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Arithmetics of $\beta-$expansions in $\mathbb{F}_q((x^{-1}))$

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Abstract. The aim of this study is to give some arithmetic properties on the set of β-polynomials in $\mathbb{F}_q((x^{-1}))$ i.e. the set of series whose β-expansion has not fractional part, where $|\beta| > 1$ is an algebraic formal power series over the finite field $\mathbb{F}_q$. We will give sufficient conditions over β to have the quantity $L_{\odot}$ is finite, where $L_{\odot}$ designates the maximal finite shift after the comma for the product of two β-polynomials.

1. Introduction

The class of β-transformation $\{T_\beta,\ \beta > 1\}$ was introduced by A. Rényi in [10]. Let $\beta > 1$ be a real number. The β-transformations is a piecewise linear transformation on $[0, 1)$ defined by

$$T_\beta : x \longrightarrow \beta x - [\beta x],$$

$$\text{If } \forall\ i \geq 1,\ x_i = [\beta T_\beta^{i-1}(x)], \quad \text{then} \quad x = \sum_{i\geq 1} \frac{x_i}{\beta^i}.$$

We define the β-expansion of x as the sequence $d_\beta(x) = 0 \bullet x_1x_2x_3\ldots$.
For any real number $x > 1$, there exists an $m > 0$ such that $\beta^{-m-1}x \in [0, 1)$. Thus we can express each x in the form

$$x = \underbrace{x_{-m}\beta^m + \cdots + x_{-1}\beta + x_0}_{[x]_\beta} + \underbrace{\frac{x_1}{\beta} + \frac{x_2}{\beta^2} + \frac{x_3}{\beta^3} + \cdots}_{\{x\}_\beta},$$

then $d_\beta(x) = (x_i)_{i\geq -m} = x_{-m}\ldots x_{-1}x_0 \bullet x_1x_2x_3\ldots$.
The part with non-negative powers of β is then called the β-integer part of x, denoted by $[x]_\beta$; the part with negative powers of β is then called the β-fractional part of x, denoted by $\{x\}_\beta = x - [x]_\beta$. This allows a natural generalization for the definition of integers in base β.

A β-expansion is finite if $(x_i)_{i\geq 1}$ is eventually 0. It is periodic if there exists $p \geq 1$ and $m \geq 1$ with $x_k = x_{k+p}$ holds for all $k \geq m$; if $x_k = x_{k+p}$ holds for all $k \geq 1$, then it is purely periodic. We denote by

$$\mathrm{Fin}(\beta) = \{x \in \mathbb{R}_+ : d_\beta(x) \text{ is finite}\}.$$

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It is proved in [1] that if $\mathbb{N} \subset \mathrm{Fin}(\beta)$, then β is a Pisot number; that is, a real algebraic integer greater than 1 with all conjugates strictly inside the unit circle. Let $\mathbb{Z}[\beta]$ be the smallest ring containing $\mathbb{Z}$ and β. Denote by $\mathbb{Z}[\beta]_{\geq 0}$ the non negative elements of $\mathbb{Z}[\beta]$. We say that the number x satisfies the finiteness property if:

$$\mathrm{Fin}(\beta) = \mathbb{Z}[\beta^{-1}]_{\geq 0}.$$

This property was introduced by Frougny and Solomyak [5]. They showed that if β satisfies the finiteness property then β is a Pisot number. Note that there are Pisot numbers without finiteness property, especially, all numbers β such that $d_\beta(1)$ is infinite.

The set of β-integers, denoted by $\mathbb{Z}_\beta$, is the set of real numbers x for which there exists $n \in \mathbb{N}$, such that $x = \pm \sum_{i=0}^{n} a_i \beta^i$, where $a_n \dots a_0 \bullet 0$ is the β-expansion of $|x|$.

In general, the sets $\mathbb{Z}_\beta$ and $\mathrm{Fin}(\beta)$ are not stable under addition and multiplication. In spite of that, it is sometimes useful in computer science to consider this operation in β-arithmetics. Therefore, it is important to study which fractional parts might appear as a result of addition and multiplication of β-integers.

The following quantities $L_\oplus$ and $L_\odot$, are introduced in [2]. They represents the maximal possible finite length of the β-fractional parts which would appear when one adds or multiplies two β-integers.

Definition 1.1. *Let $\beta > 1$. We denote*
- $L_\oplus := \min\{n \in \mathbb{N} : \forall x, y \in \mathbb{Z}_\beta, x + y \in \mathrm{Fin}(\beta) \Longrightarrow \beta^n(x + y) \in \mathbb{Z}_\beta\}$ *when this set is not empty, $+\infty$ otherwise.*
- $L_\odot := \min\{n \in \mathbb{N} : \forall x, y \in \mathbb{Z}_\beta, xy \in \mathrm{Fin}(\beta) \Longrightarrow \beta^n(xy) \in \mathbb{Z}_\beta\}$ *when this set is not empty, $+\infty$ otherwise.*

Many authors are interested in the case where $L_\oplus$ and $L_\odot$ are finite. Indeed if the sum or the product of two β-integers belongs to $\mathrm{Fin}(\beta)$, then the length of the β-fractional part of this sum or product is bounded by a constant which only depends on β. C. Frougny and B. Solomyak in [5] showed that $L_\oplus$ is finite when β is a Pisot number. The case of quadratic Pisot numbers has been studied in [4] when β is a unit. The authors gave exact values for $L_\oplus$ and $L_\odot$, when $\beta > 1$ is a solution either of equation $x^2 = mx - 1$, $m \in \mathbb{N}$, $m \geq 3$ or of equation $x^2 = mx + 1$, $m \in \mathbb{N}$. In the first case $L_\oplus = L_\odot = 1$, in the second case $L_\oplus = L_\odot = 2$, and in [6] otherwise. However, when β is of higher degree, it is a difficult problem to compute the exact value of $L_\oplus$ or $L_\odot$, and even to compute upper and lower bounds for these two constants. Several examples are studied in [2], where a method is described in order to compute upper bounds for $L_\oplus$ and $L_\odot$ for Pisot numbers satisfying additional algebraic properties.

In [3], J. Bernat determined the exact value of $L_\oplus$ for several cases of cubic Pisot units numbers: He especially proved that if we denote by $L_\oplus(k_1, k_2)$ the value of $L_\oplus$ associated to the positive root β of $P(x) = x^3 - k_1x^2 - k_2x - 1$, where $k_1, k_2 \in \mathbb{N}^2$ satisfy $\max\{1, k_2\} \leq k_1 \leq 3$. For example, $L_\oplus(1, 0) = 11$, $L_\oplus(2, 2) = 5$ and $L_\oplus(3, 2) = 4$. In particular, in the Tribonacci case, that is, when β is the positive root, of the polynomial $x^3 - x^2 - x - 1$, he proved the following result.

Proposition 1.2. *If β is the Tribonacci number, then $L_\oplus = 5$.*

Let's note that until now, we don't know the value of $L_\odot$ in the case of the Tribonacci number. It is only proven in [2] that $4 \leq L_\odot \leq 5$.

We can define analogous notions in the case of the field of formal power series over a finite field.

The main objective of this paper is to study a similar concepts over the field of formal power series over finite field. This paper is organized as follows: In section 2, we define the field of formal power series over a finite field $\mathbb{F}_q((x^{-1}))$. We will also define the β-expansion algorithm over this field. In section 3, we prove that, for any algebraic integer series β, the quantity $L_\odot$ is finite. In the sequel, we have shown that $L_\odot$ is also finite in algebraic unit basis.

2. β-expansions in $\mathbb{F}_q((x^{-1}))$

Let $\mathbb{F}_q$ be a finite field with q elements, $\mathbb{F}_q[x]$ the ring of polynomials with coefficients in $\mathbb{F}_q$ and $\mathbb{F}_q(x)$ the field of rational functions.
Let $\mathbb{F}_q((x^{-1}))$ be the field of formal power series of the form:

$$f = \sum_{k=-\infty}^{l} f_k x^k, \quad f_k \in \mathbb{F}_q,$$

where

$$l = \deg f := \begin{cases} \max\{k : f_k \neq 0\} & \text{if} \quad f \neq 0, \\ -\infty & \text{if} \quad f = 0. \end{cases}$$

Define the absolute value $|f| = q^{\deg f}$. Thus, $\mathbb{F}_q((x^{-1}))$, equipped with this absolute value, is a complete metric space, it is the completion of $\mathbb{F}_q(x)$. Since the above absolute value is not archimedean, it fulfills the strict triangle inequality:

$$|f+g| \leq \max(|f|,|g|) \quad \text{and} \quad |f+g| = \max(|f|,|g|) \quad \text{if} \quad |f| \neq |g|.$$

Consider $f \in \mathbb{F}_q((x^{-1}))$ and define the polynomial part $[f] = \sum_{k=0}^{l} f_k x^k$ where the empty sum, as usual, is defined to be zero. Therefore $[f] \in \mathbb{F}_q[x]$ and $f - [f] \in M_0$ where $M_0 = \{f \in \mathbb{F}_q((x^{-1})) : \ |f| < 1\}$.
Let $\beta \in \mathbb{F}_q((x^{-1}))$, we denote by:
- $\mathbb{F}_q(x,\beta) = \mathbb{F}_q(x)(\beta)$ the smallest field containing $\mathbb{F}_q(x)$ and β.
- $\mathbb{F}_q[x,\beta] = \mathbb{F}_q[x][\beta]$ the smallest ring containing $\mathbb{F}_q[x]$ and β.

Now, we are ready to define the β-expansions of f in the field of formal power series.

Let $\beta, f \in \mathbb{F}_q((x^{-1}))$ where $|\beta| > 1$ and $f \in M_0$. A representation in base β (or β-representation) of f is a sequence $(d_i)_{i\geq 1}$, $d_i \in \mathbb{F}_q[x]$, such that

$$f = \sum_{i\geq 1} \frac{d_i}{\beta^i}.$$

A particular β-representation of f is called the β-expansion of f and noted $d_\beta(f)$. It is obtained by using the β-transformation T_β in M_0 which is given by the mapping:

$$\begin{aligned} T_\beta : M_0 &\longrightarrow M_0 \\ f &\longmapsto \beta f - [\beta f]. \end{aligned}$$

Thus, $d_\beta(f) = 0 \bullet (d_i)_{i\geq 1}$ if and only if $d_i = [\beta T_\beta^{i-1}(f)]$. Note that $d_\beta(f)$ is finite if and only if there is a $k \geq 0$ such that $T_\beta^k(f) = 0$, $d_\beta(f)$ is ultimately periodic if and only if there is some smallest $p \geq 0$ (the pre-period length) and $s \geq 1$ (the period length) for which $T_\beta^{p+s}(f) = T_\beta^p(f)$. If $f \in M_0$ and $d_\beta(f) = 0 \bullet (d_i)_{i\geq 1}$, we often write $f = 0 \bullet d_1 d_2 d_3 \ldots$.

Now let $f \in \mathbb{F}_q((x^{-1}))$ be an element with $|f| \geq 1$. Then there is a unique $k \in \mathbb{N}$ such that $|\beta|^k \leq |f| < |\beta|^{k+1}$. Hence $|\frac{f}{\beta^{k+1}}| < 1$ and we can represent f by shifting $d_\beta(\frac{f}{\beta^{k+1}})$ by $k+1$ digits to the left. That is, if $d_\beta(\frac{f}{\beta^{k+1}}) = 0 \bullet d_1 d_2 d_3 \ldots$, then $d_\beta(f) = d_1 d_2 d_3 \ldots d_{k+1} \bullet d_{k+2} \ldots$.

Remark 2.1. *In contrast to the real case, there is no carry occurring, when we add two digits. Therefore, if* z, $w \in \mathbb{F}_q((x^{-1}))$, *we have* $d_\beta(z+w) = d_\beta(z) + d_\beta(w)$ *digitwise. We have also* $d_\beta(cf) = c d_\beta(f)$ *for every* $c \in \mathbb{F}_q$.

Theorem 2.2. *[8] A* β*-representation* $(d_j)_{j\geq 1}$ *of* f *in* M_0 *is its* β*-expansion if and only if* $|d_j| < |\beta|$ *for* $j \geq 1$.

Let us first recall some number theoretical notions.
A formal power series β is called an algebraic series over $\mathbb{F}_q(x)$, if there exists $a_n, \dots, a_0 \in \mathbb{F}_q[x]$ such that

$$P(\beta) = a_n\beta^n + a_{n-1}\beta^{n-1} + \cdots + a_1\beta + a_0 = 0.$$

If the polynomial P is of minimal degree, then P is called the minimal polynomial of β of algebraic degree n. The other roots of the minimal polynomial which are not necessarily in $\mathbb{F}_q((x^{-1}))$ are called the algebraic conjugates of β. If $a_n \in \mathbb{F}_q^*$, then β is called an algebraic integer series and if $a_0 \in \mathbb{F}_q^*$, then β is called a unit series.

Proposition 2.3. *[9] Let K be a complete field with respect to a non archimedean absolute value $|.|$ and L/K ($K \subset L$) be an algebraic extension of degree m. Then $|.|$ has a unique extension to L defined by : $|a| = \sqrt[m]{|N_{L/K}(a)|}$ and L is complete with respect to this extension.*

We apply this proposition to algebraic elements of $\mathbb{F}_q((x^{-1}))$. Since $\mathbb{F}_q[x] \subset \mathbb{F}_q((x^{-1}))$, then every algebraic element in $\mathbb{F}_q[x]$ can be valuated. However, since $\mathbb{F}_q((x^{-1}))$ is not algebraically closed, such an element needn't be necessarily a formal power series.

Lemma 2.4. *[8] Let $P(Y) = A_nY^n - A_{n-1}Y^{n-1} - \cdots - A_0$ where $A_i \in \mathbb{F}_q[x]$, for $i = 1, \dots, n$. Then P admits a unique root in $\mathbb{F}_q((x^{-1}))$ with absolute value > 1 and all other roots are with absolute value < 1 if and only if $|A_{n-1}| > |A_i|$ for $i \neq n-1$.*

If $d_\beta(f) = d_ld_{l-1}\dots d_0 \bullet d_{-1}d_{-2}\dots$, let $[f]_\beta = d_l\beta^l + d_{l-1}\beta^{l-1} + \cdots + d_0$ and $\{f\}_\beta = f - [f]_\beta$.
If $d_\beta(f)$ is finite with $f = \sum_{l}^{m} d_i\beta^{-i}$ where $m, l \in \mathbb{Z}$, then we put $\text{ord}_\beta(f) = -m$ and $\text{ord}_\beta(f) = -\infty$ otherwise. Using this last notion, we define the set of β-polynomials as follow:

$$(\mathbb{F}_q[x])_\beta = \{f \in \mathbb{F}_q((x^{-1})) : \text{ord}_\beta(f) \geq 0\}.$$

In the sequel, we will use the following notation:

$$\text{Fin}(\beta) = \{f \in \mathbb{F}_q((x^{-1})) : d_\beta(f) \text{ is finite}\}.$$

$$\text{Per}(\beta) = \{f \in \mathbb{F}_q((x^{-1})) : d_\beta(f) \text{ is periodic}\}.$$
$$\text{Pur}(\beta) = \{f \in \mathbb{F}_q((x^{-1})) : d_\beta(f) \text{ is purely periodic}\}.$$
$$\text{Per}(\beta, s) = \{f \in \mathbb{F}_q((x^{-1})) : d_\beta(f) \text{ is periodic with pre-period s}\}.$$

We define the quantity $L_\odot$ as follows:

$$L_\odot = \begin{cases} \min\ E_\beta & \text{if } E_\beta \neq \emptyset, \\ \infty & \text{if } E_\beta = \emptyset, \end{cases}$$

with $E_\beta = \{n \in \mathbb{N} : \forall p_1, p_2 \in (\mathbb{F}_q[x])_\beta,\ p_1.p_2 \in \text{Fin}(\beta) \Longrightarrow \beta^n(p_1.p_2) \in (\mathbb{F}_q[x])_\beta\}$. More precisely, we can see $L_\odot$ as follows:

$$L_\odot = \max\{-\text{ord}_\beta(p_1.p_2) : p_1, p_2 \in (\mathbb{F}_q[x])_\beta,\ p_1.p_2 \in \text{Fin}(\beta)\}.$$

Let us note that $L_\odot$ designates the maximal finite shift after the comma for the product of two β-polynomials.

Example 2.5. *Let β be the unique root of absolute value > 1 of the polynomial $P(Y) = Y^d + x^2Y^{d-1} + A_{d-2}Y^{d-2} + \cdots + A_0$, with $A_i = 1$ for all $0 \leq i \leq d-2$. Then $L_\odot = (d-1)$. Indeed, the existence of such β is due to Lemma 2.4 and his degree is 2. We have $\beta^d + x^2\beta^{d-1} + \beta^{d-2} + \cdots + 1 = 0$. Hence the β-expansion of x^2 is given by*

$$x^2 = -\beta - \frac{1}{\beta} - \frac{1}{\beta^2} - \dots - \frac{1}{\beta^{d-1}}.$$

It is clear that in this case $L_\odot = -\text{ord}_\beta(x^2) = d-1$.

3. Results

In order to prove the finiteness of $L_\odot$ in algebraic basis, we need to introduce some basic notions: Let β be an algebraic series of degree d and $\beta^{(2)}, \ldots, \beta^{(d)}$ be their conjugates.
For $f \in \mathbb{F}_q(x, \beta)$, we have $f = k_0 + k_1\beta + k_2\beta^2 + \cdots + k_{d-1}\beta^{d-1}$ with $k_i \in \mathbb{F}_q(x)$, the j-th conjugate of f is defined by $f^{(j)} = k_0 + k_1\beta^{(j)} + k_2(\beta^{(j)})^2 + \cdots + k_{d-1}(\beta^{(j)})^{d-1}$.
We define $\overline{f}$, the vector conjugate of f by $\overline{f} = \begin{pmatrix} f^{(2)} \\ \vdots \\ f^{(d)} \end{pmatrix}$ and $\|\overline{f}\| = \sup_{2\le j\le d} |f^{(j)}|$.

We begin by this lemma which is essential for the development of Theorem 3.2.

Lemma 3.1. *Let β be an algebraic series with $|\beta| > 1$ and $\beta^{(j)}$ a conjugate of β such that $|\beta^{(j)}| > 1$. If $f \in \mathbb{F}_q(x, \beta)$ with $f = \sum_{k\ge1} a_k\beta^{-k}$ and $(a_k)_{k\ge1}$ is a periodic sequence, then $f^{(j)} = \sum_{k\ge1} a_k(\beta^{(j)})^{-k}$.*

Proof. .
Let $f \in \mathbb{F}_q(x, \beta)$ where $f = \sum_{k\ge1} a_k\beta^{-k}$ and $(a_k)_{k\ge1}$ is a periodic sequence. So
$(a_k)_{k\ge1} = a_1...a_p\overline{a_{p+1}...a_{p+s}}$ with $a_p \neq a_{p+s}$. Hence we get

$$f = \frac{a_1}{\beta} + \cdots + \frac{a_p}{\beta^p} + \frac{a_{p+1}}{\beta^{p+1}} + \cdots + \frac{a_{p+s}}{\beta^{p+s}} + \frac{1}{\beta^s}(\frac{a_{p+1}}{\beta^{p+1}} + \cdots + \frac{a_{p+s}}{\beta^{p+s}}) + \frac{1}{\beta^{2s}}(\frac{a_{p+1}}{\beta^{p+1}} + \cdots + \frac{a_{p+s}}{\beta^{p+s}}) + \cdots .$$

Therefore

$$f = \frac{a_1}{\beta} + \cdots + \frac{a_p}{\beta^p} + (\frac{a_{p+1}}{\beta^{p+1}} + \cdots + \frac{a_{p+s}}{\beta^{p+s}})(1 + \frac{1}{\beta^s} + \frac{1}{\beta^{2s}} + \frac{1}{\beta^{3s}} + \cdots),$$

this gives

$$f = \frac{a_1}{\beta} + \cdots + \frac{a_p}{\beta^p} + (\frac{a_{p+1}}{\beta^{p+1}} + \cdots + \frac{a_{p+s}}{\beta^{p+s}})(\frac{1}{1 - \frac{1}{\beta^s}}).$$

For every conjugate $\beta^{(j)}$ of β, we get

$$f^{(j)} = \frac{a_1}{\beta^{(j)}} + \cdots + \frac{a_p}{(\beta^{(j)})^p} + (\frac{a_{p+1}}{(\beta^{(j)})^{p+1}} + \cdots + \frac{a_{p+s}}{(\beta^{(j)})^{p+s}})(\frac{1}{1 - \frac{1}{(\beta^{(j)})^s}}).$$

Now, for every conjugate $\beta^{(j)}$ of β such that $|\beta^{(j)}| > 1$, we have

$$f^{(j)} = \frac{a_1}{\beta^{(j)}} + \cdots + \frac{a_p}{(\beta^{(j)})^p} + (\frac{a_{p+1}}{(\beta^{(j)})^{p+1}} + \cdots + \frac{a_{p+s}}{(\beta^{(j)})^{p+s}})(1 + \frac{1}{(\beta^{(j)})^s} + \frac{1}{(\beta^{(j)})^{2s}} + \frac{1}{(\beta^{(j)})^{3s}} + \cdots).$$

Finally, we reach to our result by getting the following equality

$$f^{(j)} = \frac{a_1}{\beta^{(j)}} + \cdots + \frac{a_p}{(\beta^{(j)})^p} + \frac{a_{p+1}}{(\beta^{(j)})^{p+1}} + \cdots + \frac{a_{p+s}}{(\beta^{(j)})^{p+s}} + \frac{1}{(\beta^{(j)})^s}(\frac{a_{p+1}}{(\beta^{(j)})^{p+1}} + \cdots + \frac{a_{p+s}}{(\beta^{(j)})^{p+s}}) +$$
$$\frac{1}{(\beta^{(j)})^{2s}}(\frac{a_{p+1}}{(\beta^{(j)})^{p+1}} + \cdots + \frac{a_{p+s}}{(\beta^{(j)})^{p+s}}) + \cdots . \quad \square$$

As recently seen in the introduction, there are both quantities $L_\oplus$ and $L_\odot$, except in the case of formal series the quantity $L_\oplus$ is not interesting, because we know that the sum of two β-polynomials is always a β-polynomial. We excluded the case when $\deg(\beta) = 1$, since in this trivial case, the product of two β-polynomials is a β-polynomial. Then we have $L_\odot = 0$.
So far, we are interested in results for $L_\odot$ for general algebraic series β.

Theorem 3.2. *Let β be an algebraic integer series with $|\beta| > 1$. Then the set $((\mathbb{F}_q[x])_\beta.(\mathbb{F}_q[x])_\beta) \cap Per(\beta)$ is finite.*

Proof. . Suppose without loss of generality $d_\beta(PQ) = c_n \dots c_1 c_0 \bullet \overline{c_{-1} \dots c_{-m}}$ is ultimately periodic where P and Q are two β-polynomials such that

$$P = a_s\beta^s + a_{s-1}\beta^{s-1} + \cdots + a_0 \text{ with } |a_i| < |\beta|$$

and

$$Q = b_k\beta^k + b_{k-1}\beta^{k-1} + \cdots + b_0 \text{ with } |b_i| < |\beta|.$$

Hence

$$PQ = c_n\beta^n + \cdots + c_0 + \frac{c_{-1}}{\beta} + \frac{c_{-2}}{\beta^2} + \cdots + \frac{c_{-m}}{\beta^m} + \frac{c_{-1}}{\beta^{m+1}} + \cdots + \frac{c_{-m}}{\beta^{2m}} + \cdots .$$

So

$$\{PQ\}_\beta = PQ - c_n\beta^n - \cdots - c_0 = \frac{c_{-1}}{\beta} + \frac{c_{-2}}{\beta^2} + \cdots + \frac{c_{-m}}{\beta^m} + \frac{c_{-1}}{\beta^{m+1}} + \cdots + \frac{c_{-m}}{\beta^{2m}} + \cdots .$$

Let $(\beta^{(j)})_{2\le j\le d}$ be the conjugates of β. To show this theorem, we first distinguish these two cases:

Case 1: If $|\beta^{(j)}| > 1$.

In this case, we have $\{PQ\}_\beta \in \mathbb{F}_q(x,\beta)$ and so by Lemma 3.1, we obtain

$$\{PQ\}_\beta^{(j)} = \frac{c_{-1}}{\beta^{(j)}} + \frac{c_{-2}}{(\beta^{(j)})^2} + \cdots + \frac{c_{-m}}{(\beta^{(j)})^m} + \frac{c_{-1}}{(\beta^{(j)})^{m+1}} + \cdots + \frac{c_{-m}}{(\beta^{(j)})^{2m}} + \cdots .$$

Since $|c_i| < |\beta|$ for all $i \in \{-1, ..., -m\}$, so $|\{PQ\}_\beta^{(j)}| < |\beta|$.

Case 2: If $|\beta^{(j)}| \le 1$.

In this case, we have

$$\begin{aligned}
\{PQ\}_\beta^{(j)} &= (a_s(\beta^{(j)})^s + \cdots + a_0)(b_k(\beta^{(j)})^k + \cdots + b_0) - c_n(\beta^{(j)})^n - \cdots - c_0 \\
&= \sum_{l=0}^{s+k}(\sum_{p=0}^{l} b_p a_{l-p})(\beta^{(j)})^l - c_n(\beta^{(j)})^n - \cdots - c_0 \\
&\le \max\{|\sum_{l=0}^{s+k}(\sum_{p=0}^{l} b_p a_{l-p})(\beta^{(j)})^l|; |c_n(\beta^{(j)})^n - \cdots - c_0|\}.
\end{aligned}$$

As $|\beta^{(j)}| \le 1$, we get

$$|\sum_{l=0}^{s+k}(\sum_{p=0}^{l} b_p a_{l-p})(\beta^{(j)})^l| \le |\beta|^2 \textit{ and } |c_n(\beta^{(j)})^n - \cdots - c_0| \le |\beta|$$

Consequently,

$$|\{PQ\}_\beta^{(j)}| \le |\beta|^2.$$

Therefore the module of $\begin{pmatrix} \{PQ\}_\beta \\ \{PQ\}_\beta^{(2)} \\ \vdots \\ \{PQ\}_\beta^{(d)} \end{pmatrix}$ is less than $|\beta|^2$.

Now, since $\{PQ\}_\beta = \sum_{l=0}^{s+k}(\sum_{p=0}^{l} b_p a_{l-p})\beta^l - c_n\beta^n - \cdots - c_0$ and β is an algebraic integer of degree d, we easily deduce that $\{PQ\}_\beta = A_0 + A_1\beta + \cdots + A_{d-1}\beta^{d-1}$ with $A_i \in \mathbb{F}_q[x]$. Hence for all $j \in \{2,\ldots,d\}$; $\{PQ\}_\beta^{(j)} = A_0 + A_1(\beta^{(j)}) + \cdots + A_{d-1}(\beta^{(j)})^{d-1}$. Thus

$$\begin{pmatrix} \{PQ\}_\beta \\ \{PQ\}_\beta^{(2)} \\ \vdots \\ \vdots \\ \{PQ\}_\beta^{(d)} \end{pmatrix} = \mathrm{M} \begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ \vdots \\ A_{d-1} \end{pmatrix}, \text{ where } M = \begin{pmatrix} 1 & \beta & \ldots & \ldots & \beta^{d-1} \\ 1 & \beta^{(2)} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & (\beta^{(d-1)})^{d-1} \\ 1 & \beta^{(d)} & \ldots & \ldots & (\beta^{(d)})^{d-1} \end{pmatrix}.$$

We have $\det M = \prod_{i<j}(\beta^{(i)} - \beta^{(j)}) \neq 0$ which implies that M is invertible therefore it transforms all bounded vector in an bounded vector. From the two cases and since $|\{PQ\}_\beta| < 1$, we have $\begin{pmatrix} \{PQ\}_\beta \\ \{PQ\}_\beta^{(2)} \\ \vdots \\ \{PQ\}_\beta^{(d)} \end{pmatrix}$ is bounded, so $\begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_{d-1} \end{pmatrix}$ is also bounded and moreover belongs to $\mathbb{F}_q[x]^d$. Therefore this last vector takes a finite number of possibilities (since $\mathbb{F}_q$ is finite). □

Corollary 3.3. *Let β be an algebraic integer series with $|\beta| > 1$. Then $L_\odot$ is finite.*

In order to prove Theorem 3.5, we need the following lemma.

Lemma 3.4. *Let β be an algebraic unit series and ξ be a positive number. Then*

$$\lim_{p\to\infty} \min_{f\in X(p)} \|\overline{f}\| = \infty,$$

where

$$X(p) = \{f \in \mathrm{Fin}(\beta) : |f| \leq \xi, ord_\beta(f) = -p\}.$$

Proof. . Assume that there exists a constant B and an infinite sequence f_i (i=1,2,...) so that both

$$|f_i^{(j)}| \leq B \ \text{ for } j = 2,3,\ldots,d \ \text{ and } \ \lim_{i\to\infty} ord_\beta(f_i) = -\infty$$

holds. Since β is an algebraic series, $\mathrm{Fin}(\beta) \subset \mathbb{F}_q[x,\beta^{-1}]$. Moreover β is unit, then $\mathrm{Fin}(\beta) \subset \mathbb{F}_q[x,\beta^{-1}] \subset \mathbb{F}_q[x,\beta]$. We know that $\mathbb{F}_q[x,\beta]$ is discrete, then $\mathrm{Fin}(\beta)$ is discrete. In addition, we have $|f_i| \leq \xi$, so these f_i's are finite, a contradiction with the second condition, completing the proof. □

Theorem 3.5. *Let β be an algebraic unit series with $|\beta| > 1$. Then $L_{\odot}$ is finite.*

Proof. . By assumption, we have $d_\beta(PQ)$ is finite i.e $d_\beta(PQ) = c_n \dots c_0 \bullet c_{-1} \dots c_{-m}$, where $P, Q \in (\mathbb{F}_q[x])_\beta$ such that

$$P = a_s\beta^s + a_{s-1}\beta^{s-1} + \cdots + a_0 \text{ with } |a_i| < |\beta|$$

and

$$Q = b_k\beta^k + b_{k-1}\beta^{k-1} + \cdots + b_0 \text{ with } |b_i| < |\beta|.$$

We have

$$PQ = c_n\beta^n + \cdots + c_0 + \frac{c_{-1}}{\beta} + \frac{c_{-2}}{\beta^2} + \cdots + \frac{c_{-m}}{\beta^m}.$$

So

$$\{PQ\}_\beta = PQ - c_n\beta^n - \cdots - c_0 = \frac{c_{-1}}{\beta} + \frac{c_{-2}}{\beta^2} + \cdots + \frac{c_{-m}}{\beta^m}.$$

Let $(\beta^{(j)})_{2\leq j\leq d}$ be the conjugates of β. Now, we begin by distinguish these two cases:

Case 1: If $|\beta^{(j)}| > 1$.

In this case, we have

$$\{PQ\}_\beta^{(j)} = \frac{c_{-1}}{\beta^{(j)}} + \frac{c_{-2}}{(\beta^{(j)})^2} + \cdots + \frac{c_{-m}}{(\beta^{(j)})^m}.$$

Since $|c_i| < |\beta|$ for $i \in \{-1, ..., -m\}$, we obtain $|\{PQ\}_\beta^{(j)}| < |\beta|$.

Case 2: If $|\beta^{(j)}| \leq 1$.

$$\begin{aligned}\{PQ\}_\beta^{(j)} &= (a_s(\beta^{(j)})^s + \cdots + a_0)(b_k(\beta^{(j)})^k + \cdots + b_0) - c_n(\beta^{(j)})^n - \cdots - c_0 \\ &\leq \max\{|\sum_{l=0}^{s+k}(\sum_{p=0}^{l} b_p a_{l-p})(\beta^{(j)})^l|; |c_n(\beta^{(j)})^n - \cdots - c_0|\}.\end{aligned}$$

As $|\beta^{(j)}| \leq 1$, we get

$$|\sum_{l=0}^{s+k}(\sum_{p=0}^{l} b_p a_{l-p})(\beta^{(j)})^l| \leq |\beta|^2 \textit{ and } |c_n(\beta^{(j)})^n - \cdots - c_0| \leq |\beta|$$

Consequently,

$$|\{PQ\}_\beta^{(j)}| \leq |\beta|^2.$$

As a consequence, we deduce that for all $j \in \{2, \dots, d\}$, $\{PQ\}_\beta^{(j)}$ is bounded, wherefrom we have $\| \overline{\{PQ\}_\beta} \| \leq |\beta|^2$. Moreover, note that $|\{PQ\}_\beta| < 1$ and by Lemma 3.4, there exist $k \in \mathbb{Z}$, such that for all $PQ \in (\mathbb{F}_q[x])_\beta$ we obtain that $ord_\beta(PQ) \geq k$. Therefore, $L_{\odot}$ is finite. □

Example 3.6. *Let β be the unique root with absolute value > 1 of the polynomial $P(Y) = Y^d + x^m Y^{d-1} + A_{d-2}Y^{d-2} + \cdots + A_1 Y + x^{m-1}$, with $\deg(A_i) < m$ for $i \in \{1, \dots, d-2\}$. Then $L_{\odot} = (d-1)(m-1)$.*
Indeed, the existence of such β is due to Lemma 2.4 and his degree is m. We have

$$x^m = -\beta - \frac{A_{d-2}}{\beta} - \cdots - \frac{x^{m-1}}{\beta^{d-1}}.$$

Hence, $\mathrm{ord}_\beta(x^m) = 1 - d$. *Moreover*

$$x^{m+1} = -x\beta - \frac{xA_{d-2}}{\beta} - \cdots - \frac{x^m}{\beta^{d-1}}.$$

Therefore, $\mathrm{ord}_\beta(x^{m+1}) = 2(1-d)$ *and by induction for all positive integer s, we get* $\mathrm{ord}_\beta(x^{m+s}) = (s+1)(1-d)$. *So, in this case,* $L_\odot = -\mathrm{ord}_\beta(x^{2m-2}) = (d-1)(m-1)$.

Now, we prove that the set of periodic β-fractional part with fixed pre-period is finite where the basis β is an algebraic integer.

Theorem 3.7. *Let β be an algebraic integer series with $|\beta| > 1$. Then the set* $\mathbb{F}_q[x,\beta] \cap Per(\beta, s)$ *is finite.*

Proof. . Let $f \in \mathbb{F}_q[x,\beta] \cap M_0$ and β an algebraic integer series of degree d, then $f = \sum_{i=0}^{d-1} A_i\beta^i$, therefore $f^{(k)} = \sum_{i=0}^{d-1} A_i(\beta^{(k)})^i$ for all $k \in \{2, ..., d\}$. Thus

$$\begin{pmatrix} f \\ f^{(2)} \\ \vdots \\ \vdots \\ f^{(d)} \end{pmatrix} = \begin{pmatrix} 1 & \beta & \cdots & \cdots & \beta^{d-1} \\ 1 & \beta^{(2)} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & (\beta^{(d-1)})^{d-1} \\ 1 & \beta^{(d)} & \cdots & \cdots & (\beta^{(d)})^{d-1} \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ \vdots \\ A_{d-1} \end{pmatrix} = \mathrm{M} \begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ \vdots \\ A_{d-1} \end{pmatrix}.$$

Let now $d_\beta(f) = 0 \bullet a_1 \ldots a_s\overline{a_{s+1} \ldots a_{p+s}}$. To complete this proof, we must distinguish these two cases:

Case 1: If $|\beta^{(k)}| \le 1$. We have

$$f = \frac{a_1}{\beta} + \cdots + \frac{a_s}{\beta^s} + \frac{a_{s+1}}{\beta^{s+1}} + \cdots + \frac{a_{p+s}}{\beta^{p+s}} + \frac{1}{\beta^p}(f - \frac{a_1}{\beta} - \cdots - \frac{a_s}{\beta^s}).$$

Given that $a_1, \ldots, a_{p+s} \in \mathbb{F}_q[x]$, we find

$$f^{(k)} = \frac{a_1}{\beta^{(k)}} + \cdots + \frac{a_s}{(\beta^{(k)})^s} + \frac{a_{s+1}}{(\beta^{(k)})^{s+1}} + \cdots + \frac{a_{p+s}}{(\beta^{(k)})^{p+s}} + \frac{1}{(\beta^{(k)})^p}(f^{(k)} - \frac{a_1}{\beta^{(k)}} - \cdots - \frac{a_s}{(\beta^{(k)})^s}).$$

Then

$$f^{(k)}(1 - \frac{1}{(\beta^{(k)})^p}) = \frac{a_1}{\beta^{(k)}} + \cdots + \frac{a_s}{(\beta^{(k)})^s} + \frac{a_{s+1}}{(\beta^{(k)})^{s+1}} + \cdots + \frac{a_{p+s}}{(\beta^{(k)})^{p+s}} + \frac{1}{(\beta^{(k)})^p}(-\frac{a_1}{\beta^{(k)}} - \cdots - \frac{a_s}{(\beta^{(k)})^s}).$$

Hence

$$f^{(k)}((\beta^{(k)})^{p+s} - (\beta^{(k)})^s) = a_1(\beta^{(k)})^{p+s-1} + \cdots + a_s(\beta^{(k)})^p + a_{s+1}(\beta^{(k)})^{p-1} + \cdots + (a_{p+s} - a_s).$$

Since $|\beta^{(k)}| < 1$, we get $|f^{(k)}(\beta^{(k)})^s| < |\beta|$. As the pre-period s of f is fixed, we have $|f^{(k)}| < \frac{|\beta|}{|(\beta^{(k)})^s|}$.

Case 2: If $|\beta^{(k)}| > 1$.
In this case, we have

$$f = \frac{a_1}{\beta} + \cdots + \frac{a_s}{\beta^s} + \frac{a_{s+1}}{\beta^{s+1}} + \cdots + \frac{a_{p+s}}{\beta^{p+s}} + \frac{a_{s+1}}{\beta^{p+s+1}} + \cdots$$

and so by Lemma 3.1, we obtain

$$f^{(k)} = \frac{a_1}{\beta^{(k)}} + \cdots + \frac{a_s}{(\beta^{(k)})^s} + \frac{a_{s+1}}{(\beta^{(k)})^{s+1}} + \cdots + \frac{a_{p+s}}{(\beta^{(k)})^{p+s}} + \frac{a_{s+1}}{(\beta^{(k)})^{p+s+1}} + \cdots.$$

So $|f^{(k)}| < |\beta|$.

As a consequence, we deduce from these two cases that $f^{(k)}$ is bounded for all $k \in \{2, \ldots, d\}$. Since $\det M = \prod_{i<j}(\beta^{(i)} - \beta^{(j)}) \neq 0$, M is invertible. Hence, it transforms all bounded vector in an bounded vector. From these two cases and since $|f| < 1$, we have $\begin{pmatrix} f \\ f^{(2)} \\ \vdots \\ f^{(d)} \end{pmatrix}$ is bounded, so $\begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_{d-1} \end{pmatrix}$ is also bounded and moreover belongs to $\mathbb{F}_q[x]^d$, therefore this vector take a finite number of possibilities (since $\mathbb{F}_q$ is finite). □

Corollary 3.8. *Let β be an algebraic integer series with $|\beta| > 1$. Then the set $\mathbb{F}_q[x, \beta] \cap Pur(\beta)$ is finite.*

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