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The Moore graphs; total domination and total dominator chromatic numbers

Maliheh Alaei[a], Adel P. Kazemi[a,]

[a] *Department of Mathematics, University of Mohaghegh Ardabili, P.O. Box 5619911367, Ardabil, Iran*

Abstract. A total dominator coloring of a graph G is a proper coloring of G in which each vertex of the graph is adjacent to all vertices of a color class. The minimum number of color classes in a total dominator coloring of a graph is called its total dominator chromatic number. Here, we study the total domination and total dominator chromatic numbers of Moore graphs which are a family of k-regular graph of girth g and have the smallest order $n_0(k, g)$ (a known number in terms of k and g).

1. Introduction

Here, in a simple graph $G = (V, E)$, $deg_G(v)$, $N_G(v)$ and $N_G[v]$ denote respectively the *degree*, *open* and *closed neighborhoods* of a vertex $v \in V$, the *minimum degree*, *maximum degree* and *independence number* of G are denoted by $\delta = \delta(G)$, $\Delta = \Delta(G)$ and $\alpha = \alpha(G)$, respectively. Also a complete graph of order n by K_n and a subgraph of G *induced* by a subset $S \subseteq V$ by $G[S]$ are shown. The *girth* $g(G)$ of G is the length of the smallest cycle of the graph. Also for any positive integer k, we use $[k]$ to denote the set $\{1, 2, \cdots, k\}$.

In this paper, we will find the total dominator chromatic numbers of all of the (k, g)-Moore graphs except some Moore graphs with girth 12. For this aim, we also need their total domination numbers. These numbers are presented in Table 1. The needed definitions and terminologies are given in following.

k/g	**3**	**4**	**5**	**6**	**8**	**12**
3	(2,4)	(2,2)	(4,6)	(6,8)	(10,12)	(43,45)
4	(2,5)	(2,2)		(8,10)	(21,23)	(182,184)
7	(2,8)	(2,2)	(8,12)			
$q+1$	$(2, q+2)$	(2,2)		$(2q+2, 2q+4)$	$(2q^2+2, 2q^2+4)$	
$p+1$	$(2, p+2)$	(2,2)		$(2p+2, 2p+4)$	$2p^2+3 \leq \gamma_t \leq 2p^2+p+1$ $\chi_d^t(G) = \gamma_t(G)+2$	

Table 1: Table of the ordered pairs $(\gamma_t(G), \chi_d^t(G))$ of the known (k, g)-Moore graphs G in which q is an even prime power and p is an odd prime power.

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* Corresponding author: Adel P. Kazemi
Email addresses: malihehalaei.t@gmail.com (Maliheh Alaei), adelpkazemi@yahoo.com (Adel P. Kazemi)

1.1. Total dominating set and coloring of a graph

Total domination in graphs is now well studied in graph theory and the literature on this subject has been surveyed and detailed in the book [11]. A vertex subset of a graph with this property that every vertex of the graph is adjacent to some vertex of the set is called a *total dominating set*, briefly TDS, of the graph, and the minimum cardinality of a TDS of a graph G is called the *total domination number* $\gamma_t(G)$ of G.

If a function $c : V(G) \to [k]$ from the vertices of a graph G to a k-set $[k]$ of colors such that any two adjacent vertices have different colors, or equivalently, c is a hommorphism from G to the complete graph K_k with vertex set $[k]$, then c is called a *proper k-coloring* of G. The minimum number k of colors needed in a proper coloring of a graph G is called the *chromatic number* of G and denoted by $\chi(G)$. In a proper coloring of a graph, a set consisting of all those vertices assigned the same color is called a *color class*. Trivially every color class contains at most $\alpha(G)$ vertices. For simply, we denote a proper coloring c of a graph with ℓ color classes $V_1, \cdots, V_\ell$ by $c = (V_1, V_2, \cdots, V_\ell)$.

Motivated by the relation between coloring and total domination the concept of total dominator colorings was introduced in [14]. For more information the reader can study [10, 12–16].

Definition 1.1. A proper coloring of a graph G, which has no isolated vertex, with this property that every vertex of G is adjacent to all vertices of some color class, is called a *total dominator coloring*, briefly TDC, of G, and the minimum number of color classes in a TDC of G is called the *total dominator chromatic number* $\chi_d^t(G)$ of G.

Let $c = (V_1, V_2, \cdots, V_\ell)$ be a TDC of a graph G. We call a vertex v a *common neighbor* of V_i, or we say that v has *totally dominated* by V_i, or V_i *totally dominates* v if $V_i \subseteq N(v)$. While the set of all common neighbors of V_i, denoted by $CN_G(V_i)$ or simply by $CN(V_i)$, is called the *common neighborhood* of V_i in G with respect to c, a TDC of G with minimum number of colors is called a *min*-TDC. Since by selecting one vertex of every color class in a min-TDC of a graph, a TDS of the graph can be obtained, we have

$$\gamma_t(G) \le \chi_d^t(G) \text{ for any graph } G. \tag{1}$$

1.2. Moore graphs and generalized polygons

The cage problem asks for the construction of regular simple graphs with specified degree k and girth g and minimum order $n_0(k, g)$, which are know as (k, g)-*cages*. This problem was first considered by Tutte. The existence of a (k, g)-cage for any pair of parameters (k, g) is not immediately obvious, and it was first shown by Sachs [19]. In [6] Erdös and Sachs proved the existence of k-regular graphs with girth g for all values of k and g provided that $k \ge 2$, and also they showed that the order of such a graph is at least

$$M(k,g) = \begin{cases} \frac{2(k-1)^{\frac{g}{2}}-2}{k-2} & \text{if } g \text{ is even,} \\ \frac{k(k-1)^{\frac{g-1}{2}}-2}{k-2} & \text{if } g \text{ is odd.} \end{cases}$$

This bound is known as the *Moore bound*, and the (k, g)-cages that their order $n_0(k, g)$ achieve this bound are called as *Moore graphs*. It is known in [7] that order of the $(3, 7)$- and $(4, 5)$-cages is more than the Moore bound and so their are not Moore graphs. This shows the importance of finding the cages their orders achieve own Moore bounds. This cages are known as (k, g)-*Moore graphs* or simply *Moore graphs*, and the problem of the existence of them is closely related to the degree/diameter problem surveyed in [17], and they are almost completely characterized in [3, 4]. In addition to Table 2, which lists the orders of the known Moore graphs, next theorem summarizes the partial characterization of Moore graphs.

k/g	**3**	**4**	**5**	**6**	**8**	**12**
3	4	6	10	14	30	126
4	5	8		26	80	728
5	6	10		42	170	2730
6	7	12		62	312	7812
7	8	14	50			
$q+1$	$q+2$	$2(q+1)$		$2(q^2+q+1)$	$2(q+1)(q^2+1)$	$2(q^3+1)(q^2+q+1)$

Table 2: Table of orders of the known Moore graphs (q denotes a prime power).

Theorem 1.2. [8] *There exists a (k, g)-Moore graph G if and only if*

(1) $g \geq 3$ *and* $k = 2$ *(G is a cycle), or*

(2) $g = 3$ *and* $k \geq 2$ *(G is a complete graph), or*

(3) $g = 4$ *and* $k \geq 2$ *(G is a complete bipartite graphs), or*

(4) $g = 5$*, and* $k = 3$ *or* 7 *(G is respectively the Petersen graph or the Hoffman-Singleton graph), or possibly* $k = 57$*, or*

(5) $g \in \{6, 8, 12\}$ *and there exists a symmetric generalized* $\frac{g}{2}$*-gon of order* $k - 1$.

The fifth condition in Theorem 1.2 says that G is the incidence graph $G[\mathcal{P}, \mathcal{L}]$ of a symmetric generalized $\frac{g}{2}$-gon, and so existance of Moore graphs is equivalent to existance of some symmetric generalized polygons, which we define now.

Let a set of points $\mathcal{P}$ and a set of lines $\mathcal{L}$ be two disjoint non-empty sets, and let $I \subseteq \mathcal{P} \times \mathcal{L}$ be the *point-line incidence relation*, in this meaning that $(p, l) \in I$ means that the point p and the line l are incident. For the ordered triple $\mathcal{I} = (\mathcal{P}, \mathcal{L}, I)$, the bipartite graph with the vertex set $V = \mathcal{P} \cup \mathcal{L}$ is called the *incidence graph* $G[\mathcal{P}, \mathcal{L}]$ of $\mathcal{I}$, where pl is an edge if and only if $(p, l) \in I$. The ordered triple $\mathcal{I}$ is called a *symmetric generalized n-gon* of order q (or in some references, of order (q, q)) subject to the following regularity conditions:

GP1: For some integer $q \geq 1$, every line is incident to exactly $q + 1$ points and every point is incident to exactly $q + 1$ lines.

GP2: Any two distinct lines intersect in at most one point and there is at most one line through any two distinct points.

GP3: The incidence graph $G[\mathcal{P}, \mathcal{L}]$ has diameter n and girth $2n$.

It is known from [7] the only known symmetric generalized polygons have prime power order. For even n, an *ovoid* in a symmetric generalized n-gon $\mathcal{I}$ is a subset $\mathcal{O} \subseteq \mathcal{P}$ such that distance of every two points of $\mathcal{O}$ in $G[\mathcal{P}, \mathcal{L}]$ is n and every vertex of $G[\mathcal{P}, \mathcal{L}]$ has distance at most $\frac{n}{2}$ from some point of $\mathcal{O}$. Similarly a *spread* in $\mathcal{I}$ is a set $\mathcal{S} \subseteq \mathcal{L}$ such that distance of every two lines of $\mathcal{S}$ in $G[\mathcal{P}, \mathcal{L}]$ is n and every vertex of $G[\mathcal{P}, \mathcal{L}]$ has distance at most $\frac{n}{2}$ from some line of $\mathcal{S}$. A pair $(\mathcal{O}, \mathcal{S})$ such that every point of $\mathcal{O}$ is incident with a line of $\mathcal{S}$ is called an *ovoid-spread pairing* of $\mathcal{I}$.

2. Some preliminary results and facts

Since for $k \geq 2$ the (k, g)-cages are respectively the complete graph K_{k+1} or complete bipartite graph $K_{k,k}$ when g is 3 or 4, we have next result for the family of cages.

Theorem 2.1. *For any integer $k \geq 2$ and for $g = 3, 4$, if G is a (k, g)-cage, then*

$$\chi_d^t(G) = \begin{cases} k+1 & \text{if } g = 3, \\ 2 & \text{if } g = 4. \end{cases}$$

So, we consider (k, g)-Moore graphs with $g \geq 5$. Some properties of the (k, g)-cage graphs are given in the next fact.

Fact 2.2. *Let $c = (V_1, V_2, \cdots, V_\ell)$ be a minimal TDC of a (k, g)-cage graph G of order n with the vertex set V, girth $g \geq 5$ and independence number α in which $|V_1| \leq \cdots \leq |V_\ell|$. Also for $1 \leq i \leq \alpha$ let a_i be the number of the color classes which have cardinality i. Then the following facts hold.*

(1) $\sum_{i=1}^{\ell} |V_i| = |V|$ *implies* $\Sigma_{i=1}^{\alpha} i a_i = n$.

(2) *For some i, while $|V_i| = 1$ implies $|CN(V_i)| = k$, $|V_i| > k$ implies $|CN(V_i)| = 0$ (because G is k-regular). Also $2 \leq |V_i| \leq k$, for some i, implies $|CN(V_i)| = 0$ or 1 (because every two vertices of V_i with every two vertices of $CN(V_i)$ make a 4-cycle).*

(3) $V = \bigcup_{i=1}^{\ell} CN(V_i)$ *implies* $ka_1 + a_2 + \cdots + a_k \geq n$ *(by (1) and (2)).*

(4) $\gamma_t(G) \leq a_1 + \cdots + a_k \leq \ell$ *(the lower is obtained bound by the fact that by selecting one vertex from every color class V_i which $CN(V_i) \neq \emptyset$, we obtain a minimal TDS of the graph).*

(5) $\lceil \frac{n-\ell}{k-1} \rceil \leq a_1 \leq \min\{\ell, \lfloor \frac{\alpha\ell - n}{\alpha - 1} \rfloor\}$. *Because $n - a_1 = \sum_{|V_i| \geq 2} |V_i| \leq (\ell - a_1)\alpha$ (which is obtained by (1)) implies the upper bound, and*

$$\begin{array}{rcll} k\ell - n & \geq & k(a_1 + \cdots + a_k) - n & (by\ (4)) \\ & \geq & (k-1)(a_2 + \cdots + a_k) & \end{array}$$

implies the lower bound $a_1 \geq \frac{n-(a_2+\cdots+a_k)}{k} \geq \frac{n-\ell}{k-1}$.

We know from [2] that (k, g)-cages with even girth g (called generalized polygon graphs) exist if and only if $g = 4, 6, 8, 12$. When $g = 4$, the graph is the complete bipartite graph $K_{k,k}$ with the independence number k. Also from [7], we know that for a prime power q, while the existence of a $(q + 1, 6)$-cage is equivalent to the existence of an incidence $(q + 1)$-regular graph of a projective plane of order q and the existence of a $(q+1, 8)$-cage is equivalent to the existence of an incidence $(q+1)$-regular graph of a generalized quadrangle of order q, the existence of a $(q+1, 12)$-cage is equivalent to the existence of an incidence $(q+1)$-regular graph of a generalized 6-gon of order q. These cages have respectively the even orders $2(q^2 + q + 1)$, $2(q + 1)(q^2 + 1)$ and $2(q^3 + 1)(q^2 + q + 1)$. So, since the incidence graphs are bipartite [9], and the vertex set of a regular bipartite graph can be partitioned to two independent sets with same cardinality, we have next proposition that we will use it in many of our proofs.

Proposition 2.3. *For any prime power q and $g = 6, 8, 12$, the independence number of the $(q + 1, g)$-cage is the half of its order.*

3. The $(q + 1, 5)$-Moore graphs for $q = 2, 6$: The Petersen and Hoffman-Singleton graphs

We know the $(3, 5)$- and $(7, 5)$-Moore graphs G are known respectively as the *Petersen* and *Hoffman-Singleton graphs*, and their total domination numbers are found in [11]. Here, we find their total dominator chromatic numbers which are respectively $\gamma_t(G) + 2$ and $\gamma_t(G) + 4$.

Proposition 3.1. [11] *If G is a graph of order n, diameter 2 and girth 5, then $\gamma_t(G) = 1 + \sqrt{n-1}$.*

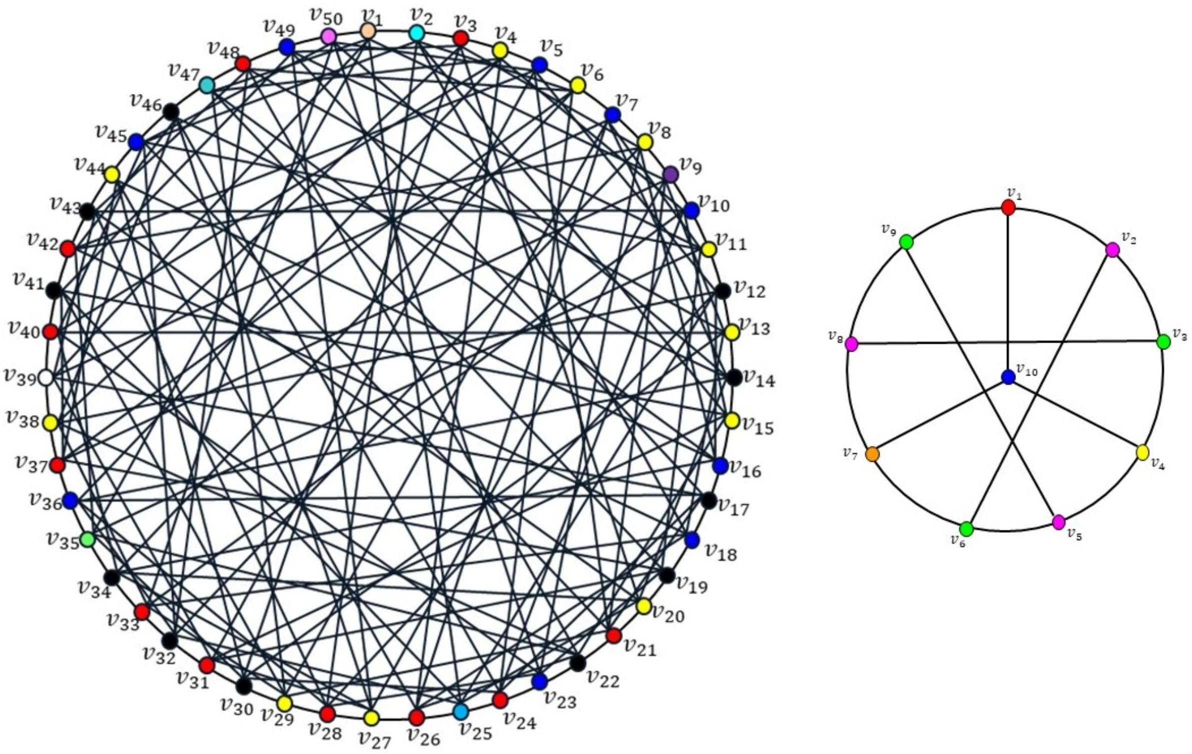

Figure 1: The Hoffman-Singleton graph (left) and the Petersen graph (right) with the min-TDCs given in Theorems 3.2 and 3.4.

Theorem 3.2. *For the Petersen graph G, $\chi_d^t(G) = 6$.*

Proof. Let G be the Petersen graph of order 10 shown in Figure 1 with $\gamma_t(G) = 4$ (by Proposition 3.1) and $\alpha(G) = 4$, and let $(V_1, V_2, \cdots, V_\ell)$ be a minimal TDC of G, in which $|V_1| \leq \cdots \leq |V_\ell|$ and $4 \leq \ell \leq 5$. Then, since at most one color class has size more than 3, we have either $\Sigma_{i=1}^3 a_i = \ell - 1 = 4$ and $|V_{\ell-1}| \leq 3 < |V_\ell| = 4$ or $\Sigma_{i=1}^3 a_i = \ell$ and $|V_\ell| \leq 3$. Since $\ell = 4$ leads us to the contradiction $3 \leq a_1 \leq 2$ by Fact 2.2 (5), we may assume $\ell = 5$, which implies $a_1 = 3$ by Fact 2.2 (5). Since also $\Sigma_{i=1}^3 a_i = 5$ implies $|V_5| \geq \frac{|V_4|+|V_5|}{2} = \frac{10-\Sigma_{i=1}^3|V_i|}{2} = \frac{7}{2} > 3$, which contradicts the fact $|V_\ell| \leq 3$, we may assume $\Sigma_{i=1}^3 a_i = 4$. Hence $(a_1, a_2, a_3) = (3, 1, 0), (3, 0, 1)$. Since $(a_1, a_2, a_3) = (3, 1, 0)$ implies the contradiction $|V_5| = 10 - \Sigma_{i=1}^4 |V_i| = 5 > \alpha$, we consider $(a_1, a_2, a_3) = (3, 0, 1)$. Since the neighborhood of every two non-adjacent vertices is not empty, we have $| \cup_{i=1}^{a_1} CN(V_i)| \leq 8$. By $CN(V_5) = \emptyset$, we must have $|CN(V_4)| \geq 2$, which is not possible (by $|V_4| = 3$ and Fact 2.2 (2)). Therefore $\chi_d^t(G) \geq 6$. As we have shown in Figure 1, since $(\{v_1\}, \{v_4\}, \{v_7\}, \{v_{10}\}, \{v_3, v_6, v_9\}, \{v_2, v_5, v_8\})$ is a TDC of G with 6 color classes, we have $\chi_d^t(G) = 6$. $\square$

Before finding the total dominator chromatic number of the Hoffman-Singleton graph, we need to recall a proposition.

Proposition 3.3. [9] *An independent set C in a Moore graph of diameter two and valency seven contains at most 15 vertices. If $|C| = 15$, then every vertex not in C has exactly three neighbours in C.*

Theorem 3.4. *For the Hoffman-Singleton graph G, $\chi_d^t(G) = 12$.*

Proof. Let G be the the Hoffman-Singleton graph of order 50 shown in Figure 1 which is the $(7, 5)$-Moore graph of diameter 2 with $\alpha(G) = 15$ (by Proposition 3.3). The set $S = \{v_1, v_2, v_9, v_{25}, v_{35}, v_{39}, v_{47}, v_{50}\}$ is a min-TDS of G by Proposition 3.1. Let $W = \{v_5, v_7, v_{10}, v_{16}, v_{18}, v_{23}, v_{36}, v_{45}, v_{49}\}$, $T = \{v_4, v_6, v_8, v_{11}, v_{13}, v_{15}, v_{20}, v_{27}, v_{29},$

$v_{38}, v_{44}\}$, $R = \{v_{12}, v_{14}, v_{17}, v_{19}, v_{22}, v_{30}, v_{32}, v_{34}, v_{41}, v_{43}, v_{46}\}$ and $Q = \{v_3, v_{21}, v_{24}, v_{26}, v_{28}, v_{31}, v_{33}, v_{37}, v_{40}, v_{42}, v_{48}\}$ be respectively the set of the blue, yellow, black and red vertices of the induced subgraph $G - S$ of G. $V(G - S) = W \cup T \cup R \cup Q$ is a partition to the independent sets. Since $(\{v_1\}, \{v_2\}, \{v_9\}, \{v_{25}\}, \{v_{35}\}, \{v_{39}\}, \{v_{47}\}, \{v_{50}\}, W, T, R, Q)$ is a TDC of G, we have $\chi_d^t(G) \leq 12$.

Now let $c = (V_1, V_2, \cdots, V_\ell)$ be a minimal TDC of G in which $8 \leq \ell \leq 11$. Since $a_1 \geq 7$ by Fact 2.2 (2),(3), we have $10 \leq \ell \leq 11$, which implies $7 \leq a_1 \leq 8$. By assumption $J = \{\, i \mid CN(V_i) \neq \emptyset\}$, let S be the TDS of G which is obtained by selecting only one vertex of every color class V_i for each $i \in J$. Then $\{\, i \mid |V_i| = 1\} \subseteq J$ and $|V_i| \leq 7$ for every $i \in J$ (by the minimality of c), and so $|J| = |S| \geq \max\{\gamma_t(G), a_1\} = 8$. Since

$$\begin{aligned} 50 - a_1 &= \sum_{i \in J,\ |V_i| \neq 1} |V_i| + \sum_{i \notin J} |V_i| \qquad (\textit{by Fact } 2.2\ (1)) \\ &\leq 7(|J| - a_1) + 15(\ell - a_1 - |J|) \end{aligned}$$

implies the contradiction $a_1 \leq \lfloor \frac{15\ell - 50 - 8|J|}{21} \rfloor \leq 2$, we have $\chi_d^t(G) = 12$. □

4. The $(q + 1, 6)$-Moore graphs when q is a prime power

Here, for any prime power q, we prove the total dominator chromatic number of the $(q + 1, 6)$-Moore graphs G is equale to $\gamma_t(G) + 2$. First we recall a lemma from [8], and find total domination numbers of these graphs.

Lemma 4.1. [8] *Let G be a $(q + 1, g)$-Moore graph of order n with vertex partition $(\mathcal{P}, \mathcal{L})$ when $g = 6, 8, 12$. Then the only maximum independent set of G is $\mathcal{P}$ or $\mathcal{L}$.*

Theorem 4.2. *For any prime power q, if G is a $(q + 1, 6)$-Moore graph, then $\gamma_t(G) = 2(q + 1)$.*

Proof. For any prime power q, let G be a $(q + 1, 6)$-Moore graph which has order $2(q^2 + q + 1)$. Since the girth of G is 6, G has a spanning tree of the form given in Figure 2, and so the set $\{v_i, u_i \mid 1 \leq i \leq q + 1\}$ is a TDS of the spanning tree, and then of G, which implies $\gamma_t(G) \leq 2(q + 1)$. On the other hand, since G is a bipartite $(q + 1)$-regular graph which its vertex set is partitioned to two independent sets with the same cardinality $q^2 + q + 1$, at least $q + 1$ vertices from every independent set are needed to totaly dominate the vertices of the other one (because $(q + 1)q < q^2 + q + 1$). Therefore, $\gamma_t(G) \geq 2(q + 1)$, and so $\gamma_t(G) = 2(q + 1)$. □

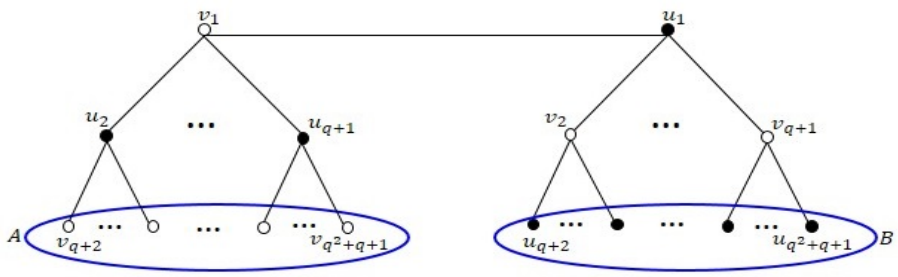

Figure 2: A spanning tree of a $(q + 1, 6)$-Moore graph. Notice by $g \nleq 5$, the sets $A = \{v_{q+2}, \cdots, v_{q^2+q+1}\}$ and $B = \{u_{q+2}, \cdots, u_{q^2+q+1}\}$ are independent sets with the same cardinality q^2 in the Moore graph.

Theorem 4.3. *For any prime power q, if G is a $(q + 1, 6)$-Moore graph, then $\chi_d^t(G) = 2q + 4$.*

Proof. For any prime power $q \geq 2$, let $(V_1, V_2, \cdots, V_\ell)$ be a minimal TDC of the $(q + 1, 6)$-Moore graph G of order $2(q^2 + q + 1)$ with $\gamma_t(G) = 2q + 2$ (by Theorem 4.2) and $\alpha(G) = q^2 + q + 1$ (by Proposition 2.3) in which $|V_1| \leq \cdots \leq |V_\ell|$ and $2q + 2 \leq \ell \leq 2q + 3$ by (1) and the contrary. Then either $\Sigma_{i=1}^{q+1} a_i = \ell - 1 = 2q + 2$ and

$|V_{\ell-1}| \leq q+1 < |V_\ell| \leq q^2+q+1$ or $\Sigma_{i=1}^{q+1} a_i = \ell$ and $|V_\ell| \leq q+1$. Notice $\ell = 2q+2$ implies $a_1 = 2q$ and $\ell = 2q+3$ implies $2q \leq a_1 \leq 2q+1$. Since by assumption $\Sigma_{i=1}^{q+1} a_i = \ell$, we have $|V_\ell| \geq \frac{\Sigma_{i=a_1+1}^{\ell}|V_i|}{\ell-a_1} = \frac{2(q^2+q+1)-\Sigma_{i=1}^{a_1}|V_i|}{\ell-a_1} > q+1$ which contradicts the fact $|V_\ell| \leq q+1$, we may assume $\Sigma_{i=1}^{q+1} a_i = 2q+2 = \ell-1$. Since for $q \geq 4$, this leads us to the contradiction $|V_\ell| = 2(q^2+q+1) - \Sigma_{i=1}^{\ell-1}|V_i| > q^2+q+1$, we continue our proof in the following two remained cases.

Case 1. $q = 2$ and $\Sigma_{i=1}^{3} a_i = 6 = \ell - 1$. Then $G = G[\mathcal{P}, \mathcal{L}]$ is the *Heawood graph* shown in Figure 3 of order 14 with $\gamma_t(G) = 6$ and $\alpha(G) = 7$, and $(a_1, a_2, a_3) = (5,1,0), (5,0,1), (4,2,0), (4,1,1), (4,0,2)$. If $(a_1, a_2, a_3) = (5,1,0)$, then $|V_7| = 7$ and so $V_7 = \mathcal{P}$ or $\mathcal{L}$ (by Lemma 4.1), which implies every vertex of $\mathcal{P} \cup \mathcal{L} - V_7$ is dominated by no color class. Therefore $(a_1, a_2, a_3) = (5,0,1)$ or $(4,2,0)$, which imply $|V_7| = 6$. By considering these facts that each of the sets $\mathcal{P}$ and $\mathcal{L}$ are independent and the graph is 3-regular and every two vertices of the graph have at most one neighbor in common, again we will have $V_7 \subseteq \mathcal{P}$ or $\mathcal{L}$, and so every vertex of $\mathcal{P} \cup \mathcal{L} - V_7$ is dominated by no color class, a contradiction. Thefore we discusse on the following remained two subcases.

1.1. $(a_1, a_2, a_3) = (4,1,1)$. Then $|V_7| = 5$. Let $V_7 \subseteq \mathcal{P}$ or $\mathcal{L}$. Since G is vertex-transitive, we may assume $V_7 \subseteq \mathcal{P}$. Again by considering these facts that each of the sets $\mathcal{P}$ and $\mathcal{L}$ are independent sets with cardinality 7 and the graph is 3-regular and every two vertices of the graph have at most one neighbor in common, we conclude that there is a vertex in $\mathcal{L}$ which is dominated by no color class, a contradiction. Hence $V_7 \cap \mathcal{P} \neq \emptyset$ and $V_7 \cap \mathcal{L} \neq \emptyset$. Since G is vertex-transitive, we may assume $|V_7 \cap \mathcal{P}| \leq |V_7 \cap \mathcal{L}|$. Then $|V_7 \cap \mathcal{P}| = 1$ and $|V_7 \cap \mathcal{L}| = 4$ (by the 3-regularity of G), and there exisits some vertex which is not dominated by any color class (by $(a_1, a_2, a_3) = (4,1,1)$).

1.2. $(a_1, a_2, a_3) = (4,0,2)$. Then $|V_i| = 1$ for $1 \leq i \leq 4$, $|V_5| = |V_6| = 3$, $|V_7| = 4$. By Fact 2.2 (2),(3), we have $|CN(V_i)| = 1$ for $i = 5,6$. Similar to the cases $(a_1, a_2, a_3) = (4,1,1)$, we may assume either $|V_7 \cap \mathcal{P}| = 1$ and $|V_7 \cap \mathcal{L}| = 3$ or $|V_7 \cap \mathcal{P}| = |V_7 \cap \mathcal{L}| = 2$. In the first case, since for $i = 5,6$, $|CN(V_i)| = 1$, we have $V_i \subseteq \mathcal{P}$ or $V_i \subseteq \mathcal{L}$, and $V_5 \cup V_6 \nsubseteq \mathcal{L}$, and it can be easily verified that some vertex of $\mathcal{P} \cup \mathcal{L}$ is dominated by no color class, a contradiction. So we assume the second case is happened, and $V_1, V_2, V_5 \subseteq \mathcal{P}$ and $V_3, V_4, V_6 \subseteq \mathcal{L}$. Let $V_7 = \{p_1, p_2, \ell_1, \ell_2\}$ for some $p_1, p_2 \in \mathcal{P}$ and some $\ell_1, \ell_2 \in \mathcal{L}$. Hence $N_G(p_1) \cup N_G(p_2) = \mathcal{L} - \{\ell_1, \ell_2\}$ and $N_G(\ell_1) \cup N_G(\ell_2) = \mathcal{P} - \{p_1, p_2\}$ (notice $|N_G(p_1) \cap N_G(p_2)| = |N_G(\ell_1) \cap N_G(\ell_2)| = 1$ by $g(G) = 6$). Let $CN(V_5) = \{\ell\}$ for some $\ell \in \mathcal{L}$. By $deg_G(\ell) = 3$, we have $\ell \in \{p_1, p_2\}$. Let $\ell = p_1$. Then $N_G(p_2) \subseteq \mathcal{L} - \{\ell_1, \ell_2\}$ implies $N_G(p_2) \cap V_5 \neq \emptyset$. Since $|N_G(p_2) \cap V_5| \geq 2$ implies the contradiction $g(G) \leq 4$, we have $|N_G(p_2) \cap V_5| = 1$ and $N_G(p_2) = \{\ell_0\} \cup V_1 \cup V_2$ for some $\ell_0 \in V_5$. Since only one vertex of $\mathcal{P}$ is dominated by V_5, the other six vertices of $\mathcal{P}$ must be dominated by V_1 or V_2, and this happens when $|N(V_1) \cup N(V_2)| = 6$. But this is impossible because of $p_2 \in N(V_1) \cap N(V_2)$.

Case 2. $q = 3$ and $\Sigma_{i=1}^{4} a_i = 8 = \ell - 1$. Then $G = G[\mathcal{P}, \mathcal{L}]$ is the graph shown in Figure 3 of order 26 with $\gamma_t(G) = 8$ and $\alpha(G) = 13$, and $(a_1, a_2, a_3, a_4) = (7,1,0,0), (7,0,1,0), (7,0,0,1), (6,1,1,0), (6,1,0,1), (6,2,0,0), (6,0,2,0), (6,0,0,2), (6,0,1,1)$. The first seven cases imply the contradiction $|V_9| = 26 - \Sigma_{i=1}^{8}|V_i| > \alpha(G)$. The case $(a_1, a_2, a_3, a_4) = (6,0,1,1)$ implies $|V_9| = 13$, and so $V_9 = \mathcal{P}$ or $\mathcal{L}$ by Lemma 4.1. Hence every vertex of $\mathcal{P} \cup \mathcal{L} - V_9$ is dominated by no color class, a contradiction. Finally let $(a_1, a_2, a_3, a_4) = (6,0,0,2)$, which implies $|V_9| = 12$. By considering these facts that the sets $\mathcal{P}$ and $\mathcal{L}$ are independent and the graph is 4-regular and every two vertices of the graph have at most one neighbor in common, again we have $V_9 \subseteq \mathcal{P}$ or $\mathcal{L}$, and so every vertex of $\mathcal{P} \cup \mathcal{L} - V_9$ is dominated by no color class, a contradiction.

Therefore, $\chi_d^t(G) \geq 2q+4$ for any prime power q, and by considering Figure 2, since $(\{v_1\}, ..., \{v_{q+1}\}, \{u_1\}, ..., \{u_{q+1}\}, A, B)$ is a TDC of G with $2q+4$ color classes, we have $\chi_d^t(G) = 2q+4$. □

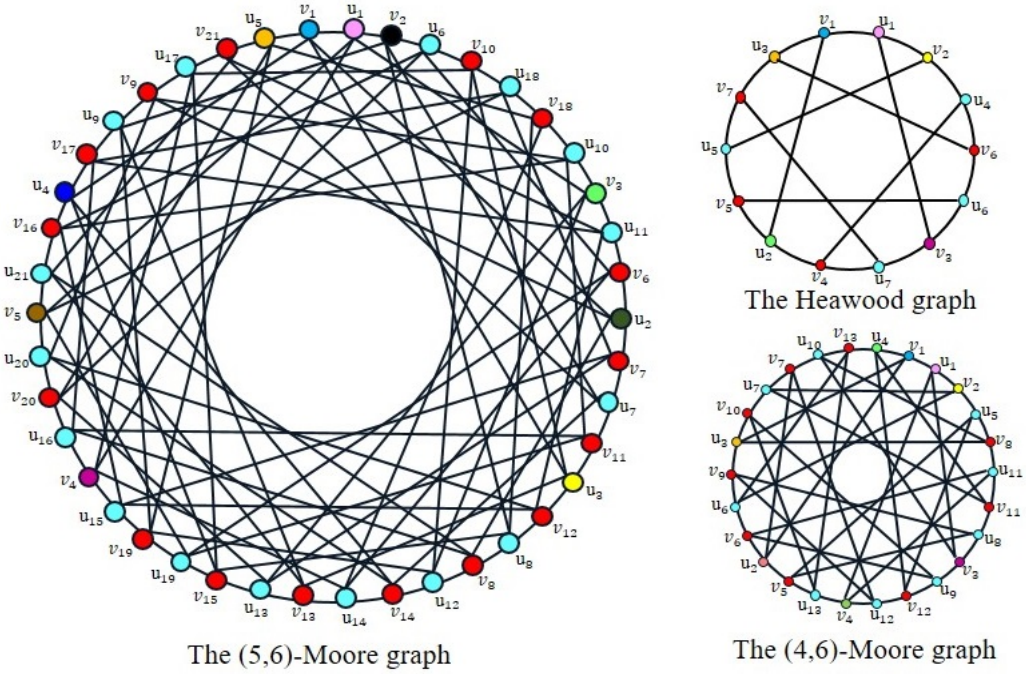

Figure 3: The $(q+1,6)$–Moore graphs with the min-TDCs given in Theorem 4.3 when $q = 2, 3, 4$.

5. The $(q+1,8)$-Moore graphs when q is a prime power

An alternate description of the known $(q+1,8)$-Moore graphs with a prime power $q \geq 2$ and the incidence $(q+1)$-regular graphs $G_q[\mathcal{P},\mathcal{L}]$ of order $2(q+1)(q^2+1)$ as follows.

Definition 5.1. [1] *For any prime power $q \geq 2$, let $\mathbb{F}_q$ be a finite field and ϱ a symbol not belonging to $\mathbb{F}_q$. Let $Gq = Gq[\mathcal{P},\mathcal{L}]$ be a bipartite graph with vertex sets*

$$\mathcal{P} = \{(a,b,c)_0, (\varrho,b,c)_0, (\varrho,\varrho,c)_0 : a,b,c \in \mathbb{F}_q\} \cup \{(\varrho,\varrho,\varrho)_0\},$$

$$\mathcal{L} = \{(a,b,c)_1, (\varrho,b,c)_1, (\varrho,\varrho,c)_1 : a,b,c \in \mathbb{F}_q\} \cup \{(\varrho,\varrho,\varrho)_1\},$$

and edge set defined as follows:
For all $a \in \mathbb{F}_q \cup \{\varrho\}$ and for all $b,c \in \mathbb{F}_q$,

$$N_{G_q}((a,b,c)_1) = \begin{cases} \{(w, aw+b, a^2w+2ab+c)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho,a,c)_0\} & \text{if } a \in \mathbb{F}_q \\ \{(c,b,w)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho,\varrho,c)_0\} & \text{if } a = \varrho \end{cases},$$

$N_{G_q}((\varrho,\varrho,c)_1) = \{(\varrho,c,w)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho,\varrho,\varrho)_0\}$,
$N_{G_q}((\varrho,\varrho,\varrho)_1) = \{(\varrho,\varrho,w)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho,\varrho,\varrho)_0\}$.
Or equivalently, for all $i \in \mathbb{F}_q \cup \{\varrho\}$ and for all $j,k \in \mathbb{F}_q$:

$$N_{G_q}((i,j,k)_0) = \begin{cases} \{(w, j-wi, w^2i-2wj+k)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho,j,i)_1\} & \text{if } i \in \mathbb{F}_q \\ \{(j,w,k)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho,\varrho,j)_1\} & \text{if } i = \varrho \end{cases},$$

$N_{G_q}((\varrho,\varrho,k)_1) = \{(\varrho,w,k)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho,\varrho,\varrho)_0\}$,
$N_{G_q}((\varrho,\varrho,\varrho)_0) = \{(\varrho,\varrho,w)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho,\varrho,\varrho)_1\}$.

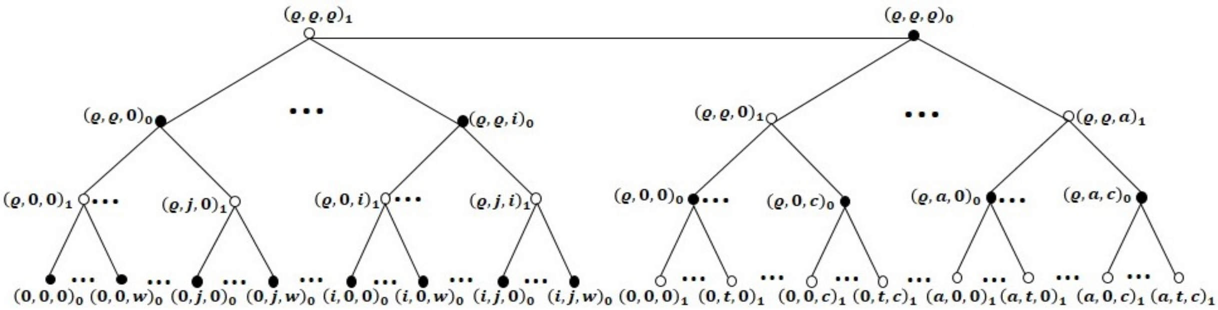

Figure 4: A spanning tree of the graph $G_q[\mathcal{P},\mathcal{L}]$. The filled vertices denote $\mathcal{P}$ and the empty vertices denote $\mathcal{L}$.

The $(q+1,8)$-Moore graphs have been constructed as the incidence graphs of generalized quadrangles $Q(4,q)$ and $W(q)$, which are known to exist for any prime power q and no example is known when $k-1$ is not a prime power. Since they are incidence graphs, these graphs are bipartite and have diameter 4. Recall that if q is even, $Q(4,q)$ is isomorphic to the dual of $W(q)$ and viceversa. Hence, the corresponding $(q+1,8)$-Moore graphs are isomorphic. The following proposition is provides an overview of results about the existence question of ovoids and spreads in known generalized quadrangles.

Proposition 5.2. [18] *For any q, the following two statements are hold.*

(1) *The generalized quadrangle $Q(4,q)$ always has ovoids. It has spreads if and only if q is even.*

(2) *The generalized quadrangle $W(q)$ always has spreads. It has ovoids if and only if q is even.*

Since $(q+1,8)$-Moore graphs have been constructed as the incidence graphs of generalized quadrangles $Q(4,q)$ and $W(q)$, and every ovoid or spread of a generalized quadrangle of order q if there exist, has cardinality q^2+1 [18], the following proposition is a result of Proposition 5.2.

Proposition 5.3. *For any even prime power q, every $(q+1,8)$-Moore graph has an ovoid and a spread of size q^2+1, while for any odd prime power q, every $(q+1,8)$-Moore graph has only one of them.*

5.1. When q is an even prime power

Here, we prove that the total dominator chromatic number of a $(q+1,8)$-Moore graph G is equal to $\gamma_t(G)+2$. First, we find their total domination numbers.

Theorem 5.4. *For any even prime power q, if G is a $(q+1,8)$-Moore graph, then $\gamma_t(G)=2(q^2+1)$.*

Proof. For any even prime power q, let G be the $(q+1,8)$-Moore graph $G_q[\mathcal{P},\mathcal{L}]$. Since G is a $(q+1)$-regular bipartite graph with the disjoint independent sets $\mathcal{P}$ and $\mathcal{L}$ with equal cardinality, at least q^2+1 vertices from each of them are needed to dominate the vertices of the other one. Also since q is even, G has an ovoid $\mathcal{O}\subset\mathcal{P}$ and a spread $\mathcal{S}\subset\mathcal{L}$ with the same cardinality q^2+1, by Propositions 5.2 and 5.3. Hence every vertex of $\mathcal{L}$ is dominated by a vertex of $\mathcal{O}$. Since also every vertex of $\mathcal{P}$ is dominated by a vertex of $\mathcal{S}$, $\mathcal{O}\cup\mathcal{S}$ is a TDS of G, and so $\gamma_t(G)=2(q^2+1)$. Figure 5 shows the $(3,8)$-Moore graph of order 30 which is known as *Tutte-Coxter graph* with the TDS $\mathcal{O}\cup\mathcal{S}$ in which $\mathcal{S}=\{(0,0,0)_1,(0,0,1)_1,(\varrho,\varrho,1)_1,(\varrho,1,0)_1,(\varrho,1,1)_1\}$ and $\mathcal{O}=\{(0,0,0)_0,(0,1,0)_0,(\varrho,\varrho,1)_0,(\varrho,0,1)_0,(\varrho,1,1)_0\}$. □

Theorem 5.5. *For any even prime power q, if G is a $(q+1,8)$-Moore graph, then $\chi_d^t(G)=2q^2+4$.*

Proof. For any even prime power q, let $(V_1,V_2,\cdots,V_\ell)$ be a minimal TDC of the $(q+1,8)$-Moore graph $G=G_q[\mathcal{P},\mathcal{L}]$ with $\gamma_t(G)=2q^2+2$ (by Theorem 5.4) and $\alpha(G)=(q^2+1)(q+1)$ (by Proposition 2.3). Since $\ell\le 2q^2+3$ leads us to the contradiction $\min\{\ell,\lfloor\frac{\alpha(G)\ell-n}{\alpha(G)-1}\rfloor\}<\lceil\frac{n-\ell}{k-1}\rceil$ by Fact 2.2 (5), we have $\ell\ge 2q^2+4$. Let $\mathcal{O}=\{v_1,...,v_{q^2+1}\}$ and $\mathcal{S}=\{u_1,...,u_{q^2+1}\}$ be respectively an ovoid and a spread of G which there exist by Proposition 5.3. Since $(\{v_1\},...,\{v_{q^2+1}\},\{u_1\},...,\{u_{q^2+1}\},\mathcal{P}-\mathcal{O},\mathcal{L}-\mathcal{S})$ is a TDC of G with $2q^2+4$ color classes, we have $\chi_d^t(G)=2q^2+4$. □

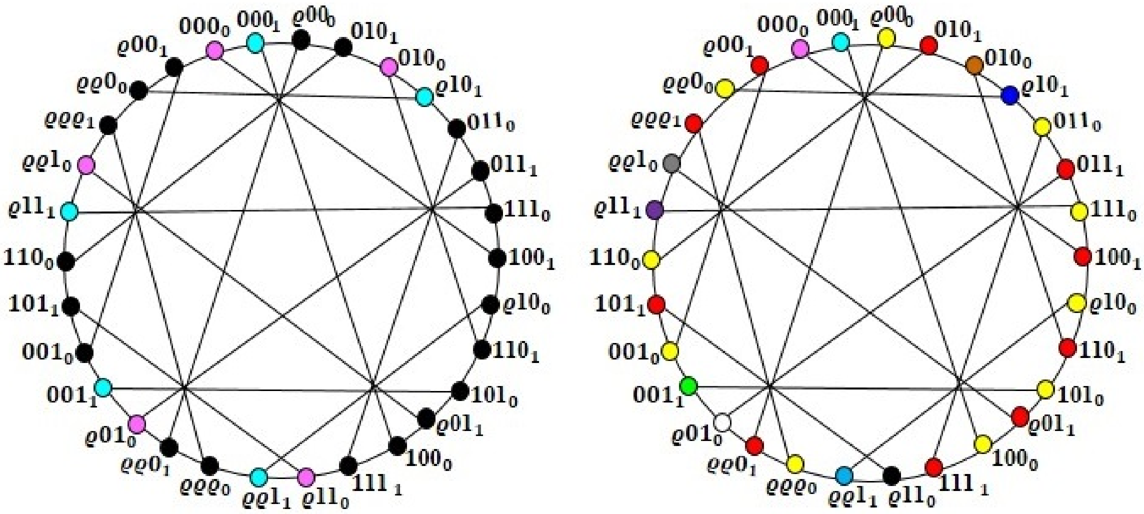

Figure 5: The $(3, 8)$-Moore graph with the min-TDS $\mathcal{O} \cup \mathcal{S}$ given in Theorem 5.4 (left) and the min-TDC given in Theorem 5.5 (right) (notice abc_d denotes vertex $(a, b, c)_d$).

5.2. *When q is an odd prime power*

Next theorem shows that total domination number of a $(q + 1, 8)$-Moore graph is restricted by some bounds when q is an odd prime power.

Theorem 5.6. *For any odd prime power q, if G is a $(q + 1, 8)$-Moore graph, then $2q^2 + 3 \leq \gamma_t(G) \leq 2q^2 + q + 1$, and the lower bound is tight when $q = 3$.*

Proof. For any odd prime power q, let G be the $(q+1, 8)$-Moore graph $G_q[\mathcal{P}, \mathcal{L}]$. Here, we can use the proof of Theorem 5.4 with this minor revision that G has only an ovoid $\mathcal{O} \subset \mathcal{P}$ or a spread $\mathcal{S} \subset \mathcal{L}$ with the cardinality $q^2 + 1$ not both of them. So, the $q^2 + 1$ vertices of the ovoid or spread dominate the vertices of one of the sets $\mathcal{P}$ or $\mathcal{L}$, and at least $q^2 + 2$ vertices are needed for dominating the vertices of the other one, which implies $\gamma_t(G) \geq 2q^2 + 3$. By Proposition 5.3, we may assume that G has an ovoid $\mathcal{O}$ which every vertex of $\mathcal{L}$ is adjacent to some vertex in $\mathcal{O}$. Since the tree in Figure 4 is a spanning tree of $G = G_q[\mathcal{P}, \mathcal{L}]$, every vertex of $\mathcal{P}$ is adjacent to some vertex in the set $T = \{(\varrho, \varrho, a)_1, (\varrho, a, b)_1 : a, b \in \mathbb{F}_q\}$. Hence $\mathcal{O} \cup T$ is a TDS of G with cardinality $2q^2 + q + 1$, and so $\gamma_t(G) \leq 2q^2 + q + 1$.

The lower bound $2q^2 + 3$ is tight for $q = 3$. Because, in the $(4, 8)$-Moore graph G of order 80 shown in Figure 6, for the ovoid $\mathcal{O} = \{(\varrho, \varrho, \varrho)_0, (0, a, 0)_0, (1, a, 2)_0, (2, a, 1)_0 : a \in \mathbb{F}_3\}$ of G and the set

$$\mathcal{K} = \{(0,0,0)_1, (0,0,1)_1, (0,0,2)_1, (\varrho,\varrho,1)_1, (\varrho,\varrho,2)_1, (\varrho,1,0)_1, (\varrho,1,1)_1, (\varrho,1,2)_1, (\varrho,2,0)_1, (\varrho,2,1)_1, (\varrho,2,2)_1\},$$

$\mathcal{O} \cup \mathcal{K}$ is a TDS of G with the cardinality $2q^2 + 3$. □

Now we find the total dominator chromatic numbers of $(q + 1, 8)$-Moore graphs in terms of their total domination numbers. For this aim, we need to recall this fact from [14] that for any graph G,

$$\gamma_t(G) \leq \chi_d^t(G) \leq \gamma_t(G) + \chi(G). \tag{2}$$

Theorem 5.7. *For any odd prime power q if G is a $(q + 1, 8)$-Moore graph, then $\chi_d^t(G) = \gamma_t(G) + 2$.*

Proof. For any odd prime power q, let $(V_1, V_2, \cdots, V_\ell)$ be a minimal TDC of a $(q + 1, 8)$-Moore graph G with the independence number $\alpha = (q^2 + 1)(q + 1)$ (by Proposition 2.3) in which $|V_1| \leq |V_2| \leq \cdots \leq |V_\ell|$ and $\gamma_t(G) \leq \ell \leq \gamma_t(G) + 1 \leq 2q^2 + q + 2$ (by (1) and Theorem 5.6). Then, since the number of color classes with cardinality more than $q + 1$ is at most one (by Fact 2.2 (2),(4)), either $\Sigma_{i=1}^{q+1} a_i = \ell - 1 = \gamma_t(G)$ and so $|V_{\ell-1}| \leq q + 1 < |V_\ell| \leq \alpha$ or $\Sigma_{i=1}^{q+1} a_i = \ell$ and so $|V_\ell| \leq q + 1$. We continue our proof in the following cases.

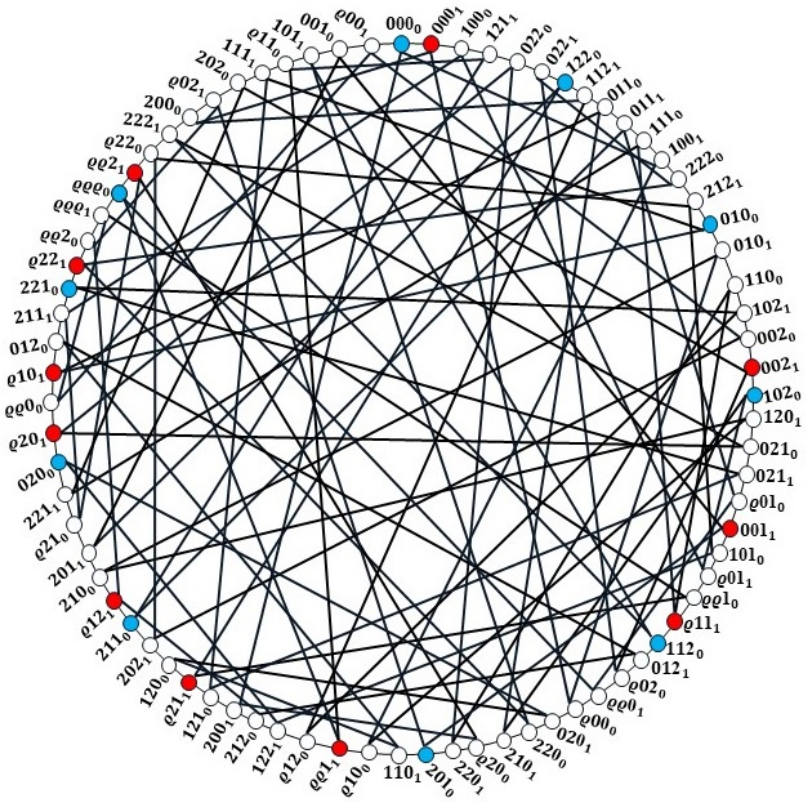

Figure 6: The (4, 8)-Moore graph with a min-TDS $\mathcal{O} \cup \mathcal{K}$, where the ovoid $\mathcal{O}$ and the set $\mathcal{K}$ in Theorem 5.6 are shown respectively by the blue and red vertices. The Hamiltonian cycle is computed using the backtracking function in the Python (notice abc_d denotes vertex $(a, b, c)_d$).

Case 1. $\ell = \gamma_t(G)$. Then $2q^2 + 2 \leq a_1 \leq 2q^2 + q - 1$ (by Fact 2.2 (4),(5) and $\ell \leq 2q^2 + q + 1$, which implies $\ell - a_1 \leq q - 1$. But this leads us to the contradiction

$$\begin{array}{rcll} 2(q^2+1)(q+1) & = & \sum_{i=1}^{\ell} |V_i| & \text{(by Fact 2.2 (1))} \\ & = & a_1 + \sum_{i=a_1+1}^{\ell} |V_i| & \\ & \leq & (2q^2+q-1) + (\ell - a_1)|V_\ell| & \\ & \leq & (2q^2+q-1) + (q-1)(q+1) & \\ & = & 3q^2 + q - 2. & \end{array}$$

Case 2. $\ell = \gamma_t(G) + 1 = 1 + \Sigma_{i=1}^{q+1} a_i$. Then $\Sigma_{i=1}^{q+1} a_i \leq 2q^2 + q + 1$ and $2q^2 + 1 \leq a_1 \leq 2q^2 + q$ (by Fact 2.2 (5)), and so $\ell - a_1 \leq q + 1$. But this leads us to the contradiction

$$\begin{array}{rcll} |V_\ell| & = & 2(q^2+1)(q+1) - a_1 - \Sigma_{i=a_1+1}^{\ell-1} |V_i| & \text{(by Fact 2.2 (1))} \\ & \geq & 2(q^2+1)(q+1) - a_1 - (\ell - a_1 - 1)|V_{\ell-1}| & \\ & \geq & 2(q^2+1)(q+1) - (2q^2+q) - q(q+1) & \\ & = & 2q^3 - q^2 + 2 & \\ & > & \alpha. & \end{array}$$

Case 3. $\ell = \gamma_t(G) + 1 = \Sigma_{i=1}^{q+1} a_i$. Then $\Sigma_{i=1}^{q+1} a_i \leq 2q^2 + q + 2$ and $2q^2 + 1 \leq a_1 \leq 2q^2 + q$, and so $\ell - a_1 \leq q + 1$.

But this leads us to the contradiction

$$\begin{array}{rcl} 2(q^2+1)(q+1) & = & \sum_{i=1}^{\ell} |V_i| \\ & = & a_1 + \sum_{i=a_1+1}^{\ell} |V_i| \\ & \leq & (2q^2+q) + (\ell - a_1)|V_\ell| \\ & \leq & (2q^2+q) + (q+1)(q+1) \\ & = & 3q(q+1)+1. \end{array}$$

Therefore $\chi_d^t(G) \geq \gamma_t(G) + 2$, and since G is bipartite, (2) implies $\chi_d^t(G) = \gamma_t(G) + 2$. $\square$

Next theorem is a result of Theorems 5.6 and 5.7.

Theorem 5.8. *For any odd prime power q, if G is a $(q+1,8)$-Moore graph, then $2q^2+5 \leq \chi_d^t(G) \leq 2q^2+q+3$, and the lower bound is tight for $q = 3$.*

6. The $(q+1, 12)$-Moore graphs for $q = 2, 3$

We know from [7] the incidence graph of a generalized 6-gon of order q is a $(q+1, 12)$-Moore graph of order $2(q+1)(q^4+q^2+1)$ whenever q is a prime power. In this section, we find first total domination numbers of the $(3,12)$-Moore graph (shown in Figure 7) and the $(4,12)$-Moore graph, which are respectively the incidence graphs of the generalized 6-gons of orders 2 and 3, and then their total dominator chromatic numbers. For this aim, we need to recall the definition of a *distance-2 ovoid* of a generalized $2n$-gon $\mathcal{I} = (\mathcal{P}, \mathcal{L}, I)$ from [8], which is a subset of the point set $\mathcal{P}$ with the property that every line of $\mathcal{L}$ contains exactly one point of the subset. Dual notion of that is a *distance-2 spread*.

Theorem 6.1. *For $q = 2, 3$, if G is the $(q+1$,12)-Moore graph, then*

$$\gamma_t(G) = \begin{cases} 43 & \text{if } q = 2, \\ 182 & \text{if } q = 3. \end{cases}$$

Proof. Let $G = G[\mathcal{P}, \mathcal{L}]$ be the $(q+1, 12)$-Moore graph when $q = 2, 3$. We continue our proof when G is one of the following graphs.

Case 1. *The (3,12)-Moore graph.* Then $G = G[\mathcal{P}, \mathcal{L}]$, shown in Figure 7 and Known as *Benson graph*, has order 126 and is the incidence graph of the generalized 6-gon of order 2. Let T be a TDS of G. Since G is 3-regular, we have $|T \cap \mathcal{P}| \geq \frac{|\mathcal{L}|}{3} = 21$ and $|T \cap \mathcal{L}| \geq \frac{|\mathcal{P}|}{3} = 21$. By assumption $|T \cap \mathcal{L}| = 21$, every vertex of $\mathcal{P}$ has exactly one neighbor in $T \cap \mathcal{L}$, and this means that $T \cap \mathcal{L}$ is a distance-2 spread of G, which contracts this fact from [5] that there is no distance-2 spread of G. Hence $|T \cap \mathcal{L}| \geq 22$. Now for the sets

$$\mathcal{O} = \{v_1, v_5, v_9, v_{13}, v_{17}, v_{21}, v_{25}, v_{29}, v_{33}, v_{51}, v_{55}, v_{61}, v_{65}, v_{75}, v_{81}, v_{85}, v_{89}, v_{93}, v_{97}, v_{105}, v_{113}\}, \text{ and}$$

$$\mathcal{S} = \{v_4, v_{12}, v_{18}, v_{22}, v_{28}, v_{32}, v_{36}, v_{44}, v_{60}, v_{66}, v_{70}, v_{74}, v_{82}, v_{84}, v_{90}, v_{94}, v_{98}, v_{104}, v_{108}, v_{114}, v_{118}, v_{124}\}$$

in Figure 7, since $\mathcal{O} \cup \mathcal{S}$ is a TDS of G, we have $\gamma_t(G) = 43$.

Case 2. *The (4,12)-Moore graph.* Then G is the $(4, 12)$-Moore graph of order 728 which is known as the incidence graph of the generalized 6-gon of order 3. Since there exist some distance-2 ovoids and distance-2 spreads of G of the same size 91 by [5], their union will be a TDS of G with the minimum cardinality 182, and so $\gamma_t(G) \leq 182$. Now by $\gamma_t(G) \geq \lceil \frac{728}{4} \rceil = 182$ we conclude $\gamma_t(G) = 182$.

$\square$

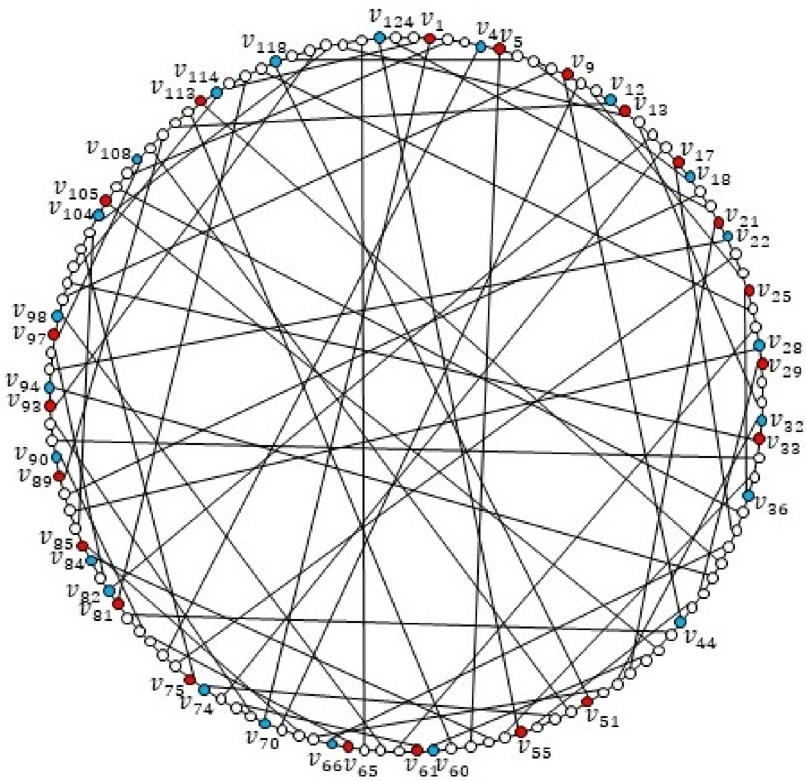

Figure 7: The Benson graph with the min-TDS $\mathcal{O} \cup \mathcal{S}$ given in Theorem 6.1 (the distance-2 ovoid $\mathcal{O}$ is shown by the blue vertices, and $\mathcal{S}$ is shown by the red vertices).

Theorem 6.2. *For $q = 2, 3$, if G is the $(q + 1, 12)$-Moore graph, then*

$$\chi_d^t(G) = \begin{cases} 45 & \text{if } q = 2, \\ 184 & \text{if } q = 3. \end{cases}$$

Proof. For $q = 2, 3$, let $(V_1, V_2, \cdots, V_\ell)$ be a minimal TDC of the $(q + 1, 12)$-Moore graph G, in which $|V_1| \le \cdots \le |V_\ell|$ and $\gamma_t(G) \le \ell \le \gamma_t(G) + 1$ by (1) and the contrary. Then either $\Sigma_{i=1}^{q+1} a_i = \ell - 1 = \gamma_t(G)$ and $|V_{\ell-1}| \le q + 1 < |V_\ell| \le \alpha(G)$ or $\Sigma_{i=1}^{q+1} a_i = \ell$ and $|V_\ell| \le q + 1$. We continue our proof when G is one of the following graphs.

Case 1. *The* $(3, 12)$-*Moore graph*. Then G is the Benson graph of order 126 shown in Figure 7 with $\gamma_t(G) = 43$ (by Theorem 6.1) and $\alpha(G) = 63$ (by Proposition 2.3). Since $\ell = 43$ leads us to the contradiction $\min\{\ell, \lfloor \frac{\alpha(G)\ell - n}{\alpha(G)-1} \rfloor\} < \lceil \frac{n-\ell}{q} \rceil$ by Fact 2.2 (5), we may assume $\ell = 44$, and so $41 \le a_1 \le 42$ by Fact 2.2 (5). Since by assumption $\Sigma_{i=1}^{3} a_i = 44$, we reach to the contradiction $|V_{44}| \ge \frac{\Sigma_{a_1+1}^{44} |V_i|}{44 - a_1} = \frac{126 - \Sigma_{i=1}^{a_1} |V_i|}{44 - a_1} > 3$ by $|V_\ell| \le q+1 = 3$, and by assumption $\Sigma_{i=1}^{3} a_i = 43$ we reach to the contradiction $|V_{44}| = 126 - \Sigma_{i=1}^{43} |V_i| > \alpha(G)$ by Fact 2.2, we have $\chi_d^t(G) \ge 45$. Consider the given min-TDS $\mathcal{O} \cup \mathcal{S}$ of G in the proof of Theorem 6.1. Since by assigning the 43 different colors of $[43]$ to the 43 vertices of $\mathcal{O} \cup \mathcal{S}$ and color 44 to the vertices of $\mathcal{P} - \mathcal{O}$ and color 45 to the vertices of $\mathcal{L} - \mathcal{S}$, we obtain a TDC of G, we have $\chi_d^t(G) = 45$.

Case 2. *The* $(4, 12)$-*Moore graph*. Then G has order 728, $\gamma_t(G) = 182$ (by Theorem 6.1) and $\alpha(G) = 364$ (by Proposition 2.3). By Fact 2.2 (5), we have $\ell \ge 184$. As we said before in the proof of Theorem 6.1, there exist a distance-2 ovoid $\mathcal{O} = \{v_1, ..., v_{91}\}$ and a distance-2 spread $\mathcal{S} = \{u_1, ..., u_{91}\}$ of G. Since $(\{v_1\}, ..., \{v_{91}\}, \{u_1\}, ..., \{u_{91}\}, \mathcal{P} - \mathcal{O}, \mathcal{L} - \mathcal{S})$ is a TDC of G with 184 color classes, we have $\chi_d^t(G) = 184$.

□

7. Conclusion and some open problems

The results of this paper are listed in the following proposition.

Proposition 7.1. *Let G be a $(q+1, g)$-Moore graph. If $(q+1, g) = (3,5), (3,12), (4,12)$, or $g = 6, 8$ and q is a prime power, then $\chi_d^t(G) = \gamma_t(G) + 2$, and if $(q+1, g) = (7,5)$, then $\chi_d^t(G) = \gamma_t(G) + 4$, where*

$$\gamma_t(G) = \begin{cases} 21 & \text{if } (q+1,g) = (4,8), \\ 43 & \text{if } (q+1,g) = (3,12), \\ 182 & \text{if } (q+1,g) = (4,12), \\ 2q+2 & \text{if } g = 6 \text{ and } q \text{is a prime power}, \\ q+2 & \text{if } (q+1,g) = (3,5),(7,5), \\ 2q^2+2 & \text{if } g = 8 \text{ and } q \text{ is an even prime power}, \end{cases}$$

and for an odd prime power q, if G is a $(q+1, 8)$-Moore graph, then $2q^2 + 3 \le \gamma_t(G) \le 2q^2 + q + 1$.

So, the following problems can be considered for some future works.

Problem 7.2. *For any odd prime power $q \ge 5$, find total domination number of the $(q+1, 8)$-Moore graph.*

Problem 7.3. *Find total domination and total dominator chromatic numbers of the $(q+1, 12)$-Moore graphs for any prime power $q \ge 4$.*

Problem 7.4. *Find total domination and total dominator chromatic numbers of the non-Moore cages.*

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