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Separation theorems and best approximation: hybrid

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Abstract. A build to applications of ”best approximation” is the main purpose of this paper. The application of approximation theory is critical in analysing obstructions and deviations of the pupil of the eye and other layers, for example, aqueous humour, lens, and vitreous body, that form them. Our paper's primary benefit is that it is built on a separation of two-best-approximations that are dependent on convex polynomials. As a result, we investigate the approximated of a convex function f to convex polynomial, then find a linear combination to verify separation. These results will be useful for studying the pupil deviation of the eye.

1. Introduction

In functional analysis, the separation theorems (STs) for convex sets in locally convex space is not only an intriguing truth in and of itself, but also serves as a helpful proof argument for key statements in a variety of subfields of mathematics. We are fortunate in that the mathematical application of STs, such as algebraic geometry, Hana-Banach theorem, Riesz representation theorem, algorithms, and numerous STs problems in economics and finance, has been established in recent years.

The last literature that discussed about STs in the best approximation an involving discrete sets were found in 1979. Singer (1979) and Papini and Singer (1979) proposed some characteristic of approximation to be used in describing second separation theorem [13]; [11]. All these characteristics were limited to the approximation theory from an element to a convex set and best approximations by elements of convex sets. But they have been yet to explore that approach in developing best approximation which stipulates ”**approximates a function f which is defined on a finite interval $[a, b]$ with preserving certain intrinsic shape properties and known as convexity or monotonicity of the functions**”.

The area of mathematics known as best approximation, which is classified to as approximations and expansions, is a subject that plays a significant role in bridging the gap between pure and applied mathematics. The work of Chebyshev (1821–1894) is considered to be the starting point for the best approximation. Chebyshev discovered the best uniform approximation of functions by polynomial. The theory of mechanics inspired Chebyshev to study best approximation since it was essential to the design of the steam engines

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that supplied energy to the Russian enterprises. The motion of these steam engines, which is roughly linear, follows a bow-like orbit because the cranks on opposite sides of the machines move in different directions as they work.

2. Preliminaries and Methodologies

The basic definitions and results related to STs and best approximation are following. Firstly, the definitions of a convex set and function are given as follow.

Definition 2.1. *[9] A nonempty set $X \subseteq \mathbb{R}^n$ is called convex provided it contains all points $x, y \in X$ of the form*

$$(1-\lambda)x + \lambda y \in X, \qquad \text{and} \qquad \lambda \in [0,1].$$

Definition 2.2. *[9] A function $f : [a,b] \to \mathbb{R}$ is called convex (or convex downward) in the interval $[a,b]$ if*

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y),$$

for all $x, y \in [a,b]$ and all $\lambda \in [0,1]$.

From [15] and [2], we take a definition.

Definition 2.3. *A polynomial p_n is called convex in the interval $[a,b]$ if*

$$p_n((1-\lambda)x + \lambda y) \leq (1-\lambda)p_n(x) + \lambda p_n(y),$$

for all $x, y \in [a,b]$ and all $\lambda \in [0,1]$.

The following is a definition of a hyperplane.

Definition 2.4. *[6] A hyperplane H in $\mathbb{R}^n$ is the set of points $x = (x_1, \ldots x_n)$ satisfying $\sum_{k=1}^{n} a_k x_k = a_\circ$, where a_k are real and not all a_k, $k = 1, \ldots, n$ are zero.*

Let's use STs to convey certain ideas to the best approximation. Let $x_1, x_2, \ldots, x_n$ be given points in any set X. We consider linear combinations of the form

$$y = \sum_{i=1}^{n} a_i x_i \,, \tag{1}$$

where a_i are scalars.

The degree of approximation and best approximation are then recalled using Definition 2.4, as stated in the definitions below.

Definition 2.5. *[6] For each $x \in X$, the degree of approximation $E_n(x)$ of x by the linear combinations y is*

$$E_n(x) = \inf_y \|x - y\|.$$

Definition 2.6. *[6] The linear combination y^* is said to be the best approximation to x, if*

$$\inf_{y^*} \|x - y^*\| \leq \inf_y \|x - y\|.$$

y^* is called a linear combination of best approximation or a polynomial of best approximation to x.

Therefore, the definition of the polynomial can be defined as follows.

Definition 2.7. *[6] Let $\phi_1, \phi_2, \ldots, \phi_n$ be given real continuous functions on $[a, b]$. A linear combination*

$$P = \sum_{i=1}^{n} a_i \phi_i$$

with real coefficients a_i will be called a polynomial in the ϕ_i.

Next, we denote $\mathbb{C}([a, b])$ as the space of all continuous functions defined in $[a, b]$ equipped with the uniform norm, that is,

$$\|f\|_{\mathbb{C}} = \max_{x \in [a,b]} |f(x)|.$$

Let us recall the notions of the best approximation of the function f and it's degree. We borrow a definitions from [2] and [7].

Definition 2.8. *Let $\mathbb{C}([a, b])$ be the space of all continuous functions and π_n be a space of all algebraic polynomials of the degree not greater than n (shortly, we write $\leq n$). One can see that $\pi_n \subset \mathbb{C}([a, b])$. A polynomial $p^* \in \pi$ is said to be the best approximation to a function $f \in \mathbb{C}([a, b])$ on the π if*

$$\|f - p^*\|_{\mathbb{C}} \leq \|f - p\|_{\mathbb{C}} \tag{2}$$

for all $p \in \pi$.

Using Definitions 2.5, 2.6, and 2.8, the degree of best approximation of the function f may now be simply stated as follows.

Definition 2.9. *[2] For a given $f \in \mathbb{C}([a, b])$ the degree of the best approximation is defined as follows:*

$$E_n(f) = \inf_{p_n \in \pi_n} \|f - p_n\|_{\mathbb{C}}.$$

Since the inequality (2) holds for all $p_n \in \pi_n$ it is true for infimum over π_n. Therefore, using Definitions 2.5, 2.6, 2.8, and 2.9, the definition of the best approximation can also be defined as follows.

Definition 2.10. *[2] A polynomial $p_n^* \in \pi_n$ is said to be the best approximation to $f \in \mathbb{C}([a, b])$ if*

$$\|f - p_n^*\|_{\mathbb{C}} \leq E_n(f).$$

Definition 2.11. *[1] If the polynomial p_n preserves the shape (or the curve) of the convex function f, it is then known as a convex polynomial.*

Definition 2.12. *[6] The following simple notion will be useful:-*

1. *The set H of points for which $\sum_{k=1}^{n} a_k x_k \leq a_\circ$ is called a closed half-space.*
2. *A compact convex set B is the intersection of all closed half-space H that contain $B = \cap_{B \subset H} H$.*
3. *For each set $B \subset \mathbb{R}^n$, there exists a smallest convex hull of B.*
4. *The convex hull of the set $B \subset \mathbb{R}^n$ consists of all points x of the form*

$$x = \sum_{i=1}^{r} p_i x^{(i)} \quad , x^{(i)} \in B, \qquad p_i > 0, \quad i = 1, \ldots, r;$$

$$\sum_{i=1}^{r} p_i = 1 \, ,$$

where $r \leq n + 1$.

Definition 2.13. *Let A be any closed convex set. Then:*

1. *If $x \notin A$, then there exists H a closed hyperplane (= strictly separates), if we choose $b \in \mathbb{R}$ such that $\sup\{f(y) : y \in A\} < b < f(x)$ [8].*
2. *An interior point of A be denoted by $int(A) \neq \emptyset$, such that x a point in $A - int(A)$. Then a support hyperplane at x exists [8].*
3. *If A and B are two disjoint closed convex sets, such that $A \neq \emptyset$, (and $B \neq \emptyset$ is a compact), then there exists H a closed hyperplane that separates them [8].*
4. *If A is a nonempty compact convex subset of $\mathbb{R}^n$, and B is a nonempty closed convex subset of $\mathbb{R}^n$, such that A and B are disjoint. Then there is H, strongly separates A and B [12].*
5. *If $A \subseteq X$ and $B \subseteq X$, and H is a closed hyperplane, then H separated two parts A and B, if*

$$f(x) \leq \alpha \leq f(y), \quad \text{for all} \;\; x \in A, \qquad y \in B$$

where f is a function from a vector space X over field $\mathbb{R}$ onto that field $\mathbb{R}$ and $\alpha \in \mathbb{R}$ [16].

Next, we recall the **First Separation Theorem** (FST).

Theorem 2.14. *[16] Two disjoint convex sets can be separated by a hyperplane.*

Here we present a formalization of the **Second Separation Theorem** (SST).

Theorem 2.15. *[16] A non-empty closed convex set can be strictly separated from a point which does not belong to it.*

Despite the fact that this strategy was successful in establishing the separation of two convex sets using hyperplane. There is still a need to define and investigate more about degrees of the best approximation of particular functions. The goal of this paper is to build new degrees for the H*-best approximation and their applications.

3. Literature Review and Study Motivation

Our discussion of the study's motivation and the related literature will take place here.

Singer (1979) has proven some hyperplane separation theorems, which reduce the problems of computing $\underline{f}(G)$ and of characterizing the elements $g_\circ \in G$ with $\underline{f}(g_\circ) = \inf(\underline{f}(G))$, where $\underline{f}$ is a continuous convex function from a vector space X over a field $\mathbb{F}$ onto that field $\mathbb{F}$, [13]. Now, we suppose $\mathbb{X}$ denote the set of all the functions $\underline{f}$ on X.

The following result shows a different method of separate without using Definition 2.1.

Lemma 3.1. *[13] Let X be a linear space, $\underline{f}$ a finite convex function on X and G a convex subset of X, for which there exists an element $x \in X$ such that $\underline{f}(x) \leq \inf(\underline{f}(G))$. If for this x there exists a function $\check{\underline{f}} \in \mathbb{X}$, $\check{\underline{f}} \neq 0$, satisfying $\sup(\check{\underline{f}}(G)) \leq \check{\underline{f}}(x)$, then for any such $\check{\underline{f}}$ and any ϑ with $\sup(\check{\underline{f}}(G)) \leq \vartheta \leq \check{\underline{f}}(x)$, we have*

$$\inf(\underline{f}(G)) \geq \inf_{y \in X}(\underline{f}(y)), \tag{3}$$

where $\check{\underline{f}}(y) = \vartheta$.

Definition 3.2. *[10] If X is a vector space that has a topology τ, then we say that X is a locally convex space if every point has a neighborhood base consisting of convex sets.*

The generalisation of Lemma 3.1 is stated in the following two theorems, which we will now go over.

Theorem 3.3. *[13] Let X be a locally convex space, $\underline{f}$ a continuous convex function on X and G a convex subset of X, $(G \neq X)$ satisfying*

$$\inf(\underline{f}(X)) < \inf(\underline{f}(G)) \tag{4}$$

and let x be any element of X satisfying

$$\underline{f}(x) < \inf(\underline{f}(G)). \tag{5}$$

Then we have

$$\inf(\underline{f}(G)) = \sup_{\wedge_1} \inf_{\vee_1} \underline{f}(y),$$

$$\wedge_1 = \{\underline{f_1} \in \mathbb{X} : \sup(\underline{f_1}(G)) \leq \underline{f_1}(x), \underline{f_1} \neq 0\},$$

$$\vee_1 = \{y \in X : \underline{f_1}(y) = \sup(\underline{f_1}(G))\},$$

and

$$\inf(\underline{f}(G)) = \sup_{\wedge_2} \inf_{\vee_2} \underline{f}(y),$$

$$\wedge_2 = \{\underline{f_2} \in \mathbb{X} : \sup(\underline{f_2}(G)) \leq \underline{f_2}(x), \underline{f_2} \neq 0\},$$

$$\vee_2 = \{y \in X : \underline{f_2}(y) = \sup(\underline{f_2}(G))\}.$$

Theorem 3.4. *[13] Let X be a locally convex space, $\underline{f}$ a continuous convex function on X and G a convex subset of X, satisfying (4) and let x be any element of X satisfying (5). An element $g_\circ \in G$ satisfying (3) if and only if there exists $\underline{\check{f}} \in \mathbb{X}$, and $\underline{\check{f}} \neq 0$, such that*

$$\underline{\check{f}}(g_\circ) = \sup(\underline{\check{f}}(G)) \leq \underline{\check{f}}(x),$$

$$\underline{\check{f}}(g_\circ) = \inf_{y \in X}(\underline{\check{f}}(y)),$$

where $\underline{\check{f}}(y) = \underline{\check{f}}(g_\circ)$.

The scientific subject in the field of "approximation, optics, and the eye" makes it challenging to locate a wealth of specialized literature.

Snell, Lemp, and Grunther (1998) go over some in-depth information about the eye and its supporting structures. Such as, the visual route, the autonomic innervation of the orbital structures, the supply to the orbit, and the related visual reflexes, (see [14]). To identify applications mathematical, however, some literature has explored the nexus between mathematics and the visual sciences. For example, Lakshminarayanan (2013) describes how optics and light analysis are done using mathematics, [5]. Iske (2018), discussed modern approximation topics and its relevant applications (see [4]). For instance, he explains how approximation theory relies heavily on the creation of algorithms for data analysis. In the sequel, Al-Muhja, Misiran and Omer (2019) reviewed some of an open problems in the application of approximation theory to this day, see [3]. Such as the eye's pupil deviations and the refraction of light as it passes through the cornea and lens inside the eye, as described by Snell's law of refraction.

4. Hybrid Theories

The eyes are crucial in the human body since they are one of the most important organs that allow a person to execute vital careers such as seeing. One of the most important mathematical subjects will use in clinical science is the theory of approximation. Approximation theory helped convert optical interpretation of the work of the eyes, as it is written in the results of this article, relying on separation theorems. Out of all the organs and layers that comprise the eye, it appears that the pupil is the primary component responsible for receiving light and initiating visual perception. In order to analyze the pupil aberrations of the eye, we provide here special theorems in approximation. By using separation theorems to expand on the idea of best approximation.

4.1. *Fundamental Findings of the Study*

By consulting Definitions 2.4, 2.8, 2.12, 2.13; and Lemma 3.1, meanwhile, we will be building our definitions.

Definition 4.1. *Let f be a function from locally convex space X onto that field $\mathbb{R}$ and $\alpha \in \mathbb{R}$.*

1. *The hyperplane $H^* = \{x \in X : X \subseteq \mathbb{R}^n, \inf_{p_n \in \pi_n} \|f - p_n\| = \alpha\}$, is a subset of X. If f is a continuous function, then H^* is closed.*
2. *H^* is said to be supporting hyperplane to a convex set X if at least one point $x_\circ$ of X lies in H^*, and $|f(x) - p_n(x)| \geq \alpha$, for all $x \in X/\{x_\circ\}$, $p_n \in \pi_n$.*

Definition 4.2. *A compact subset X of C^N is polynomially convex if for each point $x \in C^N \cap \{x_\circ\}/X$ there is a polynomial p_n such that*

$$\inf_{x \in C^N \cap \{x_\circ\}/X} \|f - p_n\| > \sup \inf_{x \in X} \|f - p_n\|.$$

Theorem 4.3. *(FST-Best Approximation). Two disjoint best convex approximations can be separated by a linear combination (hyperplane).*

Definition 2.7 and Equation (1) result in the definition below.

Definition 4.4. *Let $H^* \in \mathbb{R}^n$ be a set of all points $x = (x_1, x_2, \ldots, x_n)$, that satisfying $M = \sum_{k=1}^{n} a_k x_k = a_\circ$. M is called a linear combination and $\|M\|_{\mathbb{C}} < \infty$.*

By viewing this pupil of the eye as a convex function f, a new definition of best approximation can be developed with the aid of linear combination to describe the movement of deviation when the eye sees the surfaces of objects using the traditional explanations of the fall of light on the pupil of the eye.

Definition 4.5. *A linear combination $\dot{p}^* \in \pi$ is a said to be the H^*-best approximation to $f \in \mathbb{C}([a, b])$, if*

$$\|f - \dot{p}^*\|_{\mathbb{C}} \leq \|M\|_{\mathbb{C}} \leq \dot{E}_n(f)_{\mathbb{C}}.$$

4.2. *Proof of Theorem 4.3*

Now, we will limit our concentrate "in this subsection" to the proof of Theorem 4.3 using Definitions 2.11 and 4.5.

Proof. We first prove our theorem (Theorem 4.3), by virtue of STs properties in sections 2 and 3. Note that $\mathbb{C}([a, b])$ is a space of all continuous functions $f_1, f_2 : [a, b] \to \mathbb{R}$, such that f_1, f_2 are convex functions; and G is a convex subset of $\mathbb{C}([a, b])$. Moreover, $\dot{p}_n, \dot{q}_n, \dot{p}$ and $\dot{q}$ were convex polynomials in Δ^2 the set of all convex polynomials. To provide evidence for our assertion, let us assume that functions f_1 and f_2 have best approximations, such as; respectively, the convex polynomials $\dot{p}^*, \dot{q}^* \in \Delta^2$; $(\dot{p}^* \neq \dot{q}^*)$, such that

$$\inf_{\dot{p}_n \in \pi_n} \|f_1 - \dot{p}_n\|_{\mathbb{C}} \geq \|f_1 - \dot{p}^*\|_{\mathbb{C}} \tag{6}$$

and

$$\inf_{\dot{q}_n \in \pi_n} \|f_2 - \dot{q}_n\|_{\mathbb{C}} \geq \|f_2 - \dot{q}^*\|_{\mathbb{C}}. \tag{7}$$

Together with the prior inequalities (6) and (7), this means that for $n > 1$,

$$\dot{E}_n(f_1)_{\mathbb{C}} \geq \|f_1 - \dot{p}^*\|_{\mathbb{C}} \tag{8}$$

and

$$\dot{E}_n(f_2)_{\mathbb{C}} \geq \|f_2 - \dot{q}^*\|_{\mathbb{C}}. \tag{9}$$

As a result, it is sufficient to consider Equations (8) and (9), which have two disjoint best convex approximations of functions f_1, f_2. We finally show how to build a hyperplane due to separate of best approximations and degrees of best approximations. As a result, we select a polynomial q_n from the set of algebraic polynomials π_n. Now, based on Definitions 2.4, 2.6, 2.7, 4.4 and Equation (1), therefore, q_n has the following:

$$\dot{E}_n(f_1)_{\mathbb{C}} \geq \|q_n\|_{\mathbb{C}} \geq \|f_1 - \dot{p}^*\|_{\mathbb{C}} \tag{10}$$

and

$$\dot{E}_n(f_2)_{\mathbb{C}} \geq \|q_n\|_{\mathbb{C}} \geq \|f_2 - \dot{q}^*\|_{\mathbb{C}}. \tag{11}$$

By virtue of Definition 4.4, we find $\|M\|_{\mathbb{C}} = \|q_n\|_{\mathbb{C}}$. Then, the Equations (10) and (11), have that the H^*-best approximations to f_i and $i = 1, 2$. The proof is finished.
□

4.3. Pertaining Auxiliary Outcomes for $\mathbb{X}$

Few related outcomes are produced in this subsection using auxiliary set $\mathbb{X}$. In the sequel, our theorem 4.3 is relevant-able to the outcomes that were investigated here.

1. If $f \in \mathbb{X}$, i.e., $\underline{f} = f$. By Lemma 3.1, and Theorem 4.3, we have $[a, b] \subseteq X$, convex sets:
   (a)
   $$\exists \ \ x \in X, \ \ni \inf_{p_n \in \pi_n} \|f_{x_\circ} - p_n\|_{x_\circ \in [a,b]} \geq \{\|f\|, x \in X\}. \tag{12}$$
   (b)
   $$\exists \ \ 0 \neq \breve{f} \in \mathbb{X}, \ \ni \sup_{p_n \in \pi_n} \|\breve{f}_{x_\circ} - p_n\|_{x_\circ \in [a,b]} \leq \{\|\breve{f}\|, x \in X\}. \tag{13}$$
   Then,
   (c)
   $$\sup_{p_n \in \pi_n} \|\breve{f}_{x_\circ} - p_n\|_{x_\circ \in [a,b]} \leq \{\|\breve{f}\|, \ \forall \ \ y \in X\} \leq \{\|\breve{f}\|, x \in X\}, \tag{14}$$
   we have
   (d)
   $$\begin{cases} \inf_{p_n \in \pi_n} \|f_{x_\circ} - p_n\|_{x_\circ \in [a,b]} \geq \inf_{p_n \in \pi_n} \|f_y - p_n\|_{y \in X}, \\ E_n(f)_{[a,b]} \geq E_n(f)_X. \end{cases} \tag{15}$$

2. If X is a locally convex space, and if $f, y, p_n, \pi_n, x_\circ$ and $[a,b] \subseteq X$ (and $[a,b] \neq X$) are defined in (12). By Theorems 3.3 and 4.3, we have
   (a) By virtue of Equation (15), then Equation (12) is valid.
   (b) We have

$$E_n(f)_{[a,b]} = \sup_{\breve{\lambda}} E_n(f)_X \ ,$$

$$\breve{\lambda} = \{f \in \mathbb{X} : equ.\ (14) \ \ is \ \ valid\ \}.$$

The application of Theorem 4.3 is discussed in the next section. One possible application of our theorem is the eye theory.

5. Applications of the Study

The practical application of the essay focuses on patients with the common ailment of pupil deviation of the eye (astigmatism). The condition causes an imbalance in the direction of the eyes. In other words, one eye stares forward while the other eye deviates inward or outward. Our study's target illness condition is referred to as f. The location and angle of the pupil deviation, which vary between patients, determine the precise form of the eye's deviation f. Typically, the purpose of p_n surgery is to re-correction f the pupil eye, so that vision is balanced and functioning properly, hence improving vision and beauty. Many people notice a significant increase in their ability to see things correctly and clearly after surgery. Therefore, in this section, we will provide the experiment that was conducted on patient $\mathbb{X}$:

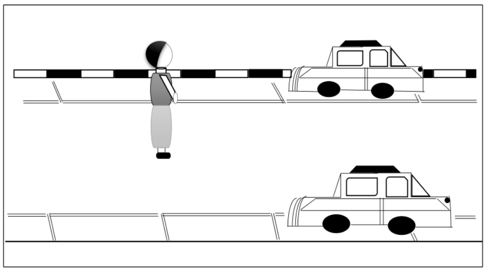

Figure 1: An illustrative example of a performed experiment that was conducted on patient $\mathbb{X}$, who was four metres away from the vehicles.

Two vehicles (A and B), distinct in type and colour, are parked on the same horizontal level in a parking garage in the Iraqi city of Diwaniyah (see Table 1). The two vehicles (A and B) are only two metres apart. In order to achieve the needed approximation, the ophthalmologist performed his experiment on patient $\mathbb{X}$ in this garage. The ophthalmologist also examined the pupil eye and its curvature of patient $\mathbb{X}$.

Patient $\mathbb{X}$ underwent an examination by the ophthalmologist. At the time of the examination, patient $\mathbb{X}$ was four metres distant from the two vehicles (A and B) see Figure 1. The patient $\mathbb{X}$ observed the two vehicles (A and B) consecutively and simultaneously. Subsequently, the ophthalmologist conducted the experiment five metres apart (see Figure 2). In conclusion, the experiment was conducted at a distance of six metres as well (see Figure 3).

The outcome is shown in the table below:

Table 1: An experiment with the pupil eye deviation of patient $\mathbb{X}$, based on Equations (10) and (11).

			Patient	$\mathbb{X}$		
Test	Distant	Time	Vehicle A	Approxi. of Vehicle A	Vehicle B	Approxi. of Vehicle B
1	4 met.		f_1	p_n	f_2	q_n
2	5 met.	same second	f_1	$\breve{p}_n$	f_2	$\breve{q}_n$
3	6 met.		f_1	$\hat{p}_n$	f_2	$\hat{q}_n$

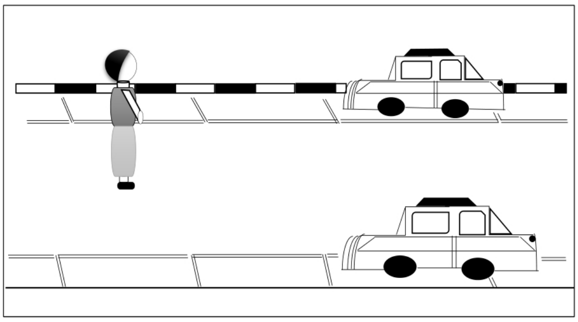

Figure 2: Re-run the experiment on the same patient $\mathbb{X}$. He was, however, only five meters away from the vehicles.

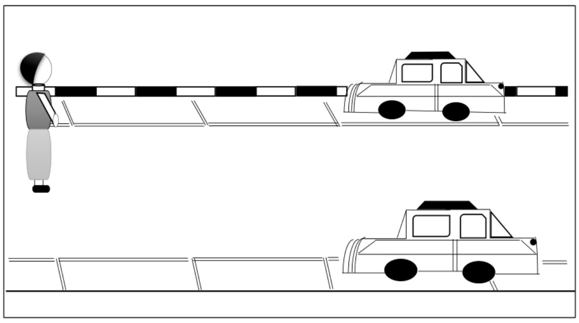

Figure 3: The pupil eye of the same patient $\mathbb{X}$ has grown significantly more deviated in which the experiment is performed six meters distant from the vehicles.

These Figures 1, 2, 3, and Definition 2.11 assist in estimating the convex of f to the pupil eye, which means that if surgery p_n preserves the shape (or curve to healthy pupil) to re-correction the pupil eye f to patient $\mathbb{X}$, visual balance (see Equations (10), (11) and Figure 4) is then recognized.

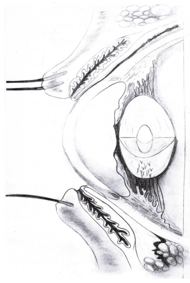

Figure 4: The shape has rendered a pencil-drawn anatomical cross-section of the eye.

6. Conclusions

A useful strategy for advancing theoretical findings in the field of best approximation is research on applications of approximation theory, as demonstrated in 1885 by the Russian scientist Chebyshev. We used the applied approach to "best approximation" in this paper by hybridizing approximation theory with separation theory, then their application in eye theory was obtained. The separation, approximation, and eye theories are regarded as the three main theories that are thought to be fundamental to this study. We explain how this study relies heavily to assist patient $\mathbb{X}$ balancing and focusing his perception of his surroundings, mean of creating a hyperplane that separates the two-best-approximations. To be more specific, we were able to obtain good definitions and concepts in this field along with significant results in best approximation. In order to facilitate the research interpretation of the findings presented in this work, we carried out a straightforward, sequential experiment, as demonstrated in Section 5.

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