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The property of presymmetry for w-distances on quasi-metric spaces

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Abstract. In this paper we extend the recently introduced notion of presymmetric w-distance on metric spaces to the context of quasi-metric spaces. We establish some of its properties and present various examples. We also show that this notion provides an efficient setting to obtain a suitable and large quasi-metric extension of a nice and elegant generalization of Banach's contraction principle due to Suzuki.

1. Introduction

In [23], Suzuki proved an elegant and fruitful improvement of Banach's contraction principle from which deduced a characterization of metric completeness. For our purposes here it will be sufficient to consider the following weak form of Suzuki's theorem.

Theorem 1.1. *([23]) Let (X, d) be a complete metric space and T be a self map of X. Suppose that there is a constant $r \in (0, 1)$ such that the following contraction condition holds for every $x, y \in X$:*

$$d(x, Tx) \leq 2d(x, y) \implies d(Tx, Ty) \leq rd(x, y). \qquad (1)$$

Then, T has a unique fixed point in X.

Our motivation in this paper comes from the difficulty in obtaining a full generalization of Suzuki's theorem in the context of quasi-metric spaces as well as in the one of w-distances on metric spaces (see Section 3 for details). Thus, and in order to mitigate the difficulties highlighted by Example 4 of [19], it was introduced the notion of a presymmetric w-distance in the setting of metric spaces and, then, a w-distance generalization of Theorem 1.1 via presymmetry of the involved w-distance was obtained [19, Theorem 2].

Here, we extend the idea of presymmetry to the framework of quasi-metric spaces. We examine some properties of presymmetric w-distances in this setting and give several examples. We also obtain a fixed point theorem whose contraction condition can be seen as an hybrid that combines conditions of Suzuki type and Samet et al. type [20], joint with presymmetry of the involved w-distance. In this way, our main result extends in several directions some relevant fixed point theorems and provides a broad generalization of Theorem 1.1, demonstrating the potential usefulness of this novel structure (although with a different approach, another precedent for the study presented here may be found in [17]).

Let us recall that the notion of a w-distance was introduced and discussed by Kada et al. in [7]. They proved prominent w-distance generalizations of important fixed point theorems as well as of Ekeland's

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variational principle and Takahashi minimization principle. Since then, the study of w-distances, their generalizations and applications have been the subject of intense research. The recent book from Rakočević [16] joint with the references therein, provide an updated and complete study of w-distances, emphasizing on its usefulness in fixed point theory. In particular, Section 6.7 of [16] constitutes a remarkable miscellany about fixed point theorems obtained using w-distances, for several generalized metric spaces and other related structures.

We stress that quasi-metric spaces constitute an important type of generalized metric spaces not only from a general topology view, but also for their applications (especially the non-T_1 quasi-metric spaces) to several branches of asymmetric functional analysis, domain theory, theories of computation and information, fractals theory, dynamical systems, machine learning, etc. (see, e.g., [2, 4, 6, 11, 18, 21, 22, 24]).

2. Preliminaries

In the sequel, by $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ we will design, respectively, the set of real numbers, the set of non-negative real numbers and the set of positive integer numbers.

By a quasi-metric on a set X we mean a function $q : X \times X \to \mathbb{R}^+$ that verifies the next two conditions for every $x, y, z \in X$:

(qm1) $q(x, y) = q(y, x) = 0$ if and only if $x = y$;

(qm2) $q(x, y) \leq q(x, z) + q(z, y)$.

By a T_1 quasi-metric on a set X we mean a quasi-metric q on X that verifies the next condition stronger than (qm1):

$q(x, y) = 0$ if and only if $x = y$.

A (T_1) quasi-metric space is a pair (X, q) where X is a (non-empty) set and q is a (T_1) quasi-metric on X.

If q is a quasi-metric on a set X, the function $q^{-1} : X \times X \to \mathbb{R}^+$ defined as $q^{-1}(x, y) = q(y, x)$ for all $x, y \in X$, is also a quasi-metric on X called the conjugate quasi-metric of q, and the function $q^s : X \times X \to \mathbb{R}^+$ defined as $q^s(x, y) = \max\{q(x, y), q^{-1}(x, y)\}$ for all $x, y \in X$, is a metric on X.

Given a quasi-metric q on a set X, put $B_q(x, \varepsilon) = \{y \in X : q(x, y) < \varepsilon\}$ for all $x \in X$ and all $\varepsilon > 0$. Then, the family $\{B_q(x, \varepsilon) : x \in X, \varepsilon > 0\}$ is a base (of open sets) for a T_0 topology Ω_q on X.

Following usual terminology, a sequence $(x_n)_{n\in\mathbb{N}}$ in a quasi-metric space (X, q) is Ω_q-convergent to $x \in X$ provided that it converges to x in the topological space (X, Ω_q). Hence, a sequence $(x_n)_{n\in\mathbb{N}}$ in a quasi-metric space (X, q) is Ω_q-convergent to $x \in X$ if and only if $q(x, x_n) \to 0$ as $n \to \infty$. Similarly, a sequence $(x_n)_{n\in\mathbb{N}}$ in a quasi-metric space (X, q) is $\Omega_{q^{-1}}$-convergent to $x \in X$ if and only if $q(x_n, x) \to 0$ as $n \to \infty$. In the rest of the paper we will simply write $q(x, x_n) \to 0$ (respectively, $q(x_n, x) \to 0$) if no confusion arises.

The lack of symmetry furnishes the existence of various notions of quasi-metric completeness in the literature (see, e.g., [5]), all of them coincide with the classical notion of completeness when dealing with a metric space. For our goals here we will consider the following very general one:

A quasi-metric space (X, q) is said to be q^{-1}-complete if every Cauchy sequence in the metric space (X, q^s) is $\Omega_{q^{-1}}$-convergent.

We will also consider bicomplete quasi-metric spaces. Remind that a quasi-metric space (X, q) is said to be bicomplete if the metric space (X, q^s) is complete.

There are many examples of q^{-1}-complete quasi-metric spaces. Next, we recall some of them, which will be considered later on.

Example 2.1. *Let q be the quasi-metric on $\mathbb{R}$ given by $q(x, y) = \max\{y - x, 0\}$ for all $x, y \in \mathbb{R}$. Note that q^s is the usual metric on $\mathbb{R}$. Hence, $(\mathbb{R}, q)$ is a bicomplete quasi-metric space and, thus, q^{-1}-complete.*

Example 2.2. *Let q be the quasi-metric on $\mathbb{N}$ given by $q(n, n) = 0$ for all $n \in \mathbb{N}$, and $q(n, m) = 1/n$ for all $n, m \in \mathbb{N}$ with $n \neq m$. Note that every non-eventually Cauchy sequence in $(\mathbb{N}, q^s)$ is $\Omega_{q^{-1}}$-convergent to any $n \in \mathbb{N}$, so $(\mathbb{N}, q)$ is q^{-1}-complete. However, (X, q) is not bicomplete because $(n)_{n\in\mathbb{N}}$ is a non-Ω_{q^s}-convergent Cauchy sequence in (X, q^s).*

Example 2.3. *Let $X = \mathbb{N} \cup \{\infty\}$ and let q be the quasi-metric on X given by $q(x,x) = 0$ for all $x \in X$, $q(\infty, n) = 1$ and $q(n, \infty) = 1/n$ for all $n \in \mathbb{N}$, and $q(n,m) = 1/n + 1/m$ for all $n, m \in \mathbb{N}$ with $n \neq m$. Note that (X,q) is q^{-1}-complete because every non-eventually constant Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in the metric space (X, q^s) satisfies $q(x_n, \infty) \to 0$. As in Example 2.2, $(n)_{n\in\mathbb{N}}$ is a non-Ω_{q^s}-convergent Cauchy sequence in (X, q^s), so that (X,q) is not bicomplete.*

In [14], Park generalized the notion of w-distance to the setting of quasi-metric spaces as follows:

A w-distance on a quasi-metric space (X,q) is a function $p : X \times X \to \mathbb{R}^+$ that verifies the next three conditions:

(w1) $p(x,y) \leq p(x,z) + p(z,y)$, for all $x,y,z \in X$;

(w2) for each $x \in X$, the function $p(x,\cdot) : X \to \mathbb{R}^+$ is $\Omega_{q^{-1}}$-lower semicontinuous;

(w3) for each $\varepsilon > 0$ there exists $\delta > 0$ such that $p(x,y) \leq \delta$ and $p(x,z) \leq \delta$ imply $q(y,z) \leq \varepsilon$.

The w-distance p is called symmetric if $p(x,y) = p(y,x)$ for all $x,y,z \in X$.

Remark 2.4. *Note that condition (w3) can be restated in an appealing way as follows (compare [12, Lemma 2.2]): For each $\varepsilon > 0$ there exists $\delta > 0$ such that $p(x,y) \leq \delta$ and $p(x,z) \leq \delta$ imply $q^s(y,z) \leq \varepsilon$.*

In [1], Al-Homidan et al. introduced and deeply analyzed the concept of a Q-function in the setting of quasi-metric spaces. In [9, Proposition 2] it was observed that the notions of w-distance and Q-function are coincident.

On the other hand, it is well known (see, e.g., [7, Example 1]) that every metric d on a set X is a w-distance on the metric space (X,d). This property does not hold for quasi-metric spaces, in general. In fact, it follows from [12, Proposition 2.3] that if a quasi-metric q on a set X is a w-distance on (X,q), then $\tau_q = \tau_{q^s}$, so (X, τ_q) is a metrizable topological space. This apparent handicap is compensated by the advantage that the use of w-distances instead of the original quasi-metrics in obtaining fixed point theorems offers, especially when the involved quasi-metric is not T_1 (see, e.g., [3, 9, 10]).

Several examples of w-distances on quasi-metric spaces may be found, for instance, in [1, 9, 10, 12, 14] (see also Section 3 below).

Remark 2.5. *Taking into account Remark 2.4, we get that every w-distance on a quasi-metric space (X,q) is a w-distance on the metric space (X,q^s). However, the converse does not hold, in general. Indeed, let $(\mathbb{R}, q)$ be the quasi-metric space of Example 2.1. We have that q^s is a w-distance on the metric space $(\mathbb{R}, q^s)$, but $q(1,0) = 0$ and $q^s(2,0) = 2 > 1 = q^s(2,1)$, so q^s does not verify condition (w2) for $(\mathbb{R}, q)$.*

3. Presymmetric w-distances on quasi-metric spaces

In the light of Theorem 1.1 the following question arises in a natural way:

Let p be a w-distance on a complete metric space (X,d), and let T be a self map of X. Suppose that there is a constant $r \in (0,1)$ such that the following contraction condition holds for every $x,y \in X$:

$$p(x,Tx) \leq 2p(x,y) \implies p(Tx,Ty) \leq rp(x,y). \tag{2}$$

Under the above assumptions, has T a fixed point in X?

In [19] it was presented an easy example showing that this question has a negative answer in general. Then, it was introduced and discussed the notion of a presymmetric w-distance, and showed that such a question has an affirmative answer when the involved w-distance is presymmetric [19, Theorem 2].

Remark 3.1. *At this point it seems appropriate to point out that the above question has a negative answer even for Banach spaces. Indeed, let $(X, \|.\|)$ be a (non-trivial) Banach space. Pick $z_0 \in X\backslash\{\mathbf{0}\}$ and $r \in (0,1)$. Define a self map T of X as follows: $T\mathbf{0} = z_0$ and $Tx = rx$ for all $x \in X\backslash\{\mathbf{0}\}$. Although T has no fixed points we are going to check that the contraction condition (2) is fulfilled for the w-distance p on $(X, \|.\|)$ given by $p(x,y) = \|y\|$ for all $x,y \in X$. To reach it, let $x,y \in X$. Then,*

- *If* $y = \mathbf{0}$, *we get* $p(x, Tx) = \|Tx\| > 0 = \|y\| = 2p(x, y)$.

- *If* $y \neq \mathbf{0}$, *we get* $p(Tx, Ty) = \|ry\| = rp(x, y)$.

Now, we generalize the concept of presymmetry introduced in [19], as follows.

Definition 3.2. *A w-distance p on a quasi-metric space (X, q) is said to be presymmetric if it verifies the next property:*

Whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $q(x_n, x) \to 0$ and $p(x_n, x) \to 0$ for some $x \in X$, then there is a subsequence $(x_{k(n)})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ fulfilling $p(x, x_{k(n)+1}) \leq p(x_{k(n)}, x)$ for all $n \in \mathbb{N}$.

Of course, if (X, q) is a metric space, the preceding notion coincides with the notion of a presymmetric w-distance as defined in [19, Definition 2]. Furthermore, we have the following result whose proof is omitted because it is identical to the one given in [19, Proposition 1]).

Proposition 3.3. *Every symmetric w-distance on a quasi-metric space is presymmetric.*

Remark 3.4. *Note that every presymmetric w-distance on a quasi-metric space (X, q) is a presymmetric w-distance on the metric space (X, q^s). The converse does not hold in general, as the example presented in Remark 2.5 shows.*

Although the quasi-metric of a quasi-metric space is not necessarily a w-distance on it, the next example shows that we can easily construct symmetric w-distances on any quasi-metric space which allow us to justify the existence of fixed point for the type of contractions that we shall introduce in Section 4.

Example 3.5. *Let (X, q) be a quasi-metric space such that there is $x_0 \in X$ verifying $q(x_0, y) > 0$ for all $y \neq x_0$, and let $c > 0$. Define $p : X \times X \to \mathbb{R}^+$ as $p(x_0, x_0) = 0$ and $p(x, y) = c$ otherwise. It is routine to check that p is a symmetric w-distance on (X, q). Now, define a self map T of X as $Tx = x_0$ for all $x \in X$. Then, we have $p(Tx, Ty) = p(x_0, x_0) = 0$ for all $x \in X$.*

We conclude this section with a representative example of (pre)symmetrics w-distances on a concrete quasi-metric space (compare [7, Examples 3 and 4], [16, Examples 2.1.4 and 2.1.5], [19, Example 6]).

Example 3.6. *Denote by q_+ the restriction of the quasi-metric q of Example 2.1 on $\mathbb{R}^+$. Let s and t be constants such that $s \geq 0$ and $t > 0$, and let $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ defined as $p(x, y) = sx + ty$ for all $x, y \in X$.*

We first show that p is a w-distance on $(\mathbb{R}^+, q_+)$. Condition (w1) is obviously fulfilled. For (w2), fix $x, y \in \mathbb{R}^+$ and let $(y_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^+$ such that $q_+(y_n, y) \to 0$. Given $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$ such that $y - y_n < \varepsilon/t$ for all $n \geq n_0$. Therefore,

$$p(x, y) = sx + ty < sx + ty_n + \varepsilon = p(x, y_n) + \varepsilon,$$

for all $n \geq n_\varepsilon$. For (w3), given $\varepsilon > 0$ choose $\delta = t\varepsilon$. Suppose that $p(x, y) \leq \delta$ and $p(x, z) \leq \delta$. Then, $ty \leq \delta$ and $tz \leq \delta$, which implies that $y \leq \varepsilon$ and $z \leq \varepsilon$, so $(q_+)^s(y, z) = |y - z| \leq \varepsilon$.

Finally, we shall discuss three cases:

Case 1. $s = t$. Then, p is a symmetric w-distance on $(\mathbb{R}^+, q_+)$.

Case 2. $s > t$. Then, p is a presymmetric w-distance on $(\mathbb{R}^+, q_+)$. Indeed, let $x \in \mathbb{R}^+$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^+$ such that $p(x_n, x) \to 0$ and $q_+(x_n, x) \to 0$. We get that $x = 0$ and $x_n \to 0$ with respect to the usual metric $(q_+)^s$. Hence, there is a subsequence $(x_{k(n)})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $x_{k(n)+1} \leq x_{k(n)}$ for all $n \in \mathbb{N}$. Consequently, $p(x, x_{k(n)+1}) = tx_{k(n)+1} \leq sx_{k(n)} = p(x_{k(n)}, x)$.

Case 3. $s < t$. Then, p is a non-presymmetric w-distance on $(\mathbb{R}^+, q_+)$. This fact follows verbatim the corresponding case in [19, Example 6].

4. Fixed point results and examples

In [20], Samet et al. unified and extended several classical and well-known fixed point theorems on metric spaces via the so-called $\alpha-\psi$ contractive type mappings. This new and appealing approach attracted the attention of many authors, who have extended and generalized these theorems to numerous settings. In this direction, Chapter 3 of the recent book by Karapinar and Argawal [8], joint with the references therein, constitutes an updated a valuable source on this topic.

We shall denote by Ψ the set of all Bianchini-Grandolfi gauge functions (see [15]), i.e., the set of all nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=0}^{\infty} \psi^n(t) < \infty$ for each $t \geq 0$. Recall that if $\psi \in \Psi$, then $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$.

Let X be a set and $\alpha : X \times X \to \mathbb{R}^+$. According to [20], a self map T of X is called α-admissible if $\alpha(Tx, Ty) \geq 1$ whenever $\alpha(x, y) \geq 1$, for all $x, y \in X$.

Let (X, q) be a quasi-metric space. Inspired on [20, Theorem 2], we say that a function α: $X \times X \to \mathbb{R}^+$ satisfies property (P_α) if for each sequence $(x_n)_{n\in\mathbb{N}}$ in X fulfilling $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $q(x_n, x) \to 0$ for some $x \in X$, we have that $\alpha(x_n, x) \geq 1$ eventually (i.e., there is $n_0 \in \mathbb{N}$ such that $\alpha(x_n, x) \geq 1$ for all $n \geq n_0$).

Definition 4.1. *Let p be a w-distance on a quasi-metric space (X, q), $\alpha : X\times X \to \mathbb{R}^+$ be a function for which property (P_α) is satisfied, T be an α-admissible self map of X for which there is $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, and let $\psi \in \Psi$. Under the preceding conditions, we say that T is a basic (p, α, ψ)-contraction of Suzuki type on (X, q) if the following contraction condition holds for every $x, y \in X$:*

$$p(x, Tx) \leq 2p(x, y) \implies \alpha(x, y)p(Tx, Ty) \leq \psi(p(x, y)). \tag{3}$$

If in Definition 4.1, the functions α and ψ are, respectively, given by $\alpha(x, y) = 1$ for all $x, y \in X$, and $\psi(t) = rt$ for all $t \in \mathbb{R}^+$, with $r \in (0, 1)$ constant, we will simply say that T is a basic p-contraction of Suzuki type (note that this case corresponds to the contraction condition (2)).

Theorem 4.2. *Let T be basic (p, α, ψ)-contraction of Suzuki type on a q^{-1}-complete quasi-metric space (X, q). If the w-distance p is presymmetric, then T has a fixed point.*

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. For each $n \in \mathbb{N} \cup \{0\}$ put $x_n = T^n x_0$. Since T is α-admissible we obtain $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

For each $n \in \mathbb{N}$ we obviously have $p(x_{n-1}, x_n) \leq 2p(x_{n-1}, x_n)$, so, by (3),

$$p(x_n, x_{n+1}) \leq \alpha(x_{n-1}, x_n)p(x_n, x_{n+1}) \leq \psi(p(x_{n-1}, x_n)),$$

for all $n \in \mathbb{N}$. Hence, recursively we get $p(x_n, x_{n+1}) \leq \psi^n(p(x_0, x_1))$ for all $n \in \mathbb{N}$.

Given $\varepsilon > 0$, let δ be the positive real number associated to ε in (w3). Without loss of generality, we assume that $\delta < \varepsilon$.

Following verbatim the part of the proof of [20, Theorem 2.1] provided on the first lines of page 2156, we deduce the existence of an $n_\delta \in \mathbb{N}$ such that $p(x_n, x_m) < \delta$ whenever $m > n > n_\delta$. Then, it follows from (w3) that $q^s(x_n, x_m) \leq \varepsilon$ whenever $m > n > n_\delta$.

Since ε is arbitrary, we conclude that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the metric space (X, q^s). Let $\xi \in X$ be such that $q(x_n, \xi) \to 0$.

We next show that $p(x_n, \xi) \to 0$. Indeed, given $\varepsilon > 0$ there exists $n_\varepsilon > n_\delta$ such that $q(x_n, \xi) < \varepsilon$ for all $n \geq n_\varepsilon$. Let $n \geq n_\varepsilon$. By (w2) we find $m > n$ for which $p(x_n, \xi) < \varepsilon + p(x_n, x_m)$. As $n > n_\delta$ we get $p(x_n, x_m) < \delta < \varepsilon$, so $p(x_n, \xi) < 2\varepsilon$ for all $n \geq n_\varepsilon$. Therefore, $p(x_n, \xi) \to 0$.

Since p is presymmetric, there is a subsequence $(x_{k(n)})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $p(\xi, x_{k(n)+1}) \leq p(x_{k(n)}, \xi)$ for all $n \in \mathbb{N}$. Then,

$$p(x_{k(n)}, x_{k(n)+1}) \leq p(x_{k(n)}, \xi) + p(\xi, x_{k(n)+1}) \leq 2p(x_{k(n)}, \xi),$$

for all $n \in \mathbb{N}$. Consequently, we can apply (3) to obtain

$$\alpha(x_{k(n)}, \xi)p(x_{k(n)+1}, T\xi) \leq \psi(p(x_{k(n)}, \xi), \tag{4}$$

for all $n \in \mathbb{N}$.

Since, by assumption, α satisfies property (P_α), we deduce the existence of an $n_0 \in \mathbb{N}$ such that $\alpha(x_{k(n)}, \xi) \geq 1$ for all $n \geq n_0$. So, by (4),

$$p(x_{k(n)+1}, T\xi) \leq p(x_{k(n)}, \xi)$$

for all $n \geq n_0$.

The preceding inequality, joint with the fact that $p(x_n, \xi) \to 0$, implies that $p(x_{k(n)+1}, T\xi) \to 0$. As $p(x_{k(n)+1}, \xi) \to 0$, we deduce from (w3) that $\xi = T\xi$. This completes the proof.

□

Corollary 4.3. *Let T be basic (p, α, ψ)-contraction of Suzuki type on a bicomplete quasi-metric space (X, q). If the w-distance p is presymmetric, then T has a fixed point.*

Corollary 4.4. *Let T be basic p-contraction of Suzuki type on a q^{-1}-complete quasi-metric space (X, q). If the w-distance p is presymmetric, then T has a unique fixed point $\xi \in X$. Moreover, $p(\xi, \xi) = 0$.*

Proof. By Theorem 4.2, T has a fixed point $\xi \in X$. Since $p(\xi, \xi) = p(\xi, T\xi) \leq 2p(\xi, \xi)$, we deduce from (3) that $p(\xi, \xi) \leq rp(\xi, \xi)$, so $p(\xi, \xi) = 0$. Finally, let $\zeta \in X$ such that $\zeta = T\zeta$. Since $p(\xi, T\xi) = 0$, we get $p(\xi, T\xi) \leq 2p(\xi, \zeta)$, so, by (3), $p(\xi, \zeta) \leq rp(\xi, \zeta)$, which implies that $p(\xi, \zeta) = 0$. From (w3) we obtain that $\xi = \zeta$. This concludes the proof. □

The following easy exemplification of Theorem 4.2 possesses some interesting peculiarities. In particular, it provides an instance where we can apply Theorem 4.2 but not Corollary 4.3. Moreover, we cannot apply Corollary 4.4 for the selected w-distance.

Example 4.5. *Let (X, q) be the q^{-1}-complete quasi-metric space of Example 2.3, and let T be the self map of X defined as $T\infty = \infty$, and $Tn = n^2$ for all $n \in \mathbb{N}$.*

Consider the functions $p : X \times X \to \mathbb{R}^+$ and $\alpha : X \times X \to \mathbb{R}^+$ defined, respectively, as

$$p(\infty, \infty) = 0,\ p(1, \infty) = 1/3 \text{ and } p(x, y) = 1 \text{ otherwise,}$$

and

$$\alpha(\infty, \infty) = \alpha(1, \infty) = 1 \text{ and } \alpha(x, y) = 0 \text{ otherwise.}$$

It is routine to check that p is a w-distance on (X, d). In fact, the proof of condition (w1) is almost trivial, and to verify (w2) notice that each $x \in \mathbb{N}$ is a $\tau_{q^{-1}}$-isolated point and that $p(x, \infty) \leq p(x, y)$ for all $x, y \in X$. Finally, for (w3) choose, for instance, $\delta = 1/4$ for any $\varepsilon > 0$. It is also clear that p is presymmetric because from $p(x_n, x) \to 0$ it follows that $x_n = x = \infty$ eventually, so, $p(x, x_n) = p(\infty, \infty) = 0$ eventually.

On the other hand, we obviously have $\alpha(\infty, T\infty) \geq 1$, and $\alpha(Tx, Ty) \geq 1$ whenever $\alpha(x, y) \geq 1$. So, T is α-admissible. Moreover, property (P_α) is also satisfied because if $(x_n)_{n\in\mathbb{N}}$ is a sequence in X satisfying $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $q(x_n, x) \to 0$ for some $x \in X$, we deduce that $x_n = x_{n+1} = \infty$ for all $n \geq 2$, so $\alpha(x_n, x) \geq 1$ eventually.

We shall show that T is a basic contraction of (p, α, ψ)-Suzuki type for any $\psi \in \Psi$.

Indeed, choose an arbitrary $\psi \in \Psi$. Let $x, y \in X$ and note that by the definitions of T, α and p, we only need to examine the case where $x = 1$ and $y = \infty$. Since $p(1, T1) = 1 > 2p(1, \infty)$ we directly conclude that T is a basic contraction of (p, α, ψ)-Suzuki type.

Therefore, all conditions of Theorem 4.2 are satisfied. In fact, T has two fixed points.

It seems interesting to emphasize that we also have the following facts:

- *T is not a basic contraction of p-Suzuki type because $p(\infty, T\infty) < 2p(\infty, 1)$ but $p(T\infty, T1) = p(\infty, 1) = 1$.*
- *T is not an $\alpha - \psi$-contractive mapping on (X, q) because $\alpha(1, \infty)q(T1, T\infty) = q(T1, T\infty) = 1 = q(1, \infty)$.*

- *The topology $\tau_{q^{-1}}$ is Hausdorff (actually, it is compact and metrizable) while τ_q is the discrete topology on X, so $\tau_q = \tau_{q^s}$.*

- *Although p is also a w-distance on the metric space (X, q^s) and property (P_α) is satisfied when we consider the metric q^s, we cannot apply Corollary 4.3 because (X, q) is not bicomplete, as noted in Example 2.3.*

In the examples that follow, the involved quasi-metrics have the structure of a weighted quasi-metric in the sense of Matthews [13].

Let us recall that a quasi-metric q on a set X is weighted provided that there exists a function $h : X \to \mathbb{R}^+$ such that $q(x, y) + h(x) = q(y, x) + h(y)$ for all $x, y \in X$. In [12, Proposition 2.10] it was proved that, in this case, the function $p : X \times X \to \mathbb{R}^+$ given by $p(x, y) = q(x, y) + h(x)$ for all $x, y \in X$, is a symmetric w-distance on (X, q) (according to [13, Theorem 4.2], p is the partial metric on X induced by q).

Example 4.6. *Let $(\mathbb{R}^+, q_+)$ be the quasi-metric space of Example 3.6. It is clear that $(\mathbb{R}^+, (q_+)^s)$ is a complete metric space, so that the quasi-metric space $(\mathbb{R}^+, q_+)$ is bicomplete.*

Let T be the self map of $\mathbb{R}^+$ defined as $Tx = x/(x + 2)$ if $x \in [0, 1]$, and $Tx = 1$ if $x \in (1, \infty)$.

Note that for $x = 1$ and $y = 5/3$ we get $q_+(Tx, Ty) = q_+(1/3, 1) = 2/3 = q_+(x, y)$, which implies that T is not a Banach contraction on $(\mathbb{R}^+, q_+)$.

However, we are going to show that T is a basic (p, α, ψ)-contraction of Suzuki type for p, α and ψ constructed as follows.

Since $q_+(x, y) + x = q_+(y, x) + y$ for all $x, y \in \mathbb{R}^+$, we deduce that q_+ is weighted via the function h given by $h(x) = x$ for all $x \in \mathbb{R}^+$. Hence, the function $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ given by $p(x, y) = q_+(x, y) + x = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$, is a symmetric w-distance on $(\mathbb{R}^+, q_+)$.

Now define $\alpha : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ as $\alpha(x, y) = 1$ for all $x, y \in \mathbb{R}^+$, and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ as $\psi(t) = t/(t + 2)$ if $t \in [0, 1]$, and $\psi(t) = 1$ if $t \in (1, \infty)$.

Obviously, α satisfies property (P_α), T is α-admissible and $\alpha(x_0 T x_0) = 1$ for all $x_0 \in \mathbb{R}^+$.

On the other hand, $\psi \in \Psi$ because $\psi^n(t) \le t/(2^n + t)$ for all $t \in \mathbb{R}^+$ and $n \ge 2$.

Finally, we show that T is a basic (p, α, ψ)-contraction of Suzuki type. Let $x, y \in \mathbb{R}^+$. By the symmetry of p it suffices to check the next cases.

Case 1. $x, y > 1$. Then, we obtain $\alpha(x, y)p(Tx, Ty) = p(1, 1) = 1 = \psi(p(x, y))$.

Case 2. $x > 1$, $y \le 1$. Then, we obtain $\alpha(x, y)p(Tx, Ty) = p(1, y/(2 + y)) = 1 = \psi(p(x, y))$.

Case 3. $x, y \le 1$. Then, we obtain $\alpha(x, y)p(Tx, Ty) = p(x/(2 + x), y/(2 + y)) = \psi(p(x, y))$.

Thus, we have proved that all conditions of Corollary 4.3 are fulfilled.

Example 4.7. *Let $X = \{0, 1\}$. Denote by X^f the set of all finite sequences (finite words) of elements of X and by X^∞ the set of all infinite sequences (infinite words) of elements of X. Put $X^\omega = X^f \cup X^\infty$.*

For each $x \in X^f$ we denote by $l(x)$ its length. Thus, if $x \in X^f$ with $x := x_1 ... x_k$, $k \in \mathbb{N}$, we have $l(x) = k$, and if $x \in X^\infty$ we have $l(x) = \infty$ and write $x := x_1 x_2 ...$

Given $x, y \in X^\omega$ we say that x is a prefix of y, and write $x \sqsubseteq y$, if it is fulfilled one of the following two conditions:
(i) $x \in X^f$, $l(x) \le l(y)$ and $x_j = y_j$ whenever $1 \le j \le l(x)$.
(ii) $x \in X^\infty$ and $x = y$.

If $x \sqsubseteq y$ with $x \ne y$, we write $x \sqsubset y$. Moreover, by $x \sqcap y$ we denote the (longest) common prefix of x and y. Note that $x \sqcap y = x$ whenever $x \sqsubseteq y$.

Now, let q be the quasi-metric on X^ω given by

$$q(x, y) = 2^{-l(x \sqcap y)} - 2^{-l(x)}$$

for all $x, y \in X^\omega$. It is well known that (X^ω, q) is bicomplete. It is also well known, and easily checked, that q is weighted via the function h given by $h(x) = 2^{-l(x)}$ for all $x \in X^\omega$.

Therefore, the function $p : X^\omega \times X^\omega \to \mathbb{R}^+$ given by $p(x, y) = 2^{-l(x \sqcap y)}$ for all $x, y \in X^\omega$, is a symmetric w-distance on (X^ω, q).

Next, we define a self map T of X^ω as follows:

If $x \in X^f$ with $l(x)$ odd, Tx is the unique $z \in X^f$ with $l(z) = 1$ and $z_1 = 0$.

If $x \in X^f$ with $l(x)$ even, Tx is the unique $y_x \in X^f$ such that $l(y_x) = l(x) + 2$, $(y_x)_1 = (y_x)_2 = 0$, and $(y_x)_j = x_{j-2}$ for $3 \leq j \leq l(x) + 2$.

If $x \in X^\infty$, Tx is the unique $y_x \in X^\infty$ such that $(y_x)_1 = (y_x)_2 = 0$, and $(y_x)_j = x_{j-2}$ for $j \geq 3$.

We first note that we cannot apply Corollary 4.4 because for $x, y \in X^f$ with $l(x)$ and $l(y)$ odd, and $x_1 = y_1$, we get $l(x \sqcap y) \geq 1$, so

$$p(Tx, Ty) = 2^{-l(z)} = 2^{-1} \geq 2^{-l(x \sqcap y)} = p(x, y).$$

However, we shall show that T is a basic contraction of (p, α, ψ)-Suzuki type, for α and ψ defined below, and thus, we will can apply Corollary 4.3.

Define $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ as $\psi(t) = t/4$ for all $t \in \mathbb{R}^+$. Then, $\psi \in \Psi$.

Now, define $\alpha : X^\omega \times X^\omega \to \mathbb{R}^+$ as follows:

$\alpha(x, y) = 1$ if $x, y \in X^f$, with $l(x)$ and $l(y)$ even, and $x \sqsubset y$;

$\alpha(x, y) = 1$ if $x \in X^f$, $y \in X^\infty$, with $l(x)$ even and $x \sqsubset y$;

and

$\alpha(x, y) = 0$ otherwise.

Let $x_0 := 00$. Then, $Tx_0 = 0000$. Hence $x_0 \sqsubset Tx_0$, and, thus, $\alpha(x_0, Tx_0) \geq 1$.

It is clear that T is α-admissible. Furthermore, property (P_α) is also satisfied: Indeed, let $(u_n)_{n\in\mathbb{N}}$ be a sequence in X^ω and $u \in X^\omega$ such that $\alpha(u_n, u_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, and $q(u_n, u) \to 0$. Then, $u_n \sqsubset u_{n+1}$ and $l(u_n)$ even for all $n \in \mathbb{N}$. Consequently, u is the unique element of X^ω verifying $u_n \sqsubset u$ for all $n \in \mathbb{N}$, which implies that $\alpha(u_n, u) = 1$ for all $n \in \mathbb{N}$.

Finally, we shall show that the contraction condition (3) holds. To this end, let $x, y \in X^\omega$. By the construction of the function α we only need to discuss the following cases.

Case 1. $x, y \in X^f$ with $l(x)$ and $l(y)$ even, and $x \sqsubset y$. We have $Tx \sqsubset Ty$, so

$$\alpha(x, y)p(Tx, Ty) = 2^{-l(Tx \sqcap Ty)} = 2^{-l(Tx)} = 2^{-(l(x)+2)} = \frac{1}{4}2^{-l(x \sqcap y)} = \frac{1}{4}p(x, y).$$

Case 2. $x \in X^f$, $y \in X^\infty$, with $l(x)$ even and $x \sqsubset y$. Exactly as in Case 1, we have $Tx \sqsubset Ty$, so

$$\alpha(x, y)p(Tx, Ty) = 2^{-l(Tx \sqcap Ty)} = 2^{-l(Tx)} = 2^{-(l(x)+2)} = \frac{1}{4}2^{-l(x \sqcap y)} = \frac{1}{4}p(x, y).$$

We conclude the paper with an example showing that the q^{-1}-completeness of the quasi-metric space (X, q) in Theorem 4.2 cannot be replaced with the following alternative notion of completeness: A quasi-metric space (X, q) is q-complete provided that every Cauchy sequence in the metric space (X, q^s) is Ω_q-convergent.

Example 4.8. *Consider the quasi-metric space $(\mathbb{N}, q)$ of Example 2.2. Denote by q' the conjugate quasi-metric q^{-1} of q. Then, $(\mathbb{N}, q')$ is q'-complete because every non-eventually constant Cauchy sequence in $(\mathbb{N}, (q')^s)$ is Ω_q-convergent to any $n \in \mathbb{N}$. Since $q'(n, m) + 1/n = q'(m, n) + 1/m$ for all $n, m \in \mathbb{N}$, we deduce that q' is weighted via the function h given by $h(n) = 1/n$ for all $n \in \mathbb{N}$. Hence, the function $p : \mathbb{N} \times \mathbb{N} \to \mathbb{R}^+$ given by $p(n, m) = 1/n + 1/m$ for all $n, m \in \mathbb{N}$, is a symmetric w-distance on $(\mathbb{N}, q')$.*

Now, define a self map T of X as $Tn = 2n$ for all $n \in \mathbb{N}$. Although T has no fixed points, it is a basic p-contraction of Suzuki type. Indeed, for each $n, m \in \mathbb{N}$ we get

$$p(Tn, Tm) = p(2n, 2m) = 1/2n + 1/2m = p(n, m)/2.$$

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