xi \in \mathrm{V}$ and $r \geq 0$, there exists a neighborhood B' of ξ and a positive $r > 0$ such that the interval $[r, r + r)B' \subseteq B$.

To know more about paratopological vector spaces see [3]. Simply paratopological vector space $(\mathrm{V}, +, ., \tau)$ is represented by (V, τ), if no confusion arises.

Definition 2.9. ([29]) Let M be a subset of real vector space V. Then

(a) M is semibalanced if for each $x \in M$, $rx \in M$ whenever $r \in [0, 1]$.
(b) M is absorbing if for each $x \in \mathrm{V}$, there is $r_0 > 0$ such that $r_0 x \in M$.

Lemma 2.10. ([29]) *Let (V, τ) be a paratopological vector space.*

(a) *If M is convex subset of V and $int(M) \neq \phi$ then $(1 - \alpha)intA + \alpha M \subseteq int(M)$, where $\alpha \in (0, 1)$ and consequently $intM$ is convex.*
(b) *If M is absorbing, convex subset of V and $intM \neq \phi$ then $\Theta \in int(M)$.*
(c) *If $\mathfrak{B}(\Theta)$ is a base of Θ-neighborhoods, then $cl(M) = \{M - B : B \in \mathfrak{B}\}$*

=

Definition 2.11. ([22]) Mapping H :V $\to U$ is open map, if the set $H(B)$ is open in U for every open set B in V.

Theorem 2.12. ([29]) *In the context of fuzzy quasi-normed space $(V, N, \star)$ it follows that $(V, \tau_N, \star)$ constitutes a quasi-metrizable paratopological vector space.*

Theorem 2.13. ([29]) *Consider fuzzy quasi-normed spaces $(V, N, \star)$ and $(U, \mathcal{N}', \star')$. Assume that $(V, N, \star)$ is right N-complete and $(U, \mathcal{N}', \star')$ is of the half second category and Hausdorff. If L: $V \to U$ is a linear, surjective, and continuous mapping, then L is open.*

Remark 2.14. ([29]) If $t - norm$ is chosen as $\star(a, b) = \min\{a, b\}$, then $B_N(\Theta)$ is convex.

Remark 2.15. ([29]) $B_N(x, r_2, t) \subseteq B_N(x, r_1, t)$, if $r_1 > r_2 > 0$ and $t_1 > t_2 > 0$ then $B_N(x, r, t_2) \subseteq B_N(x, r, t_1)$. Now the set $\{B_N(x, r_n, tn) : r_n \in (0, 1), t_n > 0, n \in \mathbb{N}\}$ forms a fundamental set of neighborhoods of x in (V, τ_N), where both sequences $\{x_n\}$ and $\{t_n\}$ converges to 0.

Lemma 2.16. ([29]) *Let $(V, F, \star)$ be fuzzy quasi normed space and $\mathfrak{B}(\Theta)$ be the collection of open balls center at origin. Then:*

(a) $B_N(\Theta, r, t)$ *is absorbing for all* $t > 0$ *and* $r \in (0, 1)$.
(b) $B_N(\Theta, r, t)$ *is semibalanced for all* $t > 0$ *and* $r \in (0, 1)$.
(c) $\lambda B_N(\Theta, r, t) = B_N\Big(\Theta, r, \lambda t\Big)$ *for every* $\lambda > 0$, $t > 0$ *and* $r \in (0, 1)$
(d) *if* $B \in \mathfrak{B}(\Theta)$, *there is* $B' \in \mathfrak{B}(\Theta)$ *such that* $B' + B' \subseteq B$.
(e) *if* $B, B' \in \mathfrak{B}(\Theta)$, *there is* $B'' \in \mathfrak{B}(\Theta)$ *such that* $B'' \subseteq B \cap B'$.

Lemma 2.17. ([29]) *Let M be a subset of fuzzy quasi normed space $(V, F, \star)$, $t > 0$. Then:*

(a) $int_N(tM) = t\, int_N(M)$
(b) $cl_N(tM) = t\, cl_N(M)$

Lemma 2.18. ([29]) *If A is absorbent and convex subset of fuzzy quasi normed space $(V, F, \star)$ then $cl(A)$ is too.*

Lemma 2.19. ([29]) *Let (V, τ), (U, τ') be two para topological spaces, $T : V \to U$ be linear mapping. Then T is open if and only if $\Theta_U \in intT(B)$ for all $B \in \mathfrak{B}$, where $\mathfrak{B}$ is a base of Θ_V- neighborhoods.*

3. Main results

Definition 3.1. Let V be a real vector space, and $\star, \circ$ be continuous $t-norm$ and $t-conorm$ respectively. Let F, G be fuzzy sets on V$\times[0, \infty)$. Then $(\mathrm{V}, \mathrm{F}, \mathrm{G}, \star, \circ)$ is said to be intuitionistic fuzzy quasi normed space, if

(a) $\mathrm{F}(x, \mathbb{0}) = 0;\ \ \forall\, x \in \mathrm{V}$
(b) $\mathrm{F}(x, t) = \mathrm{F}(-x, t) = 1\ \ \forall\, t > 0 \iff x = 0;$
(c) $\mathrm{F}(\alpha x, t) = \mathrm{F}\Big(x, \dfrac{t}{\alpha}\Big)\ \ \forall\, \alpha > 0;$
(d) $\mathrm{F}(x, t) \star \mathrm{F}(y, s) \le \mathrm{F}(x + y, t + s)\ \ \ \forall\, x, y \in \mathrm{V}$ and $s, t > 0;$
(e) $\mathrm{F}(x, -) : [0, \infty) \to [0, 1]$ is left continuous;
(f) $\lim_{t\to\infty} \mathrm{F}(x, t) = 1;$
(g) $\mathrm{G}(x, 0) = 1;\ \ \forall\, x \in \mathrm{V}$
(h) $\mathrm{G}(x, t) = \mathrm{G}(-x, t) = 0\ \ \forall\, t > 0 \iff x = 0;$
(i) $\mathrm{G}(\alpha x, t) = \mathrm{G}\Big(x, \dfrac{t}{\alpha}\Big)\ \ \forall\, \alpha > 0;$
(j) $\mathrm{G}(x, t) \circ \mathrm{G}(y, s) \ge \mathrm{G}(x + y, t + s)\ \ \ \forall\, x, y \in \mathrm{V}$ and $s, t > 0;$
(k) $\mathrm{G}(x, -) : [0, \infty) \to [0, 1]$ is right continuous;
(l) $\lim_{t\to\infty} \mathrm{G}(x, t) = 0;$

An intuitionistic fuzzy quasi norm on V is intuitionistic fuzzy norm if $\mathrm{F}(\alpha x, t) = \mathrm{F}\Big(x, \dfrac{t}{|\alpha|}\Big)$ and $\mathrm{G}(\alpha x, t) = \mathrm{G}\Big(x, \dfrac{t}{|\alpha|}\Big)$ for $\alpha \neq 0$. We will denote intuitionistic fuzzy quasi norm by $\mathcal{N} = (\mathrm{F}, \mathrm{G}, \star, \circ)$. If $\mathcal{N}$ is an intuitionistic fuzzy quasi norm on V, than $\mathcal{N}^{-1}$ is also intuitionistic fuzzy quasi norm, where $\mathcal{N}^{-1}$ is $(\mathrm{F}^{-1}, \mathrm{G}^{-1}, \star, \circ)$ and $\mathrm{F}^{-1}(x, t) = \mathrm{F}(-x, t)$ and $\mathrm{G}^{-1}(x, t) = \mathrm{G}(-x, t)$. Moreover, $\mathcal{N}^s$ defined as

$$\mathcal{N}^s = \Big(\min\{\mathrm{F}(x, t), \mathrm{F}(-x, t)\}, \max\{\mathrm{G}(x, t), \mathrm{G}(-x, t)\}, \star, \circ\Big)$$

is intuitionistic fuzzy norm on V.

Each intuitionistic fuzzy norm $\mathcal{N}$ induces a T_0 topology $\tau_{\mathcal{N}}$ on V generated by the base of open balls

$$\mathfrak{B}(x) = \{B_{\mathcal{N}}(x, r, t) : x \in \mathrm{V},\ t > 0 \text{ and } r \in (0, 1)\} \tag{1}$$

where

$$B_{\mathcal{N}}(x, r, t) = \{y \in \mathrm{V} : \mathrm{F}(x - y, t) > 1 - r,\ \mathrm{G}(x - y, t) < r\}. \tag{2}$$

From equation 1, we can define base of open balls center at Θ(origin)

$$\mathfrak{B}(\Theta) = \{B_{\mathcal{N}}(\Theta, r, t) : \Theta \in \mathrm{V},\ t > 0 \text{ and } r \in (0, 1)\} \tag{3}$$

where

$$B_{\mathcal{N}}(\Theta, r, t) = \{y \in \mathrm{V} : \mathrm{F}(y, t) > 1 - r,\ \mathrm{G}(y, t) < r\} \tag{4}$$

Definition 3.2. A sequence $\{x_n\}$ in $(\mathrm{V}, \tau_{\mathcal{N}})$ converges to x if $\lim_{n\to\infty} \mathrm{F}(x_n - x, t) = 1$ and $\lim_{n\to\infty} \mathrm{G}(x_n - x, t) = 0$ for all $t > 0$.

Definition 3.3. Let $(\mathrm{V}, \mathrm{F}, \mathrm{G}, \star, \circ)$ be intuitionistic fuzzy quasi normed space then a sequence x_n in V is left/right $\mathcal{N} - Cauchy/(\mathcal{N}^{-1} - Cauchy)$ with respect to topology $\tau_{\mathcal{N}}/(\tau_{\mathcal{N}^{-1}})$ if $x_m - x_n \to 0$ as $m, n \to \infty$ for $m > n$ respectively.

Definition 3.4. An intuitionistic fuzzy quasi normed space $(\mathrm{V}, \mathrm{F}, \mathrm{G}, \star, \circ)$ is said to be left/right complete if every left/right $\mathcal{N} - Cauchy/\mathcal{N}^{-1} - Cauchy$ sequence is convergent in V.

Definition 3.5. Let S be a subspace of a intuitionistic fuzzy quasi normed space $(\mathrm{V}, \mathrm{F}, \mathrm{G}, \star, \circ)$, then S said to be of half second category if $S = \bigcup_{n=1}^{\infty} M_n$, there exists positive integer m such that

$$int_{\mathcal{N}'}(cl_{\mathcal{N}'^{-1}} M_m) \neq \phi.$$

We denote closure and interior of a set A in $(\mathrm{V}, \tau_{\mathcal{N}})$ by $cl_{\mathcal{N}}A$ and $int_{\mathcal{N}}A$ respectively.

Example 3.6. Let $(\mathrm{V}, Q(x))$ be a quasi normed space, and $\star, \circ$ be continuous $t - norm$ and $t - conorm$ respectively. Let F, G be fuzzy sets on $\mathrm{V}\times[0, \infty)$ defined as follows;

$$\mathrm{F}(x, t) = \begin{cases} \dfrac{t}{t + Q(x)}, & t > 0 \\ 0, & t = 0 \end{cases}$$

and

$$\mathrm{G}(x, t) = \begin{cases} \dfrac{Q(x)}{t + Q(x)}, & t > 0 \\ 1, & t = 0, \end{cases}$$

where $Q(x)$ is the quasi norm on V. Then $(\mathrm{V}, \mathrm{F}, \mathrm{G}, \star, \circ)$ is intuitionistic fuzzy quasi norm on V.

Example 3.7. Let $(\ell^p, \|\cdot\|_{+p})$ be a quasi normed linear space, where $1 \leq p < \infty$ and

$$\|x\|_{+p} = \left(\sum_{n=0}^{\infty} max\{x_n, 0\}^p\right)^{1/p}$$

In the scenario where $x = (x_n) \in \ell^p$. , and for $0 < p < 1$, it constitutes a quasi-metrizable topological vector space but lacks quasi-normability. Nevertheless, each $(\ell^p, \|.\|_{+p})$ can be characterized as intuitionistic fuzzy quasi-normable through an intuitionistic fuzzy quasi-norm (F, G) defined as follows:

For $0 < p < 1$,

$$\mathrm{F}_p(x,t) = \begin{cases} \dfrac{t^p}{t^p + \sum\limits_{n=0}^{\infty}(max\{x_n, 0\})^p}, & t > 0 \\ 0, & t = 0 \end{cases} \tag{5}$$

and

$$\mathrm{G}_p(x,t) = \begin{cases} \dfrac{\sum\limits_{n=0}^{\infty}(max\{x_n, 0\})^p}{t^p + \sum\limits_{n=0}^{\infty}(max\{x_n, 0\})^p}, & t > 0 \\ 0, & t = 0 \end{cases} \tag{6}$$

for $1 \leq p < \infty$,

$$\mathrm{F}_p(x,t) = \begin{cases} \dfrac{t}{t + \Big(\sum\limits_{n=0}^{\infty}(max\{x_n, 0\})^p\Big)^{1/p}}, & t > 0 \\ 0, & t = 0 \end{cases} \tag{7}$$

$$\mathrm{G}_p(x,t) = \begin{cases} \dfrac{\Big(\sum\limits_{n=0}^{\infty}(max\{x_n, 0\})^p\Big)^{1/p}}{t + \Big(\sum\limits_{n=0}^{\infty}(max\{x_n, 0\})^p\Big)^{1/p}}, & t > 0 \\ 1, & e = 0. \end{cases} \tag{8}$$

where $\Big(\sum\limits_{n=0}^{\infty}(\max\{x_n, 0\})^p\Big)^{1/p} = \|x_n\|_{+p}$.

3.1. Open mapping theorem

In this segment, we aim to formulate the open mapping theorem within the context of intuitionistic fuzzy quasi-normed spaces.

Theorem 3.8. *Let $(V, F, G, \star, \circ)$ and $(W, F', G', \star', \circ')$ be intuitionistic fuzzy quasi normed space. Assume that $(V, F, G, \star, \circ)$ is right (F, G) – complete and $(W, F', G', \star', \circ')$ is of half second category and Hausdorff. If $L\colon V \to W$ is linear, surjective and continuous, then L is open.*

Proof. Consider the family of open balls $\mathfrak{B}(\Theta_{\mathrm{V}})$ centered at the origin Θ_{V}. According to Remark 2.14 and Lemma 2.16, for any $U = B_{\mathcal{N}}(\Theta_{\mathrm{V}}, \hat{r}, \hat{t}) \in \mathfrak{B}(\Theta_{\mathrm{V}})$, U possesses absorbent, semibalanced, and convex properties, leading to $\mathrm{V} = \bigcup\limits_{n=1}^{\infty} nU$. Given that L is both onto and linear, we have $W = L(\mathrm{V}) = \bigcup\limits_{n=1}^{\infty} nL(U)$. Since $(W, \mathrm{F}', \mathrm{G}', \star', \circ')$ falls under the category of half-second category, there exists $n \in \mathbb{N}$ such that $int_{\mathcal{N}'} cl_{\mathcal{N}'^{-1}} nL(U) \neq \phi$.

From Lemma 2.17, we deduce $int_{\mathcal{N}'} cl_{\mathcal{N}'^{-1}} L(U) \neq \phi$. Given the linearity and surjectivity of L, $L(U)$ is absorbing and convex. According to Lemma 2.18, $cl_{\mathcal{N}'^{-1}}(L(U))$ is also absorbing and convex. Referring to Lemma 2.10, $\Theta_Y \in int_{\mathcal{N}'} cl_{\mathcal{N}'^{-1}} L(U)$. Applying the definition of the interior of a set, we can assert the existence of an open ball $B_{\mathcal{N}'}(\Theta_Y, , r', t')$ such that

$$\Theta_Y \in B_{\mathcal{N}'}(\Theta_Y, r', t') \subseteq cl_{\mathcal{N}'^{-1}} L(U) \tag{9}$$

Let $U_n = B_{\mathcal{N}}\left(\Theta_{\mathrm{V}}, \frac{\hat{r}}{2^n}, \frac{\hat{t}}{2^{n+1}}\right)$. Then U_n is a local base at Θ_{V}. For any U_n, $n \in \mathbb{N}$ from equation 9, there exists $B^{(n)}_{\mathcal{N}'} = B_{\mathcal{N}'}(\Theta_Y, r'_n, t'_n)$, such that

$$B_{\mathcal{N}'}(\Theta_Y, r'_n, t'_n) \subseteq cl_{\mathcal{N}'^{-1}} L(U_n). \tag{10}$$

Where $r'_n \in (0,1)$ and $t'_n > 0$. And from remark 2.15, we have that $\lim_{n\to\infty} r'_n = 0$ and $\lim_{n\to\infty} t'_n = 0$. By definition of open map, we have to show that L maps open sets in $(\mathrm{V}, \mathrm{F}, \mathrm{G}, \star, \circ)$ onto open sets in $(W, \mathrm{F}', \mathrm{G}', \star', \circ')$ i.e. we will show that

$$B^{(1)}_{\mathcal{N}'} = B_{\mathcal{N}'}(\Theta_Y, r'_1, t'_1), \subseteq L(U). \tag{11}$$

Here, $\mathcal{N} = (\mathrm{F}, \mathrm{G})$ and $\mathcal{N}' = (\mathrm{F}', \mathrm{G}')$ is intuitionistic fuzzy quasi norms on V and W respectively. From equation 10, we have for $n = 1$

$$B_{\mathcal{N}'}(\Theta_Y, r'_1, t'_1) \subseteq cl_{\mathcal{N}'^{-1}} L(U_1).$$

Let $y \in B_{\mathcal{N}'}(\Theta_Y, r'_1, t'_1)$, there exists $x_1 \in U_1$ such that

$$\mathrm{F}'^{-1}(Lx_1 - y, t'_2) > 1 - r'_2 \quad \text{and} \quad \mathrm{G}'^{-1}(Lx_1 - y, t'_2) < r'_2$$

or

$$\mathrm{F}'(y - Lx_1, t'_2) > 1 - r'_2 \quad \text{and} \quad \mathrm{G}'(y - Lx_1, t'_2) < r'_2.$$

This implies that,

$$y - Lx_1 \in B_{\mathcal{N}'}(\Theta_Y, r'_2, t'_2) \subseteq cl_{\mathcal{N}'^{-1}} L(U_2).$$

So, there exists $x_2 \in U_2$ such that

$$\mathrm{F}'^{-1}(Lx_2 + Lx_1 - y, t'_3) > 1 - r'_3 \quad \text{and} \quad \mathrm{G}'^{-1}(Lx_2 + Lx_1 - y, t'_3) < r'_3$$

or

$$\mathrm{F}'^{-1}(y - Lx_2 - Lx_1, t'_3) > 1 - r'_3 \quad \text{and} \quad \mathrm{G}'^{-1}(y - Lx_2 - Lx_1, t'_3) < r'_3$$

On continuing this process, we have

$$\mathrm{F}'(y - Lx_n - L_{x_{n-1}} - ...... - Lx_2 - Lx_1, t'_n + 1) > 1 - r'_{n+1}$$

$$\mathrm{G}'(y - Lx_n - L_{x_{n-1}} - ...... - Lx_2 - Lx_1, t'_n + 1) < r'_{n+1}.$$

This implies that sequence $Lx_n - L_{x_{n-1}} - ...... - Lx_2 - Lx_1 \to y$ as $n \to \infty$, for $r'_n \in (0,1)$ and $t'_n > 0$. Since $x_k \in U_k = B_{\mathcal{N}}\left(\Theta_V, \frac{\hat{r}}{2^k}, \frac{\hat{t}}{2^{k+1}}\right)$ i.e., $F\left(x_k, \frac{\hat{t}}{2^{k+1}}\right) > 1 - \frac{\hat{r}}{2^k}$ and $G\left(x_k, \frac{\hat{t}}{2^{k+1}}\right) < \frac{\hat{r}}{2^k}$. Let $s_n = \sum_{k=1}^{k=n} x_k$, for $m > n$;

$$\begin{aligned}
F\left(s_m - s_n, \frac{1}{2^{n+1}}\left(1 - \frac{1}{2^{m-n}}\right)\hat{t}\right) &= F\left(\sum_{k=1}^{m} x_k - \sum_{k=1}^{n} x_k, \frac{1}{2^{n+1}}\left(1 - \frac{1}{2^{m-n}}\right)\hat{t}\right) \\
&= F\left(\sum_{k=n+1}^{m} x_k, \frac{1}{2^{n+1}}\left(1 - \frac{1}{2^{m-n}}\right)\hat{t}\right) \\
&= F\left(\sum_{k=n+1}^{m} x_k, \sum_{k=n+1}^{m} \frac{1}{2^{k+1}}\hat{t}\right) \\
&\geq \min_{n+1\leq k\leq m} F\left(x_k, \frac{1}{2^{k+1}}\hat{t}\right) \\
&\geq \min_{n+1\leq k\leq m}\left(1 - \frac{\hat{r}}{2^k}\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
G\left(s_m - s_n, \frac{1}{2^{n+1}}\left(1 - \frac{1}{2^{m-n}}\right)\hat{t}\right) &= G\left(\sum_{k=1}^{m} x_k - \sum_{k=1}^{n} x_k, \frac{1}{2^{n+1}}\left(1 - \frac{1}{2^{m-n}}\right)\hat{t}\right) \\
&= G\left(\sum_{k=n+1}^{m} x_k, \frac{1}{2^{n+1}}\left(1 - \frac{1}{2^{m-n}}\right)\hat{t}\right) \\
&= G\left(\sum_{k=n+1}^{m} x_k, \sum_{k=n+1}^{m} \frac{1}{2^{k+1}}\hat{t}\right) \\
&\leq \max_{n+1\leq k\leq m} G\left(x_k, \frac{1}{2^{k+1}}\hat{t}\right) \\
&\leq \max_{n+1\leq k\leq m}\left(\frac{\hat{r}}{2^k}\right).
\end{aligned}$$

We get that if $m, n \to \infty$ then $F \to 1$ and $G \to 0$. Thus, the sequence s_n exhibits left $\mathcal{N}$-Cauchy characteristics. By virtue of the right $\mathcal{N}$-completeness inherent in $(V, F, G, \star, \circ)$, it ensures the existence of $x \in V$ such that $s_n \xrightarrow{\mathcal{N}^{-1}} x$ as $n \to \infty$. Given the continuity of L, we can consequently affirm that

$$\sum_{k=1}^{n} Lx_k \xrightarrow{\mathcal{N}'^{-1}} Lx$$

Since $(W, \tau_{\mathcal{N}'})$ is Hausdorff. So $(U, \tau_{\mathcal{N}'^{-1}})$ will be Hausdorff. Hence, $y = Lx$.

$$
\begin{aligned}
F(x,\hat{t}) =& F\left(x - s_n + s_n, \frac{\hat{t}}{2} + \frac{\hat{t}}{2}\right)\\
&\geq \min\left\{F\left(x - s_n, \frac{\hat{t}}{2}\right), F\left(s_n, \frac{\hat{t}}{2}\right)\right\}\\
&\geq \min\left\{F^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right), F\left(s_n, \frac{\hat{t}}{2}\right)\right\}\\
&\geq \min\left\{F^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right), F\left(\sum_{k=1}^{n} x_k, \frac{\hat{t}}{2}\right)\right\}\\
&\geq \min\left\{F^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right), F\left(\sum_{k=1}^{n} x_k, \frac{\hat{t}}{2}\right)\right\}\\
&\geq \min\left\{F^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right), F\left(\sum_{k=1}^{n} x_k, \sum_{k=1}^{n} \frac{\hat{t}}{2^{k+1}}\right)\right\}\\
&\geq \min\left\{F^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right), \min_{1\leq k\leq n} F\left(x_k, \frac{\hat{t}}{2^{k+1}}\right)\right\}\\
&\geq \min\left\{F^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right), \left(1 - \frac{\hat{r}}{2^k}\right)\right\}
\end{aligned}
$$

Since $s_n \xrightarrow{\mathcal{N}^{-1}} x$ that means $F^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right) > 1 - \hat{r}$, for all $\hat{t} > 0$ and $\hat{r} \in (0,1)$. We have,

$$F(x,\hat{t}) > 1 - \hat{r}.$$

Similarly, for $G(x,\hat{t})$

$$
\begin{aligned}
G(x,\hat{t}) =& G\left(x - s_n + s_n, \frac{\hat{t}}{2} + \frac{\hat{t}}{2}\right)\\
&\leq \max\left\{G\left(x - s_n, \frac{\hat{t}}{2}\right), G\left(s_n, \frac{\hat{t}}{2}\right)\right\}\\
&\leq \max\left\{G^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right), G\left(s_n, \frac{\hat{t}}{2}\right)\right\}\\
&\leq \max\left\{G^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right), G\left(\sum_{k=1}^{n} x_k, \frac{\hat{t}}{2}\right)\right\}\\
&\leq \max\left\{G^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right), G\left(\sum_{k=1}^{n} x_k, \frac{\hat{t}}{2}\right)\right\}\\
&\leq \max\left\{G^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right), G\left(\sum_{k=1}^{n} x_k, \sum_{k=1}^{n} \frac{\hat{t}}{2^{k+1}}\right)\right\}\\
&\leq \max\left\{G^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right), \max_{1\leq k\leq n} G\left(x_k, \frac{\hat{t}}{2^{k+1}}\right)\right\}\\
&\leq \max\left\{G^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right), \left(\frac{\hat{r}}{2^k}\right)\right\}
\end{aligned}
$$

Since $s_n \xrightarrow{\mathcal{N}^{-1}} x$ that means $\mathrm{G}^{-1}\left(s_n - x, \dfrac{\hat{t}}{2}\right) < \hat{r}$, for all $\hat{t} > 0$ and $\hat{r} \in (0,1)$. Hence $x \in U = B_{\mathcal{N}}(\Theta, \hat{r}, \hat{t})$ and shows that $y = Lx \in L(U)$. So, equation 11 holds. This completes the proof. $\square$

4. Conclusion

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This article introduces the concept of the intuitionistic fuzzy quasi norm and establishes the open mapping theorem within the framework of intuitionistic fuzzy quasi normed spaces. The aforementioned theorem and findings can be extended to a broader category of intuitionistic fuzzy sets.

Acknowledgement:

The authors would like to thank the anonymous referees for their comments that helped us improve this article.

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