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Extrapolation of (p, h)-convex functions

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Abstract. In recent years, some interesting multi-term refinements of interpolated and extrapolated Jensen-type inequalities for convex functions have been established. The main objective of this paper is to utilize new techniques for generalizing these types of inequalities to (p, h)-convex functions. Some improvements of inequalities arising from the new characterization of this class of functions are also discussed. The significance of these results lies in the way they extend known results from the setting of convex functions to other classes of functions.

1. Introduction and preliminaries

Convex functions and their inequalities play an important role in economics, applied mathematics, mathematical analysis, mathematical physics, optimisation theory, potential theory, etc. Recall that a real valued function f defined on a real interval J is said to be convex if it satisfies the following inequality:

$$f\left((1-\mu)a+\mu b\right) \leq (1-\mu)f(a)+\mu f(b), \tag{1}$$

for all $a, b \in J$ and $\mu \in (0, 1)$. When this inequality is reversed, we say that the function f is concave on J. We also have the following supplementary inequality of convexity as:

$$(1+\mu)f(a)-\mu f(b) \leq f\left((1+\mu)a-\mu b\right), \tag{2}$$

where f is a convex function on $\mathbb{R}$ with $a, b \in \mathbb{R}$ and $\mu \geq 0$ (see [14, 16]).

Several researchers have studied the possibility of refining the above inequality by adding a positive term to the left-hand side, see for instance [1, 10–13], in which reversed versions of (1) have also been discussed. Nowadays, work on convex functions has developed rapidly, thanks to the use of new concepts and modern methods, such as the notions of p-convex functions, h-convex functions and (p, h)-convex functions, which have been studied by many mathematicians, see for example [2, 3, 8, 15, 17] and the references therein.

Before going any further, let us recall these new notions of convexity. Throughout this paper, the notation I stands for a p-convex subset of $\mathbb{R}$ for some real number $p \neq 0$, this means that $\left[(1-\mu)a^p+\mu b^p\right]^{\frac{1}{p}} \in I$ for all $a, b \in I$ and $\mu \in [0, 1]$.

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Definition 1.1 ([17]). *A function $f : I \to [0, +\infty)$ is said to be a p-convex function if it satisfies the following inequality*

$$f\left(\left[(1-\mu)a^p + \mu b^p\right]^{\frac{1}{p}}\right) \leq (1-\mu)f(a) + \mu f(b), \tag{3}$$

for all $a, b \in I$ and $\mu \in [0, 1]$.

In the following, $h : J \to [0, +\infty)$ is a function defined on a real interval J which contains the interval $(0, 1)$.

Definition 1.2 ([15]). *We say that $f : I \to [0, +\infty)$ is an h-convex function if it satisfies the following inequality*

$$f((1-\mu)a + \mu b) \leq h(1-\mu)f(a) + h(\mu)f(b), \tag{4}$$

for all $a, b \in I$ and $\mu \in [0, 1]$.

Definition 1.3 ([2]). *We say that $f : I \to [0, +\infty)$ is a (p, h)-convex function if it satisfies the following inequality*

$$f\left(\left[(1-\mu)a^p + \mu b^p\right]^{\frac{1}{p}}\right) \leq h(1-\mu)f(a) + h(\mu)f(b), \tag{5}$$

for all $a, b \in I$ and $\mu \in [0, 1]$.

When the inequality sign in (3), (4) and (5) is reversed, then f is called p-concave, h-concave and (p, h)-concave, respectively.

The class of (p, h)-convex functions generalises many different notions of convexity that exist in the literature. For example, if $h = id$ (id stands for the identity function) in (5), then we get the definition of p-convex functions [17]. Further, if we choose $p = 1$ (resp. $p = -1$), then we get the usual definition of the convexity (resp. harmonic convexity [7]):

$$f\Big((1-\mu)a + \mu b\Big) \leq (1-\mu)f(a) + \mu f(b),$$

$$\Big(\text{resp. } f\Big(\frac{ab}{(1-\mu)a + \mu b}\Big) \leq (1-\mu)f(a) + \mu f(b)\Big).$$

By mathematical induction, we can extend the inequality (5) to convex combinations of a finite number of points in I. This extension is known as the discrete Jensen inequality for (p, h)-convex functions.

Theorem 1.4 ([2]). *Let $\mu_1, \ldots, \mu_n$ be positive real numbers ($n \geqslant 2$) such that $\sum_{i=1}^{n} \mu_i = 1$, $a_1, \ldots, a_n \in I$ and f be a (p, h)-convex function on I. If h is a super-multiplicative function, then*

$$f\left(\Big(\sum_{i=1}^{n} \mu_i a_i^p\Big)^{\frac{1}{p}}\right) \leq \sum_{i=1}^{n} h(\mu_i) f(a_i). \tag{6}$$

The inequality sign in (6) is reversed when h is sub-multiplicative and f is (p, h)-concave.

Research related to this inequality consists of deriving new inequalities and refining the existing ones. For example, (6) was refined and reversed in [3] as follows.

Theorem 1.5 ([3]). *Under the same conditions of Theorem 1.4. If we further assume that h is super-additive, then for every $\lambda \geq 1$, we have*

$$(h(n\mu_{\min}))^{\lambda} \left(\left(h\left(\frac{1}{n}\right)\sum_{i=1}^{n} f(a_i)\right)^{\lambda} - f^{\lambda}\left(\Big(\frac{1}{n}\sum_{i=1}^{n} a_i^p\Big)^{\frac{1}{p}}\right)\right)$$

$$\leq \left(\sum_{i=1}^{n} h(\mu_i) f(a_i)\right)^{\lambda} - f^{\lambda}\left(\Big(\sum_{i=1}^{n} \mu_i a_i^p\Big)^{\frac{1}{p}}\right)$$
$$\leq (h(n\mu_{\max}))^{\lambda}\left(\left(h\left(\frac{1}{n}\right)\sum_{i=1}^{n} f(a_i)\right)^{\lambda} - f^{\lambda}\left(\Big(\frac{1}{n}\sum_{i=1}^{n} a_i^p\Big)^{\frac{1}{p}}\right)\right), \quad (7)$$

where $\mu_{\min} = \min\{\mu_k : k = 1, 2, \ldots, n\}$ *and* $\mu_{\max} = \max\{\mu_k : k = 1, 2, \ldots, n\}$.

We refer the reader to [3–6] for further discussions and refinements related to (p, h)-convex functions.

In [14], Sababheh provided the following improvement of the extrapolated version of Jensen's inequality for convex functions with many positive terms as we wish, which is in particular a refined generalisation of (2) as

$$\left(1 + \nu^{(1)}\right) f(a) - \sum_{i=1}^{n} \mu_i^{(1)} f\left(b_i^{(1)}\right)$$
$$+ \sum_{k=1}^{N} \left((n+1)\mu_{\min}^{(k)}\right)\left[\frac{1}{n+1}\Big(f(a) + \sum_{i=1}^{n} f(b_i^{(k)})\Big) - f\left(\frac{a + \sum_{i=1}^{n}(b_i^{(k)}}{n+1}\right)\right]$$
$$\leq f\left[(1 + \nu^{(1)})a - \sum_{i=1}^{n} \mu_i (b_i^{(1)})\right], \quad (8)$$

where $f : \mathbb{R} \to \mathbb{R}$ is a convex function, $a \in \mathbb{R}$, $\left(b_i^{(k)}, \mu_i^{(k)}\right) \in \mathbb{R} \times \mathbb{R}_+$ for all $1 \leq i \leq n$ and $1 \leq k \leq N$ and $\nu^{(1)} = \sum_{i=1}^{n} \mu_i^{(k)}$.

The aim of this paper is to give an extrapolated version of Jensen's inequality for (p, h)-convex functions and its refinement for many terms as we wish. More precisely, we prove the following inequality, which extends (8) to the case of (p, h)-convex functions,

$$h\left(1 + \nu^{(1)}\right) f(a) - \sum_{i=1}^{n} h(\mu_i^{(1)}) f\left(b_i^{(1)}\right)$$
$$+ \sum_{k=1}^{N} h\left((n+1)\mu_{\min}^{(k)}\right)\left[h\left(\frac{1}{n+1}\right)\Big(f(a) + \sum_{i=1}^{n} f(b_i^{(k)})\Big) - f\left(\left(\frac{a^p + \sum_{i=1}^{n}(b_i^{(k)})^p}{n+1}\right)^{\frac{1}{p}}\right)\right]$$
$$\leq f\left[\left((1 + \nu^{(1)})a^p - \sum_{i=1}^{n} \mu_i (b_i^{(1)})^p\right)^{\frac{1}{p}}\right],$$

where $a \in I$, $f : I \to [0, +\infty)$ is a (p, h)-convex function, h is super-multiplicative and super-additive function, $\mu_{\min} = \min\{\mu_i : i = 1, \ldots, n\}$, $\left(b_i^{(k)}, \mu_i^{(k)}\right) \in I \times \mathbb{R}_+$ for all $1 \leq i \leq n$ and $1 \leq k \leq N$.

Additionally, we provide an improvement of (7), as follows

$$\psi\left(\sum_{i=1}^{n} h(\mu_i) f(a_i)\right) - \psi \circ f\left(\left[\sum_{i=1}^{n} \mu_i a_i^p\right]^{\frac{1}{p}}\right)$$
$$\geq \psi\left(h(\mu_{\min})\sum_{i=1}^{n} f(a_i)\right) - \psi\left(nh(\mu_{\min}) f\left(\left[\sum_{i=1}^{n} \frac{1}{n} a_i^p\right]^{\frac{1}{p}}\right)\right)$$

$$\geq \psi\left(h\left(\mu_{\min}\right)h\left(\frac{1}{n}\right)\sum_{i=1}^{n} f(a_i)\right) - \psi\left(h\left(n\mu_{\min}\right) f\left(\left[\sum_{i=1}^{n} \frac{1}{n} a_i^p\right]^{\frac{1}{p}}\right)\right),$$

where $a_i \in I$, $\mu_i \geq 0$ such that $\sum_{i=1}^{n} \mu_i = 1$, $f : I \to \mathbb{R}_+$ is a (p, h)-convex function on I, h is super-multiplicative and super-additive function, and ψ is a strictly increasing convex function defined on $[0, +\infty)$.

The paper is organized as follows. In the second section we present the refinement of the extrapolated version of Jensen's inequality for (p, h)-convex functions. In the third section, we provide an improvement of Jensen's inequality for (p, h)-convex functions.

2. Improvement of the extrapolated version of Jensen's inequality for (p, h)-convex functions

Before stating our first result in this section, we need to recall a few definitions and notations that will be used in what follows.

Definition 2.1 ([8]). *We say that h is a super-multiplicative function, if the following inequality holds*

$$h(x)h(y) \leq h(xy) \quad \text{for all } x, y \in J. \tag{9}$$

When (9) is reversed, then h is said to be a sub-multiplicative function. If (9) is equality, then h is said to be a multiplicative function.

Definition 2.2 ([8]). *We say that h is a super-additive function, if the following inequality holds*

$$h(x) + h(y) \leq h(x + y) \quad \text{for all } x, y \in J. \tag{10}$$

Similarly, when the inequality (10) is reversed, h is called a sub-additive function. When (10) is equality, h is called an additive function.

We now give a few examples of super-additive (resp. sub-additive) and super-multiplicative (resp. sub-multiplicative) functions, which can be easily checked by the reader.

Example 2.3.

1. *For a real number s, we define the function h from $[0, +\infty)$ into itself by*

$$h(x) = \begin{cases} x^s & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$

   A simple calculation shows that h is
   (a) *additive when $s = 1$,*
   (b) *sub-additive when $s \in (-\infty, -1] \cup [0, 1)$,*
   (c) *super-additive when $s \in (-1, 0) \cup (1, \infty)$.*
2. *Let $h : [0, +\infty) \to [0, +\infty)$ defined by $h(x) = x^3 - x^2 + x$ for $x \geq 0$. It is easy to prove that*
   (a) $h(xy) - h(x)h(y) = xy(x + y)(1 - x)(1 - y)$.
   (b) $h(x + y) - h(x) - h(y) = 2xy\left[(x - 1) + (y - 1)\right]$.
   This implies that the function h is super-multiplicative on $[0, +\infty)$, super-additive on $[1, +\infty)$ and sub-additive on $[0, 1]$.
3. *If the function $h : [0, +\infty) \to [0, +\infty)$ satisfies $h(\mu x) \leq \mu h(x)$ for every $x \geq 0$ and $\mu \in [0, 1]$, then h is super-additive. Indeed, let $x, y \in [0, +\infty)$. Clearly that $h(0) = 0$, since $0 \leq h(0) \leq \mu h(0)$ for all $\mu \in [0, 1]$. So the result is trivial when either $x = 0$ or $y = 0$. Now let's suppose that $x, y > 0$. Putting $\mu_1 = \dfrac{x}{x + y}$ and $\mu_2 = \dfrac{y}{x + y}$. Obviously, $\mu_1, \mu_2 \in (0, 1)$, $h(x) = h\left(\mu_1(x + y)\right)$ and $h(y) = h\left(\mu_2(x + y)\right)$. Therefore, by combining this fact with the hypothesis, we get the desired result.*

4. *It follows from the part* (3) *that if* $h : [0,+\infty) \to [0,+\infty)$ *is a convex function such that* $h(0) = 0$, *then it is super-additive. In fact, by convexity and the fact that* $h(0) = 0$, *we can conclude that* $h(\mu x) \leq \mu h(x)$ *for every* $x \geq 0$ *and* $\mu \in (0,1)$.

Before we move on, it should be notice that if h is a super-additive function then $h(x-y) \leq h(x) - h(y)$ whenever $x - y \in J$, and if h is super-multiplicative then $h\Big(\frac{x}{y}\Big) \leq \frac{h(x)}{h(y)}$ whenever $\frac{x}{y} \in J$ and $h(y) \neq 0$, in particular, $h\Big(\frac{1}{x}\Big) \leq \frac{1}{h(x)}$ for all $x \in J \setminus \{0\}$ such that $x^{-1} \in J$ and $h(x) \neq 0$.

In the future, for $n \in \mathbb{N}^*$, we denote by $\Im_{p,n}$ the subset of $I \times I^n \times \mathbb{R}_+^n$ defined by

$$\Im_{p,n} = \left\{ (a, \mathbf{b}, \tilde{\mu}) \in I \times I^n \times \mathbb{R}_+^n : \left[\left(1 + \sum_{i=1}^{n} \mu_i \right) a^p - \sum_{i=1}^{n} \mu_i b_i^p \right]^{\frac{1}{p}} \in I \right\},$$

where $\mathbf{b} = (b_i)_{i=1}^n$ and $\tilde{\mu} = (\mu_i)_{i=1}^n$. Clearly, for every $n \in \mathbb{N}^*$, $\Im_{p,n} \neq \emptyset$. In fact, let $a, b \in I$, by letting $b_i = b$ and $\mu_i = 0$ for $i = 1, \ldots, n$, then we have $(a, \mathbf{b}, \tilde{\mu}) \in \Im_{p,n}$.

We start this section by the following result, which provides a extrapolated version of Jensen's inequality for (p,h)-convex functions.

Theorem 2.4. *Let* $(a, \mathbf{b}, \tilde{\mu}) \in \Im_{p,n}$ *and* $f : I \to [0,+\infty)$ *be a* (p,h)*-convex function. If the non-negative function* h *is super-multiplicative, then*

$$h(1+\nu) f(a) - \sum_{i=1}^{n} h(\mu_i) f(b_i) \leq f\left[\left((1+\nu) a^p - \sum_{i=1}^{n} \mu_i b_i^p \right)^{\frac{1}{p}} \right], \tag{11}$$

where $\sum_{i=1}^{n} \mu_i = \nu$.

Proof. We first treat the case where $\mu_i = 0$ for all $i \in \{1, \ldots, n\}$. Since h is super-multiplicative, we obtain that $h(1) \leq 1$. Hence, by the positivity of terms $f(a)$, $h(0)$ and $f(b_i)$ for $1 \leq i \leq n$, we get the following inequlalities

$$\begin{aligned} h(1) f(a) &\leq f(a) \\ &\leq f(a) + h(0) \sum_{i=1}^{n} f(b_i) \\ &= f\left[\left((1+\nu) a^p - \sum_{i=1}^{n} \mu_i b_i^p \right)^{\frac{1}{p}} \right] + \sum_{i=1}^{n} h(\mu_i) f(b_i), \end{aligned}$$

which give the desired result in this case.

We assume that at least one of the μ_i is non-zero. Notice first that for $(x, y, s) \in \Im_{p,1}$, we have

$$x^p = \frac{s}{s+1} y^p + \frac{1}{s+1} \left(((1+s)x^p - s y^p)^{\frac{1}{p}} \right)^p.$$

From the fact that f is (p,h)-convex, we get

$$\begin{aligned} f(x) = f\left((x^p)^{\frac{1}{p}} \right) &\leq h\Big(\frac{s}{s+1}\Big) f(y) + h\Big(\frac{1}{s+1}\Big) f\left[((1+s)x^p - s y^p)^{\frac{1}{p}} \right] \\ &\leq \frac{h(s)}{h(s+1)} f(y) + \frac{1}{h(s+1)} f\left[((1+s)x^p - s y^p)^{\frac{1}{p}} \right]. \end{aligned}$$

Hence

$$h(s+1)f(x) - f\left[((1+s)x^p - sy^p)^{\frac{1}{p}}\right] \leq h(s)f(y). \tag{12}$$

Now we put $x = a$, $y = \left(\sum_{i=1}^{n} \frac{\mu_i}{\nu} b_i^p\right)^{\frac{1}{p}}$ and $s = \nu$. Since, $(a, \mathbf{b}, \tilde{\mu}) \in \Im_{p,n}$, we obtain that $(x, y, s) \in \Im_{p,1}$. So, by applying (12), we infer that

$$\begin{aligned}
h(1+\nu)f(a) \quad - \quad & f\left[\left((1+\nu)a^p - \sum_{i=1}^{n} \mu_i b_i^p\right)^{\frac{1}{p}}\right] \\
= \quad & h(1+\nu)f(x) - f\left(\left[(1+\nu)x^p - \nu y^p\right]^{\frac{1}{p}}\right) \\
\leq \quad & h(\nu) f\left(\left[\sum_{i=1}^{n} \frac{\mu_i}{\nu} b_i^p\right]^{\frac{1}{p}}\right) \\
\leq \quad & h(\nu) \sum_{i=1}^{n} h\left(\frac{\mu_i}{\nu}\right) f(b_i) \text{ [by (6)]} \\
\leq \quad & \sum_{i=1}^{n} h\left(\mu_i\right) f(b_i).
\end{aligned}$$

This completes the proof. □

We now give the following simple inequality, which is a one-term refinement of Theorem 2.4, allowing us to obtain the general form presented in Theorem 2.7.

Theorem 2.5. *Let $(a, \mathbf{b}, \tilde{\mu}) \in \Im_{p,n}$ and the function $f : I \to [0, +\infty)$ be a (p, h)-convex. If the non-negative function h is super-multiplicative and super-additive, we then have*

$$\begin{aligned}
h(1+\nu)f(a) - \sum_{i=1}^{n} h\left(\mu_i\right) f\left(b_i\right) \leq f & \left[\left((1+\nu)a^p - \sum_{i=1}^{n} \mu_i b_i^p\right)^{\frac{1}{p}}\right] \\
& - (n+1)h\left(\mu_{\min}\right)\left[\frac{1}{n+1}\Big(f(a) + \sum_{i=1}^{n} f(b_i)\Big) - f\left(\left(\frac{a^p + \sum_{i=1}^{n}(b_i)^p}{n+1}\right)^{\frac{1}{p}}\right)\right],
\end{aligned}$$

where $\sum_{i=1}^{n} \mu_i = \nu$ *and* $\mu_{\min} = \min\{\mu_i : i = 1, 2, \ldots, n\}$.

Proof. Since h is super-multiplicative and super-additive, we have

$$\begin{aligned}
I &:= h(1+\nu)f(a) - \sum_{i=1}^{n} h(\mu_i)f(b_i) \\
&\quad + (n+1)h(\mu_{\min})\left[\frac{1}{n+1}\Big(f(a)+\sum_{i=1}^{n} f(b_i)\Big) - f\left(\left(\frac{a^p+\sum_{i=1}^{n}(b_i)^p}{n+1}\right)^{\frac{1}{p}}\right)\right] \\
&= \left(h(1+\nu)+h(\mu_{\min})\right)f(a) + \sum_{i=1}^{n}\left(-h(\mu_i)+h(\mu_{\min})\right)f(b_i) \\
&\qquad - (n+1)h(\mu_{\min})f\left(\left(\frac{a^p+\sum_{i=1}^{n}(b_i)^p}{n+1}\right)^{\frac{1}{p}}\right) \\
&\le h(1+\nu+\mu_{\min})f(a) - \sum_{i=1}^{n} h(\mu_i-\mu_{\min})f(b_i) \\
&\qquad - (n+1)h(\mu_{\min})f\left(\left(\frac{a^p+\sum_{i=1}^{n}(b_i)^p}{n+1}\right)^{\frac{1}{p}}\right).
\end{aligned}$$

We now define the $(2n+1)$-tuples $\tilde{\mu}' = (\mu_i')_{i=1}^{2n+1}$ and $\mathbf{b}' = (b_i')_{i=1}^{2n+1}$ by

$$\mu_i' = \begin{cases} \mu_i - \mu_{\min} & \text{if } 1 \le i \le n, \\ \mu_{\min} & \text{if } i \ge n+1, \end{cases} \quad \text{and} \quad b_i' = \begin{cases} b_i & \text{if } 1 \le i \le n, \\ \left(\frac{a^p+\sum_{i=1}^{n}(b_i)^p}{n+1}\right)^{\frac{1}{p}} & \text{if } i \ge n+1. \end{cases}$$

Clearly, $\sum_{i=1}^{2n+1} \mu_i' = \nu + \mu_{\min}$. By a simple calculation, we can prove that

$$\left(1+\sum_{i=1}^{2n+1}\mu_i'\right)a^p - \sum_{i=1}^{2n+1}\mu_i'(b_i')^p = \left(1+\sum_{i=1}^{n}\mu_i\right)a^p - \sum_{i=1}^{n}\mu_i b_i^p.$$

This implies that $(a, \mathbf{b}', \tilde{\mu}') \in \Im_{p,2n+1}$. Hence, it follows from Theorem 2.4 that

$$\begin{aligned}
I &\le f\left(\left[(1+\nu+\mu_{\min})a^p - \sum_{i=1}^{n}(\mu_i-\mu_{\min})b_i^p - (n+1)\mu_{\min}\left(\frac{a^p+\sum_{i=1}^{n} b_i^p}{n+1}\right)\right]^{\frac{1}{p}}\right) \\
&= f\left[\left((1+\nu)a^p - \sum_{i=1}^{n}\mu_i b_i^p\right)^{\frac{1}{p}}\right].
\end{aligned}$$

This completes the proof. □

Corollary 2.6. *Under the same notations as in Theorem 2.5, we have*

$$\begin{aligned}
h(1+\nu)h(a) - \sum_{i=1}^{n} h(\mu_i)f(b_i) &\le f\left[\left((1+\nu)a^p - \sum_{i=1}^{n}\mu_i b_i^p\right)^{\frac{1}{p}}\right] \\
&\quad - h((n+1)\mu_{\min})\left[h\left(\frac{1}{n+1}\right)\Big(f(a)+\sum_{i=1}^{n} f(b_i)\Big) - f\left(\left(\frac{a^p+\sum_{i=1}^{n}(b_i)^p}{n+1}\right)^{\frac{1}{p}}\right)\right].
\end{aligned}$$

Proof. By Theorem 2.5, it suffices to show that

$$(n+1)h(\mu_{\min})\left[\frac{1}{n+1}\Big(f(a)+\sum_{i=1}^{n}f(b_i)\Big)-f\left(\left(\frac{a^p+\sum_{i=1}^{n}(b_i)^p}{n+1}\right)^{\frac{1}{p}}\right)\right]$$
$$\geq h((n+1)\mu_{\min})\left[h\left(\frac{1}{n+1}\right)\Big(f(a)+\sum_{i=1}^{n}f(b_i)\Big)-f\left(\left(\frac{a^p+\sum_{i=1}^{n}(b_i)^p}{n+1}\right)^{\frac{1}{p}}\right)\right].$$

Indeed, using the super-multiplicative and super-additive of h, and the non-negative of f, we find that

$$\begin{aligned}
&(n+1)h(\mu_{\min})\left[\frac{1}{n+1}\Big(f(a)+\sum_{i=1}^{n}f(b_i)\Big)-f\left(\left(\frac{a^p+\sum_{i=1}^{n}(b_i)^p}{n+1}\right)^{\frac{1}{p}}\right)\right]\\
&= h(\mu_{\min})\Big(f(a)+\sum_{i=1}^{n}f(b_i)\Big)-(n+1)h(\mu_{\min})f\left(\left(\frac{a^p+\sum_{i=1}^{n}(b_i)^p}{n+1}\right)^{\frac{1}{p}}\right)\\
&\geq h\Big(\frac{(n+1)\mu_{\min}}{n+1}\Big)\Big(f(a)+\sum_{i=1}^{n}f(b_i)\Big)-h((n+1)\mu_{\min})f\left(\left(\frac{a^p+\sum_{i=1}^{n}(b_i)^p}{n+1}\right)^{\frac{1}{p}}\right)\\
&\geq h((n+1)\mu_{\min})h\Big(\frac{1}{n+1}\Big)\Big(f(a)+\sum_{i=1}^{n}f(b_i)\Big)-h((n+1)\mu_{\min})f\left(\left(\frac{a^p+\sum_{i=1}^{n}(b_i)^p}{n+1}\right)^{\frac{1}{p}}\right)\\
&= h((n+1)\mu_{\min})\Big(h\Big(\frac{1}{n+1}\Big)\Big(f(a)+\sum_{i=1}^{n}f(b_i)\Big)-f\left(\left(\frac{a^p+\sum_{i=1}^{n}(b_i)^p}{n+1}\right)^{\frac{1}{p}}\right)\Big).
\end{aligned}$$

This finishes the proof. □

Before presenting the refinement of inequality (11) by many terms, we need to introduce some notations which will be used in the sequel.

The cardinal of a subset A of $\mathbb{N}$ is denoted by $|A|$. For $\tilde{\mu}^{(1)}=\left(\mu_1^{(1)},\ldots,\mu_n^{(1)}\right)\in\mathbb{R}_+^n$, we define the sequence $\left\{\tilde{\mu}^{(k)}\right\}_{k\in\mathbb{N}^*}$ of elements of $\mathbb{R}_+^n$ by

$$\mu_i^{(k+1)}=\begin{cases}\mu_i^{(k)}-\mu_{\min}^{(k)} & \text{if } \mu_i^{(k)}\neq\mu_{\min}^{(k)}\\ \frac{(n+1)}{|A_k|}\mu_{\min}^{(k)} & \text{if } \mu_i^{(k)}=\mu_{\min}^{(k)}\end{cases}\quad(1\leq i\leq n), \tag{13}$$

where $k\in\mathbb{N}^*$, $\tilde{\mu}^{(k)}=\left(\mu_1^{(k)},\ldots,\mu_n^{(k)}\right)$, $\mu_{\min}^{(k)}=\min\limits_{1\leq i\leq n}\mu_i^{(k)}$ and $A_k=\left\{i:\mu_i^{(k)}=\mu_{\min}^{(k)}\right\}$.

We now give another sequence of elements of I^n which is associated to the sequence $\left\{\tilde{\mu}^{(k)}\right\}_{k\in\mathbb{N}^*}$. Given $\mathbf{b}^{(1)}=\left(b_1^{(1)},\ldots,b_n^{(1)}\right)\in I^n$, the sequence $\left\{\mathbf{b}^{(k)}\right\}_{k\in\mathbb{N}^*}$ of elements of I^n is defined by induction as follows:

$$b_i^{(k+1)}=\begin{cases}b_i^{(k)} & \text{if } \mu_i^{(k)}\neq\mu_{\min}^{(k)}\\ \left[\dfrac{a^p+\sum_{i=1}^{n}\left(b_i^{(k)}\right)^p}{n+1}\right]^{\frac{1}{p}} & \text{if } \mu_i^{(k)}=\mu_{\min}^{(k)}\end{cases}\quad(1\leq i\leq n), \tag{14}$$

where $k\in\mathbb{N}^*$ and $\mathbf{b}^{(k)}=\left(b_1^{(k)},\ldots,b_n^{(k)}\right)$.

We now present our improvement of the extrapolated version of Jensen's inequality for (p,h)-convex functions by many terms.

Theorem 2.7. *Let* $\left(a, \mathbf{b}^{(1)}, \tilde{\mu}^{(1)}\right) \in \Im_{p,n}$ *and* $f : I \to [0, +\infty)$ *be a* (p, h)*-convex function. If the non-negative function* h *is super-multiplicative and super-additive, then for every* $N \in \mathbb{N}^*$*, we have*

$$\begin{aligned} & h(1+\nu^{(1)})f(a) - \sum_{i=1}^{n} h(\mu_i^{(1)}) f\left(b_i^{(1)}\right) \\ & + \sum_{k=1}^{N} h\left((n+1)\mu_{\min}^{(k)}\right)\left[h\Big(\frac{1}{n+1}\Big)\Big(f(a) + \sum_{i=1}^{n} f(b_i^{(k)})\Big) - f\left(\left(\frac{a^p + \sum_{i=1}^{n}(b_i^{(k)})^p}{n+1}\right)^{\frac{1}{p}}\right)\right] \\ & \leq f\left[\left((1+\nu^{(1)})a^p - \sum_{i=1}^{n}\mu_i(b_i^{(1)})^p\right)^{\frac{1}{p}}\right], \end{aligned} \tag{15}$$

where $\nu^{(1)} = \sum_{i=1}^{n} \mu_i^{(1)}$.

Proof. Before we start our proof, it should be noted that $h(0) = 0$ and $mh(x) \leq h(mx)$ for all $x \in J$ and all $m \in \mathbb{N}$. Now, let $N \in \mathbb{N}^*$. For $k \in \{1, \ldots, N\}$, we define the following terms

$$\begin{aligned} \Gamma_k &= h\left((n+1)\mu_{\min}^{(k)}\right)\left[h\Big(\frac{1}{n+1}\Big)\Big(f(a) + \sum_{i=1}^{n} f(b_i^{(k)})\Big) - f\left(\left(\frac{a^p + \sum_{i=1}^{n}(b_i^{(k)})^p}{n+1}\right)^{\frac{1}{p}}\right)\right], \\ \Theta_k &= h\left(1+\nu^{(k)}\right) f(a) - \sum_{i=1}^{n} h(\mu_i^{(k)}) f\left(b_i^{(k)}\right), \\ \nu^{(k)} &= \sum_{i=1}^{n} \mu_i^{(k)}. \end{aligned}$$

For every $k \in \{1, \ldots, N-1\}$, we have

$$\begin{aligned} \nu^{(k+1)} &= \sum_{i=1}^{n} \mu_i^{(k+1)} \\ &= \sum_{i=1}^{n}\left(\mu_i^{(k)} - \mu_{\min}^{(k)}\right) + \sum_{i \in A_k} \frac{(n+1)\mu_{\min}^{(k)}}{|A_k|} \\ &= \nu^{(k)} - n\mu_{\min}^{(k)} + (n+1)\mu_{\min}^{(k)} \\ &= \nu^{(k)} + \mu_{\min}^{(k)}, \end{aligned}$$

and

$$\begin{aligned} \Theta_k + \Gamma_k &\leq \left(h(1+\nu^{(k)}) + h(\mu_{\min}^{(k)})\right) f(a) - \sum_{i=1}^{n}\left(h\left(\mu_i^{(k)}\right) - h\left(\mu_{\min}^{(k)}\right)\right) f\left(b_i^{(k)}\right) \\ &- h\left((n+1)\mu_{\min}^{(k)}\right) f\left(\left(\frac{a^p + \sum_{i=1}^{n}(b_i^{(k)})^p}{n+1}\right)^{\frac{1}{p}}\right) \\ &\leq h\left(1+\nu^{(k)} + \mu_{\min}^{(k)}\right) f(a) - \sum_{i=1}^{n} h\left(\mu_i^{(k)} - \mu_{\min}^{(k)}\right) f\left(b_i^{(k)}\right) \\ &- h\left((n+1)\mu_{\min}^{(k)}\right) f\left(\left(\frac{a^p + \sum_{i=1}^{n}(b_i^{(k)})^p}{n+1}\right)^{\frac{1}{p}}\right) \end{aligned}$$

$$
\begin{aligned}
&\leq h\left(1+\nu^{(k)}+\mu_{\min}^{(k)}\right) f(a)-\sum_{i \notin A_k} h\left(\mu_i^{(k)}-\mu_{\min}^{(k)}\right) f\left(b_i^{(k)}\right) \\
&- |A_k| h\left(\frac{(n+1)\mu_{\min}^{(k)}}{|A_k|}\right) f\left(\left(\frac{a^p+\sum_{i=1}^n (b_i^{(k)})^p}{n+1}\right)^{\frac{1}{p}}\right) \\
&= h\left(1+\nu^{(k)}\right) f(a)-\sum_{i=1}^n h(\mu_i^{(k)}) f\left(b_i^{(k)}\right) \\
&= \Theta_{k+1}.
\end{aligned}
$$

From this, one can obtain that

$$
\Theta_k+\sum_{l=k}^N \Gamma_l \leq \Theta_{k+1}+\sum_{l=k+1}^N \Gamma_l \quad \text{for all } 1 \leq k \leq N-1.
$$

As a consquence, we get that

$$
\Theta_1+\sum_{l=1}^N \Gamma_l \leq \Theta_N+\Gamma_N. \tag{16}
$$

On the other hand, for every $k \in \{1,\ldots,N-1\}$, we have

$$
\begin{aligned}
&\left(1+\nu^{(k+1)}\right) a^p-\sum_{i=1}^n \mu_i^{(k+1)}\left(b_i^{(k+1)}\right)^p \\
= &\left(1+\nu^{(k)}+\mu_{\min}^{(k)}\right) a^p-\sum_{i=1}^n\left(\mu_i^{(k)}-\mu_{\min}^{(k)}\right)\left(b_i^{(k)}\right)^p-(n+1)\mu_{\min}^{(k)} \frac{a^p+\sum_{i=1}^n\left(b_i^{(k)}\right)^p}{n+1} \\
= &\left(1+\nu^{(k)}\right) a^p-\sum_{i=1}^n \mu_i^{(k)}\left(b_i^{(k)}\right)^p .
\end{aligned}
$$

In particular,

$$
\left(1+\nu^{(N)}\right) a^p-\sum_{i=1}^n \mu_i^{(N)}\left(b_i^{(N)}\right)^p=\left(1+\nu^{(1)}\right) a^p-\sum_{i=1}^n \mu_i^{(1)}\left(b_i^{(1)}\right)^p . \tag{17}
$$

Hence, by combining (16) and (17) together with Theorem 2.5, we obtain the desired inequality. □

An immediate consequence of Theorem 2.7 is the following (p,h)-log-convex version. Recall that a function $f: I \to (0,+\infty)$ is said to be (p,h)-log-convex if the function $\log \circ f$ is (p,h)-convex.

Corollary 2.8. *Let $\left(a,\mathbf{b}^{(1)},\tilde{\mu}^{(1)}\right) \in \Im_{p,n}$ and $f: I \to [0,+\infty)$ be a (p,h)-log-convex function. If h is super-multiplicative and super-additive function, then for every $N \in \mathbb{N}^*$, we have*

$$
\begin{aligned}
&\frac{f^{h(1+\nu^{(1)})}(a)}{\prod_{i=1}^n f^{h(\mu_i^{(1)})}\left(b_i^{(1)}\right)} \prod_{k=1}^N\left[\frac{\left(f(a) \prod_{i=1}^n f\left(b_i^{(k)}\right)\right)^{h\left(\frac{1}{n+1}\right)}}{f\left(\left[\frac{a^p+\sum_{i=1}^n (b_i^{(k)})^p}{n+1}\right]^{\frac{1}{p}}\right)}\right]^{h\left((n+1)\mu_{\min}^{(k)}\right)} \\
&\leq f\left[\left(\left(1+\nu^{(1)}\right) a^p-\sum_{i=1}^n \mu_i^{(1)}(b_i^{(1)})^p\right)^{\frac{1}{p}}\right],
\end{aligned}
$$

where $\nu^{(1)} = \sum_{i=1}^{n} \mu_i^{(1)}$.

If we take $n = 1$ in Theorem 2.7, we obtain the following result which generalizes [3, Theorem 2.5].

Corollary 2.9. *Let* $(a, b, \mu) \in \Im_{p,1}$ *and* $f : I \to [0, +\infty)$ *be a* (p, h)*-convex function. If* h *is super-multiplicative and super-additive function, then for every* $N \in \mathbb{N}^*$*, we have*

$$\begin{aligned} & h(1+\mu)f(a) \quad - \quad h(\mu)f(b) \\ + & \sum_{k=1}^{N} h(2^i\mu)\left[h\left(\frac{1}{2}\right)\left(f(a) + f\left(\left[\frac{\left(2^{k-1}-1\right)a^p + b^p}{2^{k-1}}\right]^{\frac{1}{p}}\right)\right) - f\left(\left[\frac{\left(2^{k}-1\right)a^p + b^p}{2^{k}}\right]^{\frac{1}{p}}\right)\right] \\ & \leq f\left[((1+\mu)a^p - \mu b^p)^{\frac{1}{p}}\right]. \end{aligned}$$

3. Improved Jensen's inequality for (p, h)-convex functions

We now start this section with the following improvement of the first inequality in Theorem 1.5 when $\lambda = 1$.

Theorem 3.1. *Let* $\mu_1, \ldots, \mu_n$ *be positive real numbers* $(n \geqslant 2)$ *such that* $\sum_{i=1}^{n} \mu_i = 1$, $f : I \to \mathbb{R}_+$ *be a* (p, h)*-convex function and* $a_1, \ldots, a_n \in I$. *If* h *is a super-multiplicative and super-additive function, then*

$$\begin{aligned} \sum_{i=1}^{n} h(\mu_i) f(a_i) - f\left(\left(\sum_{i=1}^{n} \mu_i a_i^p\right)^{\frac{1}{p}}\right) \quad & \geq \quad nh(\mu_{\min})\left(\frac{1}{n}\sum_{i=1}^{n} f(a_i) - f\left(\left(\frac{1}{n}\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}}\right)\right) \\ & \geq h(n\mu_{\min})\left(\left(h\left(\frac{1}{n}\right)\sum_{i=1}^{n} f(a_i)\right) - f\left(\left(\frac{1}{n}\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}}\right)\right), \end{aligned}$$

where $\mu_{\min} = \min\{\mu_i : i = 1, \ldots, n\}$.

Proof. Since h is super-multiplicative and super-additive, we have

$$\begin{aligned} & \sum_{i=1}^{n} h(\mu_i) f(a_i) - nh(\mu_{\min})\left(\frac{1}{n}\sum_{i=1}^{n} f(a_i) - f\left(\left(\frac{1}{n}\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}}\right)\right) \\ & = \sum_{i=1}^{n} \left(h(\mu_i) - h(\mu_{\min})\right) f(a_i) + nh(\mu_{\min}) f\left(\left(\frac{1}{n}\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}}\right) \\ & \geq \sum_{i=1}^{n} h(\mu_i - \mu_{\min}) f(a_i) + nh(\mu_{\min}) f\left(\left(\frac{1}{n}\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}}\right) \\ & \geq f\left[\left(\sum_{i=1}^{n} (\mu_i - \mu_{\min}) a_i^p + n\mu_{\min}\left(\frac{1}{n}\sum_{i=1}^{n} a_i^p\right)\right)^{\frac{1}{p}}\right] \text{by (6)} \\ & = f\left(\left(\sum_{i=1}^{n} \mu_i a_i^p\right)^{\frac{1}{p}}\right). \end{aligned}$$

The second inequality follows from the fact that

$$
\begin{aligned}
& nh(\mu_{\min})\left(\frac{1}{n}\sum_{i=1}^{n} f(a_i) - f\left(\left(\frac{1}{n}\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}}\right)\right) \\
& \geq h(\mu_{\min})\sum_{i=1}^{n} f(a_i) - h(n\mu_{\min}) f\left(\left(\frac{1}{n}\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}}\right) \\
& \geq h(n\mu_{\min})\left(\left(h\left(\frac{1}{n}\right)\sum_{i=1}^{n} f(a_i)\right) - f\left(\left(\frac{1}{n}\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}}\right)\right).
\end{aligned}
$$

□

The purpose of the rest of this section is to extend Theorem 3.1 to a more general framework using the so-called weak sub-majorization theory. In the sequel, we denote by $X^* = \left(X_1^*, \ldots, X_n^*\right)$ the vector obtained from the vector $X = (X_1, \ldots, X_n) \in \mathbb{R}^n$ by rearranging the components of it in decreasing order, i.e., $X_n^* \geq \cdots \geq X_1^*$.

Let $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$ be two vectors with real components, we say that Y is weakly sub-majorized by X, written $X \succ_w Y$, if

$$
\sum_{i=1}^{k} X_i^* \geq \sum_{i=1}^{k} Y_i^* \quad \text{for all} \;\; k = 1, \ldots, n.
$$

A very important tool in weak sub-majorization, which will be used to prove our results, is given in the following lemma.

Lemma 3.2. *[9, pp. 13] Let $X = (X_i)_{i=1}^n, Y = (Y_i)_{i=1}^n \in \mathbb{R}^n$ and $J \subset \mathbb{R}$ be an interval containing the components of X and Y. If $X \succ_w Y$ and $\psi : J \to \mathbb{R}$ is a continuous increasing convex function, then*

$$
\sum_{i=1}^{n} \psi(X_i) \geq \sum_{i=1}^{n} \psi(Y_i).
$$

The following lemmas allow us to derive the general form of Theorem 3.1.

Lemma 3.3. *Let $\mu_1, \ldots, \mu_n$ be positive real numbers ($n \geqslant 2$) such that $\sum_{i=1}^{n} \mu_i = 1$, $a_1, \ldots, a_n \in I$ and $f : I \to \mathbb{R}_+$ be a (p, h)-convex function on I. Let $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ be two vectors in $\mathbb{R}^2$ with components*

$$
\begin{aligned}
X_1 &= \sum_{i=1}^{n} h(\mu_i) f(a_i), \quad X_2 = nh(\mu_{\min}) f\left(\left[\sum_{i=1}^{n} \frac{1}{n} a_i^p\right]^{\frac{1}{p}}\right) \\
Y_1 &= f\left(\left[\sum_{i=1}^{n} \mu_i a_i^p\right]^{\frac{1}{p}}\right) \;\; \text{and} \;\; Y_2 = h(\mu_{\min}) \sum_{i=1}^{n} f(a_i).
\end{aligned}
$$

If h is a super-multiplicative and super-additive function, then we have $X \succ_w Y$, namely, the vectors X^* and Y^* have components satisfying that

$$
X_1^* \geq Y_1^*, \tag{18}
$$
$$
X_1^* + X_2^* \geq Y_1^* + Y_2^*. \tag{19}
$$

Proof. The second inequality (19) comes directely from the first one in Theorem 3.1, since $X_1 + X_2 \geq Y_1 + Y_2$. To prove the first inequality (18), it is sufficient to show that $X_1 \geq Y_i$ for $i = 1, 2$. Indeed, on one hand, we have

$$\begin{aligned} X_1 - Y_2 &= \sum_{i=1}^{n} h(\mu_i) f(a_i) - h(\mu_{\min}) \sum_{i=1}^{n} f(a_i) \\ &= \sum_{i=1}^{n} \left[h(\mu_i) - h(\mu_{\min})\right] f(a_i) \\ &\geq \sum_{i=1}^{n} h(\mu_i - \mu_{\min}) f(a_i) \\ &\geq 0. \end{aligned}$$

On the other hand, by Theorem 1.4, we get that $X_1 \geq Y_1$. Hence, from this we can conclude that $X_1^* \geq Y_1^*$. □

Lemma 3.4. *Let $\mu_1, \ldots, \mu_n$ be positive real numbers ($n \geqslant 2$) such that $\sum_{i=1}^{n} \mu_i = 1$, $a_1, \ldots, a_n \in I$ and $f : I \to \mathbb{R}_+$ be a (p, h)-convex function on I. Let $T = (T_1, T_2)$ and $Z = (Z_1, Z_2)$ be two vectors in $\mathbb{R}^2$ with components*

$$T_1 = h(\mu_{\min}) \sum_{i=1}^{n} f(a_i), \quad T_2 = h(n\mu_{\min}) f\left(\left[\sum_{i=1}^{n} \frac{1}{n} a_i^p\right]^{\frac{1}{p}}\right)$$

$$Z_1 = h(n\mu_{\min}) h\left(\frac{1}{n}\right) \sum_{i=1}^{n} f(a_i) \ \ \text{and} \ \ Z_2 = nh(\mu_{\min}) f\left(\left[\sum_{i=1}^{n} \frac{1}{n} a_i^p\right]^{\frac{1}{p}}\right).$$

If h is a super-multiplicative and super-additive function, then we have $T \succ_w Z$, namely, the vectors T^* and Z^* have components satisfying that

$$T_1^* \geq T_1^*, \tag{20}$$
$$T_1^* + T_2^* \geq Z_1^* + Z_2^*. \tag{21}$$

Proof. Since h is supermultipicative and superadditive, we have $T_1 \geq Z_1$ and $T_2 \geq Z_2$. This implies that $T_1^* \geq Z_i$ for $i = 1, 2$. Consequently, we get (18). The second inequality (19) comes directly from the second inequality of Theorem 3.1. □

Theorem 3.5. *Let $\mu_1, \ldots, \mu_n$ be positive real numbers ($n \geqslant 2$) such that $\sum_{i=1}^{n} \mu_i = 1$, $a_1, \ldots, a_n \in I$, $f : I \to \mathbb{R}_+$ be a (p, h)-convex function on I and ψ be a strictly increasing convex function defined on $[0, +\infty)$. If h is super-multiplicative and super-additive function, then we have*

$$\begin{aligned} &\psi\left(\sum_{i=1}^{n} h(\mu_i) f(a_i)\right) - \psi \circ f\left(\left[\sum_{i=1}^{n} \mu_i a_i^p\right]^{\frac{1}{p}}\right) \\ &\qquad \geq \psi\left(h(\mu_{\min}) \sum_{i=1}^{n} f(a_i)\right) - \psi\left(nh(\mu_{\min}) f\left(\left[\sum_{i=1}^{n} \frac{1}{n} a_i^p\right]^{\frac{1}{p}}\right)\right) \\ &\qquad \geq \psi\left(h(\mu_{\min}) h\left(\frac{1}{n}\right) \sum_{i=1}^{n} f(a_i)\right) - \psi\left(h(n\mu_{\min}) f\left(\left[\sum_{i=1}^{n} \frac{1}{n} a_i^p\right]^{\frac{1}{p}}\right)\right). \end{aligned}$$

Proof. Consider the vectors $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$ $T = (T_1, T_2)$ and $Z = (Z_1, Z_1)$ defined in Lemmas 3.3 and 3.4. These lemmas assert that $X \succ_w Y$ and $T \succ_w Z$. Hence, by applying Lemma 3.2, we infer that

$$\psi(X_1) + \psi(X_2) \geq \psi(Y_1) + \psi(Y_2)$$

and

$$\psi(T_1) + \psi(T_2) \geq \psi(Z_1) + \psi(Z_2).$$

These inequalities give the required result. □

Replacing f by $\log f$ in Theorem 3.5, we derive the log-convex version of the previous result as follows.

Theorem 3.6. *Let $\mu_1, \dots, \mu_n$ be positive real numbers ($n \geqslant 2$) such that $\sum_{i=1}^{n} \mu_i = 1$, $a_1, \dots, a_n \in I$, $f : I \to \mathbb{R}_+$ be a (p, h)-log-convex function on I and ψ be a strictly increasing convex function defined on $[0, +\infty)$. If h is a super-multiplicative and super-additive function, then we have*

$$\begin{aligned}
&\psi \circ \log\left(\prod_{i=1}^{n} f^{h(\mu_i)}(a_i)\right) - \psi \circ \log f\left(\left[\sum_{i=1}^{n} \mu_i a_i^p\right]^{\frac{1}{p}}\right) \\
&\quad \geq \psi \circ \log\left(\prod_{i=1}^{n} f^{h(\mu_{\min})}(a_i)\right) - \psi\left(\log f^{nh\left(\mu_{\min}\right)}\left(\left[\sum_{i=1}^{n} \frac{1}{n} a_i^p\right]^{\frac{1}{p}}\right)\right) \\
&\quad \geq \psi \circ \log\left(\prod_{i=1}^{n} f^{h(\mu_{\min})h\left(\frac{1}{n}\right)}(a_i)\right) - \psi\left(\log f^{h\left(n\mu_{\min}\right)}\left(\left[\sum_{i=1}^{n} \frac{1}{n} a_i^p\right]^{\frac{1}{p}}\right)\right).
\end{aligned}$$

Now, by letting $\psi(x) = x^\lambda$ ($\lambda \geq 1, x \geq 0$) in Theorem 3.5, we get the following improvement of Theorem 1.5.

Theorem 3.7. *Let $\mu_1, \dots, \mu_n$ be positive real numbers ($n \geqslant 2$) such that $\sum_{i=1}^{n} \mu_i = 1$, $a_1, \dots, a_n \in I$, $f : I \to \mathbb{R}_+$ be a (p, h)-convex function on I and ψ be a strictly increasing convex function defined on $[0, +\infty)$. If h is super-multiplicative and super-additive function, then we have*

$$\begin{aligned}
&\left(\sum_{i=1}^{n} h(\mu_i) f(a_i)\right)^{\lambda} - f^{\lambda}\left(\left[\sum_{i=1}^{n} \mu_i a_i^p\right]^{\frac{1}{p}}\right) \\
&\qquad \geq \left(nh\left(\mu_{\min}\right)\right)^{\lambda}\left[\left(\sum_{i=1}^{n} f(a_i)\right)^{\lambda} - f^{\lambda}\left(\left[\sum_{i=1}^{n} \frac{1}{n} a_i^p\right]^{\frac{1}{p}}\right)\right] \\
&\qquad \geq \left(h\left(\mu_{\min}\right) h\left(\frac{1}{n}\right) \sum_{i=1}^{n} f(a_i)\right)^{\lambda} - \left(h\left(n\mu_{\min}\right) f\left(\left[\sum_{i=1}^{n} \frac{1}{n} a_i^p\right]^{\frac{1}{p}}\right)\right)^{\lambda}.
\end{aligned}$$

By taking $\psi(x) = \exp(x)$ ($x \in \mathbb{R}$) in Theorem 3.6, we find the following result.

Theorem 3.8. *Let $\mu_1, \dots, \mu_n$ be positive real numbers ($n \geqslant 2$) such that $\sum_{i=1}^{n} \mu_i = 1$, $a_1, \dots, a_n \in I$, $f : I \to \mathbb{R}_+$ be a (p, h)-log-convex function on I and ψ be a strictly increasing convex function defined on $[0, +\infty)$. If h is a* *super-multiplicative and super-additive function, then we have*

$$\begin{aligned}\left(\prod_{i=1}^{n} f^{h(\mu_i)}(a_i)\right) - f\left(\left[\sum_{i=1}^{n} \mu_i a_i^p\right]^{\frac{1}{p}}\right) &\geq \prod_{i=1}^{n} f^{h(\mu_{\min})}(a_i) - f^{nh\left(\mu_{\min}\right)}\left(\left[\sum_{i=1}^{n} \frac{1}{n} a_i^p\right]^{\frac{1}{p}}\right) \\ &\geq \prod_{i=1}^{n} f^{h(\mu_{\min})h\left(\frac{1}{n}\right)}(a_i) - f^{h\left(n\mu_{\min}\right)}\left(\left[\sum_{i=1}^{n} \frac{1}{n} a_i^p\right]^{\frac{1}{p}}\right).\end{aligned}$$

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