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Modified inertial projection methods for bilevel quasi-monotone variational inequalities

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Abstract. In this paper, we propose two projection-based methods with different inertial steps to solve the bilevel quasi-monotone variational inequality problem in real Hilbert spaces. Our proposed algorithms need to compute the projection on the feasible set only once in each iteration with Armijo line search methods. Strong convergence theorems of the proposed algorithms are established under some suitable and mild conditions. Some numerical examples are given to illustrate the comparison of our proposed algorithms with some already known algorithms.

1. Introduction

Let C be a nonempty, closed, and convex subset of a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. Recall that the bilevel variational inequality problem (shortly, BVIP) is described as follows:

$$\text{Find } x^* \in \Omega \text{ such that } \langle Fx^*, y - x^* \rangle \geq 0, \quad \forall y \in \Omega, \tag{1}$$

where $F : C \to H$ is an operator and Ω denotes the set of all solutions of the following variational inequality problem (shortly, VIP):

$$\text{Find } y^* \in C \text{ such that } \langle My^*, z - y^* \rangle \geq 0, \quad \forall z \in C. \tag{2}$$

where $M : C \to H$ is an operator. Since the bilevel variational inequality problems include a number of problems, such as, quasi-variational inequality problems, complementary problems, and so on. For more details on the theory, algorithms and applications of bilevel optimization problems. We refer reader to the recent monograph [10]. It is therefore necessary to develop some fast and efficient numerical approaches to solve the BVIP. In this paper, we are concerned with projection-based methods for solving the BVIP.

The simplest projection-type method is the projected gradient method (shortly, PGM), which starting from any $x_0 \in C$, iteratively updates x_{n+1} according to the formula

$$x_{n+1} = P_C(x_n - \lambda A x_n).$$

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This projected gradient method can be easily implemented because it only needs to calculate the function value and the projection onto C once in each iteration. However, the projected gradient method requires a restrictive hypothesis on M for the convergence, that is, M is strongly monotone and Lipschitz continuous. To relax the strong assumptions required by the projected gradient method and thus broaden the class of the problems that we can solve, the extragradient method (shortly, EGM) proposed by Korpelevich [16]. Taking the initial value $x_0 \in C$, we generate a succession x_n such that

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n). \end{cases}$$

In recent years, the EGM was extensively studied by scholars, and they proposed a large number of improved versions of the EGM for solving variational inequalities in infinite-dimensional Hilbert spaces, see, e.g., [3, 6, 7, 22] and the references therein. However, we note that the EGM needs to perform two projection calculations on the feasible set C in each iteration, which may seriously affect the computational performance, especially when C is a general closed convex set. To overcome this disadvantage, Censor, Gibali and Reich [20] introduced the subgradient extragradient method (shortly, SEGM), which can be seen as a modification of the EGM. They replaced the second projection onto C with a projection onto a half-space. More precisely, their algorithm is expressed as follows:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ T_n = \{x \in H \mid \langle x_n - \lambda A x_n - y_n, x - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda A y_n), \end{cases}$$

where mapping M is L-Lipschitz continuous monotone and fixed step size λ is in $(0, 1/L)$. They confirmed that the SEGM is weakly convergent in a Hilbert space. It is worth noting that the projection onto a half-space T_n can be calculated by an explicit formula. This greatly improves the computational performance of the EGM. Today, many scholars have found that the linear search accelerates the convergence rate. Iusem [9] proposed a new iterative algorithm that is based on the EGM and the Armijo line search method for solving variational inequality problems in finite-dimensional spaces. Note that the convergence of Iusem's method is proved under the assumption that the mapping M is not Lipschitz continuous. It should be noted that Iusem's method may need to compute multiple projections on the feasible set in each iteration due to its use of the Armijio line search criterion. Solodov and Svaiter [24] introduced an improved algorithm with a new Armijo-type step size to overcome this obstacle. They constructed a new hyperplane which separates the current iterate from the solution of the VIP. The convergence of the method is also confirmed under the condition that the mapping M is uniformly continuous. Moreover, the method of Solodov and Svaiter [24] requires only one projection onto the feasible set in each iteration, which greatly improves the computational efficiency of the method of Iusem [9]. A large number of scholars improved algorithms of Solodov and Svaiter [24] to solve monotone VIPs (see, e.g., [18, 20, 25]) and pseudo-monotone VIPs (see, e.g., [2, 16, 19, 30, 31]). The convergence of these methods is established under the assumption of the mapping M without Lipschitz continuity.

The researchers tried to change their algorithms to increase the convergence rate of the VIP and its related problems. Algorithms with fast convergence rate has been of utmost interest. Introduction of the inertial term into an iterative scheme has been shown to be efficient technique for accelerating the convergence rate of such iterative method. The inertial technique stemmed from a discrete analogue of a second order dissipative dynamical system. In convex optimization, the well-known Polyak's heavy ball algorithm in [1], which is an inertial extrapolation process for minimizing a smooth convex function is the first of such method. Therefore, many scholars used this popular method to solve the VIP and its related problems using different methods (see [4, 13, 15, 23] for details).

Shehu et al.[21] presented a modified subgradient extragradient method with single inertial step and self-adaptive step size to solve the VIP: Given $\lambda_1 > 0, x_0, x_1 \in H$ and $\mu \in (0,1)$,

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda_n a w_n), \\ T_n = \{w \in H \mid \langle w_n - \lambda_n a w_n - y_n, w - y_n\rangle \le 0\}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_{T_n}(w_n - \lambda_n a y_n), \end{cases}$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|}, \lambda_n\right\}, & if Aw_n \ne Ay_n, \\ \lambda_n, & otherwise, \end{cases}$$

and $0 \le \theta_n \le \theta_{n+1} \le 1$, $0 < \alpha \le \alpha_n \le \alpha_{n+1} \le \frac{1}{2+\delta}(\delta > 0)$. If the operator A is monotone and L-Lipschitz continuous, then the sequence $\{x_n\}$ generated by the algorithm converges weakly to a solution of the VIP.

Inspired by Shehu et al.[21], Yao et al.[33] proposed a relaxed the SEGM with double inertial extrapolation steps. It is of the form: Given $\lambda_1 > 0, x_0, x_1 \in H$ and $\mu \in (0,1)$,

$$\begin{cases} z_n = x_n + \delta(x_n - x_{n-1}), \\ w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda_n A w_n), \\ T_n := \{w \in H \mid \langle w_n - \lambda_n A w_n - y_n, w - y_n\rangle \le 0\}, \\ x_{n+1} = (1-\alpha_n)z_n + \alpha_n P_{T_n}(w_n - \lambda_n A y_n), \end{cases}$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|}, \lambda_n\right\}, & if Aw_n \ne Ay_n, \\ \lambda_n, & otherwise, \end{cases}$$

and $0 \le \theta_n \le \theta_{n+1} \le 1$, $0 \le \delta < \min\{\frac{\epsilon - \sqrt{2\epsilon}}{\epsilon}, \theta_1\}$, $0 < \alpha \le \alpha_n \le \alpha_{n+1} < \frac{1}{1+\epsilon}(2 < \epsilon < \infty)$. The sequence $\{x_n\}$ converges weakly to a solution of the VIP when A is pseudo-monotone and L-Lipschitz continuous. Moreover, in the Numerical Examples section of Yao et al.[33] has been shown that this method is more efficient and implementable.

Motivated by the works of Peng et al. [14], Tan et al.[26, 27], Thong et al.[29] and Li et al.[11] in this direction, in this paper, we propose a projection and contraction algorithm with different inertial extrapolation steps for approximating the solution of the bilevel quasi-monotone variational inequality problems in the framework of real Hilbert spaces. We incorporate different inertial steps into our algorithms with a better relaxation on the cost operator M. In our case, M is assumed to be quasi-monotone instead of the usual condition that M is pseudo-monotone as seen in [26, 29] and most literature. We obtain a strong convergence result for the sequence generated by this method under certain mild assumptions on the algorithm parameters. We give some numerical examples to show the applicability and efficiency of our proposed methods.

This paper is organized as follows. In Section 2, we recall some fundamental definitions and preliminary lemmas for further use. In Section 3, we describe the form of the algorithms and analyze the convergence of the proposed algorithms. In Section 4, several numerical experiments are performed to illustrate the superior performance of the proposed algorithms against the previously known algorithms. In Section 5, we have performed a brief summary.

2. Preliminaries

Let C be a nonempty, closed, and convex subset of a real Hilbert space H. The weak convergence and strong convergence of $\{x_n\}$ to x are represented by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. Let $P_C : H \to C$ denote the metric (nearest point) projection from H onto C, characterized by $P_C(x) := \arg\min \|x-y\|, y \in C$. It is known that $P_C x$ is nonexpansive and $P_C \in C$ for all $x \in H$.

Definition 2.1. *The operator $M : H \to H$ is said to be*
(i) *L-Lipschitz continuous, if there exists a constant $L > 0$ such that*

$$\|Mx - My\| \le L\|x - y\|, \ \ \forall x, y \in H.$$

(ii) *η-strongly monotone, if there exists a constant $\eta > 0$ such that*

$$\langle Mx - My, y - x\rangle \ge \eta\|y - x\|^2, \ \ \forall x, y \in H.$$

(iii) *monotone, if*

$$\langle Mx - My, y - x\rangle \ge 0, \ \ \forall x, y \in H.$$

(iv) *γ-strongly pseudo-monotone, if there exists a constant $\gamma > 0$ such that*

$$\langle Mx, y - x\rangle \ge 0 \Rightarrow \langle My, y - x\rangle \ge \gamma\|y - x\|^2, \ \ \forall x, y \in H.$$

(v) *pseudo-monotone, if*

$$\langle Mx, y - x\rangle \ge 0 \Rightarrow \langle My, y - x\rangle \ge 0, \ \ \forall x, y \in H.$$

(vi) *quasi-monotone, if*

$$\langle Mx, y - x\rangle > 0 \Rightarrow \langle My, y - x\rangle \ge 0, \ \ \forall x, y \in H.$$

Clearly, (ii) $\Rightarrow$ (iii) $\Rightarrow$ (v) $\Rightarrow$ (vi) and (ii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi), but the converses are not always true.

Lemma 2.2. *The following statements hold in H:*
(i) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \quad \forall x, y \in H.$
(ii) $\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H.$
(iii) $\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2, \quad \forall x, y \in H, \lambda \in \mathbb{R}.$

Lemma 2.3. *Let C be a nonempty closed and convex subset of H and P_C be the metric projection from H onto C. Then for any $x, y \in H$ and $z \in C$, the following hold:*
(i) $\|P_C x - P_C y\| \le \langle P_C x - P_C y, x - y\rangle$.
(ii) $\|P_C x - z\|^2 \le \|x - z\|^2 - \|P_C x - x\|^2$.

Lemma 2.4. *For any $x \in H$ and $z \in C$, then $z = P_C(x)$ if and only if*

$$\langle x - z, y - z\rangle \le 0, \ \ \forall y \in C.$$

Lemma 2.5. *[5] Assume that C is a closed and convex subset of a real Hilbert space H. Let operator $M : C \to H$ be continuous and pseudo-monotone. Then x^* is a solution of the VIP if and only if $\langle Mx, x - x^*\rangle \ge 0, \quad \forall x \in C$.*

Lemma 2.6. *[8] Assume that C is a convex and closed nonempty subset of a real Hilbert space H. Let h be a real-valued function on H and define $K = \{x \in C : h(x) \le 0\}$. If K is nonempty and h is θ-Lipschitz continuous on C, then*

$$dist(x, K) \ge \theta^{-1}\max\{h(x), 0\}, \ \ \forall x \in C,$$

where $dist(x, K)$ denotes the distance function from x to K.

Lemma 2.7. *[32] Let $\gamma > 0$ and $\alpha \in (0, 1]$. Let $F : H \to H$ be a β-strongly monotone and L-Lipschitz continuous mapping with $0 < \beta \le L$. Associating with a nonexpansive mapping $T : H \to H$, define a mapping $T^\gamma : H \to H$ by $T^\gamma x = (I - \alpha\gamma F)(Tx)$, $\forall x \in H$. Then, T^γ is a contraction mapping provided $\gamma < \frac{2\beta}{L^2}$, that is*

$$\|T^\gamma x - T^\gamma y\| \le (1 - \alpha\eta)\|x - y\|, \ \ \forall x, y \in H,$$

where $\eta = 1 - \sqrt{1 - \gamma(2\beta - \gamma L^2)} \in (0, 1)$.

Lemma 2.8. *[17] Let $\{p_n\}$ be a positive sequence, $\{q_n\}$ be a sequence of real numbers, and $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $\sum_{n=1}^{\infty}\alpha_n = \infty$. Assume that*

$$p_{n+1} \leq (1-\alpha_n)p_n + \alpha_n q_n, \quad \forall n \geq 1.$$

If $\limsup_{k\to\infty} q_{n_k} \leq 0$ for every subsequence $\{p_{n_k}\}$ of $\{p_n\}$ satisfying $\liminf_{k\to\infty}(p_{n_k+1} - p_{n_k}) \geq 0$, then $\lim_{n\to\infty} p_n = 0$.

3. Main results

In this section, we introduce two new algorithms for finding the solutions of the bilevel quasi-monotone variational inequality problem. The following assumptions are assumed to be satisfied.
(A1) The feasible C is a nonempty closed and convex subset of the real Hilbert space H.
(A2) The solution set of the VIP is nonempty, that is, $\Omega \neq \emptyset$.
(A3) The operator $M : H \to H$ is quasi-monotone, L_0-Lipschitz continuous on H, and the operator $M : H \to H$ satisfies the following assumption

$$\text{whenever } \{x_n\} \subset C, \ x_n \rightharpoonup z, \ \text{one has } \|Mz\| \leq \liminf_{n\to\infty}\|Mx_n\|.$$

(A4) The mapping $F : H \to H$ is L_F-Lipschitz continuous and β-strongly monotone on H such that $L_F \geq \beta$.
(A5) Let $\{\epsilon_n\}$ be a positive sequence such that $\lim_{n\to\infty}\frac{\epsilon_n}{\alpha_n} = 0$, and $\lim_{n\to\infty}\frac{\beta_n}{\alpha_n} = 0$, where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$ satisfies $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$.
(A6) The solution set of the BVIP (1),

$$\Gamma := \{x^* \in \Omega \ \text{such that} \ \langle Fx^*, y - x^*\rangle \geq 0, \ \forall x \in \Omega\} \neq \emptyset.$$

Algorithm 3.1 Modified inertial extragradient method for solving the BVIP.

Initialization. Let $\theta > 0$, $\ell \in (0,1)$, $\mu > 0$, $\gamma \in (0, \frac{2\beta}{L_F^2})$, $\lambda_n \in (0, \frac{1}{\mu})$, $\{\phi_n\} \subset (0,1)$, $\lim_{n\to\infty}\frac{\phi_n}{\alpha_n} = 0$, and let $x_0, x_1 \in H$. Choose a nonnegative real sequence $\{p_n\}$ such that $\sum_{n=0}^{\infty} p_n < +\infty$.

Iterative Step. Given the iterates x_{n-1} and x_n for each $n \geq 1$, calculate x_{n+1} as follows:
Step 1. Compute $w_n = (1-\phi_n)[x_n + \theta_n(x_n - x_{n-1})]$, where

$$\theta_n = \begin{cases} \min\Big\{\theta, \dfrac{\epsilon_n}{\|x_n - x_{n-1}\|}\Big\}, & if\, x_n \neq x_{n-1}, \\ \theta, & otherwise. \end{cases} \tag{3}$$

Step 2. Compute $y_n = P_C(w_n - \lambda_n M w_n)$,

$$\lambda_{n+1} = \begin{cases} \min\Big\{\dfrac{\mu\|w_n - y_n\|}{\|Mw_n - My_n\|}, \lambda_n + p_n\Big\}, & if\, Mw_n \neq My_n, \\ \lambda_n + p_n, & otherwise. \end{cases}$$

Step 3. Compute $t_n = w_n - \tau_n(w_n - y_n)$, where $\tau_n = \ell^{m_n}$ and m_n is the smallest non-negative integer m satisfying

$$\langle Mw_n - M(w_n - \ell^m(w_n - y_n)), w_n - y_n\rangle \leq \mu\|w_n - y_n\|^2. \tag{4}$$

Step 4. Compute $z_n = P_{H_n}(w_n)$, where the half-space H_n is defined by

$$H_n = \{x \in C : h_n(x) \leq 0\} \ \textit{and} \ \ h_n(x) = \langle Mt_n, x - t_n\rangle. \tag{5}$$

Step 5. Compute $x_{n+1} = z_n - \alpha_n \gamma F z_n$.
Set $n := n + 1$ and return to **Step 1**.

Remark 3.1. *We note here that inertial calculation criterion is easy to implement because the term $\|x_n - x_{n-1}\|$ is known before calculating θ_n. It follows from (3) and the assumptions on $\{\alpha_n\}$ that $\lim\limits_{n\to\infty} \frac{\theta_n\|x_n - x_{n-1}\|}{\alpha_n} = 0$. Furthermore, the assumption $(A5)$ is easily satisfied by, for example, taking $\alpha_n = \frac{1}{n+1}$ and $\epsilon_n = \frac{1}{(n+1)^2}$.*

Remark 3.2. *Suppose that Condition (A3) holds. Then the sequence $\{\lambda_n\}$ generated by Algorithm 3.1 is well defined and $\lim\limits_{n\to\infty} \lambda_n = \lambda$ and $\lambda \in \Big[\min\{\frac{\mu}{L_0}, \lambda_1\}, \lambda_1 + \sum\limits_{n=1}^{\infty} p_n\Big]$.*

Proof. *We can easily prove the remark by means of [12]. We omit the proof here to avoid repetitive expressions.* □

The following lemmas are crucial for the convergence analysis of Algorithm 3.1.

Lemma 3.3. *Suppose that Assumptions $(A1) - (A3)$ hold. The Armijo line search rule (5) is well defined.*

Proof. Since mapping M is uniformly continuous on C and $\ell \in (0, 1)$, one obtains

$$\lim_{m\to\infty} \langle Mw_n - M(w_n - \ell^m(w_n - y_n)), w_n - y_n\rangle = 0$$

Moreover, it can be easily seen that $\|w_n - y_n\| > 0$ (otherwise, y_n is a solution of the VIP). Thus, there exists a non-negative integer m_n satisfying (5). □

Lemma 3.4. *Suppose that Assumptions $(A1) - (A3)$ hold. Let x^* be a solution of the VIP. Then $h_n(x^*) \leq 0$ and $h_n(w_n) \geq \tau_n(\lambda_n^{-1} - \mu)\|w_n - y_n\|^2$. In particular, if $w_n - y_n \neq 0$ then $h_n(w_n) > 0$.*

Proof. From $x^* \in \Omega$, $t_n \in C$ and Lemma 2.5, one obtains $h_n(x^*) = \langle Mt_n, x^* - t_n\rangle \leq 0$. Using the definitions of h_n and t_n, one sees that

$$h_n(w_n) = \langle Mt_n, w_n - t_n\rangle = \langle Mt_n, \tau_n(w_n - y_n)\rangle = \tau_n\langle Mt_n, w_n - y_n\rangle. \tag{6}$$

By using the property of the projection $\|x - P_C(y)\|^2 \leq \langle x - y, x - P_C(y)\rangle$, $\forall x \in C$, $y \in H$ and taking $x = w_n$ and $y = w_n - \lambda_n M w_n$, we obtain

$$\|w_n - P_C(w_n - \lambda_n M w_n)\|^2 \leq \lambda_n \langle Mw_n, w_n - P_C(w_n - \lambda_n M w_n)\rangle,$$

which yields that $\langle Mw_n, w_n - y_n\rangle \geq \lambda_n^{-1}\|w_n - y_n\|^2$. From (5), one has

$$\begin{aligned} \langle Mt_n, w_n - y_n\rangle \ &\geq \ \langle Mw_n, w_n - y_n\rangle - \mu\|w_n - y_n\|^2 \\ &\geq \ (\lambda_n^{-1} - \mu)\,\|w_n - y_n\|^2. \end{aligned} \tag{7}$$

Combining (6) and (7), we observe that $h_n(w_n) \geq \tau_n(\lambda_n^{-1} - \mu)\|w_n - y_n\|^2$. □

Lemma 3.5. *Suppose that Assumption $(A1) - (A3)$ hold. Let $\{y_n\}$ and $\{w_n\}$ be sequences generated by Algorithm 3.1. Let $\{y_{n_k}\}$ and $\{w_{n_k}\}$ be subsequences of $\{y_n\}$ and $\{w_n\}$ respectively such that $\{y_{n_k}\}$ converge weakly to a point $z \in H$ and $\lim\limits_{k\to\infty} \|w_{n_k} - y_{n_k}\| = 0$, then $z \in VI(C, A)$.*

Proof. *Since $y_{n_k} = P_C(w_{n_k} - \lambda_{n_k} M w_{n_k})$, from the characterization of the projection operator, we have*

$$\langle w_{n_k} - \lambda_{n_k} M w_{n_k} - y_{n_k}, x - y_{n_k}\rangle \le 0, \quad \forall x \in C.$$

It implies that

$$\langle y_{n_k} - w_{n_k}, x - y_{n_k}\rangle \le \lambda_{n_k}\langle M w_{n_k}, x - y_{n_k}\rangle = \lambda_{n_k}\langle M w_{n_k}, w_{n_k} - y_{n_k}\rangle + \lambda_{n_k}\langle M w_{n_k}, x - w_{n_k}\rangle.$$

Since $\lambda_{n_k} > 0$, we have

$$\frac{1}{\lambda_{n_k}}\langle y_{n_k} - w_{n_k}, x - y_{n_k}\rangle + \langle M w_{n_k}, y_{n_k} - w_{n_k}\rangle \le \langle M w_{n_k}, x - w_{n_k}\rangle. \tag{8}$$

From the hypothesis, the sequence $\{y_{n_k}\}$ converges weakly to a point $z \in H$ and $\lim_{k\to\infty} \|w_{n_k} - y_{n_k}\| = 0$. It follows that, $\{w_{n_k}\}$ and $\{M w_{n_k}\}$ are bounded sequences. From Remark 3.2, we know that $\lim_{k\to\infty} \lambda_{n_k} = \lambda > 0$. Thus, from (8), we get

$$0 \le \liminf_{k\to\infty}\langle M w_{n_k}, x - w_{n_k}\rangle \le \limsup_{k\to\infty}\langle M w_{n_k}, x - w_{n_k}\rangle < \infty, \quad \forall x \in C. \tag{9}$$

We equally observe that

$$\begin{aligned}\langle M y_{n_k}, x - y_{n_k}\rangle &= \langle M y_{n_k}, x - w_{n_k} + w_{n_k} - y_{n_k}\rangle \\ &= \langle M y_{n_k} - M w_{n_k}, x - w_{n_k}\rangle + \langle M w_{n_k}, x - w_{n_k}\rangle + \langle M y_{n_k}, w_{n_k} - y_{n_k}\rangle.\end{aligned} \tag{10}$$

Since the operator M is L_0-Lipschitz continuous, we have that

$$\lim_{k\to\infty} \|M w_{n_k} - M y_{n_k}\| \le \lim_{k\to\infty}(L_0\|w_{n_k} - y_{n_k}\|) = 0. \tag{11}$$

Thus, combining (9), (10) and (11), we get

$$0 \le \liminf_{k\to\infty}\langle M y_{n_k}, x - y_{n_k}\rangle \le \limsup_{k\to\infty}\langle M y_{n_k}, x - y_{n_k}\rangle < \infty, \quad \forall z \in C. \tag{12}$$

From (12), we can examine two cases as follows:
Case 1: *Suppose that $\limsup_{k\to\infty}\langle M y_{n_k}, x - y_{n_k}\rangle > 0, \quad \forall x \in C$. Then, there exists a subsequence $\{y_{n_{k_m}}\}$ of sequence $\{y_{n_k}\}$ such that $\lim_{m\to\infty}\langle M y_{n_{k_m}}, x - y_{n_{k_m}}\rangle > 0$. It follows that we can find $m_0 \ge 1$ such that*

$$\langle M y_{n_{k_m}}, x - y_{n_{k_m}}\rangle > 0, \quad \forall m \ge m_0.$$

Since the mapping M is quasi-monotone, it follows that

$$\langle M x, x - y_{n_{k_m}}\rangle \ge 0, \quad \forall x \in C, m \ge m_0. \tag{13}$$

Passing limit as $m \to \infty$ in (13) we have

$$\lim_{m\to\infty}\langle M x, x - y_{n_{k_m}}\rangle = \langle M x, x - z\rangle \ge 0, \quad \forall x \in C.$$

Hence, $z \in VI(C, A)$.
Case 2: *Suppose in (12) that*

$$\limsup_{k\to\infty}\langle M y_{n_k}, x - y_{n_k}\rangle = 0. \tag{14}$$

We consider $\{\delta_k\}$ to be a non-increasing positive sequence given as

$$\delta_k := |\langle M y_{n_k}, x - y_{n_k}\rangle| + \frac{1}{k+1}. \tag{15}$$

It is obvious that $\delta_k \to 0$ as $k \to \infty$.
Now by combining (14) and (15) we have

$$\langle M y_{n_k}, x - y_{n_k}\rangle + \delta_k > 0. \tag{16}$$

And for all $k \geq 1$, since $\{y_{n_k}\} \subset C$, it implies that $\{My_{n_k}\}$ is strictly non-zero and let $\lim_{k\to\infty} \|My_{n_k}\| = M_0 > 0$. It follows that we can find $k_0 \geq 1$ such that

$$\|My_{n_k}\| > \frac{M_0}{2}, \quad \forall k \geq k_0. \tag{17}$$

Also, $\{\varepsilon_{n_k}\}$ denotes a sequence defined by $\varepsilon_{n_k} = \frac{My_{n_k}}{\|My_{n_k}\|^2}$. It implies that

$$\langle My_{n_k}, \varepsilon_{n_k}\rangle = 1. \tag{18}$$

Combining (16) and (18), we obtain

$$\langle My_{n_k}, x - y_{n_k}\rangle + \delta_k\langle My_{n_k}, \varepsilon_{n_k}\rangle > 0. \tag{19}$$

Consequently

$$\langle My_{n_k}, x + \delta_k\varepsilon_{n_k} - y_{n_k}\rangle > 0.$$

By the quasi-monotonicity of the operator M on H we get

$$\langle M(x + \delta_k\varepsilon_{n_k}), x + \delta_k\varepsilon_{n_k} - y_{n_k}\rangle \geq 0. \tag{20}$$

But observe that

$$\begin{aligned}\langle Mx, x + \delta_k\varepsilon_{n_k} - y_{n_k}\rangle &= \langle Mx - M(x + \delta_k\varepsilon_{n_k}) + M(x + \delta_k\varepsilon_{n_k}), x + \delta_k\varepsilon_{n_k} - y_{n_k}\rangle \\ &= \langle Mx - M(x + \delta_k\varepsilon_{n_k}), x + \delta_k\varepsilon_{n_k} - y_{n_k}\rangle \\ &\quad + \langle M(x + \delta_k\varepsilon_{n_k}), x + \delta_k\varepsilon_{n_k} - y_{n_k}\rangle.\end{aligned} \tag{21}$$

Combining (20) and (21), applying Cauchy-Schwartz's inequality we obtain

$$\begin{aligned}\langle Mx, x + \delta_k\varepsilon_{n_k} - y_{n_k}\rangle &\geq \langle Mx - M(x + \delta_k\varepsilon_{n_k}), x + \delta_k\varepsilon_{n_k} - y_{n_k}\rangle \\ &\geq -\|Mx - M(x + \delta_k\varepsilon_{n_k})\|\,\|x + \delta_k\varepsilon_{n_k} - y_{n_k}\|.\end{aligned} \tag{22}$$

Since the operator M is L_0-Lipschitz continuous, we have

$$\langle Mx, x + \delta_k\varepsilon_{n_k} - y_{n_k}\rangle + L_0\|\delta_k\varepsilon_{n_k}\|\,\|x + \delta_k\varepsilon_{n_k} - y_{n_k}\| \geq 0. \tag{23}$$

Combining (17) and (23), noting the definition of $\{\varepsilon_{n_k}\}$, we get

$$\langle Mx, x + \delta_k\varepsilon_{n_k} - y_{n_k}\rangle + \frac{2L_0}{M_0}\delta_k\|x + \delta_k\varepsilon_{n_k} - y_{n_k}\| \geq 0, \quad \forall k \geq k_0. \tag{24}$$

Passing limit in (24) and since $\delta_k \to 0$ as $k \to \infty$,we have

$$\lim_{k\to\infty}(\langle Mx, x + \delta_k\varepsilon_{n_k} - y_{n_k}\rangle + \frac{2L_0}{M_0}\delta_k\|x + \delta_k\varepsilon_{n_k} - y_{n_k}\|) = \langle Mx, x - z\rangle \geq 0, \quad \forall x \in C. \tag{25}$$

Thus, $z \in VI(C, A)$. □

Lemma 3.6. *Suppose that Assumptions $(A1) - (A3)$ hold. Let the sequences $\{w_n\}$ and $\{y_n\}$ be created by Algorithm 3.1. If $\lim_{n\to\infty} \tau_n\|w_n - y_n\|^2 = 0$, then $\lim_{n\to\infty} \|w_n - y_n\| = 0$.*

Proof. *We show that $\lim_{n\to\infty} \|w_n - y_n\| = 0$ by considering two cases of τ_n. First, we assume that $\lim\inf_{n\to\infty} \tau_n > 0$. Then, there exists a positive number τ such that $\tau_n \geq \tau > 0, \quad \forall n \in N$. Moreover, one sees that*

$$\|w_n - y_n\|^2 = \frac{1}{\tau_n}\tau_n\|w_n - y_n\|^2 \leq \frac{1}{\tau}\tau_n\|w_n - y_n\|^2.$$

Therefore, we obtain $\lim_{n\to\infty} \|w_n - y_n\| = 0$ by the hypothesis. On the other hand, one supposes that $\lim\inf_{n\to\infty} \tau_n = 0$. In this situation, we suppose that $\{n_k\}$ is a subsequence of $\{n\}$ such that

$$\lim_{k\to\infty} \tau_{n_k} = 0 \quad \text{and} \quad \lim_{k\to\infty} \|w_{n_k} - y_{n_k}\| = a > 0. \tag{26}$$

Let $g_{n_k} = w_{n_k} - \ell^{-1}\tau_{n_k}(w_{n_k} - y_{n_k})$. *It follows that*

$$\lim_{k\to\infty} \|g_{n_k} - w_{n_k}\|^2 = \lim_{k\to\infty} \frac{1}{\ell^2}\tau_{n_k} \cdot \tau_{n_k}\|w_{n_k} - y_{n_k}\|^2 = 0,$$

which together with the fact that M *is uniformly continuous, gives* $\lim_{k\to\infty} \|Mw_{n_k} - Mg_{n_k}\| = 0$. *From the definition of* g_{n_k} *and (5), we obtain*

$$\langle Mw_{n_k} - Mg_{n_k}, w_{n_k} - y_{n_k}\rangle > \mu\|w_{n_k} - y_{n_k}\|^2, \tag{27}$$

which further yields that $\lim_{k\to\infty} \|w_{n_k} - y_{n_k}\| = 0$. *This contradicts the hypothesis. Thus we conclude that* $\lim_{n\to\infty} \|w_n - y_n\| = 0$. *This proof is completed.* □

Theorem 3.7. *Assume that Assumption* $(A1) - (A5)$ *hold. Then the sequence* $\{x_n\}$ *generated by Algorithm 3.1 converges to the unique solution of the BVIP in norm.*

Proof. *We divide the proof into four claims.*
Claim 1 *The sequence* $\{x_n\}$ *is bounded. Taking* $x = w_n$, $y = p$, $C = H_n$ *and* $p \in \Gamma$, *from Algorithm 3.1 and the property of the projection* $\|P_C(x) - y\|^2 \le \|x - y\|^2 - \|x - P_C(x)\|^2, \quad \forall y \in C$, *we deduce*

$$\begin{aligned} \|z_n - p\|^2 &\le \|w_n - p\|^2 - \|w_n - P_{H_n}(w_n)\|^2 \\ &= \|w_n - p\|^2 - dist^2(w_n, H_n), \end{aligned} \tag{28}$$

which means that

$$\|z_n - p\| \le \|w_n - p\|, \ \ \forall n \ge 1. \tag{29}$$

By the definition of w_n, *one has*

$$\begin{aligned} \|w_n - p\| &= \|(1-\phi_n)[x_n + \theta_n(x_n - x_{n-1})] - (1-\phi_n)p - \phi_n p\| \\ &\le (1-\phi_n)\|x_n - p\| + (1-\phi_n)\theta_n\|x_n - x_{n-1}\| + \phi_n\|p\| \\ &\le \|x_n - p\| + \alpha_n \cdot \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \alpha_n \cdot \frac{\phi_n}{\alpha_n}\|p\|. \end{aligned} \tag{30}$$

According to Remark 3.1, we have $\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \to 0$ *as* $n \to \infty$. *Therefore, there exists a constant* $M_1 > 0$ *such that*

$$\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \le M_1, \ \ \forall n \ge 1. \tag{31}$$

Since $\lim_{n\to\infty} \frac{\phi_n}{\alpha_n} = 0$, *we have*

$$\frac{\phi_n}{\alpha_n} \le M_2, \ \ \forall n \ge 1,$$

for some $M_2 > 0$. *Combining (29), (30) and (31), we obtain*

$$\|z_n - p\| \le \|w_n - p\| \le \|x_n - p\| + \alpha_n M_3. \tag{32}$$

Here $M_3 := M_1 + M_2\|p\|$. *Using Lemma 2.7 and (32), it follows that*

$$\begin{aligned} \|x_{n+1} - p\| &= \|(I - \alpha_n\gamma F)z_n - (I - \alpha_n\gamma F)p - \alpha_n\gamma Fp\| \\ &\le (1 - \alpha_n\eta)\|z_n - p\| + \alpha_n\gamma\|Fp\| \\ &\le (1 - \alpha_n\eta)\|x_n - p\| + \alpha_n\eta \cdot \frac{M_3}{\eta} + \alpha_n\eta \cdot \frac{\gamma}{\eta}\|Fp\| \\ &\le \max\Big\{\frac{M_3 + \gamma\|Fp\|}{\eta}, \|x_n - p\|\Big\} \le \ldots \le \max\Big\{\frac{M_3 + \gamma\|Fp\|}{\eta}, \|x_1 - p\|\Big\}, \end{aligned}$$

where $\eta = 1 - \sqrt{1 - \gamma(2\beta - \gamma L_F^2)} \in (0, 1)$. *This implies that the sequence* $\{x_n\}$ *is bounded. We obtain that the sequences* $\{w_n\}$, $\{y_n\}$, $\{t_n\}$, *and* $\{z_n\}$ *are also bounded.*

Claim 2

$$\|z_n - w_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M_6,$$

and

$$[D^{-1}\tau_n(\lambda_n^{-1} - \mu)\|w_n - y_n\|^2]^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M_6,$$

for some $M_4 > 0$. *It follows from (32) that*

$$\begin{aligned} \|w_n - p\|^2 &\le \|x_n - p\|^2 + 2\alpha_n M_3\|x_n - p\| + \alpha_n^2 M_3^2 \\ &\le \|x_n - p\|^2 + \alpha_n M_4 \end{aligned} \tag{33}$$

for some $M_4 > 0$. *Using the inequality* $\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle$, $\forall x, y \in H$, *one has*

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|(I - \alpha_n\gamma F)z_n - (I - \alpha_n\gamma F)p - \alpha_n\gamma Fp\|^2 \\ &\le (1 - \alpha_n\eta)^2\|z_n - p\|^2 + 2\alpha_n\gamma\langle Fp, p - x_{n+1}\rangle \\ &\le \|z_n - p\|^2 + \alpha_n M_5, \end{aligned} \tag{34}$$

for some $M_5 > 0$. *Combining (28), (33) and (34) we obtain*

$$\begin{aligned} \|x_{n+1} - p\|^2 &\le \|w_n - p\|^2 - \|z_n - w_n\|^2 + \alpha_n M_5 \\ &\le \|x_n - p\|^2 - \|z_n - w_n\|^2 + \alpha_n M_6, \end{aligned} \tag{35}$$

where $M_6 := M_4 + M_5$. *The first inequality can be obtained by a simple conversion.*
From $\{Mt_n\}$ *is bounded, there is* $M_7 > 0$ *such that* $\|Mt_n\| \le M_7$, $\forall n \ge 1$. *For any* $u, v \in H$, *we derive*

$$\|h_n(u) - h_n(v)\| = \|\langle Mt_n, u - v\rangle\| \le \|Mt_n\|\,\|u - v\| \le M_7\|u - v\|,$$

which means that $h_n(x)$ *is* M_7*-Lipschitz continuous on* H. *From Lemma 2.6 and Lemma 3.4, we find that*

$$dist(w_n, H_n) \ge M_7^{-1}h_n(w_n) \ge M_7^{-1}\tau_n(\lambda_n^{-1} - \mu)\|w_n - y_n\|^2.$$

This together with (28) gives

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - [M_7^{-1}\tau_n(\lambda_n^{-1} - \mu)\|w_n - y_n\|^2]^2.$$

From (34), (33) and the last inequality, we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\le \|z_n - p\|^2 + \alpha_n M_5 \\ &\le \|x_n - p\|^2 - [M_7^{-1}\tau_n(\lambda_n^{-1} - \mu)\|w_n - y_n\|^2]^2 + \alpha_n M_6. \end{aligned}$$

The second inequality can be obtained by a simple conversion.

Claim 3

$$\|x_{n+1} - p\|^2 \le (1 - \alpha_n\eta)\|x_n - p\|^2 + \alpha_n\eta\Big[\frac{2\gamma}{\eta}\langle Fp, p - x_{n+1}\rangle + \frac{3M_8\theta_n}{\alpha_n\eta}\|x_n - x_{n-1}\|\Big],$$

for some $M_8 > 0$. *Indeed, we have*

$$\begin{aligned} \|w_n - p\|^2 \le\ & \|(1 - \phi_n)x_n - p\|^2 + 2(1 - \phi_n)\theta_n\|(1 - \phi_n)x_n - p\|\,\|x_n - x_{n-1}\| \\ & + (1 - \phi_n)^2\theta_n^2\|x_n - x_{n-1}\|^2. \end{aligned} \tag{36}$$

Combining (29) and (34), we deduce

$$\|x_{n+1} - p\|^2 \le (1 - \alpha_n\eta)\|w_n - p\|^2 + 2\alpha_n\gamma\langle Fp, p - x_{n+1}\rangle. \tag{37}$$

Substituting (36) into (37), we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1-\alpha_n\eta)\|(1-\phi_n)x_n-p\|^2+2\alpha_n\gamma\langle Fp,p-x_{n+1}\rangle \\ &\quad +\theta_n\|x_n-x_{n-1}\|[2(1-\phi_n)\|(1-\phi_n)x_n-p\|+(1-\phi_n)^2\theta_n\|x_n-x_{n-1}\|] \\ &\le (1-\alpha_n\eta)\|(1-\phi_n)x_n-p\|^2+\alpha_n\eta\Big[\frac{2\gamma}{\eta}\langle Fp,p-x_{n+1}\rangle+\frac{3M_8\theta_n(1-\phi_n)^2}{\alpha_n\eta}\|x_n-x_{n-1}\|\Big],\end{aligned}$$

where $M_8:=\sup\{\|x_n-p\|,\theta_n\|x_n-x_{n-1}\|\}>0$.

Claim 4

The sequence $\{\|x_n-p\|\}$ *converges to zero. By Lemma 2.8, it needs to show that* $\limsup_{k\to\infty}\langle Fp,p-x_{n_k+1}\rangle\le 0$ *for every subsequence* $\{\|x_{n_k}-p\|\}$ *of* $\{\|x_n-p\|\}$ *satisfying*

$$\liminf_{k\to\infty}(\|x_{n_k+1}-p\|-\|x_{n_k}-p\|)\ge 0. \tag{38}$$

For this purpose, one assumes that $\{\|x_{n_k}-p\|\}$ *is a subsequence of* $\{\|x_n-p\|\}$ *such that (38) holds. Then*

$$\liminf_{k\to\infty}(\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2)=\liminf_{k\to\infty}[(\|x_{n_k+1}-p\|-\|x_{n_k}-p\|)(\|x_{n_k+1}-p\|+\|x_{n_k}-p\|)]\ge 0.$$

By Claim 2 and the assumption on $\{\alpha_n\}$*, one obtains*

$$\begin{aligned}\limsup_{k\to\infty}(\|w_{n_k}-z_{n_k}\|^2) &\le \limsup_{k\to\infty}[\alpha_{n_k}M_6+\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2] \\ &\le \limsup_{k\to\infty}\alpha_{n_k}M_6+\limsup_{k\to\infty}[\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2] \\ &= -\liminf_{k\to\infty}[\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2]\le 0,\end{aligned}$$

and

$$\limsup_{k\to\infty}[M_7^{-1}\tau_{n_k}(\lambda_{n_k}^{-1}-\mu)\|w_{n_k}-y_{n_k}\|^2]^2\le 0.$$

These imply that

$$\lim_{k\to\infty}\|w_{n_k}-z_{n_k}\|=0,$$

and

$$\lim_{k\to\infty}\tau_{n_k}\|w_{n_k}-y_{n_k}\|^2=0.$$

It follows from Lemma 3.6 that $\lim_{k\to\infty}\|w_{n_k}-y_{n_k}\|=0$. *Moreover, we can show that*

$$\|x_{n_k+1}-z_{n_k}\|=\alpha_{n_k}\gamma\|Fz_{n_k}\|\to 0 \text{ as } k\to\infty, \tag{39}$$

and

$$\begin{aligned}\|x_{n_k}-w_{n_k}\| &= \|\phi_{n_k}x_{n_k}+\phi_{n_k}\theta_{n_k}x_{n_k}-\phi_{n_k}\theta_{n_k}x_{n_k-1}-\theta_{n_k}(x_{n_k}-x_{n_k-1})\| \\ &\le \alpha_{n_k}\cdot\frac{\theta_{n_k}}{\alpha_{n_k}}\|\phi_{n_k}x_{n_k}-\phi_{n_k}x_{n_k-1}\|+\phi_{n_k}\|x_{n_k}\|+\alpha_{n_k}\cdot\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|\to 0.\end{aligned} \tag{40}$$

Combining (39) and (40), we arrive at

$$\|x_{n_k+1}-x_{n_k}\|\le\|x_{n_k+1}-z_{n_k}\|+\|z_{n_k}-w_{n_k}\|+\|w_{n_k}-x_{n_k}\|\to 0 \text{ as } k\to\infty. \tag{41}$$

Since the sequence $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$, which converges weakly to some $z \in H$. By (40), we obtain $w_{n_{k_j}} \rightharpoonup z$ as $j \to \infty$. This together with $\lim_{k\to\infty} \|w_{n_k} - y_{n_k}\| = 0$ and Lemma 3.5 yields that $z \in \Omega$. From the assumption that p is the unique solution of the BVIP, we deduce

$$\limsup_{k\to\infty} \langle Fp, p - x_{n_k} \rangle = \lim_{j\to\infty} \langle Fp, p - x_{n_{k_j}} \rangle = \langle Fp, p - z \rangle \leq 0. \tag{42}$$

Using (41) and (42), we obtain

$$\limsup_{k\to\infty} \langle Fp, p - x_{n_k+1} \rangle = \limsup_{k\to\infty} \langle Fp, p - x_{n_k} \rangle \leq 0. \tag{43}$$

From $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0$ and (43), we observe

$$\limsup_{k\to\infty} \Big[\frac{2\gamma}{\eta} \langle Fp, p - x_{n_k+1} \rangle + \frac{3M_8 \theta_{n_k}}{\alpha_{n_k} \eta} \|x_{n_k} - x_{n_k-1}\| \Big] \leq 0. \tag{44}$$

Combining Claim 3, Assumption (A5) and (44), in the light of Lemma 2.8, one concludes that $\lim_{n\to\infty} \|x_n - p\| = 0$. That is, $x_n \to p$ as $n \to \infty$. This completes the proof.
□

Algorithm 3.2 Double inertial extragradient method for solving the BVIP.

Initialization. Let $\theta > 0$, $\delta > 0$, $\ell \in (0, 1)$, $\mu > 0$, $\gamma \in (0, \frac{2\beta}{L_F^2})$, $\lambda_n \in (0, \frac{1}{\mu})$ and let $x_0, x_1 \in H$. Choose a nonnegative real sequence $\{p_n\}$ such that $\sum_{n=0}^{\infty} p_n < +\infty$.

Iterative Step Given the iterates x_{n-1} and x_n for each $n \geq 1$, calculate x_{n+1} as follows:

Step 1. Compute

$$\begin{cases} w_n = x_n + \theta_n (x_n - x_{n-1}), \\ v_n = x_n + \delta_n (x_n - x_{n-1}), \end{cases}$$

where

$$\theta_n = \begin{cases} \min\Big\{\theta, \dfrac{\epsilon_n}{\|x_n - x_{n-1}\|}\Big\}, & if x_n \neq x_{n-1}, \\ \theta, & otherwise, \end{cases}$$

$$\delta_n = \begin{cases} \min\Big\{\delta, \dfrac{\epsilon_n}{\|x_n - x_{n-1}\|}\Big\}, & if x_n \neq x_{n-1}, \\ \delta, & otherwise. \end{cases}$$

Step 2. Compute $y_n = P_C(w_n - \lambda_n M w_n)$,

$$\lambda_{n+1} = \begin{cases} \min\Big\{\dfrac{\mu\|w_n - y_n\|}{\|Mw_n - My_n\|}, \lambda_n + p_n\Big\}, & if Mw_n \neq My_n, \\ \lambda_n + p_n, & otherwise. \end{cases}$$

Step 3. Compute $t_n = w_n - \tau_n(w_n - y_n)$, where $\tau_n = \ell^{m_n}$ and m_n is the smallest non-negative integer m satisfying

$$\langle Mw_n - M(w_n - \ell^m r_\lambda(w_n)), w_n - y_n \rangle \geq \mu \|w_n - y_n\|^2.$$

Step 4. Compute $z_n = P_{H_n}(w_n)$, where the half-space H_n is defined by

$$H_n = \{x \in C : h_n(x) \leq 0\} \quad \text{and} \quad h_n(x) = \langle Mt_n, x - t_n \rangle.$$

Step 5. Compute $x_{n+1} = \beta_n v_n + (1 - \beta_n) z_n - \alpha_n \gamma F z_n$.
Set $n := n + 1$ and return to **Step 1**.

Remark 3.8. *We note here that inertial calculation criterion is easy to implement since the term* $\|x_n - x_{n-1}\|$ *is known before calculating* θ_n*. It follows from (4) and the assumptions on* $\{\alpha_n\}$ *that* $\lim_{n\to\infty} \frac{\theta_n \|x_n - x_{n-1}\|}{\alpha_n} = 0$ *and* $\lim_{n\to\infty} \frac{\delta_n \|x_n - x_{n-1}\|}{\alpha_n} = 0$.

Theorem 3.9. *Assume that Assumption* $(A1) - (A5)$ *hold. Then the sequence* $\{x_n\}$ *generated by Algorithm 3.2 converges to the unique solution of the BVIP in norm.*

Proof. *We divide the proof into four claims.*
Claim 1 *The sequence* $\{x_n\}$ *is bounded. Let* $p \in \Gamma$*. From Algorithm 3.2 and the property of the projection* $\|P_C(x) - y\|^2 \leq \|x - y\|^2 - \|x - P_C(x)\|^2$, $\forall y \in C$, *and take* $x = w_n$, $y = p$ *and* $C = H_n$, *we deduce*

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \|w_n - P_{H_n}(w_n)\|^2 \leq \|w_n - p\|^2 - dist^2(w_n, H_n), \tag{45}$$

which means that

$$\|z_n - p\| \leq \|w_n - p\|, \quad \forall n \geq 1. \tag{46}$$

By the definition of w_n, *one has*

$$\|w_n - p\| \leq \alpha_n \cdot \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| + \|x_n - p\|. \tag{47}$$

According to Remark 3.8, we have $\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \to 0$ *as* $n \to \infty$*. Therefore, there exists a constant* $Q_1 > 0$ *such that*

$$\frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \leq Q_1, \quad \forall n \geq 1. \tag{48}$$

Combining (46), (47) and (48), we obtain

$$\|z_n - p\| \leq \|w_n - p\| \leq \|x_n - p\| + \alpha_n Q_1, \quad \forall n \geq 1. \tag{49}$$

By the definition of v_n, *one has*

$$\|v_n - p\| \leq \alpha_n \cdot \frac{\delta_n}{\alpha_n} \|x_n - x_{n-1}\| + \|x_n - p\|. \tag{50}$$

According to Remark 3.8, we have $\frac{\delta_n}{\alpha_n}\|x_n - x_{n-1}\| \to 0$ *as* $n \to \infty$*. Therefore, there exists a constant* $Q_2 > 0$ *such that*

$$\frac{\delta_n}{\alpha_n} \|x_n - x_{n-1}\| \leq Q_2, \quad \forall n \geq 1. \tag{51}$$

Combining (50), (51), we obtain

$$\|v_n - p\| \leq \|x_n - p\| + \alpha_n Q_2, \quad \forall n \geq 1. \tag{52}$$

Using Lemma 2.7, (49) and (52), it follows that

$$\begin{aligned}\|x_{n+1}-p\| &= \|\beta_n v_n + (1-\beta_n)z_n - \alpha_n\gamma F z_n - p\| \\ &= \|\beta_n v_n - \beta_n p + (1-\beta_n)z_n - \alpha_n\gamma F z_n + \alpha_n\gamma F p - \alpha_n\gamma F p + \beta_n p - p\| \\ &\le \|[(1-\beta_n)z_n - \alpha_n\gamma F z_n] - [(1-\beta_n)p - \alpha_n\gamma F p]\| + \beta_n\|v_n - p\| + \alpha_n\gamma\|Fp\| \\ &\le (1-\beta_n-\alpha_n\tau)\|z_n-p\| + \beta_n\|v_n-p\| + \alpha_n\gamma\|Fp\| \\ &\le (1-\beta_n-\alpha_n\tau)\|x_n-p\| + \beta_n\|x_n-p\| + \alpha_n Q_1 + \alpha_n\beta_n Q_2 + \alpha_n\gamma\|Fp\| \\ &= (1-\alpha_n\tau)\|x_n-p\| + \alpha_n\tau\frac{\beta_n Q_2 + \gamma\|Fp\| + Q_1}{\tau} \\ &\le \max\Big\{\frac{Q_1+\gamma\|Fp\|+\beta_n Q_2}{\tau}, \|x_n-p\|\Big\} \\ &\le \dots \le \max\Big\{\frac{Q_1+\gamma\|Fp\|+\beta_n Q_2}{\tau}, \|x_1-p\|\Big\},\end{aligned}$$

where $\tau = 1 - \sqrt{1-\gamma(2\beta-\gamma L_F^2)} \in (0,1)$. *This implies that the sequence* $\{x_n\}$ *is bounded. We obtain that the sequences* $\{w_n\}$, $\{y_n\}$, $\{t_n\}$ *and* $\{z_n\}$ *are also bounded.*

Claim 2

$$\|z_n-w_n\|^2 \le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + \alpha_n Q_5 + 2\beta_n\langle v_n-p, x_{n+1}-p\rangle,$$

and

$$[Q_6^{-1}\tau_n(\lambda_n^{-1}-\mu)\|w_n-y_n\|^2]^2 \le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + \alpha_n Q_5 + 2\beta_n\langle v_n-p, x_{n+1}-p\rangle,$$

for some $Q_5 > 0$. *It follows from (49) that*

$$\begin{aligned}\|w_n-p\|^2 &\le \|x_n-p\|^2 + \alpha_n(2Q_1\|x_n-p\| + \alpha_n Q_1^2) \\ &\le \|x_n-p\|^2 + \alpha_n Q_3,\end{aligned} \tag{53}$$

for some $Q_3 > 0$. *Using the inequality* $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \ \forall x,y\in H$, *one has*

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\beta_n v_n - \beta_n p + (1-\beta_n)z_n - \alpha_n\gamma F z_n + \alpha_n\gamma F p - \alpha_n\gamma F p + \beta_n p - p\|^2 \\ &= \|[(1-\beta_n)z_n - \alpha_n\gamma F z_n] - [(1-\beta_n)p - \alpha_n\gamma F p] + \beta_n v_n - \beta_n p - \alpha_n\gamma F p\|^2 \\ &\le (1-\beta_n-\alpha_n\tau)^2\|z_n-p\|^2 + 2\langle\beta_n v_n - \beta_n p - \alpha_n\gamma F p, x_{n+1}-p\rangle \\ &\le (1-\alpha_n\tau)^2\|z_n-p\|^2 + 2\alpha_n\gamma\langle Fp, p-x_{n+1}\rangle + 2\beta_n\langle v_n-p, x_{n+1}-p\rangle \\ &\le \|z_n-p\|^2 + \alpha_n Q_4 + 2\beta_n\langle v_n-p, x_{n+1}-p\rangle,\end{aligned} \tag{54}$$

for some $Q_4 > 0$. *Combining the property of the projection* $\|P_C(x)-y\|^2 \le \|x-y\|^2 - \|x-P_C(x)\|^2$ *and (54), we obtain*

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \|z_n-p\|^2 + \alpha_n Q_4 + 2\beta_n\langle v_n-p, x_{n+1}-p\rangle \\ &\le \|w_n-p\|^2 - \|z_n-w_n\|^2 + \alpha_n Q_4 + 2\beta_n\langle v_n-p, x_{n+1}-p\rangle \\ &\le \|x_n-p\|^2 - \|z_n-w_n\|^2 + \alpha_n Q_5 + 2\beta_n\langle v_n-p, x_{n+1}-p\rangle,\end{aligned}$$

where $Q_5 := Q_3 + Q_4$. *The first inequality can be obtained by a simple conversion.*
From Mt_n *is bounded, there is* $Q_6 > 0$ *such that* $\|Mt_n\| \le Q_6, \forall n \ge 1$. *For any* $u, v \in H$, *we derive*

$$\|h_n(u) - h_n(v)\| = \|\langle Mt_n, u-v\rangle\| \le \|Mt_n\|\,\|u-v\| \le Q_6\|u-v\|,$$

which means that $h_n(x)$ *is* Q_6*-Lipschitz continuous on* H. *From Lemma 2.6 and Lemma 3.4, we find that*

$$dist(w_n, H_n) \ge Q_6^{-1}h_n(w_n) \ge Q_6^{-1}\tau_n(\lambda_n^{-1}-\mu)\|w_n-y_n\|^2.$$

This together with (45) gives

$$\|z_n-p\|^2 \le \|w_n-p\|^2 - [Q_6^{-1}\tau_n(\lambda_n^{-1}-\mu)\|w_n-y_n\|^2]^2. \tag{55}$$

From (54), (53) and (55), we have

$$\|x_{n+1}-p\|^2 \le \|z_n-p\|^2+\alpha_n Q_4+2\beta_n\langle v_n-p, x_{n+1}-p\rangle$$
$$\le \|x_n-p\|^2-[Q_6^{-1}\tau_n(\lambda_n^{-1}-\mu)\|w_n-y_n\|^2]^2+\alpha_n Q_5+2\beta_n\langle v_n-p, x_{n+1}-p\rangle.$$

The second inequality can be obtained by a simple conversion.

Claim 3

$$\|x_{n+1}-p\|^2 \le (1-\alpha_n\tau)\|x_n-p\|^2 +\alpha_n\tau\Big[\frac{2\gamma}{\tau}\langle Fp, p-x_{n+1}\rangle+\frac{3Q_7\theta_n}{\alpha_n\tau}\|x_n-x_{n-1}\|+\frac{2\beta_n\langle v_n-p, x_{n+1}-p\rangle}{\alpha_n\tau}\Big],$$

for some $Q_7>0$. *Indeed, we have*

$$\|w_n-p\|^2 \le \|x_n-p\|^2+2\theta_n\|x_n-p\|\,\|x_n-x_{n-1}\|+\theta_n^2\|x_n-x_{n-1}\|^2. \tag{56}$$

Combining (46) and (54), we deduce

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1-\alpha_n\tau)^2\|z_n-p\|^2+2\alpha_n\gamma\langle Fp, p-x_{n+1}\rangle+2\beta_n\langle v_n-p, x_{n+1}-p\rangle\\ &\le (1-\alpha_n\tau)\|w_n-p\|^2+2\alpha_n\gamma\langle Fp, p-x_{n+1}\rangle+2\beta_n\langle v_n-p, x_{n+1}-p\rangle.\end{aligned} \tag{57}$$

Substituting (56) into (57), we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1-\alpha_n\tau)\|x_n-p\|^2+2\alpha_n\gamma\langle Fp, p-x_{n+1}\rangle\\ &\quad+\theta_n\|x_n-x_{n-1}\|(2\|x_n-p\|+\theta_n\|x_n-x_{n-1}\|)+2\beta_n\langle v_n-p, x_{n+1}-p\rangle.\\ &\le (1-\alpha_n\tau)\|x_n-p\|^2+\alpha_n\tau\Big[\frac{2\gamma\langle Fp, p-x_{n+1}\rangle}{\tau}+\frac{3Q_7\theta_n\|x_n-x_{n-1}\|}{\alpha_n\tau}\\ &\quad+\frac{2\beta_n\langle v_n-p, x_{n+1}-p\rangle}{\alpha_n\tau}\Big],\end{aligned}$$

where $Q_7:=\sup\{\|x_n-p\|, \theta_n\|x_n-x_{n-1}\|\}>0$.

Claim 4 *The sequence* $\{\|x_n-p\|\}$ *converges to zero. By Lemma 2.8, it needs to show that* $\limsup_{k\to\infty}\langle Fp, p-x_{n_{k+1}}\rangle\le 0$ *for every subsequence* $\{\|x_{n_k}-p\|\}$ *of* $\{\|x_n-p\|\}$ *satisfying*

$$\liminf_{k\to\infty}(\|x_{n_{k+1}}-p\|-\|x_{n_k}-p\|)\ge 0. \tag{58}$$

For this purpose, one assumes that $\{\|x_{n_k}-p\|\}$ *is a subsequence of* $\{\|x_n-p\|\}$ *such that (58) holds. Then*

$$\liminf_{k\to\infty}(\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2)=\liminf_{k\to\infty}[(\|x_{n_k+1}-p\|-\|x_{n_k}-p\|)(\|x_{n_k+1}-p\|+\|x_{n_k}-p\|)]\ge 0.$$

By Claim 2 and the assumption on $\{\alpha_n\}$, *one obtains*

$$\begin{aligned}\limsup_{k\to\infty}(\|w_{n_k}-z_{n_k}\|^2) &\le \limsup_{k\to\infty}[\alpha_{n_k}Q_5+\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2+2\beta_{n_k}\langle v_{n_k}-p, x_{n_k+1}-p\rangle]\\ &\le \limsup_{k\to\infty}\alpha_{n_k}Q_5+\limsup_{k\to\infty}[\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2]\\ &\quad+\limsup_{k\to\infty}2\beta_{n_k}\langle v_{n_k}-p, x_{n_k+1}-p\rangle]\\ &=-\liminf_{k\to\infty}[\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2]\le 0,\end{aligned}$$

and

$$\limsup_{k\to\infty}[Q_6^{-1}\tau_{n_k}(\lambda_{n_k}^{-1}-\mu)\|w_{n_k}-y_{n_k}\|^2]^2\le 0.$$

These imply that

$$\lim_{k\to\infty} \|w_{n_k} - z_{n_k}\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \tau_{n_k}\|w_{n_k} - y_{n_k}\|^2 = 0. \tag{59}$$

It follows from Lemma 3.6 that $\lim_{k\to\infty} \|w_{n_k} - y_{n_k}\| = 0$. *Moreover, we can show that*

$$\|x_{n_k+1} - z_{n_k}\| = \|\beta_{n_k} v_{n_k} - \beta_{n_k} z_{n_k} - \alpha_{n_k}\gamma F z_{n_k}\| \to 0 \quad \text{as} \quad k \to \infty, \tag{60}$$

and

$$\|x_{n_k} - w_{n_k}\| = \alpha_{n_k} \cdot \frac{\theta_{n_k}}{\alpha_{n_k}} \|x_{n_k} - x_{n_k-1}\| \to 0 \quad \text{as} \quad k \to \infty. \tag{61}$$

Combining (59), (60) and (61) we arrive at

$$\|x_{n_k+1} - x_{n_k}\| \le \|x_{n_k+1} - z_{n_k}\| + \|z_{n_k} - w_{n_k}\| + \|w_{n_k} - x_{n_k}\| \to 0 \quad \text{as} \quad k \to \infty. \tag{62}$$

Since the sequence $\{x_{n_k}\}$ *is bounded, there exists a subsequence* $\{x_{n_{k_j}}\}$ *of* $\{x_{n_k}\}$*, which converges weakly to some* $z \in H$. *By (61), we obtain* $w_{n_{k_j}} \rightharpoonup z$ *as* $j \to \infty$. *This together with* $\lim_{k\to\infty} \|w_{n_k} - y_{n_k}\| = 0$ *and Lemma 3.5 yields that* $z \in \Omega$. *From the assumption that p is the unique solution of the BVIP, we deduce*

$$\limsup_{k\to\infty} \langle Fp, p - x_{n_k}\rangle = \lim_{j\to\infty} \langle Fp, p - x_{n_{k_j}}\rangle = \langle Fp, p - z\rangle \le 0. \tag{63}$$

Using (62) and (63), we obtain

$$\limsup_{k\to\infty} \langle Fp, p - x_{n_k+1}\rangle = \limsup_{k\to\infty} \langle Fp, p - x_{n_k}\rangle \le 0. \tag{64}$$

From Remark 3.8, (64), (A5), and $\langle v_n - p, x_{n+1} - p\rangle$ *have bounded, we observe*

$$\limsup_{k\to\infty} \Big[\frac{2\gamma}{\tau}\langle Fp, p - x_{n_k+1}\rangle + \frac{3Q_7\theta_{n_k}}{\alpha_{n_k}\tau}\|x_{n_k} - x_{n_k-1}\| + \frac{2\beta_n\langle v_n - p, x_{n+1} - p\rangle}{\alpha_n \tau}\Big] \le 0. \tag{65}$$

Combining Claim3, Assumption (A5) and (65), in the light of Lemma 2.8, one concludes that $\lim_{n\to\infty} \|x_n - p\| = 0$. *That is* $x_n \to p$ *as* $n \to \infty$. *This completes the proof.* □

4. Numerical experiments

Under this section, we shall present numerical illustrations to show the behavior of our proposed iterative method. For the purpose of these illustrations, all codes are written in Matlab R2023a and executed on a PC Desktop Intel(R) Core(TM) i5-6300U CPU @ 2.40GHz 2.50GHz, RAM 8.00GB.

Example 4.1. *[28] Consider the following fractional programming problem:*

$$\min f(x) = \frac{x^T Q x + a^T x + a_0}{b^T x + b_0},$$

subject to $x \in X := \{x \in R^4 : b^T x + b_0 > 0\}$, *where*

$$Q = \begin{pmatrix} 5 & -1 & 2 & 0 \\ -1 & 5 & -1 & 3 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 0 & 5 \end{pmatrix},$$

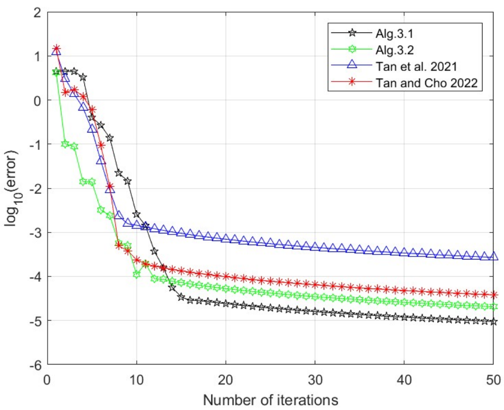

Figure 1: $x_1 = 2rand(4,1)$ for Example 4.1

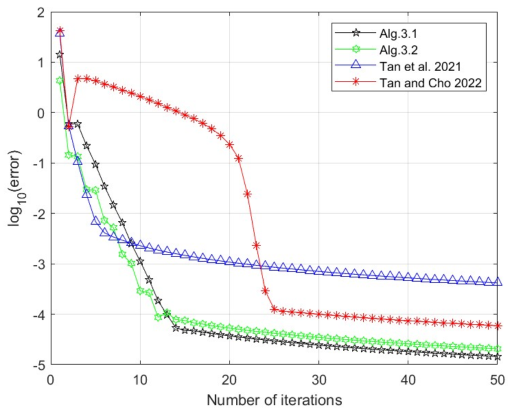

Figure 2: $x_1 = 4rand(4,1)$ for Example 4.1

$a = (1,-2,-2,1)^T$, $b = (2,1,1,0)^T$, $a_0 = -2$, $b_0 = 4$. It is easy to verify that Q is symmetric, positive definite in R^4, consequently f is pseudo-convex on $X = \{x \in R^4 : b^T x + b_0 > 0\}$. Let $A(x) := \nabla f(x)$. We minimize f over a nonempty, closed and convex subset $C := \{x \in R^4 : 1 \le x_i \le 10, i = 1,\dots,4\} \subset X$ by using $A(x) := \nabla f(x) = ((b^T x + b_0)(2Qx + a) - b(x^T Qx + a^T x + a_0))/(b^T x + b_0)^2$. This problem has a unique solution $x^* = (1,1,1,1)^T \in C$. It is known that a differentiable function f is pseudo-convex if and only if its gradient is pseudo-monotone, thus A is pseudo-monotone, then A is quasi-monotone. We now consider the mapping $F : R^m \to R^m (m = 4)$ define by $F(x) = Gx + q$, where $G = BB^T + D + K$, and B is a $m \times m$ with their entries being generated in $(-1,1)$, K is a $m \times m$ diagonal matrix whose diagonal entries are positive in $(0,1)$ (so G is positive semidefinite), $q \in R^m$ is a vector with entries being generated in $(0,1)$. It is clear that F is L_F-Lipschitz continuous and β-strongly monotone with $L_F = \max\{eigG\}$ and $\beta = \min\{eigG\}$, where $eigG$ represents all eigenvalues of G.

We use the proposed Algorithm 3.1 and 3.2 to solve the BVIP with M, F, and C given above, and compare them with the algorithm introduced by B. Tan, S. Y. Cho in [28] and the algorithm introduced by B. Tan, et al in [26]. The parameters of all algorithms are set as follows. Take $\alpha_n = 1/(n+1)$, $\gamma = 1.7\beta/L_F^2$, $\epsilon = 100/(s+1)^2$, and $\phi = 1/(10000s+1)$ for all algorithms. The remaining parameters are shown in the table below. We use $D_n = \|x_{n+1} - x_n\|$ to measure the error of the nth iteration because we do not know the exact solution to the BVIP. The maximum number of iterations 50 is used as a common stopping criterion for all algorithms. Numerical results of all algorithms with four different initial values $x_0 = x_1$ are reports in Fig. 1. Fig. 2. Fig. 3. Fig. 4.

Table 1: Methods parameters for Example 4.1

Alg 3.1	$\ell = 0.25$	$\mu = 0.25$	$\theta = 0.8$	$\lambda = 0.2$
	$\gamma = 0.5$			
Alg 3.2	$\ell = 0.25$	$\mu = 0.25$	$\theta = 0.3$	$\delta = 0.8$
	$\lambda = 0.1$	$\gamma = 0.5$		
Tan et al. 2021	$\ell = 0.25$	$\mu = 0.25$	$\theta = 0.3$	$\lambda = 0.2$
	$\gamma = 0.5$			
Tan and Cho 2022	$\ell = 0.25$	$\mu = 0.25$	$\theta = 0.8$	$\lambda = 0.2$
	$\gamma = 0.5$			

Example 4.2. *[28] Let $H = L^2([0,1])$ be an infinite-dimensional Hilbert space with inner product $\langle x, y\rangle = \int_0^1 x(t)y(t)dt$ and induced norm $\|x\| = (\int_0^1 |x(t)|^2 dt)^{1/2}$. Assume that the feasible set is given by $C = \{x \in H : \|x\| \le 2\}$. Define a mapping $h : C \to \mathbb{R}$ by $h(m) = 1/(1 + \|m\|^2)$. Recall that the Volterra integration operator $V : H \to H$ is given by $V(m)(t) = \int_0^t m(s)ds$, $\forall t \in [0,1]$, $m \in H$.*

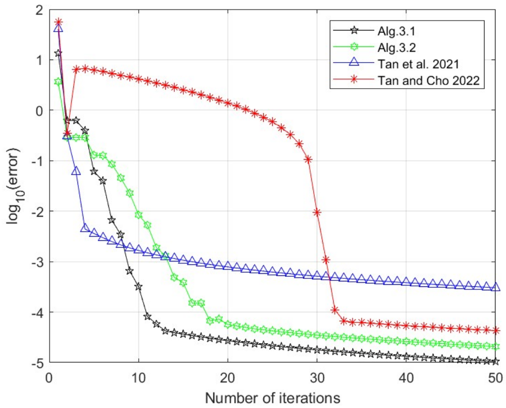

Figure 3: $x_1 = 6rand(4,1)$ for Example 4.1

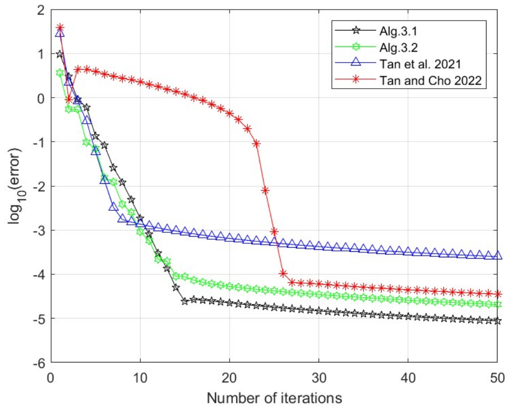

Figure 4: $x_1 = 8rand(4,1)$ for Example 4.1

Now, we define the mapping $M : C \to H$ as follows: $M(m)(t) = h(m)V(m)(t)$, $\forall t \in [0,1]$, $m \in C$. Notice that the operator M is Lipschitz continuous and pseudo-monotone but not monotone (see [18], Example 6.10).

We use the proposed Algorithms 3.1 and 3.2 to solve the BVIP with M and C given above, and compare them with several previously known Shehu et al. in [21] and Yao et al. in [33]. The parameters of all algorithms are set as follows. The numerical behavior of all algorithms are shown in Fig. 5. Fig. 6. Fig. 7. Fig. 8.

Table 2: Methods parameters for Example 4.2

our Alg. 1	$\alpha_n = 0.5$	$\mu = 0.9$	$\lambda_n = 0.6$	$\phi_n = 0.4$
	$\gamma = 0.4$	$\theta_n = 1 - 1/(n+1)$		
our Alg. 2	$\alpha_n = 0.5$	$\mu = 0.9$	$\lambda_n = 0.6$	$\phi_n = 0.4$
	$\gamma = 0.4$	$\theta_n = 1 - 1/(n+1)$		
Shehu et al. 2022	$\mu = 0.9$	$\alpha_n = 0.5$	$\lambda_n = 0.6$	$\theta_n = 1 - 1/(n+1)$
Yao et al. 2022	$\mu = 0.9$	$\alpha_n = 0.5$	$\lambda_n = 0.6$	$\theta_n = 1 - 1/(n+1)$

5. Conclusions

In this paper, we introduce two projection-based methods with different inertial steps to solve the bilevel quasi-monotone variational inequality problem in real Hilbert spaces. Our proposed algorithms are based on the inertial method, the projection method, the contraction mapping and the extragradient algorithms. A new non-monotonic step-size and the Armijo linesearch are embedded into the proposed algorithm so that it can work well without knowing the prior knowledge of the Lipschitz constant of mapping. Furthermore, we establish strong convergence of algorithms under some mild assumptions. The incorporation of the inertial term and the relaxation of the cost operator (quasi-monotone) generally enhance efficiency and applicability of our iterative methods. Finally, numerical experiments demonstrate the algorithms proposed in this paper have a faster convergence speed and higher accuracy than the algorithms known in this literature.

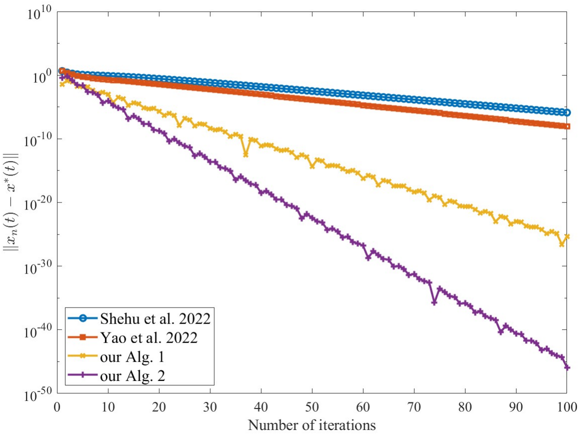

Figure 5: $x_0(t) = x_1(t) = 30t^5$ for Example 4.2

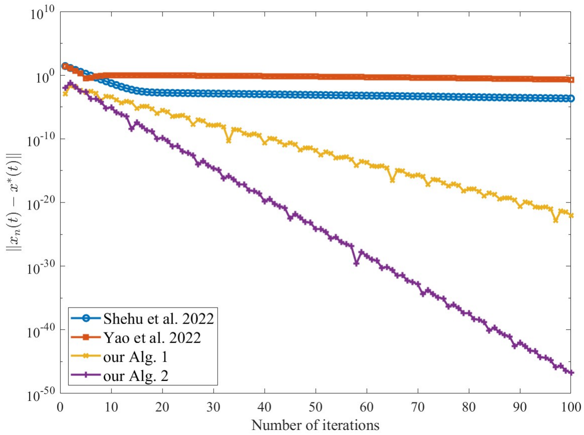

Figure 6: $x_0(t) = x_1(t) = 30\log(t)$ for Example 4.2

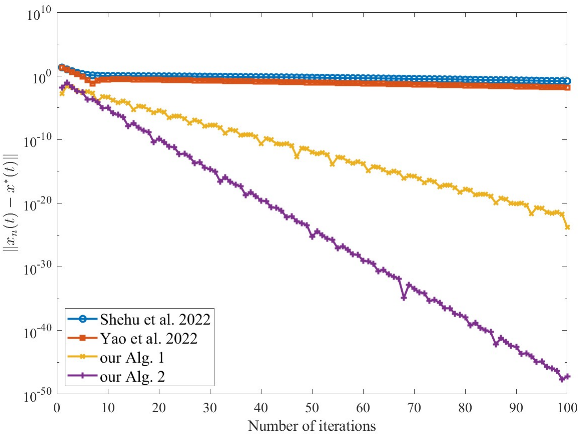

Figure 7: $x_0(t) = x_1(t) = 30\exp(t)$ for Example 4.2

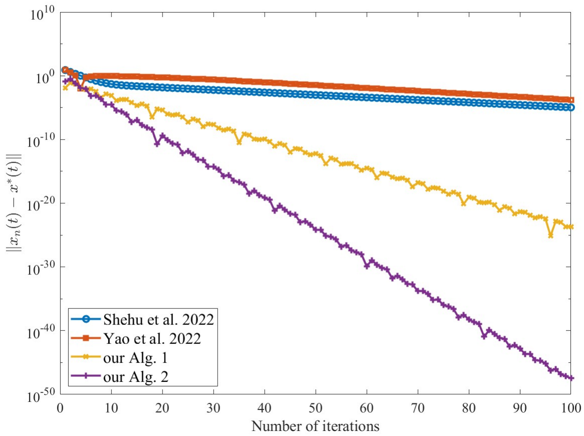

Figure 8: $x_0(t) = x_1(t) = 30\sin(t)$ for Example 4.2

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