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On the filter bornological convergence of the nested sequences of sets

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Abstract. In this work, we show the equivalence of filter bornological convergence and bornological convergence for the nested sequences of sets on topological vector spaces. Then we investigate the filter bornological limit of the nested sequence of sets.

1. Introduction

In 1951, Fast ([13]) and Steinhaus ([27]) defined the concept of statistical convergence, which is a weaker type of convergence than classical convergence. In 2000, Kostyrko et al ([19]) defined the concept of ideal convergence (briefly, $\mathcal{I}$-convergence), which is a more general convergence type that includes classical convergence and statistical convergence (see also [20]). [6] and [7] provide some results regarding statistical convergence and ideal convergence of sequences of functions. Ideals and filters are dual concepts to each other. The reader can refer to [2, 4, 16] for filter convergence (briefly, $\mathcal{F}$-convergence) equivalent to ideal convergence. Today, many researchers continue to work on statistical convergence, ideal convergence and filter convergence.

Many convergence methods have been defined for sequences of sets. Among these convergence types, we can mention Hausdorff convergence, Wijsman convergence and Kuratowski convergence (see [22, 23, 30, 31]). For generalizations of these concepts in terms of statistical convergence and ideal convergence, the reader may refer to [18, 26, 28]. Apreutesei [3] showed that Wijsman convergence and Hausdorff convergence are equivalent to each other for monotone sequences of sets. In [1], it has been shown that ideal Hausdorff convergence and Hausdorff convergence are equivalent, and ideal Wijsman convergence and Wijsman convergence are equivalent for nested sequence of sets.

The concept of bornology has the structure of ideals, the only difference is that it creates a cover. In the literature, there are studies on bornological spaces, bornological convergence (briefly, $\mathcal{B}$-convergence) and different types of convergence on bornologies (see [8, 9, 11, 24]). In [5], it was introduced the concept of filter bornological convergence (briefly, $\mathcal{FB}$-convergence) for sequences of sets on topological vector spaces.

In the second part of this study, we will give basic information about filters, bornologies and their associated convergence types. In the third part, we will present some results related to bornological convergence and filter bornological convergence. We show that bornological convergence and filter bornological convergence are equivalent for nested set sequences.

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2. Preliminaries

Let's start with the definition of topological vector space. Let X be a vector space on the real numbers field $\mathbb{R}$ and let τ be a linear topology on X (that means the operations addition and scalar multiplication are τ-continuous on X). In this case, the pair (X, τ) is called a topological vector space (or linear topological space) and it is denoted by TVS for short (see [17, 21, 29]). Throughout this paper, $\mathrm{Cl}(X)$ and $\mathcal{K}(X)$ denote the family of all nonempty closed subsets and the family of all nonempty compact subsets of X, respectively.

We will denote the zero element of the space X by θ. Each linear topologies on a TVS has a base $\mathcal{N}$ of neighbourhoods of zero, providing the following properties (we use such a base in our proofs):

a. Each $V \in \mathcal{N}$ is a *balanced set* (i.e., $\lambda x \in V$ for each $x \in V$ and each $\lambda \in \mathbb{N}$ with $|\lambda| \leq 1$).

b. Each $V \in \mathcal{N}$ is an *absorbing set* (i.e., for each $x \in X$ there is a $\lambda > 0$ such that $\lambda x \in V$).

c. For each $V \in \mathcal{N}$ there is a set $W \in \mathcal{N}$ such that $W + W \subseteq V$. Here, the operation $W + W$ is defined as $W + W := \{x + y : x, y \in W\}$.

Every normed space is a topological vector space. Also, if a TVS has a countable base of neighbourhoods of θ then it is metrizable.

Let $A \subseteq X$, the closure of A is defined by

$$\mathrm{cl}(A) = \{x \in X : \text{For every } U \in \mathcal{N} \text{ there exists a } y \in A \text{ such that } x - y \in U\}.$$

If (X, τ) is a Hausdorff TVS then the intersection of all neighborhoods of θ has only θ, that is, $\bigcap_{U \in \mathcal{N}} U = \{\theta\}$ (see [17]).

A nonempty family $\mathcal{F}$ of subsets of $\mathbb{N}$ is said to be a filter on $\mathbb{N}$, if it provides the following conditions (see [12, 32]):

i. $\emptyset \notin \mathcal{F}$,

ii. If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$,

iii. If $A \in \mathcal{F}$ and $A \subseteq B$ then $B \in \mathcal{F}$.

If $\mathcal{F} = \{\mathbb{N}\}$ then $\mathcal{F}$ is called a trivial filter, otherwise a non-trivial filter. The family $\mathcal{F} = \{B : A \subseteq B\}$ is called a principal filter generated by A where A is a nonempty subset of $\mathbb{N}$ ($\mathcal{F}$ is nonprincipal otherwise). $\mathcal{F}$ is said to be a free filter if the intersection of all its members is empty (that is, $\bigcap_{A \in \mathcal{F}} A = \emptyset$). If a filter $\mathcal{F}$ is not free then it is called fixed.

Definition of filter convergence of a sequence may be given for an arbitrary topological space and, then in particular, for a topological vector space. The reader can look at [4] and [16] for filter convergence.

Definition 2.1. *Let (X, τ) be a TVS. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X, let $x_0 \in X$ and let $\mathcal{F}$ be a filter on $\mathbb{N}$. The sequence (x_n) is said to be filter convergent (or $\mathcal{F}$-convergent) to the point x_0, if for every neighborhood U of θ we have*

$$\{n \in \mathbb{N} : x_n - x_0 \in U\} \in \mathcal{F}.$$

Then we write $\mathcal{F} - \lim x_n = x_0$ or $x_n \xrightarrow{\mathcal{F}} x_0$.

In the following, we give some examples of filters and filter convergence. $|A|$ denotes the cardinality of the set A.

1. Fréchet Filter: The family $\mathcal{F}_r = \{A \subseteq \mathbb{N} : |\mathbb{N} \setminus A| < \infty\}$ is called the *Fréchet filter*. $\mathcal{F}_r$-convergence coincides with the ordinary convergence. $\mathcal{F}_r$ is the minimum filter by the inclusion relation. So the free filters are characterized as follows:

$$\mathcal{F} \text{ is free} \Longleftrightarrow \mathcal{F} \supseteq \mathcal{F}_r.$$

2. Statistical Convergence Filter: Let $A \subseteq \mathbb{N}$. Let $A(n) = |\{1, ..., n\} \cap A|$ indicate the number of elements in the set A from 1 to n. The functions

$$\underline{\delta}(A) = \liminf_{n\to\infty} \frac{A(n)}{n} \text{ and } \overline{\delta}(A) = \limsup_{n\to\infty} \frac{A(n)}{n}$$

are called the *lower asymptotic density* and *upper asymptotic density* of the set A, respectively. If $\underline{\delta}(A) = \overline{\delta}(A)$, that is, the limit

$$\lim_{n\to\infty} \frac{A(n)}{n}$$

exists, then the value of this limit is called the *asymptotic density* of the set A, and it is denoted by $\delta(A)$ ([10, 14, 25]). The family $\mathcal{F}_{st} = \{A \subseteq \mathbb{N} : \delta(A) = 1\}$ is called the *statistical convergence filter*. $\mathcal{F}_{st}$-convergence coincides with the statistical convergence ([13, 27]).

Now, we give the concept of bornological convergence of a sequence of sets on TVS. For the bornological convergence of nets of sets in metric spaces, the reader may look to [11] and [24]. Let's first give the definition of bornology.

A family $\mathcal{B}$ of subsets of a set X is said to be a bornology, if it provides the following conditions (see [15]):

i. $\mathcal{B}$ is a cover of X, i.e. $X = \bigcup_{\check{B}\in\mathcal{B}} \check{B}$,

ii. $\mathcal{B}$ is closed under subsets, i.e. if $\check{B} \in \mathcal{B}$ and $\check{A} \subseteq \check{B}$ then $\check{A} \in \mathcal{B}$,

iii. $\mathcal{B}$ is closed under finite unions, i.e. if $\check{A}, \check{B} \in \mathcal{B}$ then $\check{A} \cup \check{B} \in \mathcal{B}$.

Items (ii) and (iii) together form the definition of the ideal. That's why every bornology is also an ideal. The converse of this statement is not true, as ideals do not need to form a cover. If an ideal is a cover of a set X, then we call it a bornology on X.

Example 2.2. *The following (1)-(3) families are bornologies on any set X, and families (4) and (5) on* $\mathbb{R}^2$.
1) The power set $\mathcal{P}(X)$.
2) $\mathcal{B}_{\text{fin}} = \left\{\check{B} \subseteq X : \check{B} \textit{ is finite}\right\}$.
3) $\mathcal{B}_{\text{bnd}} = \left\{\check{B} \subseteq X : \check{B} \textit{ is bounded}\right\}$.
4) $\mathcal{B}_{\text{vert}} = \left\{\check{B} \subseteq \mathbb{R}^2 : \check{B} \textit{ is contained in a finite union of vertical lines}\right\}$.
5) $\mathcal{B}_{\text{horz}} = \left\{\check{B} \subseteq \mathbb{R}^2 : \check{B} \textit{ is contained in a finite union of horizontal lines}\right\}$.

In the following, the concept of ε-enlargement of a set on metric spaces is generalized to the concept of U-enlargement on topological spaces.

Definition 2.3. *Let (X, τ) be a TVS and let U be a neighborhood of θ. For $A \subseteq X$, the set*

$$A^U = \{x \in X : x - y \in U \text{ for some } y \in A\}$$

is called U-enlargement of the set A ([5]).

Definition 2.4. *Let (X, τ) be a TVS and let $\mathcal{B}$ be a bornology on X. Take a sequence $(A_n)_{n\in\mathbb{N}}$ of nonempty subsets of X and a set $A \subseteq X$.*

1. *$(A_n)_{n\in\mathbb{N}}$ is said to be lower bornological convergent to the set A, if for each neighborhood U of θ and each $\check{B} \in \mathcal{B}$ there is a $n_0 \in \mathbb{N}$ such that*

   $$A \cap \check{B} \subseteq A_n^U \text{ for every } n \geq n_0.$$

   Then we write $\mathcal{B}^- - \lim A_n = A$.
2. *$(A_n)_{n\in\mathbb{N}}$ is said to be upper bornological convergent to the set A, if for each neighborhood U of θ and each $\check{B} \in \mathcal{B}$ there is a $n_0 \in \mathbb{N}$ such that*

   $$A_n \cap \check{B} \subseteq A^U \text{ for every } n \geq n_0.$$

   Then we write $\mathcal{B}^+ - \lim A_n = A$.
3. *If the sequence $(A_n)_{n\in\mathbb{N}}$ is both lower bornological convergent and upper bornological convergent to the set A then $(A_n)_{n\in\mathbb{N}}$ is called bornological convergent to the set A. In this case, it is denoted by $\mathcal{B} - \lim A_n = A$. (see [5, 11, 24])*

Example 2.5. *Consider the topological vector space $\left(\mathbb{R}^2, \tau\right)$ equipped with Euclidean topology.*
Let $A_n = \left\{(x, y) \in \mathbb{R}^2 : y = x/n\right\}$ for each $n \in \mathbb{N}$ and $A = \left\{(x, y) \in \mathbb{R}^2 : y = 0\right\}$.
Where $\varepsilon > 0$ and $U = U_\varepsilon = \left\{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < \varepsilon\right\}$, the U-enlargement of sets A and A_n are

$$A^U = \left\{(x, y) \in \mathbb{R}^2 : -\varepsilon < y < \varepsilon\right\}$$

and

$$A_n^U = \left\{(x, y) \in \mathbb{R}^2 : \frac{x}{n} - \varepsilon < y < \frac{x}{n} + \varepsilon\right\},$$

respectively.

- *Let $\check{B}$ be any element of the bornology $\mathcal{B}_{\text{vert}}$. Then, there is a set $B = \left\{(x, y) \in \mathbb{R}^2 : x \in \{a_1, a_2, ..., a_m\}\right\}$ such that $\check{B} \subseteq B$ where $a_1, a_2, ..., a_m$ are real constants.*
  Let's take an arbitrary $\varepsilon > 0$ and let U be the neighborhood of $\theta = (0, 0)$ associated with ε. We have

  $$A_n \cap B = \left\{\left(a_1, \frac{a_1}{n}\right), ..., \left(a_m, \frac{a_m}{n}\right)\right\}$$

  for every $n \in \mathbb{N}$. Let $a = \max_{1\leq i\leq m} |a_i|$. Then, there exist an $n_\varepsilon \in \mathbb{N}$ such that $n_\varepsilon > \frac{a}{\varepsilon}$. Hence we get

  $$\left|\frac{a_i}{n}\right| \leq \frac{a}{n} \leq \frac{a}{n_\varepsilon} < \varepsilon$$

  for every $i \in \{1, 2, ..., m\}$ and every $n \geq n_\varepsilon$, and so

  $$A_n \cap \check{B} \subseteq A_n \cap B \subseteq A^U$$

for every $n \geq n_\varepsilon$. Thus, we obtain $\mathcal{B}^+_{\text{vert}} - \lim A_n = A$.
For the same sets and numbers (i.e., $\check{B}, B, U, \varepsilon, a_i, a$), we have

$$A \cap B = \{(a_1, 0), ..., (a_m, 0)\}.$$

Then, there exist an $n_\varepsilon \in \mathbb{N}$ such that $n_\varepsilon > \frac{a}{\varepsilon}$. Hence we get

$$\frac{a}{n} - \varepsilon < 0 < \frac{a}{n} + \varepsilon \Longrightarrow \frac{a_i}{n} - \varepsilon < 0 < \frac{a_i}{n} + \varepsilon$$

for every $i \in \{1, 2, ..., m\}$ and every $n \geq n_\varepsilon$. Hence, for every $n \geq n_\varepsilon$ we get $(a_i, 0) \in A_n^U$ and so

$$A \cap \check{B} \subseteq A \cap B \subseteq A_n^U.$$

Therefore, we obtain $\mathcal{B}^-_{\text{vert}} - \lim A_n = A$. Consequently, $\mathcal{B}_{\text{vert}} - \lim A_n = A$.

- *If we specifically choose the set $\check{B}_1 = \left\{(x, y) \in \mathbb{R}^2 : y = 1\right\} \in \mathcal{B}_{\text{horz}}$, we get*

  $$A_n \cap \check{B}_1 \nsubseteq A^U \text{ for every } n \in \mathbb{N}$$

  for each the neighborhood U of $\theta = (0, 0)$ associated with $0 < \varepsilon < 1$. Therefore, the sequence (A_n) is not $\mathcal{B}^+_{\text{horz}}$-convergent to A.
  Similarly, if we specifically choose the set $\check{B}_2 = A = \left\{(x, y) \in \mathbb{R}^2 : y = 0\right\} \in \mathcal{B}_{\text{horz}}$, we get

  $$A \cap \check{B}_2 = A \nsubseteq A_n^U \text{ for every } n \in \mathbb{N}$$

  for each the neighborhood U of $\theta = (0, 0)$ associated with $\varepsilon > 0$. Therefore, the sequence (A_n) is not $\mathcal{B}^-_{\text{horz}}$-convergent to A. Consequently, (A_n) is not $\mathcal{B}_{\text{horz}}$-convergent to A. □

In the following, the concept of filter bornological convergence, which is a generalization of bornological convergence, is given.

Definition 2.6. *Let (X, τ) be a TVS, let $\mathcal{B}$ be a bornology on X and let $\mathcal{F}$ be a filter on $\mathbb{N}$. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of nonempty subsets of X and let $A \subseteq X$.*

1. *The sequence $(A_n)_{n\in\mathbb{N}}$ is said to be filter lower bornological convergent to the set A, if for each neighborhood U of θ and each $\check{B} \in \mathcal{B}$ we have*

   $$\left\{n \in \mathbb{N} : A \cap \check{B} \subseteq A_n^U\right\} \in \mathcal{F}$$

   and then we write $\mathcal{F}\mathcal{B}^-\text{-}\lim A_n = A$.
2. *The sequence $(A_n)_{n\in\mathbb{N}}$ is said to be filter upper bornological convergent to the set A, if for each neighborhood U of θ and each $\check{B} \in \mathcal{B}$ we have*

   $$\left\{n \in \mathbb{N} : A_n \cap \check{B} \subseteq A^U\right\} \in \mathcal{F}$$

   and then we write $\mathcal{F}\mathcal{B}^+\text{-}\lim A_n = A$.
3. *If the sequence $(A_n)_{n\in\mathbb{N}}$ is both filter lower bornological convergent and filter upper bornological convergent to the set A, that is, for each neighborhood U of θ and each $\check{B} \in \mathcal{B}$ we have*

   $$\left\{n \in \mathbb{N} : A \cap \check{B} \subseteq A_n^U \text{ and } A_n \cap \check{B} \subseteq A^U\right\} \in \mathcal{F}$$

   then the sequence $(A_n)_{n\in\mathbb{N}}$ is called filter bornological convergent to the set A. In this case, it is denoted by $\mathcal{F}\mathcal{B} - \lim A_n = A$.
   (see [5])

For $\mathcal{F} = \mathcal{F}_r$, $\mathcal{F}_r\mathcal{B}$-convergence of a sequence of sets is equivalent to its bornological convergence.

Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of nonempty subsets of X. If $A_n \subseteq A_{n+1}$ (resp. $A_{n+1} \subseteq A_n$) for every $n \in \mathbb{N}$ then $(A_n)_{n\in\mathbb{N}}$ is called a monotone increasing sequence (resp. monotone decreasing sequence). We say that $(A_n)_{n\in\mathbb{N}}$ is a nested sequence if it is monotonically increasing or decreasing.

In the following example, $\mathbb{P}$ denotes the set of all prime numbers. It is a known fact that $\delta(\mathbb{P}) = 0$ (hence, $\mathbb{N} \setminus \mathbb{P} \in \mathcal{F}_{st}$).

Example 2.7. *Let $\mathcal{B}$ be any bornology on the TVS $\left(\mathbb{R}^2, \tau\right)$ with Euclidean topology. Let's consider the sequence $(A_n)_{n\in\mathbb{N}}$ defined as*

$$A_n = \begin{cases} \left\{(x,y) \in \mathbb{R}^2 : x^2 + ny^2 \leq 16\right\} & , n \notin \mathbb{P} \\ \left\{(x,y) \in \mathbb{R}^2 : \left(1 + \frac{30}{n}\right)x^2 + y^2 \leq 16\right\} & , n \in \mathbb{P} \end{cases}$$

and let $A = \left\{(x,y) \in \mathbb{R}^2 : -4 \leq x \leq 4 \text{ and } y = 0\right\}$.
For every neighborhood U of $\theta = (0,0)$ and every $\check{B} \in \mathcal{B}$ we have

$$\begin{aligned} A \cap \check{B} &\subseteq A \subseteq A_n \subseteq A_n^U \text{ for each } n \in \mathbb{N} \setminus \mathbb{P} \\ A_n \cap \check{B} &\subseteq A_n \subseteq A^U \text{ for each } n \in \mathbb{N} \setminus (\mathbb{P} \cup M) \end{aligned}$$

where $M = M\left(U, \check{B}\right)$ is a finite subset of $\mathbb{N}$. Then, we get

$$\left\{n \in \mathbb{N} : A \cap \check{B} \subseteq A_n^U \text{ and } A_n \cap \check{B} \subseteq A^U\right\} \supseteq \mathbb{N} \setminus (\mathbb{P} \cup M) \in \mathcal{F}_{st}$$

and so

$$\left\{n \in \mathbb{N} : A \cap \check{B} \subseteq A_n^U \text{ and } A_n \cap \check{B} \subseteq A^U\right\} \in \mathcal{F}_{st}$$

for every U and every $\check{B}$. Therefore, we obtain $\mathcal{F}_{st}\mathcal{B} - \lim A_n = A$.
But this sequence is not bornologically convergent to A. For every neighborhood U of $\theta = (0,0)$ and every $\check{B} \in \mathcal{B}$ we have

$$A \cap \check{B} \subseteq A \subseteq A_n^U \text{ for each } n \in \mathbb{N} \setminus M$$

where $M = M\left(U, \check{B}\right)$ is a finite subset of $\mathbb{N}$. Then, we get $\mathcal{B}^- - \lim A_n = A$.
Specifically, let's choose an neighborhood U_1 of $\theta = (0,0)$ associated with $\varepsilon = 1$ and choose $\check{B}_1 \in \mathcal{B}$ with $(0,2) \in \check{B}_1$. We have

$$A_n \cap \check{B}_1 \nsubseteq A^{U_1} \text{ for each } n \in \mathbb{P} \cup M_1$$

where $M_1 = M_1\left(U_1, \check{B}_1\right)$ is a finite subset of $\mathbb{N}$. Since $\mathbb{P} \cup M_1$ is an infinite set, we get $\mathcal{B}^+ - \lim A_n \neq A$. □

3. Main Results

In this section, we first show that the $\mathcal{FB}$-limit is unique if the limit sets are closed. We give the bornological limit of nested sequences of sets. As our main result, we show that bornological convergence and filter bornological convergence are equivalent for nested sequences of sets.

Theorem 3.1. *Let* (X, τ) *be a TVS,* $\mathcal{B}$ *be a bornology on X, and* $\mathcal{F}$ *be a free filter on* $\mathbb{N}$*. If* $\mathcal{F}\mathcal{B} - \lim A_n = A$ *and* $\mathcal{F}\mathcal{B} - \lim A_n = B$ *where A and B are nonempty sets then* $\mathrm{cl}(A) = \mathrm{cl}(B)$*.*

Proof. Let U be an arbitrary neighborhood of θ and let W be another neighborhood of θ such that $W+W+W \subseteq U$. Let $x \in \mathrm{cl}(A)$. There is a point $y \in A$ such that $x - y \in W$. If $\mathcal{F}\mathcal{B} - \lim A_n = A$ and $\mathcal{F}\mathcal{B} - \lim A_n = B$ then we have

$$K\left(W, \check{B}\right) = \left\{n \in \mathbb{N} : A_n \cap \check{B} \subseteq A^W \text{ and } A \cap \check{B} \subseteq A_n^W\right\} \in \mathcal{F}$$

and

$$L\left(W, \check{B}\right) = \left\{n \in \mathbb{N} : A_n \cap \check{B} \subseteq B^W \text{ and } B \cap \check{B} \subseteq A_n^W\right\} \in \mathcal{F}$$

for each $\check{B} \in \mathcal{B}$. Let's choose a $\check{B}_1 \in \mathcal{B}$ such that $y \in \check{B}_1$. For every $n \in K_1$ we get

$$y \in A \cap \check{B}_1 \subseteq A_n^W \Longrightarrow y \in A_n^W$$

where $K_1 = K_1\left(W, \check{B}_1\right) = \left\{n \in \mathbb{N} : A_n \cap \check{B}_1 \subseteq A^W \text{ and } A \cap \check{B}_1 \subseteq A_n^W\right\} \in \mathcal{F}$. For each $n \in K_1$ there exist $z_n \in A_n$ such that $y - z_n \in W$. None of the points z_n may belong to $\check{B}_1$. Since $K_1 \cap L\left(W, \check{B}\right) \in \mathcal{F}$ for every $\check{B} \in \mathcal{B}$, we can choose an $n_* \in K_1$ and a $\check{B}_2 \in \mathcal{B}$ such that $n_* \in L_2\left(W, \check{B}_2\right)$ and $z_{n_*} \in \check{B}_2$ where $L_2 = L_2\left(W, \check{B}_2\right) = \left\{n \in \mathbb{N} : A_n \cap \check{B}_2 \subseteq B^W \text{ and } B \cap \check{B}_2 \subseteq A_n^W\right\} \in \mathcal{F}$. Then we have

$$z_{n_*} \in A_{n_*} \cap \check{B}_2 \subseteq B^W \Longrightarrow z_{n_*} \in B^W.$$

There exists $t \in B$ such that $z_{n_*} - t \in W$. Hence we get

$$x - t = (x - y) + (y - z_{n_*}) + (z_{n_*} - t) \in W + W + W \subseteq U.$$

Then we have $x \in B^U$. Since U is an arbitrary neighborhood of θ, we get $x \in \mathrm{cl}(B)$. That is, $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$. The inclusion $\mathrm{cl}(B) \subseteq \mathrm{cl}(A)$ can be shown similarly. Consequently, we obtain $\mathrm{cl}(A) = \mathrm{cl}(B)$. □

Theorem 3.2. *Let* (X, τ) *be a first countable TVS and* $\mathcal{B}$ *be a bornology on X. Let* $A_n \in \mathcal{K}(X)$ *for every* $n \in \mathbb{N}$*.*

1. *If the sequence* $(A_n)_{n\in\mathbb{N}}$ *is monotone increasing and* $\mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} A_n\right)$ *is compact then*

$$\mathcal{B} - \lim A_n = \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} A_n\right).$$

2. *If the sequence* $(A_n)_{n\in\mathbb{N}}$ *is monotone decreasing then*

$$\mathcal{B} - \lim A_n = \bigcap_{n\in\mathbb{N}} A_n.$$

Proof.

1. Let's assume that $(A_n)_{n\in\mathbb{N}}$ is a monotone increasing sequence. Let $A := \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} A_n\right)$.
   - Let's choose an arbitrary $\check{B} \in \mathcal{B}$ and an arbitrary neighborhood U of θ. For each $n \in \mathbb{N}$ we have

$$A_n \cap \check{B} \subseteq A_n \subseteq \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} A_n\right) = A \subseteq A^U.$$

   Then we get $\mathcal{B}^+ - \lim A_n = A$.

- Again, let U be an arbitrary neighborhood of θ and let $\check{B} \in \mathcal{B}$. Since A is compact, there are finitely many $y_1, y_2, ..., y_m \in \bigcup_{n\in\mathbb{N}} A_n$ such that $A \subseteq \bigcup_{i=1}^{m} (y_i + U)$. Let's index the y_j's again such that $A \cap \check{B} \subseteq \bigcup_{j=1}^{k} \left(y_j + U\right)$, $k \leq m$. For each $x \in A \cap \check{B}$, there exists a $y_j \in \bigcup_{n\in\mathbb{N}} A_n$ $(j \in \{1, 2, ..., k\})$ such that $x \in y_j + U$, i.e. $x - y_j \in U$. From $y_j \in \bigcup_{n\in\mathbb{N}} A_n$, there is an $n_j \in \mathbb{N}$ such that $y_j \in A_{n_j}$ for each $j \in \{1, 2, ..., k\}$. Let $n_0 := \max\left\{n_j : j \in \{1, 2, ..., k\}\right\}$. Since $(A_n)_{n\in\mathbb{N}}$ is monotone increasing, we have $y_j \in A_n$ for each $j \in \{1, 2, ..., k\}$ and every $n \geq n_0$. If $x - y_j \in U$ and $y_j \in A_n$ then we have $x \in A_n^U$. Hence we have

  $$A \cap \check{B} \subseteq A_n^U \text{ for every } n \geq n_0$$

  (Here n_0 depends on U and $\check{B}$, but not on $x \in A \cap \check{B}$). Therefore we get $\mathcal{B}^- - \lim A_n = A$.
  Consequently, we obtain $\mathcal{B} - \lim A_n = \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} A_n\right)$.

2. Now, let's assume that $(A_n)_{n\in\mathbb{N}}$ is a monotone decreasing sequence and let $A := \bigcap_{n\in\mathbb{N}} A_n$.
   - Let's choose an arbitrary $\check{B} \in \mathcal{B}$ and an arbitrary neighborhood U of θ. We have

     $$A \cap \check{B} \subseteq A = \bigcap_{n\in\mathbb{N}} A_n \subseteq A_n \subseteq A_n^U$$

     for each $n \in \mathbb{N}$. Then we get $\mathcal{B}^- - \lim A_n = A$.
   - Let U be an arbitrary neighborhood of θ. We show that there exists an $n_0 \in \mathbb{N}$ such that

     $$A_n \subseteq A^U$$

     for every $n \geq n_0$. Let's assume that this is not true. That is, there is an infinite set $K = \{n_1 < n_2 < ... < n_k < ...\} \subseteq \mathbb{N}$ such that

     $$A_{n_k} \nsubseteq A^U$$

     for every $k \in \mathbb{N}$. Then, for each $k \in \mathbb{N}$ there exists a point x_k such that $x_k \in A_{n_k}$ and $x_k \notin A^U$. Since $(A_n)_{n\in\mathbb{N}}$ is monotone decreasing, $(x_k)_{k\in\mathbb{N}}$ is a sequence in the set A_1. From the compactness of A_1, the sequence $(x_k)_{k\in\mathbb{N}}$ has a convergent subsequence $\left(x_{k_j}\right)_{j\in\mathbb{N}}$ (say x_0 to it's limit). In this case, we have $x_0 \in A$, and for the neighborhood U there is a $j_0 \in \mathbb{N}$ such that

     $$x_{k_j} - x_0 \in U$$

     for every $j \geq j_0$. Then we get $x_{k_j} \in A^U$ and that is a contradiction. Therefore, there exists an $n_0 \in \mathbb{N}$ such that

     $$A_n \subseteq A^U$$

     for every $n \geq n_0$.
     Now, for an arbitrary $\check{B} \in \mathcal{B}$ and an arbitrary neighborhood U of θ we can say that there exists an $n_0 \in \mathbb{N}$ such that

     $$A_n \cap \check{B} \subseteq A_n \subseteq A^U$$

     for every $n \geq n_0$. Hence we get $\mathcal{B}^+ - \lim A_n = A$.
     Consequently, we obtain $\mathcal{B} - \lim A_n = \bigcap_{n\in\mathbb{N}} A_n$.

□

Now, we give the equivalence of bornological convergence and filter bornological convergence.

Theorem 3.3. *Let (X,τ) be a Hausdorff TVS, $\mathcal{B}$ be a bornology on X, and $\mathcal{F}$ be a free filter on $\mathbb{N}$. Let $(A_n)_{n\in\mathbb{N}}$ be a nested sequence of closed subsets of X and $A \in \mathrm{Cl}(X)$. Then we have:*

$$\mathcal{B}-\lim A_n = A \Longleftrightarrow \mathcal{F}\mathcal{B}-\lim A_n = A.$$

Proof.
$(\Longrightarrow)$: It is easily obtained from the inclusion $\mathcal{F} \supseteq \mathcal{F}_r$.
$(\Longleftarrow)$: Let's assume that $\mathcal{F}\mathcal{B}-\lim A_n = A$. We will prove in two cases according to whether $(A_n)_{n\in\mathbb{N}}$ is increasing or decreasing.
Case 1: Let $(A_n)_{n\in\mathbb{N}}$ be an increasing sequence such that $A_n \subseteq A_{n+1}$ for every $n \in \mathbb{N}$.
Firstly, we will show that $A_n \subseteq A$ for every $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ be an arbitrary constant index and let $u \in A_n$. Since (A_n) is increasing, we have $u \in A_m$ for every $m \geq n$. Since $\mathcal{F}\mathcal{B}-\lim A_n = A$, if for each neighborhood U of θ and each $\check{B} \in \mathcal{B}$ we have

$$K\left(U,\check{B}\right) := \left\{m \in \mathbb{N} : A \cap \check{B} \subseteq A_m^U \text{ and } A_m \cap \check{B} \subseteq A^U\right\} \in \mathcal{F}.$$

Let's choose a $\check{B}_0 \in \mathcal{B}$ such that $u \in \check{B}_0$. Then, for each neighborhood U of θ and each $m \in K\left(U,\check{B}_0\right) \setminus \{1,2,...,n\} \in \mathcal{F}$ we get

$$u \in A_m \cap \check{B}_0 \subseteq A^U \Longrightarrow u \in A^U.$$

Due to A is closed, we obtain $u \in A$. Hence we get $A_n \subseteq A$ for every $n \in \mathbb{N}$.
Now, we show that $\mathcal{B}-\lim A_n = A$. Let U be an arbitrary neighborhood of θ and let $\check{B} \in \mathcal{B}$. Since $A_n \subseteq A$ for every $n \in \mathbb{N}$, we get

$$A_n \cap \check{B} \subseteq A_n \subseteq A \subseteq A^U$$

for every $n \in \mathbb{N}$. This implies that $\mathcal{B}^+-\lim A_n = A$.
From our assumption of $\mathcal{F}\mathcal{B}-\lim A_n = A$, we have

$$L\left(U,\check{B}\right) := \left\{n \in \mathbb{N} : A \cap \check{B} \subseteq A_n^U \text{ and } A_n \cap \check{B} \subseteq A^U\right\} \in \mathcal{F}.$$

Let $n_0 := \min L\left(U,\check{B}\right)$. Since (A_n) is increasing, we have

$$A \cap \check{B} \subseteq A_{n_0}^U \subseteq A_n^U$$

for each $n \geq n_0$. Therefore, we get $\mathcal{B}^- -\lim A_n = A$.
Consequently, we obtain $\mathcal{B}-\lim A_n = A$.
Case 2: Let $(A_n)_{n\in\mathbb{N}}$ be a decreasing sequence such that $A_{n+1} \subseteq A_n$ for every $n \in \mathbb{N}$.
Firstly, we will show that $A \subseteq A_n$ for every $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ be an arbitrary constant index and let $u \in A$. Let U be an arbitrary neighborhood of θ and let $\check{B}_0 \in \mathcal{B}$ with $u \in \check{B}_0$. Since $\mathcal{F}\mathcal{B}-\lim A_n = A$, we have

$$K\left(U,\check{B}_0\right) := \left\{m \in \mathbb{N} : A \cap \check{B}_0 \subseteq A_m^U \text{ and } A_m \cap \check{B}_0 \subseteq A^U\right\} \in \mathcal{F}.$$

Then we get

$$u \in A \cap \check{B}_0 \subseteq A_m^U \subseteq A_n^U \Longrightarrow u \in A_n^U$$

for each U and each $m \in K\left(U,\check{B}_0\right) \setminus \{1,2,...,n\} \in \mathcal{F}$. Due to A_n is closed, we obtain $u \in A_n$. Hence we get $A \subseteq A_n$ for every $n \in \mathbb{N}$.
Now, we show that $\mathcal{B}-\lim A_n = A$. Let U be an arbitrary neighborhood of θ and let $\check{B} \in \mathcal{B}$. Since $A \subseteq A_n$ for every $n \in \mathbb{N}$, we get

$$A \cap \check{B} \subseteq A \subseteq A_n \subseteq A_n^U$$

for every $n \in \mathbb{N}$. This implies that $\mathcal{B}^- - \lim A_n = A$.
From our assumption of $\mathcal{F}\mathcal{B} - \lim A_n = A$, we have

$$L\left(U, \check{B}\right) := \left\{n \in \mathbb{N} : A \cap \check{B} \subseteq A_n^U \text{ and } A_n \cap \check{B} \subseteq A^U\right\} \in \mathcal{F}.$$

Let $n_0 := \min L\left(U, \check{B}\right)$. Since (A_n) is decreasing, we have

$$A_n \cap \check{B} \subseteq A_{n_0} \cap \check{B} \subseteq A^U$$

for each $n \geq n_0$. Therefore, we get $\mathcal{B}^+ - \lim A_n = A$.
Consequently, we obtain $\mathcal{B} - \lim A_n = A$. □

From Theorem 3.2 and Theorem 3.3, we can give the following corollary.

Corollary 3.4. *Let (X, τ) be a Hausdorff TVS, $\mathcal{B}$ be a bornology on X, and $\mathcal{F}$ be a free filter on $\mathbb{N}$. Let $(A_n)_{n\in\mathbb{N}}$ be a nested sequence where $A_n \in \mathcal{K}(X)$ for every $n \in \mathbb{N}$.*

1. *If $(A_n)_{n\in\mathbb{N}}$ is an increasing sequence and $\text{cl}\left(\bigcup_{n\in M} A_n\right)$ is compact then*

$$\mathcal{F}\mathcal{B} - \lim A_n = \text{cl}\left(\bigcup_{n\in\mathbb{N}} A_n\right).$$

2. *If $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence then*

$$\mathcal{F}\mathcal{B} - \lim A_n = \bigcap_{n\in\mathbb{N}} A_n.$$

Definition 3.5. *Let (X, τ) be a TVS and $\mathcal{F}$ be a filter on $\mathbb{N}$. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of nonempty subsets of X. We say that the sequence $(A_n)_{n\in\mathbb{N}}$ is $\mathcal{F}$-monotone increasing if there exist a set $M = \{n_1 < n_2 < ... < n_k < ...\} \in \mathcal{F}$ such that $A_{n_k} \subseteq A_{n_{k+1}}$ for every $k \in \mathbb{N}$. $(A_n)_{n\in\mathbb{N}}$ is said to be $\mathcal{F}$-monotone decreasing if there exist a set $M = \{n_1 < n_2 < ... < n_k < ...\} \in \mathcal{F}$ such that $A_{n_{k+1}} \subseteq A_{n_k}$ for every $k \in \mathbb{N}$.*
If $(A_n)_{n\in\mathbb{N}}$ is $\mathcal{F}$-monotone increasing or $\mathcal{F}$-monotone decreasing then we say that $(A_n)_{n\in\mathbb{N}}$ is an $\mathcal{F}$-nested sequence.

Theorem 3.6. *Let (X, τ) be a first countable TVS and $\mathcal{B}$ be a bornology on X. Let $A_n \in \mathcal{K}(X)$ for every $n \in \mathbb{N}$ and let $M = \{n_1 < n_2 < ... < n_k < ...\} \in \mathcal{F}$ where $\mathcal{F}$ is a free filter.*

1. *If the subsequence $(A_n)_{n\in M}$ of $(A_n)_{n\in\mathbb{N}}$ is monotone increasing and $\text{cl}\left(\bigcup_{n\in M} A_n\right)$ is compact then*

$$\mathcal{F}\mathcal{B} - \lim A_n = \text{cl}\left(\bigcup_{n\in M} A_n\right).$$

2. *If the subsequence $(A_n)_{n\in M}$ of $(A_n)_{n\in\mathbb{N}}$ is monotone decreasing then*

$$\mathcal{F}\mathcal{B} - \lim A_n = \bigcap_{n\in M} A_n.$$

Proof.

1. According to Theorem 3.2, since $(A_n)_{n\in M}$ is monotone increasing we have

$$\mathcal{B}-\lim A_{n_k} = \mathrm{cl}\left(\bigcup_{n\in M} A_n\right) := A.$$

Let U be an arbitrary neighborhood of θ and let $\check{B} \in \mathcal{B}$. Then there is a $k_0 \in \mathbb{N}$ such that

$$A \cap \check{B} \subseteq A_{n_k}^U \text{ and } A_{n_k} \cap \check{B} \subseteq A^U$$

for every $k \geq k_0$. Hence we get

$$\left\{n \in \mathbb{N} : A \cap \check{B} \subseteq A_n^U \text{ and } A_n \cap \check{B} \subseteq A^U\right\} \supseteq M \setminus \{1, 2, ..., k_0 - 1\}.$$

Since $\mathcal{F}$ is a free filter, we have $M \setminus \{1, 2, ..., k_0 - 1\} \in \mathcal{F}$. Therefore we obtain

$$\left\{n \in \mathbb{N} : A \cap \check{B} \subseteq A_n^U \text{ and } A_n \cap \check{B} \subseteq A^U\right\} \in \mathcal{F}$$

and so $\mathcal{FB}-\lim A_n = A$.

2. It can be proved similarly.

□

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