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Quasi-Cauchy sequences on asymmetric metric spaces

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Abstract. In recent years, quasi-Cauchy sequences have been studied by many authors in the real line and in metric spaces. In this paper, we investigate the concepts of quasi Cauchyness of sequences, ward compactness and ward continuity in asymmetric metric spaces. We prove that forward totally boundedness coincides with upward compactness, backward totally boundedness coincides with downward compactness, an upward continuous function on a subset E of an asymmetric metric space X to an asymmetric metric space Y is forward continuous under the condition that forward convergence implies backward convergence on X. We also prove some other interesting theorems.

1. Introduction

Sequences, sequential compactness, sequential continuity of functions are very important concepts which play a very important role, not only in pure mathematics, but also in other branches of science involving mathematics especially in computer science, combinatorics, information theory, biological science, geographic information systems, population modelling, and motion planning in robotics. Cauchy sequence is a concept that involves far more than that the distance between successive terms is tending to zero, and more generally speaking, than that the distance between successive terms is tending to zero, where by successive terms, we mean x_n and x_{n+1}. Nevertheless, sequences which satisfy this weaker property are more interesting in their own right in asymmetric metric spaces than in metric spaces.

The asymmetric distance function was first defined in 1914 by F. Hausdorff. Hausdorff introduced the concept of asymmetry by defining a distance that expresses the distance between any two subsets of a metric space which does not have symmetry property. Asymmetric metric spaces were then studied by Wilson by calling the concept as quasi-metric ([13]). Later, Albert (1941) and Stoltenberg (1969) studied quasi-metric spaces, while Ribeiro (1943) named these spaces as weak metric spaces. Reilly and Künzi worked on quasi-pseudo metric spaces ([10, 12]). Kočinac and Künzi studied selection principles in quasi-metric and quasi-uniform spaces ([9]). In addition, Colins and Zimmer (2007) have investigated the concepts of convergence, compactness, and total boundedness in asymmetric metric spaces. For more information about asymmetric metric structures see the book [5].

An asymmetric metric is a non-negative real valued function defined on $X \times X$ which does not have to satisfy the symmetry property of classical metric conditions. In this case (X, d) is called an asymmetric metric

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space, where X is a non-empty set. The absence of symmetry in asymmetric metric spaces has revealed that there are two types of topology in these spaces, and therefore two types of examination of basic concepts such as convergence of a sequence, compactness, completeness and total boundedness of a set. The set $B^+(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ is a *forward ball* at center x with radius ε, the set $B^-(x, \varepsilon) = \{y \in X : d(y, x) < \varepsilon\}$ is a *backward ball* at center x with radius ε. The topology generated by the forward balls is called the *forward topology*, and the topology generated by the backward balls is called the *backward topology*.

A function f on an asymmetric metric space is forward (backward) continuous if it preserves forward (backward) convergent sequences ([6]). Using the idea of continuity of a function in terms of sequences in the sense that a function preserves a certain kind of sequences, many kinds of continuities were introduced and investigated. Recently, in [2], a concept of ward continuity, and a concept of ward compactness have been introduced in the senses that a real function is called ward continuous if $\lim_{n\to\infty} \Delta f(x_n) = 0$ whenever $\lim_{n\to\infty} \Delta x_n = 0$, and a subset E of the set of real numbers is called ward compact if whenever (x_n) is a sequence of points in E there is a subsequence $(y_k) = (x_{n_k})$ of (x_n) with $\lim_{k\to\infty} \Delta y_k = 0$, where $\Delta y_k = y_{k+1} - y_k$. We note that forward continuity and forward compactness terms were used in place of the terms ward continuity and ward compactness in [2], respectively.

Since theorems and proofs related to backward topology are similar to those related to forward topology, we will not give the proofs of theorems for backward case, only express statements. .

The aim of this paper is to investigate the concept of quasi-Cauchyness and prove interesting theorems in asymmetric metric spaces.

2. Preliminaries

Now we give some definitions and notation which will be needed throughout the paper. X, Y, $\mathbb{R}$ and $\mathbb{N}$ will denote an asymmetric metric space with an asymmetric metric d_X, an asymmetric metric space with an asymmetric metric d_Y, the set of real numbers, and the set of positive integers, respectively. We will use d for both d_X and d_Y when confusion does not arise. We will use boldface letters $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, ... for sequences $\mathbf{x} = (x_n)$, $\mathbf{y} = (y_n)$, $\mathbf{z} = (z_n)$, ... of points in X or in Y.

Definition 2.1. A sequence (x_n) is called *forward* (*backward*) *Cauchy* if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for $m \geq n \geq n_0$, $d(x_n, x_m) < \varepsilon$ ($d(x_m, x_n) < \varepsilon$) .

We recall the definition of forward (backward) convergence which was given in [6].

Definition 2.2. A sequence (x_n) in an asymmetric metric space X forward (backward) converges to $x \in X$ if $\lim_{n\to\infty} d(x, x_n) = 0$ ($\lim_{n\to\infty} d(x_n, x) = 0$).

3. Ward compactness in asymmetric metric spaces

The term quasi-Cauchy was used by Burton and Coleman in real case ([1]), and whereas Cakalli used the term forward in real case in [2] and metric spaces in [**?**].

Palladino ([11]) introduced a concept of upward half Cauchyness, and a concept of downward half Cauchyness in the following way: a sequence (x_n) of points in $\mathbb{R}$ is called *upward half Cauchy* if for every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ so that $x_n - x_m < \varepsilon$ for $m \geq n \geq n_0$, and *downward half Cauchy* if for every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ so that $x_m - x_n < \varepsilon$ for $m \geq n \geq n_0$. Using the idea of the definition of an upward half Cauchy sequence, a concept of upward half quasi-Cauchy sequence is introduced in [3]. A sequence (x_n) of points in $\mathbb{R}$ is called *upward half quasi-Cauchy* if for every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $x_n - x_{n+1} < \varepsilon$ for $n \geq n_0$. A subset E of $\mathbb{R}$ is called *upward compact* if any sequence of points in E has an upward half quasi-Cauchy subsequence. A sequence (x_n) of points in $\mathbb{R}$ is called *downward half quasi-Cauchy* if for every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $x_{n+1} - x_n < \varepsilon$ for $n \geq n_0$ ([3]). A subset E of $\mathbb{R}$ is called *downward compact* if any sequence of points in E has a downward half quasi-Cauchy subsequence. A subset E of $\mathbb{R}$ is called *up half compact* if any sequence of points in E has an up half Cauchy subsequence, and is called *down half compact* if any sequence of points in E has a down half Cauchy subsequence ([3]).

In this section, we give the definition of a concept of forward (backward) quasi-Cauchy sequence, and investigate new types of compactness on asymmetric metric spaces. Now we give definition of a forward (backward) quasi-Cauchy sequence.

Definition 3.1. A sequence (x_n) in an asymmetric metric space X is called forward (backward) quasi Cauchy if $(\Delta^+ x_n)$ $((\Delta^- x_n))$ is a null sequence, where $\Delta^+ x_n = d(x_n, x_{n+1})$ $(\Delta^- x_n = d(x_{n+1}, x_n))$ for each positive integer n.

$\Delta^+(X)$ $(\Delta^-(X))$ will stand for the set of all forward (backward) quasi-Cauchy sequences of points in X. We note that left (right) quasi-K-Cauchy sequence term is used in [4] instead of forward (backward) quasi-Cauchy sequence.

According to this definition, it is clear that any forward Cauchy sequence is forward quasi Cauchy sequence, but the converse is not true as it is demonstrated in the following.

Example 3.2. *(Sorgenfrey asymmetric)* The function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$d(x, y) = \begin{cases} y - x, & \text{if } y \geq x; \\ 1, & \text{if } y < x. \end{cases}$$

is an asymmetric metric. It is easy to see that $(x_n) = \sqrt{n}$ is forward quasi Cauchy but not forward Cauchy in this asymmetric metric space.

Forward (backward) Cauchy sequences have the property that any subsequence of a forward (backward) Cauchy sequence is forward (backward) Cauchy in asymmetric metric spaces. The analogous property fails for forward (backward) quasi-Cauchy sequences. In fact, it is precisely the forward (backward) Cauchy sequences which have the property that all of their subsequences are forward (backward) quasi-Cauchy. One of the most important concepts in metric spaces is no doubt compactness. Many kinds of compactness were introduced and investigated in real case and in metric spaces; ward compactness ([2]), statistical compactness. Two kinds of quasi-Cauchness was investigated for the real spaces which made us to introduce the concept in asymetric metric spaces ([3]). There are two kinds of compactness in asymetric metric spaces. Now we introduce a concept of upward (downward) compactness of a subset of X.

Definition 3.3. A subset E of X is called upward (downward) compact if any sequence $\mathbf{x} = (x_n)$ of points in E has a forward (backward) quasi-Cauchy subsequence.

According to this definition, we see that any finite subset of X is both upward and downward compact. It is easy to see that any subset of a upward (downward) compact set of X is upward (downward) compact, union of finite number of upward (downward) compact subsets of X is upward (downward) compact and intersection of family of any upward (downward) compact subsets of X is upward (downward) compact. These observations above suggested to us the following.

Theorem 3.4. ([7]) *A subset E of X is forward totally bounded if and only if any sequence of points in E has a forward Cauchy subsequence.*

It is easy to see that a subset E of X is backward totally bounded if and only if any sequence of points in E has a backward Cauchy subsequence. Now we prove that forward totally boundedness coincides with upward compactness in the following

Theorem 3.5. *A subset E of X is forward totally bounded if and only if it is upward compact.*

Proof. It is clear that forward totally boundedness of E implies upward compactness of E.

To prove the converse suppose that E is not forward totally bounded. In that case, there is an $\varepsilon > 0$ such that E has not a finite ε-net. Now take any element of E and say x_1. Since E is not forward totally bounded, $S_\varepsilon(x_1) \neq E$. Otherwise $\{x_1\}$ would be a finite ε-net of E. So there is an $x_2 \in E$ such that $x_2 \notin S_\varepsilon(x_1)$, i.e. $d(x_1, x_2) \geq \varepsilon$. Since $\{x_1, x_2\}$ is not a finite ε-net, $S_\varepsilon(x_1) \cup S_\varepsilon(x_2) \neq E$. Now let $x_3 \notin S_\varepsilon(x_1) \cup S_\varepsilon(x_2)$. So we obtain $d(x_1, x_3) \geq \varepsilon$ and $d(x_2, x_3) \geq \varepsilon$. Continuing in this way we can generate the sequence (x_n) such that $x_n \notin \cup_{i=1}^{n-1} S_\varepsilon(x_i)$ for $n = 2, 3, ...$ So we get $d(x_i, x_n) \geq \varepsilon$ for $i = 1, 2, ..., n-1$ and for $n = 2, 3, ...$ As a result of this, for all $n, m \in \mathbf{N}$ which satisfy $n < m$ we have $d(x_n, x_m) \geq \varepsilon$. Therefore $d(x_n, x_{n+1}) \geq \varepsilon$. Thus the sequence (x_n) constructed has not any forward quasi-Cauchy subsequence. This contradiction completes the proof. □

Following a similar way to the proof of the preceding theorem, one can prove that A subset of X is backward totally bounded if and only if it is downward compact.

4. Ward continuity of functions in asymmetric metric spaces

In this section, we introduce a concept of an upward (downward) continuous function on asymmmetric metric spaces and examine the relationship between upward continuity and forward uniform continuity.

Some examples are given in [6, 7] that forward convergence does not imply forward Cauchyness, and now we give an another simple example of this situation.

Example 4.1. In the Sorgenfrey asymmetric metric, if we take $x_n = (x + 1/n)$, we see that (x_n) is forward convergent but it is not forward Cauchy.

It is also true that backward convergence does not imply backward Cauchyness, and one can give an example of a sequence which is backward convergent, not backward Cauchy.

Lemma 4.2. *Let (X, d) be an asymmetric metric space which has the property that forward convergence implies backward convergence. Then any forward convergent sequence is forward Cauchy.*

It is also true that any forward convergent sequence is forward Cauchy in an asymmetric metric space which has the property that backward convergence implies forward convergence.

Definition 4.3. A function f is called upward (downward) continuous on a subset E of X to Y if f preserves forward (backward) quasi-Cauchy sequences, i.e. the sequence $f(\mathbf{x}) = (f(x_n))$ is forward (backward) quasi-Cauchy in Y whenever $\boldsymbol{x} = (x_n)$ is forward (backward) quasi-Cauchy in X.

Theorem 4.4. *Assume that f be an upward continuous function on a subset E of X to Y and that forward convergence implies backward convergence on X , then it is forward continuous.*

Proof. Take any upward continuous function f on E to Y. Let (x_n) be any forward convergent sequence of points in E with forward limit ℓ. Then the sequence

$$(x_1, \ell, x_2, \ell, ..., x_n, \ell, ...)$$

is also forward convergent to ℓ. Hence

$$(x_1, \ell, x_2, \ell, ..., x_n, \ell, ...)$$

is forward quasi Cauchy. As f is upward continuous from E to Y, the sequence

$$(f(x_1), f(\ell), f(x_2), f(\ell), ..., f(x_n), f(\ell), ...)$$

if forward quasi Cauchy in Y . Thus $\lim_{n\to\infty} d_Y(f(\ell), f(x_n)) = 0$. This completes the proof of the theorem. □

Corollary 4.5. *Assume that f be an upward continuous on a subset E of X to Y and that backward convergence implies forward convergence on X, then it is backward continuous.*

It turns out that if f is downward continuous on a subset E of X to Y and if backward convergence implies forward convergence on X , then it is backward continuous.

Theorem 4.6. *Let f be an upward continuous function from X to Y. Then upward continuous image of any upward compact subset of X is upward compact.*

Proof. Let f be an upward continuous function and E be an upward compact subset of X. Take any sequence $\mathbf{y} = (y_n)$ of terms in $f(E)$. Write $y_n = f(x_n)$, where $x_n \in E$ for each $n \in \mathbf{N}$. Upward compactness of E implies that there is a subsequence $\mathbf{t} = (t_k) = (x_{n_k})$ of $\mathbf{x}$ with $\lim_{k\to\infty} \Delta^+ t_k = 0$. Since f is upward continuous, $f(\mathbf{t}) = (f(t_k))$ is forward quasi-Cauchy. As $f(\mathbf{t})$ is a subsequence of the sequence $f(\mathbf{x})$ with $\lim_{k\to\infty} \Delta^+ f(t_k) = 0$, the proof follows. □

Using similar steps to the proof of the preceding theorem, one can easily obtain that downward continuous image of any downward compact subset of X is downward compact.

We recall that a subset E of X is called forward compact if any sequence of point in E has a forward convergent subsequence with a forward limit in E .

Corollary 4.7. *Upward continuous image of any forward compact subset of X is forward compact.*

Proof. The proof follows from the preceding theorem, so is omitted. □

It is easy to see that downward continuous image of any backward compact subset of X is backward compact.

Notation 4.8. Let Y^X denote the space of all functions from X to Y. The uniform asymmetric metric on Y^X is

$$\overline{\rho}(f, g) := \sup\{\overline{d}(f(x), g(x)) : x \in X\},$$

where $\overline{d}(x, y) = \min\{d(x, y), 1\}$ and d is the asymmetric metric defined on Y. This asymmetric metric generates the uniform topology on Y^X.

Now we state that any forward (backward) uniformly continuous function preserves forward (backward) quasi-Cauchy sequences. This important result proved by Jao and Kanibir on quasi-pseudo metric spaces.

Theorem 4.9. ([4]) *If f is forward uniformly continuous on a subset E of X, then it is upward continuous on E.*

It is also true that any backward uniformly continuous function on a subset E of X is downward continuous on E.

It is well known that uniform limit of a sequence of continuous functions is continuous. This is also true in the case of upward (downward) continuity, i.e. forward (backward) uniform limit of a sequence of upward (downward) continuous functions is upward (downward) continuous under the condition that Y is sequentially compact.

Theorem 4.10. *Suppose that Y is forward sequentially compact. If (f_n) is a sequence of upward continuous functions defined on a subset E of X to Y and (f_n) is backward uniformly convergent to a function f, then f is upward continuous on E.*

Proof. Let $\mathbf{x} = (x_n)$ be any forward quasi-Cauchy sequence of points in E and $\varepsilon > 0$. Since (f_n) is backward uniformly convergent to f, there exists a positive integer N_1 such that $d_Y(f_n(x), f(x)) < \frac{\varepsilon}{3}$ for all $x \in E$ whenever $n \geq N_1$. Since Y is forward sequentially compact, $(f_n(x))$ forward converges to $f(x)$ for all $x \in E$. So there exists a positive integer N_2 such that $d_Y(f(x), (f_n(x)) < \frac{\varepsilon}{3}$ for all $x \in E$ whenever $n \geq N_2$. Take $N = max\{N_1, N_2\}$. As f_N is upward continuous, there exists a positive integer N_0, depending on ε and greater than N such that $d(f_N(x_n), f_N(x_{n+1})) < \frac{\varepsilon}{3}$ for $n \geq N_0$. Now for $n \geq N_0$ we have
$d(f(x_n), f(x_{n+1})) \leq d(f(x_n), f_N(x_n)) + d(f_N(x_n), f_N(x_{n+1})) + d(f_N(x_{n+1}), f(x_{n+1})) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$.
This completes the proof of the theorem. □

Suppose that Y is backward sequentially compact. If (f_n) is a sequence of downward continuous functions defined on a subset E of X to Y and (f_n) is forward uniformly convergent to a function f, then f is downward continuous on E.

Theorem 4.11. *Suppose that forward convergence is equivalent to backward convergence in Y. If (f_n) is a sequence of upward continuous functions defined on a subset E of X to Y and (f_n) is forward uniformly convergent to a function f, then f is upward continuous on E.*

Proof. Let $\mathbf{x} = (x_n)$ be any forward quasi-Cauchy sequence of points in E and $\varepsilon > 0$. Then there exists a positive integer N_1 such that $d_Y(f(x), f_n(x)) < \frac{\varepsilon}{3}$ for all $x \in X$ whenever $n \geq N_1$. Since forward convergence is equivalent to backward convergence in Y, $(f_n(x))$ backward converges to $f(x)$ for all $x \in E$. So there exists a positive integer N_2 such that $d_Y(f_n(x), f(x)) < \frac{\varepsilon}{3}$ for all $n \geq N_2$. Take $N = max\{N_1, N_2\}$. As f_N is upward continuous, there exists a positive integer N_0, depending on ε and greater than N such that $d(f_N(x_n), f_N(x_{n+1})) < \frac{\varepsilon}{3}$ for $n \geq N_0$. Now for $n \geq N_0$ we have
$d(f(x_n), f(x_{n+1})) \leq d(f(x_n), f_N(x_n)) + d(f_N(x_n), f_N(x_{n+1})) + d(f_N(x_{n+1}), f(x_{n+1})) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$.
This completes the proof of the theorem. □

Suppose that forward convergence is equivalent to backward convergence in Y. If (f_n) is a sequence of downward continuous functions defined on a subset E of X to Y and (f_n) is backward uniformly convergent to a function f, then f is downward continuous on E.

Lemma 4.12. *If the asymmetric metric space (Y, d_Y) is forward compact and has the property that forward convergence implies backward convergence, then the space Y^X is forward complete in the uniform asymmetric metric $\overline{\rho}$ regarding to d_Y.*

One can prove that the space Y^X is backward complete in the uniform asymmetric metric $\overline{\rho}$ regarding to d_Y if the asymmetric metric space (Y, d_Y) is backward compact and has the property that backward convergence implies forward convergence.

Theorem 4.13. *Suppose that (Y, d_Y) is forward compact, and that forward convergence implies backward convergence in Y. The set of all upward continuous functions defined on X to Y is a forward closed subset of the space Y^X in the uniform asymmetric metric $\overline{\rho}$ regarding to d_Y.*

Proof. Let us denote the set of all upward continuous functions on X by $\Delta^+C(X)$ and f be any element in the forward closure of $\Delta^+C(X)$. Then there exists a sequence of points in $\Delta^+C(X)$ that forward uniformly converges to f. To show that f is upward continuous take any forward quasi Cauchy sequence (x_n). Let $\varepsilon > 0$. Since (f_n) forward converges to f, there exists a positive integer N_1 such that $d_Y(f(x), f_n(x)) < \frac{\varepsilon}{3}$ for all $x \in X$ whenever $n \geq N_1$. Also there exists a positive integer N_2 such that $d_Y(f_n(x), f(x)) < \frac{\varepsilon}{3}$ for all $n \geq N_2$. Take $N = max\{N_1, N_2\}$. As f_N is upward continuous, there exists a positive integer N_0, depending on ε and greater than N such that $d(f_N(x_n), f_N(x_{n+1})) < \frac{\varepsilon}{3}$ for $n \geq N_0$. Hence for all $n \geq N_0$, we have
$d(f(x_n), f(x_{n+1})) \leq d(f(x_n), f_N(x_n)) + d(f_N(x_n), f_N(x_{n+1})) + d(f_N(x_{n+1}), f(x_{n+1})) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. This completes the proof of the theorem. □

If (Y, d_Y) is backward compact, and that backward convergence implies forward convergence in Y, then the set of all downward continuous functions defined on X to Y is a backward closed subset of the space Y^X in the uniform asymmetric metric $\overline{\rho}$ regarding to d_Y.

Corollary 4.14. *Suppose that (Y, d_Y) is forward compact, and that forward convergence implies backward convergence in Y. The set of all upward continuous functions from X to Y is forward complete in the uniform asymmetric metric $\overline{\rho}$ regarding to d_Y.*

Proof. The proof follows from the Lemma 4.12 and the preceding theorem. □

Supposing that (Y, d_Y) is backward compact, and that backward convergence implies forward convergence in Y. The set of all downward continuous functions from X to Y is backward complete in the uniform asymmetric metric $\overline{\rho}$ regarding to d_Y.

5. Conclusion

In this paper, we introduce and investigate not only upward and downward continuities of functions, but also some other kinds of continuities defined via upward and downward quasi-Cauchy sequences in an asymetric metric space X. We also investigate necessary and sufficient conditions for a subset of X to be upward totally bounded. We prove results related to these newly defined continuities, newly defined compactness, and some other kinds of continuities and compactness; namely compactness, continuity, and uniform continuity on asymetric metric spaces. It turns out that not only the set of all upward continuous functions defined on X to Y is a forward complete subspace of the space Y^X, but also the set of all downward continuous functions from X to Y is backward complete subspace of the space Y^X in the uniform asymmetric metric $\overline{\rho}$ regarding to d_Y under certain conditions. We suggest to investigate upward and downward quasi-Cauchy sequences of fuzzy points, soft points or neutrosophic on asymetric metric spaces.

We also suggest to investigate upward and downward quasi-Cauchy double sequences (see for example [8] for the definitions and related concepts in the double sequences case).

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