Filomat 38:25 (2024), 8925–8935
https://doi.org/10.2298/FIL2425925C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Several fundamental inequalities for submanifolds immersed in Riemannian manifolds equipped with golden structure

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Abstract. This article's main purpose is to derive Chen's inequalities for submanifolds immersed in golden Riemannian manifolds having constant sectional curvatures and equipped with a semi-symmetric metric connection. It also demonstrates the associations relating to the sectional curvatures, scalar curvature, Ricci curvatures, and the mean curvature linked to the semi-symmetric metric connection. The cases about equality are considered.

1. Introduction

The basic idea of Golden structure [14] was laid by polynomial structures on a manifold, which was addressed in [18]. In [20], invariant submanifolds were studied for their various features in a golden Riemannian manifold, and integrability was demonstrated in [17]. In [32], the Golden sectional curvature has been investigated, and the geometry of submanifolds of locally decomposable Golden Riemannian manifolds with constant Golden sectional curvature has been investigated.

The concept of the metallic structure was introduced in 2013 due to [20], as a generalization of the golden structure defined on Riemannian manifolds. Since then, various aspects of metallic Riemannian manifolds with respect to curvature have been discussed in [4, 5].

Moreover, Chen [12] studied submanifolds of real space form in 1993 and presented the fundamental concept of the sharp inequality linking the intrinsic and extrinsic invariants. Chen-like inequalities were subsequently investigated in numerous other ambient spaces, as well as in the references [13, 15, 16, 24, 25].

Semi-symmetric metric connections (ssmc) on Riemannian manifolds were conceptualised by H.A. Hayden in [21]. In [36], K. Yano investigated some characteristics of a Riemannian manifold with a ssmc. T. Imai discovered certain characteristics of a Riemannian manifold and a hypersurface of a Riemannian manifold equipped with a ssmc in [22] and [23]. Submanifolds of a Riemannian manifold endowed with ssc were examined by Z. Nakao [28]. However, establishing basic relationships between a submanifold's intrinsic and extrinsic invariants is one of the fundamental issues in submanifold theory. In this regard, B. Y. Chen [6, 7, 10] introduced inequalities known as Chen inequalities. Subsequently, numerous geometers examined comparable issues for distinct submanifolds in different ambient spaces; as evidenced by the works of [1–3, 26, 27] and [29].

2020 *Mathematics Subject Classification*. Primary 53B05; Secondary 53B20, 53C25, 53C40.
Keywords. Golden Riemannian manifolds, slant submanifolds, curvature.
Received: 22 January 2024; Revised: 30 April 2024; Accepted: 11 May 2024
Communicated by Mića S. Stanković
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Motivated by the aforementioned advancements, the purpose of this note is to obtain sharp inequalities for submanifolds of locally decomposable golden Riemannian manifolds that possess constant sectional curvatures and are equipped with ssmc. Relationships between the sectional curvatures, scalar curvature, Ricci curvatures, and the mean curvature linked to the ssmc have also been derived. The equality cases are taken into account.

2. Preliminaries

2.1. Golden Riemannian manifolds

Any $(1,1)$-tensor field φ generates a Golden-type structure on a manifold N^{n+p} if [18]:

$$\varphi^2 - \varphi - I = 0,$$

(N, φ) is referred to as the Golden manifold. Moreover, in case φ on the Riemannian manifold (N, g) satisfies

$$g(\varphi\tilde{X}, \tilde{Y}) = g(\tilde{Y}, \varphi\tilde{X}),$$

N is termed as Golden Riemannian manifold [14]. Substituting $\varphi\tilde{X}$ in place of $\tilde{X}$ in above relation produces the subsequent

$$g(\varphi\tilde{X}, \varphi\tilde{Y}) = g(\varphi^2\tilde{X}, \tilde{Y}) = g(\varphi\tilde{X}, \tilde{Y}) + g(\tilde{X}, \tilde{Y}).$$

Consider a Golden Riemannian manifold (N, g, φ). If we have

$$\nabla\varphi = 0,$$

that is, φ is covariantly parallel, then N is a locally decomposable Golden Riemannian manifold.

The curvature tensor of a locally decomposable Golden Riemannian manifold (N, g, φ) with a constant Golden sectional curvature is therefore defined as follows: [32]

$$\begin{aligned}\tilde{R}\left(\tilde{X}, \tilde{Y}\right)\tilde{Z} \quad &= \frac{c}{3}\Big\{g\left(\tilde{Y}, \tilde{Z}\right)\tilde{X} - g\left(\tilde{X}, \tilde{Z}\right)\tilde{Y} \\ &-g\left(\tilde{Y}, \varphi\tilde{Z}\right)\tilde{X} - g\left(\tilde{Y}, \tilde{Z}\right)\varphi\tilde{X} + 2g\left(\tilde{Y}, \varphi\tilde{Z}\right)\varphi\tilde{X} \\ &+g\left(\tilde{X}, \varphi\tilde{Z}\right)\tilde{Y} + g\left(\tilde{X}, \tilde{Z}\right)\varphi\tilde{Y} - 2g\left(\tilde{X}, \varphi\tilde{Z}\right)\varphi\tilde{Y}\Big\}\end{aligned} \tag{2.1}$$

regarding any vector fields $\tilde{X}$, $\tilde{Y}$, and $\tilde{Z}$ on N.

2.2. Riemannian Invariants

Let $N^{n+p}(c)$ be a Riemannian manifold and $\tilde{\nabla}$ be a linear connection on it. If $\tilde{T}$, the torsion tensor of $\tilde{\nabla}$, is written as [11]

$$\tilde{T}(\tilde{X}, \tilde{Y}) = \tilde{\nabla}_{\tilde{X}}\tilde{Y} - \tilde{\nabla}_{\tilde{Y}}\tilde{X} - [\tilde{X}, \tilde{Y}],$$

satisfying

$$\tilde{T}(\tilde{X}, \tilde{Y}) = \phi(\widetilde{Y})\tilde{X} - \phi(\tilde{X})\tilde{Y}.$$

Then, the connection $\tilde{\nabla}$ is referred to as semi-symmetric for a 1 -form ϕ.

Assume any Riemannian metric g and connection $\widetilde{\nabla}$ on N. $\widetilde{\nabla}$ is referred to as a ssmc on N if

$$\widetilde{\nabla}g = 0.$$

In accordance with [36], $\tilde{\nabla}$ is an ssmc on N such that

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = \overset{\circ}{\nabla}_{\tilde{X}}\tilde{Y} + \phi(\tilde{Y})\tilde{X} - g(\tilde{X}, \tilde{Y})P$$

in this case, the Levi-Civita connection with respect to g is denoted by $\overset{\circ}{\widetilde{\nabla}}$ and for each vector field $\tilde{X}$,

$$g(P,\widetilde{X}) = \phi(\tilde{X})$$

defines a vector field P.

We will take a Riemannian manifold N that is equipped with $\overset{\circ}{\widetilde{\nabla}}$, and a ssmc $\tilde{\nabla}$.

Suppose a Riemannian manifold N has a n-dimensional submanifold called M^n. We take into consideration the induced Levi-Civita connection indicated by $\overset{\circ}{\nabla}$ and the induced ssmc indicated by ∇ on M^n.

The curvature tensor of N^{n+p} with respect to $\tilde{\nabla}$ is $\widetilde{R}$, and the curvature tensor of N with respect to $\overset{\circ}{\widetilde{\nabla}}$ is $\overset{\circ}{\widetilde{R}}$. Additionally, we indicate the curvature tensors of ∇ and $\overset{\circ}{\nabla}$ on M^n by the symbols R and $\overset{\circ}{R}$, respectively. The Gauss formulas for ∇ and $\overset{\circ}{\nabla}$, respectively, can be expressed as follows:

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y),$$
$$\overset{\circ}{\tilde{\nabla}}_X \, Y = \overset{\circ}{\nabla}_X \, Y + \overset{\circ}{h} (X,Y),$$

$\forall X, Y \in \chi(M^n)$.

Let h is a (0,2)-tensor on M^n, and $\overset{\circ}{h}$ is the second fundamental form of M^n. The formula (7) from [28] indicates that h is symmetric as well. One represents the mean curvature vector of M^n in N^{n+p} by $\overset{\circ}{H}$.

Consider the golden Riemannian manifold $N^{n+p}(c)$, which has a constant golden sectional curvature and a ssmc $\widetilde{\nabla}$. With the aid of (2.1), the curvature tensor $\overset{\circ}{\widetilde{R}}$ with regard to $\overset{\circ}{\widetilde{\nabla}}$ on $N^{n+p}(c)$ is written by

$$\begin{aligned}\overset{\circ}{\widetilde{R}} (X,Y,Z,W) \quad = \quad & \frac{c}{3}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\ & -g(Y,\varphi Z)g(X,W) - g(Y,Z)g(\varphi X,W) \\ & +2g(Y,\varphi Z)g(\varphi X,W) + g(X,\varphi Z)g(Y,W) \\ & +g(X,Z)g(\varphi Y,W) - 2g(X,\varphi Z)g(\varphi Y,W)\}.\end{aligned} \tag{2.2}$$

Following that, we can write the curvature tensor $\bar{R}$ with respect to the ssmc $\bar{\nabla}$ on $N^{n+p}(c)$ as [23]

$$\begin{aligned}\bar{R}(X,Y,Z,W) \quad = & \overset{\circ}{\bar{R}} (X,Y,Z,W) - \alpha(Y,Z)g(X,W) + \alpha(X,Z)g(Y,W) \\ & -\alpha(X,W)g(Y,Z) + \alpha(Y,W)g(X,Z),\end{aligned} \tag{2.3}$$

$\forall X, Y, Z, W \in \chi(M^n)$, where the $(0,2)$-tensor field α is expressed as

$$\alpha(X,Y) = \left(\overset{\circ}{\bar{\nabla}}_X \, \phi\right) Y - \phi(X)\phi(Y) + \frac{1}{2}\phi(P)g(X,Y).$$

The curvature tensor $\bar{R}$ can be expressed as follows from (2.2) and (2.3)

$$\begin{aligned}\bar{R}(X,Y,Z,W) \quad = \quad & \frac{c}{3}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W) - g(Y,\varphi Z)g(X,W) \\ & -g(Y,Z)g(\varphi X,W) + 2g(Y,\varphi Z)g(\varphi X,W) + g(X,\varphi Z)g(Y,W) \\ & +g(X,Z)g(\varphi Y,W) - 2g(X,\varphi Z)g(\varphi Y,W)\} - \alpha(Y,Z)g(X,W) \\ & +\alpha(X,Z)g(Y,W) - \alpha(X,W)g(Y,Z) + \alpha(Y,W)g(X,Z),\end{aligned} \tag{2.4}$$

λ represents the trace of α.

For the submanifold M^n, the Gauss equation into the real space form $N^{n+p}(c)$ van be recalled as

$$\overset{\circ}{\bar{R}}(X, Y, Z, W) = \overset{\circ}{R}(X, Y, Z, W) + g(\overset{\circ}{h}(X, Z), \overset{\circ}{h}(Y, W)) - g(\overset{\circ}{h}(X, W), \overset{\circ}{h}(Y, Z)).$$

Let $\pi \subset T_xM^n, x \in M^n$ represent a section in 2-plane. A sectional curvature of M^n about the induced ssmc ∇ is denoted by $K(\pi)$. The scalar curvature τ at x for any orthonormal basis $\{e_1, \ldots, e_m\}$ of T_xM^n is given by

$$\tau(x) = \sum_{1 \le i < j \le n} K\left(e_i \wedge e_j\right).$$

We recollect the algebraic lemma given below:

Lemma 2.1. *[12] When the $(n + 1)$ real numbers are represented by $a_1, \ldots, a_n, b$ for $n \ge 2$*

$$\Big(\sum_{i=1}^{n} a_i\Big)^2 = (n-1)\Big(\sum_{i=1}^{n} a_i^2 + b\Big),$$

then,

$$2a_1a_2 \ge b$$

and equality hold if and only if

$$a_1 + a_2 = a_3 = \cdots = a_n.$$

Suppose M^n be a Riemannian manifold and X be a unit vector in L, L is a k-plane section of T_xM^n, *where*, $x \in M^n$. To ensure that $e_1 = X$, we select an orthonormal basis $\{e_1, \ldots, e_k\}$ of L.

The k-Ricci curvature, also known as the Ricci curvature, of L at X is defined by [8]

$$\mathrm{Ric}_L(X) = K_{12} + K_{13} + \ldots + K_{1k},$$

where the sectional curvature of the 2-plane section spanned by e_i, e_j is indicated by K_{ij}, as usual. The Riemannian invariant Θ_k on M^n for each integer $k, 2 \le k \le n$ is defined as follows:

$$\Theta_k(x) = \frac{1}{k-1} \inf_{L,X} \mathrm{Ric}_L(X), \quad x \in M^n,$$

X varies over all of the unit vectors in L and L runs over every k-plane section in T_xM^n.

3. Chen First Inequality

Let M^n be a Riemannian manifold, τ the scalar curvature at x, $\pi \subset T_xM^n, x \in M^n$, and $K(\pi)$ be the sectional curvature of M^n associated with a 2-plane section. Remember that the first invariant of Chen is provided by

$$\delta_M(x) = \tau(x) - \inf\{K(\pi) \mid \pi \subset T_xM^n, x \in M^n, \dim \pi = 2\},$$

(for instance, [10]).

From here on, let us set $N^{n+p}(c)$ for a golden Riemannian manifold that possesses a ssmc and has constant golden sectional curvature. We prove the optimal inequality that we will refer to as Chen's first inequality:

Theorem 3.1. *Let $M^n, n \ge 3$, be a submanifold immersed in $N^{n+p}(c)$ and λ be the trace of α. Then, for each $x \in M^n$, we get*

$$\begin{aligned} \tau - K(\pi) \quad &\le (n-2)\left[\tfrac{n^2}{2(n-1)}\|H\|^2 - \lambda\right] - \mathrm{trace}\left(\alpha_{|_{\pi^\perp}}\right) \\ &-\tfrac{c}{6}\left\{2 + 3n - n^2 - 2\|\varphi\|^2 - 2\|\varphi\| + 2n\|\varphi\|\right\}, \end{aligned} \tag{3.1}$$

π represents a 2-plane section of T_xM^n.

Proof. The Gauss equation for ssmc is derived from [28].

$$\widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(h(X,Z),h(Y,W)) - g(h(Y,Z),h(X,W)). \tag{3.2}$$

Let $x \in M^n$, and the orthonormal bases of T_xM^n and $T_x^{\perp}M^n$, respectively, be $\{e_1, e_2, \ldots, e_n\}$ and $\left\{e_{n+1}, \ldots, e_{n+p}\right\}$. From the equations (2.4) and (3.2), for $X = W = e_i, Y = Z = e_j, i \neq j$, one determines

$$2\tau + \|h\|^2 - n^2\|H\|^2 = -2(n-1)\lambda + \frac{c}{3}\left\{n^2 - 3n - 2n\|\varphi\| + 2\|\varphi\|^2\right\}, \tag{3.3}$$

where λ is the trace of α and

$$\sum_{i=1}^{n} g(e_i, \varphi e_i) = \|\varphi\|,$$

indicated by

$$\|h\|^2 = \textstyle\sum_{i,j=1}^{n} g\left(h\left(e_i, e_j\right), h\left(e_i, e_j\right)\right),$$
$$H = \tfrac{1}{n}\,\text{trace}\, h.$$

One takes

$$\varepsilon = 2\tau - \frac{n^2(n-2)}{n-1}\|H\|^2 + 2(n-1)\lambda - \frac{c}{3}\left\{n^2 - 3n - 2n\|\varphi\| + 2\|\varphi\|^2\right\}. \tag{3.4}$$

Next, we obtain from (3.3) and (3.4)

$$n^2\|H\|^2 = (n-1)\left(\|h\|^2 + \varepsilon\right). \tag{3.5}$$

Consider the following:

$$x \in M^n, \pi \subset T_xM^n, \dim \pi = 2, \pi = sp\left\{e_1, e_2\right\}.$$

By defining $e_{n+1} = \frac{H}{\|H\|}$, we may derive the following from the relation (3.5):

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = (n-1)\left[\sum_{i,j=1}^{n}\sum_{r=n+1}^{n+p}\left(h_{ij}^{r}\right)^2 + \varepsilon\right],$$

or in the same manner,

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = (n-1)\left\{\sum_{i=1}^{n}\left(h_{ii}^{n+1}\right)^2\right\} + (n-1)\left\{\sum_{i\neq j}\left(h_{ij}^{n+1}\right)^2 + \sum_{i,j=1}^{n}\sum_{r=n+2}^{n+p}\left(h_{ij}^{r}\right)^2 + \varepsilon\right\}. \tag{3.6}$$

We obtain from (3.6) by applying Lemma 2.1:

$$2h_{11}^{n+1}h_{22}^{n+1} \geq \sum_{i\neq j}\left(h_{ij}^{n+1}\right)^2 + \sum_{i,j=1}^{n}\sum_{r=n+2}^{n+p}\left(h_{ij}^{r}\right)^2 + \varepsilon. \tag{3.7}$$

For the case $X = W = e_1, Y = Z = e_2$, the Gauss equation yields

$$\begin{aligned}
K(\pi) =& R(e_1, e_2, e_2, e_1) \\
=& \frac{c}{3}\{1 - \|\varphi\|\} - \alpha(e_1, e_1) - \alpha(e_2, e_2) + \sum_{r=n+1}^{p}\left[h_{11}^r h_{22}^r - \left(h_{12}^r\right)^2\right] \\
\geq& \frac{c}{3}\{1 - \|\varphi\|\} - \alpha(e_1, e_1) - \alpha(e_2, e_2) + \sum_{r=n+2}^{n+p} h_{11}^r h_{22}^r - \sum_{r=n+1}^{n+p}\left(h_{12}^r\right)^2 \\
&+ \frac{1}{2}\left[\sum_{i\neq j}\left(h_{ij}^{n+1}\right)^2 + \sum_{i,j=1}^{n}\sum_{r=n+2}^{n+p}\left(h_{ij}^r\right)^2 + \varepsilon\right] \\
=& \frac{c}{3}\{1 - \|\varphi\|\} - \alpha(e_1, e_1) - \alpha(e_2, e_2) + \sum_{r=n+2}^{n+p} h_{11}^r h_{22}^r - \sum_{r=n+1}^{n+p}\left(h_{12}^r\right)^2 \\
&+ \frac{1}{2}\sum_{i\neq j}\left(h_{ij}^{n+1}\right)^2 + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{r=n+2}^{n+p}\left(h_{ij}^r\right)^2 + \frac{1}{2}\varepsilon \\
=& \frac{c}{3}\{1 - \|\varphi\|\} - \alpha(e_1, e_1) - \alpha(e_2, e_2) + \frac{1}{2}\sum_{i\neq j}\left(h_{ij}^{n+1}\right)^2 + \frac{1}{2}\sum_{r=n+2}^{n+p}\sum_{i,j>2}\left(h_{ij}^r\right)^2 \\
&+ \frac{1}{2}\sum_{r=n+2}^{n+p}\left(h_{11}^r + h_{22}^r\right)^2 + \sum_{j>2}\left[\left(h_{1j}^{n+1}\right)^2 + \left(h_{2j}^{n+1}\right)^2\right] + \frac{1}{2}\varepsilon \\
\geq& \frac{c}{3}\{1 - \|\varphi\|\} - \alpha(e_1, e_1) - \alpha(e_2, e_2) + \frac{\varepsilon}{2},
\end{aligned}$$

that implies

$$K(\pi) \geq \frac{c}{3}\{1 - \|\varphi\|\} - \alpha(e_1, e_1) - \alpha(e_2, e_2) + \frac{\varepsilon}{2}.$$

We note that

$$\alpha(e_1, e_1) + \alpha(e_2, e_2) = \lambda - \mathrm{trace}\left(\alpha_{|_{\pi^\perp}}\right)$$

Based on (3.4), we obtain

$$\begin{aligned}
K(\pi) \quad &\geq \tau + (n-2)\left[-\tfrac{n^2}{2(n-1)}\|H\|^2 + \lambda\right] + \mathrm{trace}\left(\alpha_{|_{\pi^\perp}}\right) \\
&+\tfrac{c}{6}\left\{2 + 3n - n^2 - 2\|\varphi\|^2 - 2\|\varphi\| + 2n\|\varphi\|\right\},
\end{aligned}$$

This stands for the inequality that needs to be demonstrated.

Remember the following significant finding from [22] (Proposition 1.2).

Proposition 3.2. *The mean curvature H of M^n with regard to ssmc and the mean curvature $\overset{\circ}{H}$ of M^n corresponding to Levi-Civita connection coincide if and only if P is tangent to M^n.*

Remark 3.1. *Formula (7) from [28] indicates that if P is tangent to M^n, then $h = \overset{\circ}{h}$.*

Relation (3.1) in this instance becomes

Corollary 3.3. *Similar to the Theorem 3.1, if P is tangent to M^n, then we get*

$$\begin{aligned}
\tau - K(\pi) \quad &\leq (n-2)\left[\tfrac{n^2}{2(n-1)}\|\overset{\circ}{H}\|^2 - \lambda\right] - \mathrm{trace}\left(\alpha_{|_{\pi^\perp}}\right) \\
&-\tfrac{c}{6}\left\{2 + 3n - n^2 - 2\|\varphi\|^2 - 2\|\varphi\| + 2n\|\varphi\|\right\}.
\end{aligned} \tag{3.8}$$

Theorem 3.4. *Given that M^n is tangent to P. The equality case in (3.1) is true if and only if with orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of T_xM^n and orthonormal basis $\left\{e_{n+1}, \ldots, e_{n+p}\right\}$ of $T_x^{\perp}M^n$, shape operators A at $x \in M^n$, take the following identities:*

$$A_{e_{n+1}} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu \end{pmatrix}, \quad a + b = \mu,$$

$$A_{e_{n+i}} = \begin{pmatrix} h_{11}^{n+i} & h_{12}^{n+i} & 0 & \cdots & 0 \\ h_{12}^{n+i} & -h_{11}^{n+i} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad 2 \leq i \leq p,$$

where we indicate $n + 1 \leq r \leq n + p$ and $h_{ij}^r = g\left(h\left(e_i, e_j\right), e_r\right), 1 \leq i, j \leq n$.

Proof. When all prior inequalities are equal and the equality in the lemma is achieved, the equality case is said to hold at $x \in M^n$:

$$h_{ij}^{n+1} = 0, \quad \forall i \neq j, i, j > 2,$$

$$h_{ij}^r = 0, \quad \forall i \neq j, i, j > 2, r = n+1, \ldots, n+p,$$

$$h_{11}^r + h_{22}^r = 0, \quad \forall r = n+2, \ldots, n+p,$$

$$h_{1j}^{n+1} = h_{2j}^{n+1} = 0, \quad \forall j > 2,$$

$$h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+1} = \ldots = h_{nn}^{n+1}.$$

Assuming $h_{12}^{n+1} = 0$, we can select $\{e_1, e_2\}$. We then denote with $a = h_{11}^r, b = h_{22}^r, \mu = h_{33}^{n+1} = \ldots = h_{nn}^{n+1}$. As a result, A adopt the desired forms.

4. Ricci Curvature along a Unit Tangent Vector's Direction

In this part, we prove a sharp relationship between the mean curvature H and the Ricci curvature in the direction of a unit tangent vector X with regard to $\tilde{\nabla}$. Indicate by

$$N(x) = \{X \in T_xM^n \mid h(X, Y) = 0, \quad \forall Y \in T_xM^n\}.$$

Theorem 4.1. *Let $M^n, n \geq 3$ be Riemannian manifold immersed in $N^{n+p}(c)$. Following holds:*
(i) $\forall X \in T_xM$, we possess

$$\begin{aligned} \|H\|^2 \quad \geq \quad & \frac{4}{n^2}[\mathrm{Ric}(X) - \frac{c}{3}\{(2-n) + \|\varphi\|\} + (2n-3)\lambda] \\ & - \frac{4}{n^2}[(n-2)\alpha(X, X)]. \end{aligned} \tag{4.1}$$

(ii) In the case where $H(x) = 0$, the equality condition of (4.1) is satisfied by X at x if and only if $X \in N(x)$.

Proof. (i) Assume that $X \in T_xM^n$. Let $e_1, e_2, \ldots, e_n, e_{n+1}, \ldots e_{n+p}$ be an orthonormal basis that we select so that $e_1, e_2, \ldots, e_n$ are tangent to M^n at x, with $e_1 = X$. We derive from (3.3) that

$$n^2\|H\|^2 \quad = \quad 2\tau + \frac{1}{2}\sum_{r=n+1}^{n+p}\left[\left(h_{11}^r + \ldots + h_{nn}^r\right)^2 + \left(h_{11}^r - h_{22}^r - \ldots - h_{nn}^r\right)^2\right]$$

$$+2\sum_{r=n+1}^{n+p}\sum_{1\leq i<j\leq n}\left(h_{ij}^{r}\right)^{2}-2\sum_{r=n+1}^{n+p}\sum_{2\leq i<j\leq n}\left(h_{ii}^{r}h_{jj}^{r}\right) \tag{4.2}$$
$$+2(n-1)\lambda-\frac{c}{3}\left\{n^{2}-3n-2n\|\varphi\|+2\|\varphi\|^{2}\right\}.$$

After performing some basic calculations, we obtain from the Gauss equation for $X = W = e_i, Y = Z = e_j$, and $i \neq j$,

$$\sum_{2\leq i<j\leq n}K_{ij} = \sum_{r=n+1}^{n+p}\sum_{2\leq i<j\leq n}\left[h_{ii}^{r}h_{jj}^{r}-\left(h_{ij}^{r}\right)^{2}\right]-(n-2)\left[\lambda-\alpha\left(e_{1},e_{1}\right)\right]$$
$$+\frac{c}{6}\left\{(n-1)(n-4)-2(n-1)\|\varphi\|+2\|\varphi\|^{2}\right\}. \tag{4.3}$$

Substitution of (4.3) into (4.2) yields

$$n^{2}\|H\|^{2}\geq\frac{1}{2}n^{2}\|H\|^{2}+2\left(\tau-\sum_{2\leq i<j\leq n}K_{ij}\right)+2\sum_{r=n+1}^{n+p}\sum_{j=2}^{n}\left(h_{1j}^{r}\right)^{2}$$
$$-\frac{2c}{3}\left\{(2-n)+\|\varphi\|\right\}+2(2n-3)\lambda-2(n-2)\alpha\left(e_{1},e_{1}\right),$$

that provides us

$$\frac{1}{2}n^{2}\|H\|^{2}\geq 2\,\mathrm{Ric}(X)-\frac{2c}{3}\left\{(2-n)+\|\varphi\|\right\}$$
$$+2(2n-3)\lambda-2(n-2)\alpha(X,X).$$

The inequality (4.1) is demonstrated by this.

(ii) Let us assume that $H(x) = 0$. In (4.1), equality exists if and only if

$$h_{12}^{r}=\ldots=h_{1n}^{r}=0,$$
$$h_{11}^{r}=h_{22}^{r}+\ldots+h_{nn}^{r},\quad r\in\{n+1,\ldots,n+p\}.$$

So $h_{1j}^{r} = 0, \forall j \in \{1, \ldots, n\}, r \in \{n + 1, \ldots, n + p\}$, i.e. $X \in N(x)$.

Corollary 4.2. *If M^n is tangent to P, then for all unit tangent vectors at x, the equality case of (4.1) holds identically if and only if either $n = 2$ and x is totally umbilical point, or x is a totally geodesic point.*

Proof. For all unit tangent vectors at x, the equality case of (4.1) holds if and only if

$$h_{ij}^{r}=0,\quad i\neq j,\quad r\in\{n+1,\ldots,n+p\},$$
$$h_{11}^{r}+\ldots+h_{nn}^{r}-2h_{ii}^{r}=0,\quad i\in\{1,\ldots,n\},\quad r\in\{n+1,\ldots,n+p\}.$$

We separate these two instances:

- x is a totally geodesic point if $n \neq 2$;
- Given $n = 2$, x is a totally umbilical point.

Converse is trivial.

5. *k*-RICCI CURVATURE

Consider a Riemannian manifold M^n immersed in $N^{n+p}(c)$. First, we establish a connection between the sectional curvature of M^n and $||H||^2$. We utilize this inequality to establish a connection between $||H||^2$ (extrinsic invariant) and the k-Ricci curvature of M^n (intrinsic invariant). We assume that P is tangent to M^n in this section.

Theorem 5.1. *Let $M^n, n \geq 3$ be Riemannian submanifold of $N^{n+p}(c)$. Further, let M^n be tangent to P. Subsequently, we have*

$$||H||^2 \geq \frac{2\tau}{n(n-1)} - \frac{c}{3n(n-1)}\left\{n^2 - 3n - 2n||\varphi|| + 2||\varphi||^2\right\} + \frac{2}{n}\lambda. \tag{5.1}$$

Proof. Let T_xM^n have an orthonormal basis defined by $\{e_1, e_2, \dots, e_n\}$ and for any $x \in M^n$. The relationship (3.4) is identical as

$$n^2||H||^2 = 2\tau + ||h||^2 + 2(n-1)\lambda - \frac{c}{3}\left\{n^2 - 3n - 2n||\varphi|| + 2||\varphi||^2\right\}. \tag{5.2}$$

At x, we select an orthonormal basis $\left\{e_1, \dots, e_n, e_{n+1}, \dots, e_{n+p}\right\}$ such that $H(x)$ and e_{n+1} are parallel to each other, where $A_{e_{n+1}}$ is diagonalized by $e_1, \dots, e_n$. The forms then taken by the shape operators are

$$A_{e_{n+1}} = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix},$$

$A_{e_r} = \left(h^r_{ij}\right), i, j = 1, \dots, n; r = n+2, \dots, n+p, \operatorname{trace} A_r = 0.$

We obtain from (5.2)

$$\begin{aligned} n^2||H||^2 &= 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{n+p}\sum_{i,j=1}^{n}\left(h^r_{ij}\right)^2 \\ &\quad +2(n-1)\lambda - \frac{c}{3}\left\{n^2 - 3n - 2n||\varphi|| + 2||\varphi||^2\right\}. \end{aligned} \tag{5.3}$$

On the contrary, since

$$0 \leq \sum_{i<j}\left(a_i - a_j\right)^2 = (n-1)\sum_i a_i^2 - 2\sum_{i<j} a_i a_j.$$

We acquire

$$n^2||H||^2 = \left(\sum_{i=1}^{n} a_i\right)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{i<j} a_i a_j \leq n\sum_{i=1}^{n} a_i^2,$$

which implies

$$\sum_{i=1}^{n} a_i^2 \geq n||H||^2$$

Based on (5.3), we possess

$$n^2||H||^2 \geq 2\tau + n||H||^2 + 2(n-1)\lambda - \frac{c}{3}\left\{n^2 - 3n - 2n||\varphi|| + 2||\varphi||^2\right\}$$

Likewise, in the same way,

$$||H||^2 \geq \frac{2\tau}{n(n-1)} - \frac{c}{3n(n-1)}\left\{n^2 - 3n - 2n||\varphi|| + 2||\varphi||^2\right\} + \frac{2}{n}\lambda.$$

Applying Theorem 5.1, we get the subsequent

Theorem 5.2. *Let $M^n, n \geq 3$ be a Riemannian manifold immersed in $N^{n+p}(c)$. Then, for any $x \in M^n$ and any integer $k, 2 \leq k \leq n$, we obtain*

$$||H||^2(p) \geq \Theta_k(p) - \frac{c}{3n(n-1)}\left\{n^2 - 3n - 2n||\varphi|| + 2||\varphi||^2\right\} + \frac{2}{n}\lambda, \tag{5.4}$$

where we have considered M^n is tangent to P.

Proof. For T_xM^n, let $\{e_1, \ldots, e_n\}$ be an orthonormal basis. The k-plane section spanned by $e_{i_1}, \ldots, e_{i_k}$ is denoted by $L_{i_1 \ldots i_k}$. According to definitions, one possesses

$$\tau\left(L_{i_1\ldots i_k}\right) = \frac{1}{2}\sum_{i\in\{i_1,\ldots,i_k\}} \mathrm{Ric}_{L_{i_1\ldots i_k}}(e_i),$$
$$\tau(x) = \frac{1}{C_{n-2}^{k-2}}\sum_{1\leq i_1<\ldots<i_k\leq n} \tau\left(L_{i_1\ldots i_k}\right).$$

From (5.1), one obtains

$$\tau(x) \geq \frac{n(n-1)}{2}\Theta_k(p),$$

implying (5.4).

6. Golden sectional curvature as a function of golden scalar curvature

Assume that N is a Golden Riemannian manifold whose Golden sectional curvature (GSC) is constant. Next, as described in [32], the Golden Ricci tensor in terms of GSC of N is as follows:

$$\begin{aligned} S^G\left(\tilde{X}, \tilde{Y}\right) &= S\left(\tilde{X}, \varphi\tilde{Y}\right) \\ &= \frac{c}{3}\left\{g\left(\tilde{X}, \varphi\tilde{Y}\right)[||\varphi|| - 3] + g\left(\tilde{X}, \tilde{Y}\right)[2||\varphi|| - n]\right\}. \end{aligned}$$

Moreover, [32] defines the Golden scalar curvature in terms of the GSC.

$$r^G = \frac{c}{3}\left\{||\varphi||(3||\varphi|| - 3 - n)\right\}. \tag{6.1}$$

We demonstrate the following sharp inequality using the Golden scalar curvature in terms of GSC:

Theorem 6.1. *Let $M^n, n \geq 3$ be a Riemannian manifold immersed in $N^{n+p}(c)$ and allow M^n to be tangent to P. Following that, we have*

$$||H||^2 \geq \frac{2\tau}{n(n-1)} - \frac{c}{3n(n-1)}\left\{||\varphi||(3||\varphi|| - 3 - n)\right\} + \frac{2}{n}\lambda. \tag{6.2}$$

Applying Theorem 6.1, we derive the subsequent

Theorem 6.2. *For any Riemannian manifold $M^n, n \geq 3$ immersed in $N^{n+p}(c)$ and tangent to P. We obtain*

$$||H||^2(p) \geq \Theta_k(p) - \frac{c}{3n(n-1)}\left\{||\varphi||(3||\varphi|| - 3 - n)\right\} + \frac{2}{n}\lambda, \tag{6.3}$$

where $k, 2 \leq k \leq n$ is any integer and $x \in M^n$.

7. Acknowledgement

The first author expresses gratitude to Maulana Azad National Urdu University, Hyderabad, India, for the financial support via Minor Research Project (MiRP) No. MANUU/DR&C/F.11/2023-24/97, facilitating the completion of this research. Furthermore, the authors wish to express their gratitude to the referees for their valuable and constructive comments.

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