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Spectrality and non-spectrality of self-affine measures with five-element digit sets on $\mathbb{R}^2$

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Abstract. Let $M = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}$ be an expanding real matrix, and let $\mathcal{D} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ be a digit set. In this paper, we mainly study the properties of spectra of self-affine measure $\mu_{M,\mathcal{D}}$ generated by M and $\mathcal{D}$. We showed that $\mu_{M,\mathcal{D}}$ is a spectral measure if and only if $5 \mid \rho$. Furthermore, by extending the maximal mapping to plane, we gives a characterization for $E(\Lambda)$ to be a maximal orthogonal family in $L^2(\mu_{M,\mathcal{D}})$. Based on these, we also obtained some sufficient conditions for the maximal orthogonal set to be an orthogonal basis of $L^2(\mu_{M,\mathcal{D}})$.

1. Introduction

One of the fundamental problems in harmonic analysis is whether $E(\Lambda) := \{e^{-2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\}$ forms an orthonormal basis for $L^2(\mu)$, the space of all square-integrable functions with respect to a probability measure μ. Let μ be a probability measure with compact support on $\mathbb{R}^d$, then μ is called a *spectral measure* if there exists a countable set $\Lambda \subseteq \mathbb{R}^d$ such that the set of exponential functions $E(\Lambda)$ forms an orthonormal basis for $L^2(\mu)$. If such Λ exists, then Λ is called a *spectrum* of μ, and (μ, Λ) is called a *spectral pair*.

The origin of the question could date back to Fuglede[20] and his famous conjecture: a measurable set is a spectral set if and only if it tiles the whole Euclidean space by translation. Although, it was proved to be false by Tao and others in dimension three or higher (see[26, 32, 36]), but it is still a hot topic in one and two dimensions. After the original work of Fuglede, the study of spectral measures is also blooming.

In this paper, we mainly consider the properties of spectra for a class of self-affine measure $\mu_{M,\mathcal{D}}$ on $\mathbb{R}^2$, which generated by the following iterated function systems (IFS)

$$\{\phi_d(x) = M^{-1}(x + d)\}_{d\in\mathcal{D}},$$

where M is an expanding matrix (that is, all eigenvalues of M are strictly larger than one in modulus) and $\mathcal{D} \subset \mathbb{R}^2$ is a finite subset of cardinality $\#\mathcal{D}$. By Hutchinson's theorem[24], there exists a unique probability

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measure $\mu_{M,\mathcal{D}}$ satisfying the following equation

$$\mu_{M,\mathcal{D}}(\cdot) = \frac{1}{\#\mathcal{D}} \sum_{d\in\mathcal{D}} \mu_{M,\mathcal{D}} \circ \phi_d^{-1}(\cdot), \tag{1}$$

which is supported on $T(M,\mathcal{D})$, where

$$T(M,\mathcal{D}) = \bigcup_{d\in\mathcal{D}} \phi_d(T(M,\mathcal{D})).$$

The set $T(M,\mathcal{D})$ is called a *self-affine set* and the measure $\mu_{M,\mathcal{D}}$ is called a *self-affine measure*.

Moreover, the measure $\mu_{M,\mathcal{D}}$ can be expressed by the infinite convolution of Dirac measures as follows

$$\mu_{M,\mathcal{D}} = \delta_{M^{-1}\mathcal{D}} * \delta_{M^{-2}\mathcal{D}} * \delta_{M^{-3}\mathcal{D}} * \cdots, \tag{2}$$

where $\delta_{\mathcal{D}} = \frac{1}{\#\mathcal{D}} \sum_{d\in\mathcal{D}} \delta_d$, δ_d is the Dirac measure at the point d and the convergence is in weak sense.

The first example of a singular, non-atomic, spectral measure was given by Jorgensen and Pedersen in [25] , this measure $\mu_{4,\{0,2\}}$ generated by the IFS $\{4^{-1}(x+d) : d \in \{0,2\}\}$, and the spectrum for this measure is as follows

$$\Lambda = \left\{ \sum_{k=0}^{n} 4^k l_k \,:\, l_k \in \{0,1\},\, n \in \mathbb{N} \right\}.$$

This surprising discovery received a lot of attention, and the research on the spectrality or non-spectrality of singular measures has become a hot topic (see [2–4, 8, 11, 13–18, 22, 27, 28] and the references therein for recent advances). As well as, the convergence of $\mu_{4,\{0,2\}}$ with different spectra has different results (see [16, 34, 35]), and the spectral properties of various classes of spectral measures have been analyzed (see [1, 10, 12, 23, 31] and the references therein for details). In these researches, we can find that the construction of these fractal spectral measures stem from the existence of compatible pairs.

Definition 1.1. *Let $M \in M_2(\mathbb{Z})$ be an expanding matrix with integer entries, and let $\mathcal{D}, C \subset \mathbb{Z}^2$ be two finite subsets of integer vectors with $\#\mathcal{D} = \#C$. We say that $(M^{-1}\mathcal{D}, C)$ forms a compatible pair (or $(M,\mathcal{D},C)$ forms a Hadamard triple) if the matrix*

$$H := \frac{1}{\sqrt{\#\mathcal{D}}} \left[e^{2\pi i \langle M^{-1}d, c\rangle} \right]_{d\in\mathcal{D}, c\in C}$$

is unitary, that is, $H^*H = HH^* = I$, where H^* denotes the transposed conjugate of H.

Recall that, Łaba and Wang [29] showed that $\mu_{M,\mathcal{D}}$ is a spectral measure if $(M^{-1}\mathcal{D}, C)$ forms a compatible pair for $C \subseteq \mathbb{Z}$, $M > 1$ and $\mathcal{D} \subseteq \mathbb{Z}$. Then many researchers tried to study the similar case on higher dimension. Recently, Dutkay et al.[19] proved that the compatible pair always generate self-affine spectral measures.

Unlike the one-dimensional case, the study on the spectrality of self-affine measures in higher dimensions is more complicated. In [13], Deng and Lau considered the self-similar Sierpinski-type measures generated by a real matrix $M = \mathrm{diag}(\rho,\rho)$ $(\rho < 1)$ and $\mathcal{D} = \{(0,0)^t, (1,0)^t, (0,1)^t\}$, they proved that $\mu_{M,\mathcal{D}}$ is a spectral measure if and only if $|\rho| = \frac{1}{3p}$ for some $p \in \mathbb{N}$. Later, Dai et al.[11] investigated a general case of $M = \mathrm{diag}(\rho_1,\rho_2)$ $(\rho_1,\rho_2 > 1)$ and $\mathcal{D} = \{(0,0)^t, (1,0)^t, (0,1)^t\}$, they showed that the measure $\mu_{M,\mathcal{D}}$ is a spectral measure if and only if $3 \mid \rho_i$, $i = 1,2$. After then, Chen and Yan[7] considered the self-affine measure $\mu_{M,\mathcal{D}}$ generated by the expanding matrix $M = \mathrm{diag}(\rho,\rho)$ $(\rho > 1)$ and $\mathcal{D} = \{(0,0)^t, (1,0)^t, (0,1)^t, (-1,-1)^t\}$, they proved that $\mu_{M,\mathcal{D}}$ is a spectral measure if and only if $2 \mid \rho$. Also, Chen and Tang considered the similar case, see[6].

Motivated by the above work, in this paper, our main purpose is to study the spectrality of the self-affine measures $\mu_{M,\mathcal{D}}$ on $\mathbb{R}^2$, which is generated by

$$M = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix} \text{ and } \mathcal{D} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}, \tag{3}$$

where $\rho > 1$ is a real number.

Our main results are as follows.

Theorem 1.2. *Let $\mu_{M,\mathcal{D}}$ be the self-affine measure defined by (1), and $M, \mathcal{D}$ are as in* (3)*. Then $\mu_{M,\mathcal{D}}$ is a spectral measure if and only if* $5 \mid \rho$.

For the Theorem 1.2, it can be seen that the proof of the sufficiency of this theorem can be achieved by constructing a compatible pair. However, the proof of the necessity of this theorem is the difficult part. For the necessity, we first relate it to the one-dimensional case by its infinite orthogonal set.

Theorem 1.3. *Let $\mu_{M,\mathcal{D}}$ be the self-affine measure defined as in (1), and $M, \mathcal{D}$ are as in* (3)*. Then $\mu_{M,\mathcal{D}}$ admits an infinite orthogonal set if and only if $\rho = \sqrt[r]{\frac{5p}{q}}$ for some $r, p, q \in \mathbb{N}$ with* $\gcd(5p, q) = 1$.

According to Theorem 1.3, we can prove that $\mu_{M,\mathcal{D}}$ is not a spectral measure in the following two cases:
Case I : $\rho = \sqrt[r]{\frac{5p}{q}}$ and $r > 1$ (see Proposition 4.1);
Case II : $\rho = \frac{5p}{q}$ and $q > 1$ (see Proposition 4.5).
Throughout the paper, we assume that r is the smallest integer such that $\rho^r \in \mathbb{Q}$ (for example, $\rho = \sqrt[4]{\frac{25}{16}} = \sqrt[2]{\frac{5}{4}}$, we take $r = 2$).

It is well known that most of singularly spectral measures have uncountable spectra which contains 0, that is to say, all spectra have complicated structures. Naturally, we want to find the answer to the following question:

what is the family of spectra of a given spectral self-affine measure ?

Motivated by above question, An et al. [5] studied the spectral structure of planar Sierpinski measure, where $M = \mathrm{diag}(3q, 3q)$ ($q > 1$ is an integer) and $\mathcal{D} = \{(0,0)^t, (1,0)^t, (0,1)^t\}$. They gives a characterization for $E(\Lambda)$ to be a maximal orthogonal family in $L^2(\mu_{M,\mathcal{D}})$, and also give some sufficient conditions for a maximal orthogonal family $E(\Lambda)$ to be or not to be an orthogonal basis of $L^2(\mu_{M,\mathcal{D}})$. Later, Li et al. [30] studied the spectral structure of the planar self-similar measures, where $M = \mathrm{diag}(2q, 2q)$ (q is a positive integer) and $\mathcal{D} = \{(0,0)^t, (1,0)^t, (0,1)^t, (-1,-1)^t\}$. They also obtained some sufficient conditions for the maximal orthogonal set to be or not to be a basis for $L^2(\mu_{M,\mathcal{D}})$.

Let τ be a maximal mapping defined by Definition 3.1 and Σ_5^τ be defined as in (13). Set

$$\tau^*(\Sigma_5^\tau) = \left\{ \tau^*(I) = \sum_{j=1}^{\infty} (5p)^{j-1} \tau(I|_j) \,:\, I \in \Sigma_5^\tau \right\}.$$

Following, we give the final major result of this paper.

Theorem 1.4. *Let τ be a maximal mapping and $\Lambda = \tau^*(\Sigma_5^\tau)$. If for each $I \in \Sigma_5^*$ there exists $J_I \in \Sigma_5^\infty$ such that $IJ_I \in \Sigma_5^\tau$ and*

$$\sup_{I \in \Sigma_5^*} N_I(J_I) < \infty,$$

where $N_I(J_I)$ is defined as in (16)*. Then Λ is a spectrum of $\mu_{M,\mathcal{D}}$.*

The paper is organized as follows. In Section 2, we introduce some basic concepts and lemmas, and the proof of Theorem 1.3. In Section 3, we mainly prove the sufficiency of Theorem 1.2 and Theorem 1.4, and give some important propositions and lemmas. In Section 4, we prove the necessity of Theorem 1.2.

2. Preliminaries and proof of Theorem 1.3

Let μ be a Borel probability measure with compact support on $\mathbb{R}^2$. The Fourier transform of μ is defined by

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^2} e^{-2\pi i\langle \xi, x\rangle}\, \mathrm{d}\mu(x), \tag{4}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product. Suppose $\mu_{M,\mathcal{D}}$ be the self-affine measure which is generated by M and $\mathcal{D}$, where $M, \mathcal{D}$ are as in (3), then we have

$$\hat{\mu}_{M,\mathcal{D}}(\xi) = \prod_{k=1}^{\infty} m_{\mathcal{D}}(M^{-k}\xi), \quad \xi = \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} \in \mathbb{R}^2, \tag{5}$$

where

$$m_{\mathcal{D}}(\xi) = \frac{1}{5}\sum_{d\in\mathcal{D}} e^{-2\pi i\langle d,\xi\rangle} = \frac{1}{5}\left(1 + e^{-2\pi i\xi_1} + e^{-2\pi i\xi_2} + e^{-2\pi i(\xi_1-\xi_2)} + e^{-2\pi i(\xi_2-\xi_1)}\right)$$

is the *mask polynomial* of $\mathcal{D}$. By a direct calculation, we know that

$$\mathcal{Z}(m_{\mathcal{D}}) = \left(\pm\ \frac{1}{5}\ \begin{pmatrix}1\\ -1\end{pmatrix} + \mathbb{Z}^2\right) \cup \left(\pm\ \frac{2}{5}\ \begin{pmatrix}1\\ -1\end{pmatrix} + \mathbb{Z}^2\right). \tag{6}$$

Denote

$$A_1 = \frac{1}{5}\begin{pmatrix}1\\ -1\end{pmatrix} + \mathbb{Z}^2, \quad A_2 = \frac{1}{5}\begin{pmatrix}-1\\ 1\end{pmatrix} + \mathbb{Z}^2, \quad A_3 = \frac{2}{5}\begin{pmatrix}1\\ -1\end{pmatrix} + \mathbb{Z}^2, \quad A_4 = \frac{2}{5}\begin{pmatrix}-1\\ 1\end{pmatrix} + \mathbb{Z}^2. \tag{7}$$

Then by (5), we have

$$\mathcal{Z}\left(\hat{\mu}_{M,\mathcal{D}}\right) = \bigcup_{k=1}^{\infty} M^k \mathcal{Z}\left(m_{\mathcal{D}}\right) = \bigcup_{k=1}^{\infty} \rho^k (A_1 \cup A_2 \cup A_3 \cup A_4). \tag{8}$$

Let $\Lambda \subset \mathbb{R}^2$ be a countable set. Recall that Λ is called an *orthogonal set* (*a spectrum*) of μ if $E(\Lambda) = \{e^{-2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\}$ forms an orthogonal set (an orthonormal basis) for $L^2(\mu)$. It is easy to check that the orthogonality of $E(\Lambda)$ is equivalent to the following condition

$$(\Lambda - \Lambda) \setminus \{0\} \subset \mathcal{Z}(\hat{\mu}), \tag{9}$$

where $\mathcal{Z}(\hat{\mu}) = \{\xi \in \mathbb{R}^2 : \hat{\mu}(\xi) = 0\}$. We assume that $0 \in \Lambda$ because all orthogonal sets (or spectra) are invariant under translations. For any $\xi \in \mathbb{R}^2$, we define

$$Q_\Lambda(\xi) = \sum_{\lambda\in\Lambda} |\hat{\mu}(\xi + \lambda)|^2.$$

The following criterion is a universal test for a set $\Lambda \subseteq \mathbb{R}^2$, which is a basic tool to determine whether Λ is an orthogonal set (a spectrum) of μ.

Theorem 2.1. *[25] Let μ be a Borel probability measure with compact support on $\mathbb{R}^2$, and let $\Lambda \subset \mathbb{R}^2$ be a countable subset. Then*

(i) Λ is an orthogonal set of μ if and only if $Q_\Lambda(\xi) \leq 1$ for $\xi \in \mathbb{R}^2$. In this case, $Q_\Lambda(z)$ is an entire function in $\mathbb{C}^2$.

(ii) Λ is a spectrum of μ if and only if $Q_\Lambda(\xi) \equiv 1$ for $\xi \in \mathbb{R}^2$.

The following useful lemma is given by [10], which is an important tool for spectrality and non-spectrality of the measure with convolution structure.

Lemma 2.2. *[10] Let $\mu = \mu_0 * \mu_1$ be the convolution of two probability measures μ_i, $i = 0, 1$, and they are not Dirac measures. Suppose that Λ is an orthogonal set of μ_0 with $0 \in \Lambda$, then Λ is also an orthogonal set of μ, but cannot be a spectrum of μ.*

Following, we recall the famous Ramsey's Theorem. An et al.[2] first introduced the idea of this theorem into spectral theory.

Theorem 2.3. (Ramsey's Theorem)*[33] Let $\mathcal{A}$ be a countable infinite set and let $\mathcal{A}^{(k)}$ be the set of all k elements subsets of $\mathcal{A}$. For any splitting of $\mathcal{A}^{(k)}$ into r classes, there exists an infinite subset $\mathcal{T} \subseteq \mathcal{A}$ such that $\mathcal{T}^{(k)}$ is contained in the same class.*

The following lemma gives a relationship of orthogonal sets between $\mu_{M,\mathcal{D}}$ and the Bernoulli measure $\mu_{\rho^{-1},5}$, which was proved by Deng and Lau [13]. For $\rho > 1$, the Bernoulli measure $\mu_{\rho^{-1},5}$ is a self-similar measure on $\mathbb{R}$, which is defined by

$$\mu_{\rho^{-1},5}(\cdot) = \frac{1}{5}\sum_{i=0}^{4} \mu_{\rho^{-1},5}(\rho(\cdot) - i). \tag{10}$$

Lemma 2.4. *Let Λ be an infinite orthogonal set of $\mu_{M,\mathcal{D}}$, and let $\psi_i(\Lambda)$ be the collection of the i-th coordinates of Λ for $i = 1, 2$. Then $\psi_1(\Lambda)$ and $\psi_2(\Lambda)$ are infinite sets.*

Proof. From (5) and (10), we have

$$\hat{\mu}_{\rho^{-1},5}(x) = \prod_{j=1}^{\infty} m_{\mathcal{D}}(\rho^{-j}x) = \prod_{j=1}^{\infty}\Big(\frac{1}{5}(1 + e^{-2\pi i\rho^{-j}x} + e^{-4\pi i\rho^{-j}x} + e^{-6\pi i\rho^{-j}x} + e^{-8\pi i\rho^{-j}x})\Big).$$

Then

$$\mathcal{Z}(\hat{\mu}_{\rho^{-1},5}) = \bigcup_{j=1}^{\infty} \rho^j \Big((\pm\frac{1}{5} + \mathbb{Z}) \cup (\pm\frac{2}{5} + \mathbb{Z})\Big). \tag{11}$$

Since Λ be an orthogonal set of $\mu_{M,\mathcal{D}}$, we have

$$(\Lambda - \Lambda) \setminus \{0\} \subseteq \mathcal{Z}(\hat{\mu}_{M,\mathcal{D}}) = \bigcup_{j=1}^{\infty} \rho^j (A_1 \cup A_2 \cup A_3 \cup A_4). \tag{12}$$

Therefore,

$$(\psi_i(\Lambda) - \psi_i(\Lambda)) \setminus \{0\} \subseteq \bigcup_{j=1}^{\infty} \rho^j \Big((\pm\frac{1}{5} + \mathbb{Z}) \cup (\pm\frac{2}{5} + \mathbb{Z})\Big).$$

This means that $\psi_i(\Lambda)$ is an orthogonal set of $\mu_{\rho^{-1},5}$.

Now, we first show that $\psi_1(\Lambda)$ is an infinite set. Suppose on the contrary that $\psi_1(\Lambda)$ is finite. By the pigeonhole principle, there exist $\lambda = \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}, \lambda' = \begin{pmatrix}\lambda_1\\ \lambda_2'\end{pmatrix} \in \Lambda$ with $\lambda_2 \neq \lambda_2'$. Thus $\lambda - \lambda' = \begin{pmatrix}0\\ \lambda_2 - \lambda_2'\end{pmatrix} \notin \mathcal{Z}(\hat{\mu}_{M,\mathcal{D}})$, this is a contradiction with the assumption of Λ is an orthogonal set of $\mu_{M,\mathcal{D}}$. Similarly, we also have $\psi_2(\Lambda)$ is infinite. We complete the proof. □

In the rest of this section, we will prove the Theorem 1.3. Before that, we give the following lemma, which is useful to proving Theorem 1.3.

Lemma 2.5. *[12, 23] Let $\mu_{\rho^{-1},5}(\rho > 1)$ be the Bernoulli measure defined as in* (10)*. Then $\mu_{\rho^{-1},5}$ admits an infinite orthogonal set if and only if $\rho = \sqrt[r]{\frac{5p}{q}}$ for some $r, p, q \in \mathbb{N}$ with* $\gcd(5p, q) = 1$.

Proof of Theorem 1.3. Firstly, we prove the necessity. Let Λ be an infinite orthogonal set for $\mu_{M,\mathcal{D}}$, then

$$(\Lambda - \Lambda)\backslash\{0\} \subseteq \bigcup_{j=1}^{\infty} \rho^j (A_1 \cup A_2 \cup A_3 \cup A_4).$$

Denote $\mathcal{A}_i = \bigcup_{j=1}^{\infty} \rho^j A_i$ for $i = 1, 2, 3, 4$, then by Ramsey's Theorem, there exists an infinite subset $\Lambda' \subseteq \Lambda$ and $i_0 \in \{1, 2, 3, 4\}$ such that

$$(\Lambda' - \Lambda')\backslash\{0\} \subseteq \mathcal{A}_{i_0}.$$

According to the Lemma 2.4, we know $\psi_1(\Lambda')$ and $\psi_2(\Lambda')$ are infinite sets and

$$(\psi_k(\Lambda') - \psi_k(\Lambda'))\backslash\{0\} \subseteq \bigcup_{j=1}^{\infty} \rho^j \Big((\pm\frac{1}{5} + \mathbb{Z}) \cup (\pm\frac{2}{5} + \mathbb{Z}) \Big),$$

for $k = 1, 2$. This means that $\psi_k(\Lambda')$ is an infinite orthogonal set. By Lemma 2.5, we can get $\rho = \sqrt[r]{\frac{5p}{q}}$ for some $p, q, r \in \mathbb{N}$ with $\gcd(5p, q) = 1$.

Conversely, suppose that $\rho = \sqrt[r]{\frac{5p}{q}}$ for some $r, p, q \in \mathbb{N}$ with $\gcd(5p, q) = 1$, and let

$$\Lambda = \left\{ \sum_{j=1}^{m} (5p)^j \mathbf{c} \,:\, m \in \mathbb{N},\, \mathbf{c} = \frac{1}{5}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} \cup \{\mathbf{0}\}.$$

Then Λ is an infinite set. Now, we need to show that the orthogonality of Λ. For any two distinct vectors $\lambda_{\mathbf{1}}, \lambda_{\mathbf{2}} \in \Lambda$, by the definition of Λ, we can express them by

$$\lambda_{\mathbf{1}} = \sum_{j=1}^{m} (5p)^j \mathbf{c},\;\; \lambda_{\mathbf{2}} = \sum_{j=1}^{n} (5p)^j \mathbf{c}$$

with $m, n \in \mathbb{N}$ and $m > n$. Then

$$\lambda_{\mathbf{1}} - \lambda_{\mathbf{2}} = \sum_{j=n+1}^{m} (5p)^j \mathbf{c} \in M^{r(n+1)} \mathcal{Z}(\hat{\delta}_{\mathcal{D}}) \subset \mathcal{Z}(\hat{\mu}_{M,\mathcal{D}}).$$

Hence we have Λ is an infinite orthogonal set of $\mu_{M,\mathcal{D}}$. We complete the proof. $\square$

3. Proof of the sufficiency of Theorem 1.2 and Theorem 1.4

In this section, we will first proof the sufficiency of Theorem 1.2. Furthermore, we also consider under what conditions the maximal orthogonal set is a spectrum of $\mu_{M,\mathcal{D}}$.

Proof of the sufficiency of Theorem 1.2. Suppose $\rho = 5p$ with $p \in \mathbb{N}$, and let

$$C_p = p\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \end{pmatrix} \right\}.$$

By the simple calculations, we have that $(M^{-1}\mathcal{D}, C_p)$ is a compatible pair. By Theorem 1.3 in [19], we know that $\mu_{M,\mathcal{D}}$ is a spectral measure. $\square$

Up to now, we have already known that if $5 \mid \rho$, then $\mu_{M,\mathcal{D}}$ is a spectral measure. In this case, we want to know the structure of the spectra of $\mu_{M,\mathcal{D}}$. Let $\rho = 5p$, and let

$$T_p = M\left[-\frac{1}{2}, \frac{1}{2}\right)^2 \cap \mathbb{Z}^2 = \left\{(m,n)^t \in \mathbb{Z}^2 \, : \, -\frac{5p}{2} \le m, n < \frac{5p}{2}\right\},$$

which is a complete residual system module M in $\mathbb{Z}^2$. We can decompose T_p into the following disjoint union

$$T_p = \bigcup_{a \in B_p} (a + C_p) \pmod M,$$

where $B_p = \{(x,y)^t \in T_p : -p/2 \le x < p/2, -5p/2 \le y < 5p/2\}$, and

$$C_p = p\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}, \begin{pmatrix}2\\-2\end{pmatrix}, \begin{pmatrix}-2\\2\end{pmatrix}\right\}.$$

For any $\gamma \in \mathbb{Z}^2$, we have

$$\gamma = \sum_{i=1}^{\infty} M^{i-1} c_i = \sum_{i=1}^{\infty} (5p)^{i-1} c_i,$$

where all $c_i \in T_p$ and $c_i = 0$ for sufficient large i.

Following, we introduce the concept of the maximal mapping, which will be used to study the structure of the maximal orthogonal sets of $\mu_{M,\mathcal{D}}$. Before that, we give some descriptions of the symbols. Let $\Sigma_5 = \{-2,-1,0,1,2\}$, and let $\Sigma_5^n = \{I = i_1 i_2 \cdots i_n : \text{ all } i_j \in \Sigma_5\}$ be the set of all words with length $n > 0$ and by convention we note $\Sigma_5^0 = \{\emptyset\}$. Let $\Sigma_5^* = \cup_{n=0}^{\infty} \Sigma_5^n$ be the set of all finite words, and denote the set of all infinite words by $\Sigma_5^{\infty} = \{I = i_1 i_2 \cdots : \text{ all } i_j \in \Sigma_5\}$. And for any $I \in \Sigma_5^*$, $J \in \Sigma_5^* \cup \Sigma_5^{\infty}$, we denote IJ as the concatenation of I and J. Moreover, we adopt the notations $I^{\infty} = II\cdots$, $I^s = \underbrace{II\cdots I}_{s}$ for each $I \in \Sigma_5^*$. And we define $I|_k$ is the prefix word of I with length k ($k \ge 1$).

Definition 3.1. *A mapping τ form Σ_5^* to T_p is called a maximal mapping if*
(i) $\tau(0^k i_{k+1}) = i_{k+1} p\mathbf{v}$ *for all $k \ge 1$, and $i_{k+1} \in \Sigma_5$;*
(ii) *for any $Ij \in \Sigma_5^*$, $\tau(Ij) = e_I + jp\mathbf{v} \pmod M$, where $e_I \in B_p$;*
(iii) *for any $I \in \Sigma_5^*$, there exists $J \in \Sigma_5^{\infty}$ such that $\tau((IJ)|_j) = \mathbf{0}$ for sufficient large j,* where $\mathbf{v} = (1,-1)^t$.

Let τ be a maximal mapping from Σ_5^* to T_p. Set

$$\Sigma_5^{\tau} = \left\{I \in \Sigma_5^{\infty} \, : \, \tau(I|_n) = 0 \text{ for sufficient large } n\right\}. \tag{13}$$

Then we can define a mapping τ^* from Σ_5^{τ} to $\mathbb{Z}^2$ by

$$\tau^*(I) = \sum_{i=1}^{\infty} (5p)^{i-1} \tau(I|_i), \ \forall I \in \Sigma_5^{\tau}. \tag{14}$$

The following theorem provide the relationship between a maximal mapping and a maximal orthogonal set of $\mu_{M,\mathcal{D}}$.

Theorem 3.2. *Let $\rho = 5p$ and Λ be a subset of $\mathbb{R}^2$ with $0 \in \Lambda$. Then Λ is a maximal orthogonal set of $\mu_{M,\mathcal{D}}$ if and only if there exists a maximal mapping τ such that $\Lambda = \tau^*(\Sigma_5^{\tau})$.*

Proof. Let τ be a maximal mapping, and

$$\tau^*(\Sigma_5^{\tau}) = \left\{\tau^*(I) = \sum_{i=1}^{\infty} (5p)^{i-1} \tau(I|_i) : I \in \Sigma_5^{\tau}\right\}.$$

We first show that the orthogonality of $\tau^*(\Sigma_5^\tau)$ for $\mu_{M,\mathcal{D}}$. For any two distinct elements $I, J \in \Sigma_5^\tau$, let k be the index of the first place in which I, J disagree, then we can deduce that

$$\tau^*(I) - \tau^*(J) = \sum_{i=1}^{\infty}(5p)^{i-1}\left(\tau(I|_i) - \tau(J|_i)\right) = \sum_{i=k}^{\infty}(5p)^{i-1}\left(\tau(I|_i) - \tau(J|_i)\right) \in (5p)^{k-1}\left((i_k - j_k)p\mathbf{v} + 5p\mathbb{Z}^2\right).$$

Hence $\tau^*(I) - \tau^*(J) \in \mathcal{Z}(\hat{\mu}_{M,\mathcal{D}})$, which means that $\tau^*(\Sigma_5^\tau)$ is an orthogonal set of $\mu_{M,\mathcal{D}}$.

Now, we consider its maximality. Assume $\tau^*(\Sigma_5^\tau)$ is not maximal, towards a contradiction. Then there exists an element $\xi \notin \tau^*(\Sigma_5^\tau)$, such that $\tau^*(\Sigma_5^\tau) \cup \{\xi\}$ is also an orthogonal set of $\mu_{M,\mathcal{D}}$. According to the orthogonality of $\tau^*(\Sigma_5^\tau) \cup \{\xi\}$ and $0 \in \tau^*(\Sigma_5^\tau)$, we can obtain that $\xi \in \mathcal{Z}(\hat{\mu}_{M,\mathcal{D}})$. There exists $\{c_i\}_{i=1}^m \subseteq T_p$ and ξ can be written as

$$\xi = \sum_{i=1}^{m}(5p)^{i-1}c_i.$$

It is enough to prove that there exists $I \in \Sigma_5^\tau$ such that $\xi = \tau^*(I)$. Firstly, we need to show that there exists $i_1 \in \Sigma_5$ such that $c_1 = \tau(i_1)$. If not, by the (iii) in Definition 3.1, for each $i \in \Sigma_5$, there is a $J_i \in \Sigma_5^\infty$ such that $iJ_i \in \Sigma_5^\tau$. Then $\tau^*(iJ_i) - \xi \in \mathcal{Z}(\hat{\mu}_{M,\mathcal{D}})$. And

$$\tau^*(iJ_i) - \xi = \tau(i) - c_1 + 5pZ_0$$

with $Z_0 \in \mathbb{Z}^2$. Since

$$\tau(i) - c_1 \in (T_p - T_p)\backslash\{0\} \subset (-5p, 5p)^2\backslash\{0\},$$

we have

$$\tau(i) - c_1 \in \mathcal{Z}(\hat{\delta}_{M^{-1}\mathcal{D}}).$$

This means that $\{\tau(i) : i \in \Sigma_5\} \cup \{c_1\}$ is an orthogonal set of $\delta_{M^{-1}\mathcal{D}}$, which contradicts to the fact that $\dim(L^2(\delta_{M^{-1}\mathcal{D}})) = 5$. Therefore $c_1 = \tau(i_1)$ for some $i_1 \in \Sigma_5$.

Similarly, there exist some $i_2 \in \Sigma_5$ such that $c_2 = \tau(i_1 i_2)$. By finite steps, we can get that there exists $I \in \Sigma_5^\tau$ such that $\xi = \tau^*(I)$. This is a contradiction. Therefore, $\tau^*(\Sigma_5^\tau)$ is a maximal orthogonal set.

Conversely, suppose that Λ is a maximal orthogonal set of $\mu_{M,\mathcal{D}}$ and $0 \in \Lambda$, then $\Lambda \setminus \{0\} \subset \mathcal{Z}(\hat{\mu}_{M,\mathcal{D}})$. We can write $\Lambda = \{\lambda_n\}_{n=0}^\infty$ with $\lambda_0 = 0$, then λ_n has a unique expression, that is

$$\lambda_n = \sum_{i=1}^{\infty} c_{n,i}(5p)^{i-1},$$

where all $c_{n,i} \in T_p$ and $c_{n,i} = 0$ for large enough i.

Let $\Lambda_\emptyset = \{c_{n,1} : n \geq 0\}$, then $\Lambda_\emptyset$ is nonempty because $0 \in \Lambda_\emptyset$. If $c_{n,1}$ and $c_{m,1}$ are any two distinct vectors in $\Lambda_\emptyset \subset T_p$, then there exist λ_m and λ_n such that

$$\lambda_n - \lambda_m = c_{n,1} - c_{m,1} + 5pZ_0$$

with $Z_0 \in \mathbb{Z}^2$. By the orthogonality of Λ, one has

$$c_{n,1} - c_{m,1} \in \mathcal{Z}(\hat{\delta}_{M^{-1}\mathcal{D}}).$$

Then $\Lambda_\emptyset$ is an orthogonal set of $\delta_{M^{-1}\mathcal{D}}$. Moreover, we know that $\Lambda_\emptyset$ is maximal. If not, there is a $\xi \in T_p\backslash\Lambda_\emptyset$ such that $\Lambda_\emptyset \cup \{\xi\}$ is also an orthogonal set of $\delta_{M^{-1}\mathcal{D}}$, that is $c_{n,1} - \xi \in \mathcal{Z}(\hat{\delta}_{M^{-1}\mathcal{D}})$ for any $n \geq 0$. Then, there exists $\lambda_n \in \Lambda$ such that

$$\lambda_n - \xi = c_{n,1} - \xi + \sum_{i=2}^{\infty} c_{n,i}(5p)^{i-1} \in \mathcal{Z}(\hat{\delta}_{M^{-1}\mathcal{D}}) \subset \mathcal{Z}(\hat{\mu}_{M,\mathcal{D}}).$$

This contradicts to the maximality of Λ. Hence $\Lambda_\emptyset = C_p$. We can define a mapping τ from Σ_5 to T_p by

$$\tau(i_1) = i_1 p\mathbf{v}.$$

Now, we can set $\Lambda_{i_1} = \{c_{n,2} : \tau(i_1) = c_{n,1}, n \geq 0\}$, then Λ_{i_1} is nonempty. Similarly, we can also get that Λ_{i_1} is a spectrum of $\delta_{M^{-1}\mathcal{D}}$. Then there exists a unique $t_{i_1} \in B_p$ such that

$$\Lambda_{i_1} = \{c_{n,2} : \tau(i_1) = c_{n,1}, n \geq 0\} = t_{i_1} + C_p \pmod M.$$

We define $\tau(i_1 i_2) = t_{i_1} + i_2 p\mathbf{v} \pmod M$ for $i_2 \in \Sigma_5$. By induction, we can define τ on Σ_5^* by

$$\tau(Ij) = t_I + jp\mathbf{v} \pmod M,$$

where $t_I \in B_p$.

Now, we show that τ is a maximal mapping. By the construction of τ, we have (i) and (ii) of Definition 3.1 are hold. As for (iii), for any $I = i_1 i_2 \cdots i_m \in \Sigma_5^*$, there exists $\lambda_n \in \Lambda$ such that $\tau(i_1 i_2 \cdots i_k) = c_{n,k}$ for $k = 1, 2, \cdots, m$. As $\lambda_n = \sum_{i=1}^{\infty} c_{n,i}(5p)^{i-1}$ for all $c_{n,i} \in T_p$ and $c_{n,i} = 0$ for large enough i, then we can find $J \in \Sigma_5^\infty$ such that $\tau((IJ)|_j) = \mathbf{0}$ for sufficient large j.

Finally, we prove that $\Lambda = \tau^*(\Sigma_5^\tau)$. For each $\lambda_n \in \Lambda$, there is a integer N_n such that $\lambda_n = \sum_{i=1}^{N_n} c_{n,i}(5p)^{i-1}$ with $c_{n,i} \in T_p$ and $c_{n,N_n} \neq 0$. Then there exists $I \in \Sigma_5^\infty$ such that $\tau(I|_k) = c_{n,k}$ for $1 \leq k \leq N_n$, and $\tau(I|_k) = 0$ for $k > N_n$. So $\Lambda \subset \tau^*(\Sigma_5^\tau)$. On the other hand, $\tau^*(\Sigma_5^\tau)$ is an orthogonal set of $\mu_{M,\mathcal{D}}$, then $\tau^*(\Sigma_5^\tau) \subset \Lambda$ because Λ is the maximal orthogonal set. Therefore, we obtain $\Lambda = \tau^*(\Sigma_5^\tau)$ and the proof is completed. □

Based on the above facts, a question emerged in my mind: what conditions can be restricted on a maximal orthogonal set to make it the spectrum of $\mu_{M,\mathcal{D}}$.

Definition 3.3. *Let $I = i_1 i_2 \cdots \in \Sigma_5^* \cup \Sigma_5^\infty$. If there exists an integer N such that $i_N \neq 0$ but $i_k = 0$ for all $k > N$, then N is called the efficient length of the word I, which is denoted by $l(I) = N$. In particular, we set $l(I) = 0$ if $I = 0^n$ or $I = 0^\infty$.*

For the sake of brevity, we will refer to the following notations.

Let $I = i_1 i_2 \cdots$ be a word in $\Sigma_5^* \cup \Sigma_5^\infty$, denote

$$I_{n,m} = i_n i_{n+1} \cdots i_{m-1}$$

for $n < m$ and $I_{n,n} = \emptyset$.

For any $I \in \Sigma_5^*$, we set

$$I^\tau = \tau(I|_1)\tau(I|_2)\tau(I|_3)\cdots.$$

Let τ be a maximal mapping, then for each $I \in \Sigma_5^n$, there is a $J \in \Sigma_5^\infty$ such that $IJ \in \Sigma_5^\tau$. Then the word $(IJ)^\tau$ can be decomposed by

$$(IJ)^\tau = I^\tau (IJ)^\tau_{n_0,n_1} (IJ)^\tau_{n_1,n_2} \cdots (IJ)^\tau_{n_m,n_{m+1}} \tag{15}$$

where $n+1 = n_0 < n_1 \cdots < n_m < n_{m+1} = \infty$ such that $\tau((IJ)|_{n_k}) \neq 0$ but the last word of $(IJ)|_{n_k} = 0$ for $1 \leq k \leq m$. Denote

$$N_I(J) = \sum_{i=0}^{m} l((IJ)^\tau_{n_i,n_{i+1}}), \tag{16}$$

which depend on the partition $\{n_0, n_1, \cdots, n_m\}$. By the definition of the maximal mapping τ, one has $N_I(J) < \infty$.

Denote

$$\mu_n = \delta_{M^{-1}\mathcal{D}} * \delta_{M^{-2}\mathcal{D}} * \cdots * \delta_{M^{-n}\mathcal{D}}, \ n \geq 1.$$

Proposition 3.4. *Let τ be a maximal mapping from Σ_5^* to T_p. Then $\tau^*(\Sigma_5^n)$ is a spectrum of μ_n for any $n \geq 1$.*

Proof. Firstly, we will show that $\tau^*(\Sigma_5^n)$ is an orthogonal set of μ_n. For any different elements I and J in Σ_5^n, let k be the smallest index such that $I|_k \neq J|_k$, then

$$\begin{aligned}\tau^*(I) - \tau^*(J) &= \sum_{i=1}^{n}(5p)^{i-1}\tau(I|_i) - \sum_{i=1}^{n}(5p)^{i-1}\tau(J|_i)\\ &= \sum_{i=k}^{n}(5p)^{i-1}(\tau(I|_i) - \tau(J|_i))\\ &\in (5p)^{k-1}\left(\tau(I|_k) - \tau(J|_k) + 5p\mathbb{Z}^2\right) \subset \mathcal{Z}(\hat{\mu}_n).\end{aligned}$$

This means that $\tau^*(\Sigma_5^n)$ is an orthogonal set of μ_n. Since $\#(\tau^*(\Sigma_5^n)) = 5^n = \dim(L^2(\mu_n))$, we have $\tau^*(\Sigma_5^n)$ is a spectrum of μ_n. □

The following lemma is important for the proof of Theorem 1.4.

Lemma 3.5. *Let τ be a maximal mapping from Σ_5^* to T_p and let $\xi \in \left[-\frac{1}{2}, \frac{1}{2}\right]^2$. Suppose that for each $I \in \Sigma_5^*$, there exists $J_I \in \Sigma_5^\infty$ such that $IJ_I \in \Sigma_5^\tau$ and $\sup_{I\in\Sigma_5^*} N_I(J_I) < \infty$. Then there is a positive constant c and $J \in \Sigma_5^\infty$ (may be not J_I) for each $I \in \Sigma_5^n$ such that $IJ \in \Sigma_5^\tau$ and*

$$\left|\hat{\mu}_{M,\mathcal{D}}\left(\frac{\xi + \tau^*(IJ)}{(5p)^n}\right)\right|^2 \geq c.$$

Proof. For any $n \geq 1$ and $I \in \Sigma_5^n$, it is enough to prove the conclusion by following two cases.
Case 1. $\tau(I0^k) = 0$ for all $k \geq 1$. Set $J = 0^\infty \in \Sigma_5^\infty$, and

$$\hat{\mu}_{M,\mathcal{D}}\left(\frac{\xi + \tau^*(IJ)}{(5p)^n}\right) = \hat{\mu}_{M,\mathcal{D}}\left(\frac{\xi + \tau^*(I)}{(5p)^n}\right).$$

As $\tau(I|_k) \in T_p \subset \left[-\frac{5p}{2}, \frac{5p}{2}\right]^2$, we have

$$\left|\left(\frac{\xi + \tau^*(I)}{(5p)^n}\right)^{(i)}\right| \leq \frac{\frac{1}{2} + \frac{5p}{2}(1 + 5p + \cdots + (5p)^{n-1})}{(5p)^n} \leq \frac{5}{8}$$

for $i = 1, 2$. Let

$$\beta_0 = \min\left\{\left|\hat{\mu}_{M,\mathcal{D}}(\xi)\right|^2 : \xi \in \left[-\frac{5}{8}, \frac{5}{8}\right]^2\right\},$$

then $0 < \beta_0 < 1$, and $\left|\hat{\mu}_{M,\mathcal{D}}\left(\frac{\xi+\tau^*(I)}{(5p)^n}\right)\right|^2 \geq \beta_0$. That is $\left|\hat{\mu}_{M,\mathcal{D}}\left(\frac{\xi+\tau^*(IJ)}{(5p)^n}\right)\right|^2 \geq \beta_0$. The Case 1 follows.
Case 2. $\tau(I0^k) \neq 0$ for some $k \geq 1$. Let k be the first integer such that $\tau(I0^k) \neq 0$. Without loss of generality, we can assume that $k = 1$. Otherwise, we can replace I by $I0^{k-1}$. Suppose that τ be a maximal mapping from Σ_5^* to T_p, then there is a $J_{I0} \in \Sigma_5^\infty$ such that $I0J_{I0} \in \Sigma_5^\tau$ and $N_{I0}(J_{I0}) < \infty$. That is, set $J = 0J_{I0} \in \Sigma_5^\infty$, we have $IJ \in \Sigma_5^\tau$ and $N_I(J) < \infty$. Using the decomposition of (15), we have $(IJ)^\tau = I^\tau(IJ)^\tau_{n_0,n_1}(IJ)^\tau_{n_1,n_2}\cdots(IJ)^\tau_{n_m,n_{m+1}}$,

where $n+1 = n_0 < n_1 < \cdots < n_m < n_{m+1} < \infty$ and $\tau((IJ)|_{n_j}) \neq 0$. Denote $n'_k = l\Big((IJ)^{\tau}_{n_k,n_{k+1}}\Big)$, then

$$\begin{aligned}\tau^*(IJ) &= \sum_{i=1}^{\infty}(5p)^{i-1}\tau((IJ)|_i) = \tau^*(I) + \sum_{i=n+1}^{\infty}(5p)^{i-1}\tau((IJ)|_i)\\ &= \tau^*(I) + \sum_{k=0}^{m}\sum_{i=n_k}^{n_{k+1}-1}(5p)^{i-1}\tau((IJ)|_i)\\ &= \tau^*(I) + \sum_{k=0}^{m}\sum_{i=n_k}^{n_k+n'_k-1}(5p)^{i-1}\tau((IJ)|_i).\end{aligned}$$

In particular, if $m = 1$, then $\tau^*(IJ) = \tau^*(I) + \sum_{i=n_0}^{n_0+n'_0-1}(5p)^{i-1}\tau((IJ)|_i) + \sum_{i=n_1}^{n_1+n'_1-1}(5p)^{i-1}\tau((IJ)|_i)$. Then

$$\begin{aligned}\left|\hat{\mu}_{M,\mathcal{D}}\left(\frac{\xi+\tau^*(IJ)}{(5p)^n}\right)\right|^2 &= \left|\hat{\mu}_{M,\mathcal{D}}\left(\frac{\xi+\tau^*(I)}{(5p)^n} + \sum_{i=1}^{n'_0}(5p)^{i-1}\tau((IJ)|_{n+i}) + (5p)^{n_1-n-1}\sum_{i=1}^{n'_1}(5p)^{i-1}\tau((IJ)|_{n_1+i-1})\right)\right|^2\\ &= \left|\hat{\mu}_{M,\mathcal{D}}\left(\xi_0 + v_0 + (5p)^{n_1-n-1}\sum_{i=1}^{n'_1}(5p)^{i-1}\tau((IJ)|_{n_1+i-1})\right)\right|^2\\ &= \left|\hat{\mu}_{n_1-n-1}(\xi_0+v_0)\right|^2\left|\hat{\mu}_{M,\mathcal{D}}(\xi_1 + \sum_{i=1}^{n'_1}(5p)^{i-1}\tau((IJ)|_{n_1+i-1}))\right|^2\\ &= \left|\hat{\mu}_{n_1-n-1}(\xi_0+v_0)\right|^2\left|\hat{\mu}_{M,\mathcal{D}}(\xi_1+v_1)\right|^2\\ &\geq \left|\hat{\mu}_{M,\mathcal{D}}(\xi_0+v_0)\right|^2\left|\hat{\mu}_{M,\mathcal{D}}(\xi_1+v_1)\right|^2,\end{aligned}$$

where

$$\xi_0 = \frac{\xi+\tau^*(I)}{(5p)^n}, v_0 = \sum_{i=1}^{n'_0}(5p)^{i-1}\tau((IJ)|_{n+i}),$$

$$\xi_1 = \frac{\xi_0+v_0}{(5p)^{n_1-n-1}}, v_1 = \sum_{i=1}^{n'_1}(5p)^{i-1}\tau((IJ)|_{n_1+i-1}).$$

From $\sup_{I\in\Sigma_5^*} N_I(J) < \infty$, there exists a positive integer M_τ such that $n'_0 + n'_1 \leq M_\tau$. As $\tau((IJ)|_{n+1}) \in B_p\backslash\{0\}$, we have $v_0 = (v_0^{(1)}, v_0^{(2)})^t \notin \mathcal{Z}(\hat{\mu}_{M,\mathcal{D}})$. Recall that $\tau((IJ)|_k) \in T_p \subset \left[-\frac{5p}{2}, \frac{5p}{2}\right]^2$, then for $i = 1, 2$, we have

$$|v_0^{(i)}| \leq \frac{5p}{2}(1 + 5p + \cdots + (5p)^{n'_0-1}) \leq (5p)^{M^\tau}$$

and

$$|\xi_0^{(i)}| \leq \frac{\frac{1}{2} + \frac{5p}{2}(1 + 5p + \cdots + (5p)^{n-1})}{(5p)^n} \leq \frac{5}{8}.$$

Similarly, we also get $\xi_1 \in \left[-\frac{5}{8}, \frac{5}{8}\right]^2$, $v_1 \in [-(5p)^{M^\tau}, (5p)^{M^\tau}]^2 \cap \mathbb{Z}^2$ and $v_1 \notin \mathcal{Z}(\hat{\mu}_{M,\mathcal{D}})$.

Let

$$\mathcal{S} = \bigcup\left\{v + \xi : \xi \in \left[-\frac{5}{8}, \frac{5}{8}\right]^2, v \in [-(5p)^{M^\tau}, (5p)^{M^\tau}]^2 \cap \mathbb{Z}^2, v \notin \mathcal{Z}(\hat{\mu}_{M,\mathcal{D}})\right\}$$

and

$$\beta_0 = \min\left\{\left|\hat{\mu}_{M,\mathcal{D}}(\xi)\right|^2 : \xi \in \mathcal{S}\right\},$$

then $0 < \beta_0 < 1$, and thus

$$\left|\hat{\mu}_{M,\mathcal{D}}\left(\frac{\xi + \tau^*(IJ)}{(5p)^n}\right)\right|^2 \geq \beta_0 \left|\hat{\mu}_{M,\mathcal{D}}\left(\xi_1 + v_1\right)\right|^2 \geq \beta_0^2.$$

Generally, if $m \leq M_\tau$, similar to the above discussions, we can get

$$\left|\hat{\mu}_{M,\mathcal{D}}\left(\frac{\xi + \tau^*(IJ)}{(5p)^n}\right)\right|^2 \geq \beta_0^{m+1} \geq \beta_0^{M_\tau+1}.$$

Let $c := \beta_0^{M_\tau+1}$, we can get the conclusion. So the proof is completed. □

At the end of this section, we prove the Theorem 1.4.

Proof of Theorem 1.4. From Theorem 3.2, we have shown that $\Lambda = \tau^*(\Sigma_5^\tau)$ is an orthogonal set of $\mu_{M,\mathcal{D}}$. This means that

$$Q_\Lambda(\xi) := \sum_{I\in\Sigma_5^\tau} \left|\hat{\mu}_{M,\mathcal{D}}(\xi + \tau^*(I))\right|^2 = \sum_{k=0}^{\infty} \sum_{\{I\in\Sigma_5^\tau:\, l(I^\tau)=k\}} \left|\hat{\mu}_{M,\mathcal{D}}(\xi + \tau^*(I))\right|^2 \leq 1.$$

We first need to show that $Q_\Lambda(\xi) \geq 1$ at a small domain. Let $\xi \in \left(-\frac{1}{2}, \frac{1}{2}\right)^2$, then for any $0 < \varepsilon < 1$, there exists a integer $N := N(\varepsilon)$ such that

$$\sum_{I\in\Sigma_5^\tau,\, l(I^\tau)>N} \left|\hat{\mu}_{M,\mathcal{D}}(\xi + \tau^*(I))\right|^2 < \varepsilon.$$

For $n > N$, one has

$$\begin{aligned} Q_\Lambda(\xi) &= \sum_{k=0}^{\infty} \sum_{I\in\Sigma_5^\tau, l(I^\tau)=k} \left|\hat{\mu}_{M,\mathcal{D}}(\xi + \tau^*(I))\right|^2 \\ &\geq \sum_{k=0}^{N} \sum_{I\in\Sigma_5^\tau, l(I^\tau)=k} \left|\hat{\mu}_{M,\mathcal{D}}(\xi + \tau^*(I))\right|^2 \\ &= \sum_{k=0}^{N} \sum_{I\in\Sigma_5^\tau, l(I^\tau)=k} \left|\hat{\mu}_n(\xi + \tau^*(I))\right|^2 \left|\hat{\mu}_{M,\mathcal{D}}\left(\frac{\xi + \tau^*(I)}{(5p)^n}\right)\right|^2. \end{aligned}$$

Set $\Omega_N = \{I \in \Sigma_5^\tau : l(I^\tau) \leq N\}$, then for each $I \in \Omega_N$, we have

$$\left|\left(\frac{\xi + \tau^*(I)}{(5p)^n}\right)^{(i)}\right| \leq \frac{\frac{1}{2} + \frac{5p}{2}(1 + 5p + \cdots + (5p)^{N-1})}{(5p)^n} < \frac{1}{2(5p-1)(5p)^{n-N-1}}$$

for $i = 1, 2$. Together with the continuity of $\hat{\mu}_{M,\mathcal{D}}$ and $\hat{\mu}_{M,\mathcal{D}}(0) = 1$, then there exists N_1 such that

$$\left|\hat{\mu}_{M,\mathcal{D}}\left(\frac{\xi + \tau^*(I)}{(5p)^n}\right)\right|^2 > 1 - \varepsilon$$

whenever $n \geq N_1 > N$. Therefore,

$$Q_\Lambda(\xi) > (1 - \varepsilon) \sum_{I\in\Omega_N} \left|\hat{\mu}_n(\xi + \tau^*(I))\right|^2$$

for all $\xi \in \left(-\frac{1}{2}, \frac{1}{2}\right)^2$. By Proposition 3.4, we have

$$\begin{aligned} 1 = \sum_{I \in \Sigma_5^n} \left|\hat{\mu}_n(\xi + \tau^*(I))\right|^2 &= \sum_{I \in \Sigma_5^n, l(I^\tau) \le N} \left|\hat{\mu}_n(\xi + \tau^*(I))\right|^2 + \sum_{I \in \Sigma_5^n, N < l(I^\tau) \le n} \left|\hat{\mu}_n(\xi + \tau^*(I))\right|^2 \\ &= \sum_{I \in \Sigma_5^n, l(I^\tau) \le N} \left|\hat{\mu}_n(\xi + \tau^*(IJ))\right|^2 + \sum_{I \in \Sigma_5^n, N < l(I^\tau) \le n} \left|\hat{\mu}_n(\xi + \tau^*(IJ))\right|^2 \\ &= \sum_{I \in \Omega_N} \left|\hat{\mu}_n(\xi + \tau^*(I))\right|^2 + \sum_{I \in \Sigma_5^\tau, N < l(I^\tau) \le n} \left|\hat{\mu}_n(\xi + \tau^*(I))\right|^2, \end{aligned} \tag{17}$$

where the second equation holds because τ is a maximal mapping, then for $I \in \Sigma_5^n$, there is a $J \in \Sigma_5^\infty$ such that $IJ \in \Sigma_5^\tau$. By Lemma 3.5, there is a positive constant c such that

$$\begin{aligned} \sum_{I \in \Sigma_5^n, N < l(I^\tau) \le n} \left|\hat{\mu}_n(\xi + \tau^*(I))\right|^2 &= \sum_{I \in \Sigma_5^n, N < l(I^\tau) \le n} \left|\hat{\mu}_n(\xi + \tau^*(IJ))\right|^2 \\ &\le \frac{1}{c} \sum_{I \in \Sigma_5^\tau, N < l(I^\tau) \le n} \left|\hat{\mu}_{M,\mathcal{D}}(\xi + \tau^*(I))\right|^2 \\ &\le \frac{1}{c} \sum_{I \in \Sigma_5^\tau, l(I^\tau) > N} \left|\hat{\mu}_{M,\mathcal{D}}(\xi + \tau^*(I))\right|^2 < \frac{\varepsilon}{c}. \end{aligned}$$

Combining with (17), we can get

$$\sum_{I \in \Omega_N} \left|\hat{\mu}_n(\xi + \tau^*(I))\right|^2 = 1 - \sum_{I \in \Sigma_5^\tau, N < l(I^\tau) \le n} \left|\hat{\mu}_n(\xi + \tau^*(I))\right|^2 > 1 - \frac{\varepsilon}{c}.$$

Hence, $Q_\Lambda(\xi) > (1-\varepsilon)\left(1 - \frac{\varepsilon}{c}\right)$. Letting $\varepsilon \to 0$, we have $Q_\Lambda(\xi) \ge 1$. Therefore, $Q_\Lambda(\xi) = 1$ for $\xi \in \left(-\frac{1}{2}, \frac{1}{2}\right)^2$.

Now, we need to show that $Q_\Lambda(\xi) = 1$ for $\xi \in \mathbb{R}^2$. By the Theorem 2.1, we have $Q_\Lambda(\xi)$ is an entire function on the complex plane. For $\xi \in \mathbb{R}^2$, we have known that $Q_\Lambda(\alpha\xi) = 1$ for $\alpha \in (-\delta, \delta)$ with $\delta > 0$. Then $Q_\Lambda(\alpha\xi) = 1$ for $\alpha \in \mathbb{R}$. Therefore, $Q_\Lambda(\xi) = 1$ for $\xi \in \mathbb{R}^2$. And we complete the proof. □

4. Proof of the necessity of Theorem 1.2

In this section, we mainly prove the necessity of Theorem 1.2. Assume that $\mu_{M,\mathcal{D}}$ is a spectral measure with a spectrum Λ and $0 \in \Lambda$. By Theorem 1.3, we have $\rho = \sqrt[r]{\frac{5p}{q}}$ for $p, q, r \in \mathbb{N}$ and $\gcd(5p, q) = 1$. So we will side step the necessity by showing that the following two cases are not spectral measure.

4.1. $\rho = \sqrt[r]{\frac{5p}{q}}$ and $r > 1$

Proposition 4.1. *If $\rho = \sqrt[r]{\frac{5p}{q}}$ for $p, q \in \mathbb{N}$ with $\gcd(5p, q) = 1$ and $r > 1$, then $\mu_{M,\mathcal{D}}$ is not a spectral measure.*

We first introduction the following lemma, which was given by Deng and Lau in [13]. This is a useful lemma to prove the Proposition 4.1.

Lemma 4.2. *[13] Assume that $b \in \mathbb{R}$ admits a minimal integer polynomial $px^r - q$ and satisfies that $a_1 b^l + a_2 b^m = a_3 b^n$, where $l, m, n \ge 0$ and $a_1, a_2, a_3 \in \mathbb{Z} \setminus \{0\}$. Then $l \equiv m \equiv n \pmod{r}$.*

Proof of Proposition 4.1. Suppose that $\rho = \sqrt[r]{\frac{5p}{q}}$ with $r > 1$, then ρ admits the minimal integer polynomial $qx^r - 5p = 0$. For each $\xi \in \mathbb{R}^2$, we write

$$\hat{\mu}_{M,\mathcal{D}}(\xi) = \prod_{k=1}^{\infty} \hat{\delta}_{\mathcal{D}}(M^{-k}\xi) = \prod_{i=1}^{r}\prod_{j=0}^{\infty} \hat{\delta}_{\mathcal{D}}(M^{-(jr+i)}\xi) = \prod_{i=1}^{r}\prod_{j=0}^{\infty} \hat{\delta}_{\mathcal{D}}(\rho^{-(jr+i)}\xi).$$

Denote $\mu_i = *_{j=0}^{\infty}\delta_{M^{-(jr+i)}\mathcal{D}}$ for $1 \le i \le r$, then

$$\hat{\mu}_i(\xi) = \prod_{j=0}^{\infty} \hat{\delta}_{\mathcal{D}}(\rho^{-(jr+i)}\xi) \tag{18}$$

for $1 \le i \le r$ and we have $\mu_{M,\mathcal{D}} = \mu_1 * \mu_2 \cdots * \mu_r$. According to Theorem 1.3, we have $\mu_{M,\mathcal{D}}$ admits an infinite orthogonal set. Let Λ be an orthogonal set of $\mu_{M,\mathcal{D}}$. We claim that Λ be an orthogonal set of μ_i for some $1 \le i \le r$. In fact, there are two distinct elements λ_1 and λ_2 in Λ such that $\lambda_{12} := \lambda_1 - \lambda_2 \in \mathcal{Z}(\hat{\mu}_{M,\mathcal{D}})$. And, we can express them by

$$\lambda_1 = \rho^s\alpha_1,\ \lambda_2 = \rho^t\alpha_2,\ \lambda_{12} = \rho^l\alpha_{12},$$

where $s, t, l \in \mathbb{N}$ and $\alpha_1, \alpha_2, \alpha_{12} \in \mathcal{Z}(m_{\mathcal{D}}) = \cup_{i=1}^4 A_i$. Hence $\rho^l\alpha_{12} = \rho^s\alpha_1 - \rho^t\alpha_2$. By Lemma 4.2, one has $s \equiv t \equiv l \pmod r$. Then we deduce that Λ is an orthogonal set of μ_i for some $1 \le i \le r$. By Lemma 2.2, we know that Λ cannot be a spectrum of $\mu_{M,\mathcal{D}}$. The desired conclusion get. □

Furthermore, we can get the following result.

Proposition 4.3. *Let $\rho = \frac{5p}{q}$ for some $p, q \in \mathbb{N}$ with* gcd$(5p, q) = 1$. *If Λ is an orthogonal set of $\mu_{M,\mathcal{D}}$, then there exists $t \in \mathbb{N}$ such that*

$$(\Lambda - \Lambda) \setminus \{0\} \subseteq \rho^t \bigcup_{j=0}^{\infty} (5p)^j (A_1 \cup A_2 \cup A_3 \cup A_4). \tag{19}$$

Proof. For any $\lambda_1 \in \Lambda$, and let t be the smallest index such that $(\Lambda - \Lambda) \cap \rho^t\left(\bigcup_{i=1}^4 A_i\right) \neq \emptyset$, then there exists $\lambda_2 \in \Lambda$ and $\lambda_1 \neq \lambda_2$ such that $\lambda_1 - \lambda_2 = \rho^t\alpha_{12}$ for some $\alpha_{12} \in A_1 \cup A_2 \cup A_3 \cup A_4$. Then for any $\lambda_3 \in \Lambda \setminus \{\lambda_1, \lambda_2\}$, by the orthogonality of Λ, we have

$$\lambda_1 - \lambda_3 = \rho^s\alpha_{13} \text{ and } \lambda_2 - \lambda_3 = \rho^l\alpha_{23},$$

where $\alpha_{13}, \alpha_{23} \in \bigcup_{i=1}^4 A_i$ and $s \ge t, l \in \mathbb{N}$. Therefore,

$$\rho^s\alpha_{13} = \rho^t\alpha_{12} + \rho^l\alpha_{23}. \tag{20}$$

Now, we will proof the proposition by the following cases.
Case 1: if $s = t$, then $\lambda_1 - \lambda_3 = \rho^s\alpha_{13} = \rho^t\alpha_{13} \in \rho^t(A_1 \cup A_2 \cup A_3 \cup A_4)$.
Case 2: if $s > t$, then $\alpha_{12} + \rho^{l-t}\alpha_{23} = \rho^{s-t}\alpha_{13}$. We claim that $l - t = 0$. Indeed, suppose that $l - t > 0$. Then

$$q^{l+s-2t}\alpha_{12} + q^{s-t}(5p)^{l-t}\alpha_{23} = q^{l-t}(5p)^{s-t}\alpha_{13}.$$

But the right-side of the above equation belongs in $\mathbb{Z}^2$ while the left-side is not, which is a contradiction. Similarly, $l - t < 0$ can also get contradiction. So, the claim follows. Then we get $\rho^{s-t}\alpha_{13} = \alpha_{12} + \alpha_{23} \in \frac{\mathbb{Z}^2}{5}$. Since $\rho = \frac{5p}{q}$ and gcd$(5p, q) = 1$, we have $\alpha_{13} \in \frac{q^{s-t}\mathbb{Z}^2}{5} \cap \left(\bigcup_{i=1}^4 A_i\right)$. Then we have

$$q^{t-s}\alpha_{13} \in A_1 \cup A_2 \cup A_3 \cup A_4.$$

In fact, as $\alpha_{13} \in \frac{q^{s-t}\mathbb{Z}^2}{5} \cap \left(\bigcup_{i=1}^4 A_i\right)$, we have $\alpha_{13} = \frac{q^{s-t}\mathbf{z}}{5} = \beta$, where $\beta \in \bigcup_{i=1}^4 A_i$ and $\mathbf{z} \in \mathbb{Z}^2$. Therefore,

$$5\alpha_{13} = q^{s-t}\mathbf{z} = 5\beta \equiv \left\{\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}, \begin{pmatrix}2\\-2\end{pmatrix}, \begin{pmatrix}-2\\2\end{pmatrix}\right\} \pmod{5\mathbb{Z}^2}, \tag{21}$$

so $q^{t-s}\alpha_{13} \in A_1 \cup A_2 \cup A_3 \cup A_4$. Hence

$$\lambda_1 - \lambda_3 = \rho^s\alpha_{13} = \rho^t(5p)^{s-t}q^{t-s}\alpha_{13} \in \rho^t(5p)^{s-t}(A_1 \cup A_2 \cup A_3 \cup A_4).$$

This completes the proof. □

Together with Theorem 3.2, Lemma 2.2, Lemma 2.5 and Proposition 4.3, we obtain the following proposition.

Proposition 4.4. *Let $\rho = \frac{5p}{q}$ for some $p, q \in \mathbb{N}$ with $\gcd(5p, q) = 1$, and let Λ be a set in $\mathbb{R}^2$ with $0 \in \Lambda$. Then Λ is a maximal orthogonal set of $\mu_{M,\mathcal{D}}$ if and only if there exists a maximal mapping τ with respect to $\rho = 5p$ such that*

$$\Lambda = \frac{\rho^t}{5p}\tau^*(\Sigma_5^\tau), \quad t \in \mathbb{N}.$$

4.2. $\rho = \frac{5p}{q}$ and $q > 1$

Proposition 4.5. *If $\rho = \frac{5p}{q}$ for $p, q \in \mathbb{N}$ with $\gcd(5p, q) = 1$ and $q > 1$, then $\mu_{M,\mathcal{D}}$ is not a spectral measure.*

We will prove this proposition by the following technical lemmas.

Lemma 4.6. *Let $\rho = \frac{5p}{q}$ for some $p, q \in \mathbb{N}$ with $\gcd(5p, q) = 1$ and $1 < q < 5p$. Fix $j \in \{1, 2\}$, then for any $x = \begin{pmatrix}x_1\\x_2\end{pmatrix} \in \mathbb{R}^2$ with $|x_j| > 1$, there exists $y = \begin{pmatrix}y_1\\y_2\end{pmatrix} \in \mathbb{R}^2$ such that*

$$\rho^{-2}|x_j|^{\log_{5p} q} \le |y_j| \le \rho^{-1}|x_j|,$$

and

$$|\hat{\mu}_{M,\mathcal{D}}(x)| \le c|\hat{\mu}_{M,\mathcal{D}}(y)|,$$

where $c = \max\left\{|m_{\mathcal{D}}(x)| : x \notin \left(-\frac{1}{10p}, \frac{1}{10p}\right)^2\right\}$.

Proof. Without loss of generality, it is enough to prove that it holds when $j = 1$. Denote that $\{t\}$ is the decimal part of $t \in \mathbb{R}$, and $\{t\} \in \left(-\frac{1}{2}, \frac{1}{2}\right]$. If $|\{\rho^{-1}x_1\}| \ge \frac{1}{10p}$, then

$$|\hat{\mu}_{M,\mathcal{D}}(x)| = |m_{\mathcal{D}}(\rho^{-1}x)||\hat{\mu}_{M,\mathcal{D}}(\rho^{-1}x)| \le c|\hat{\mu}_{M,\mathcal{D}}(\rho^{-1}x)|. \tag{22}$$

Taking $y = \rho^{-1}x$, then we have $|\hat{\mu}_{M,\mathcal{D}}(x)| \le c|\hat{\mu}_{M,\mathcal{D}}(y)|$. If $|\{\rho^{-1}x_1\}| < \frac{1}{10p}$, then

$$\rho^{-1}x_1 - \{\rho^{-1}x_1\} = (5p)^s z_0 \tag{23}$$

where $s \ge 0$ and $z_0 \in \mathbb{Z}$ with $5p \nmid z_0$. Then

$$\rho^{-(s+2)}x_1 = \frac{q^{s+1}z_0}{5p} + \rho^{-(s+1)}\{\rho^{-1}x_1\}.$$

Since $|\rho^{-(s+1)}\{\rho^{-1}x_1\}| < \frac{1}{10p}$ and $\frac{1}{5p} \leq |\{\frac{q^{s+1}z_0}{5p}\}| \leq 1 - \frac{1}{5p}$, we have $|\{\rho^{-(s+2)}x_1\}| \notin \left(-\frac{1}{10p}, \frac{1}{10p}\right)$. By the similar argument to (22), we obtain that

$$\begin{aligned}|\hat{\mu}_{M,\mathcal{D}}(x)| &= \prod_{j=1}^{s+2} |m_{\mathcal{D}}(\rho^{-j}x)||\hat{\mu}_{M,\mathcal{D}}(\rho^{-(s+2)}x)| \\ &\leq |m_{\mathcal{D}}(\rho^{-(s+2)}x)||\hat{\mu}_{M,\mathcal{D}}(\rho^{-(s+2)}x)| \\ &\leq c|\hat{\mu}_{M,\mathcal{D}}(\rho^{-(s+2)}x)|.\end{aligned}$$

Now taking $y = \rho^{-(s+2)}x = \begin{pmatrix}\rho^{-(s+2)}x_1\\ \rho^{-(s+2)}x_2\end{pmatrix}$, then we have

$$|y_1| = |\rho^{-(s+2)}x_1| \leq \rho^{-1}|x_1|. \tag{24}$$

Together with (23), we have $|x_1| \geq (5p)^s$, this implies $\rho^{-s} \geq |x_1|^{\log_{5p} q-1}$, then

$$|y_1| = \rho^{-(s+2)}|x_1| \geq \rho^{-2}|x_1|^{\log_{5p} q}.$$

Hence, we obtain

$$\rho^{-2}|x_1|^{\log_{5p} q} \leq |y_1| \leq \rho^{-1}|x_1|.$$

Therefore, we complete the proof. □

Lemma 4.7. *Let $\rho = \frac{5p}{q}$ for some $p, q \in \mathbb{N}$ with $\gcd(5p, q) = 1$ and $1 < q < 5p$. Fix $j \in \{1, 2\}$, then there exist $\beta > 0$ such that*

$$|\hat{\mu}_{M,\mathcal{D}}(x)| \leq (\ln |x_j|)^{-\beta}$$

for each $x = \begin{pmatrix}x_1\\ x_2\end{pmatrix} \in \mathbb{R}^2$ with $|x_j| > 1$.

Proof. Without loss of generality, we only need to prove the case $j = 1$. Let $x = \begin{pmatrix}x_1\\ x_2\end{pmatrix} \in \mathbb{R}^2$ with $|x_1| > 1$. Set $d^{(0)} = x$, and applying Lemma 4.6 iteratively, we obtain a sequence $\left\{d^{(i)} = \begin{pmatrix}d_1^{(i)}\\ d_2^{(i)}\end{pmatrix}\right\}_{i=0}^{l}$, where l is the smallest number such that $|d_1^{(l)}| \leq 1$, which satisfies

$$\rho^{-2}|d_1^{(i-1)}|^{\log_{5p} q} \leq |d_1^{(i)}| \leq \rho^{-1}|d_1^{(i-1)}|, \quad i = 1, 2, \cdots, l \tag{25}$$

and

$$|\hat{\mu}_{M,\mathcal{D}}(d^{(i-1)})| \leq c|\hat{\mu}_{M,\mathcal{D}}(d^{(i)})|, \quad i = 1, 2, \cdots, l. \tag{26}$$

Then it follows from (26) that

$$|\hat{\mu}_{M,\mathcal{D}}(x)| \leq c^l|\hat{\mu}_{M,\mathcal{D}}(d^{(l)})| \leq c^l. \tag{27}$$

Moreover, denote $\omega = \log_{5p} q$, by (25), we obtain

$$\rho^{-2(1-\omega)^{-1}}|x_1|^{\omega^l} \leq |d_1^{(l)}| \leq 1. \tag{28}$$

Thus, we have

$$\ln |x_1| \leq 2(1-\omega)^{-1}\omega^{-l} \ln 5p \quad \text{and} \quad l \geq \frac{1}{\ln \omega} \ln\left(\frac{2(1-\omega)^{-1}\ln 5p}{\ln |x_1|}\right).$$

Since

$$\frac{1}{\ln\omega}\cdot\ln\left(\frac{2(1-\omega)^{-1}\ln 5p}{\ln|x_1|}\right)\ln c=\ln\left((\ln|x_1|)^{-\frac{\ln c}{\ln\omega}}\cdot(2(1-\omega)^{-1}\ln 5p)^{\frac{\ln c}{\ln\omega}}\right),\tag{29}$$

according to (27) and (29) ,we have

$$|\hat{\mu}_{M,\mathcal{D}}(x)|\le(\ln|x_1|)^{-\frac{\ln c}{\ln\omega}}(2(1-\omega)^{-1}\ln 5p)^{\frac{\ln c}{\ln\omega}}.$$

Hence, for each $\beta\in\left(0,\frac{\ln c}{\ln\omega}\right)$ and $x=\begin{pmatrix}x_1\\x_2\end{pmatrix}\in\mathbb{R}^2$ with $|x_1|>1$, we have

$$|\hat{\mu}_{M,\mathcal{D}}(x)|\le(\ln|x_1|)^{-\beta}.$$

Therefore, we complete the proof. □

Lemma 4.8. *Let $\rho=\frac{5p}{q}$ for some $p,q\in\mathbb{N}$ with $\gcd(5p,q)=1$. Then for all $\xi\in\mathbb{R}^2$,*

$$\sum_{I\in\Sigma_5^\tau,l(I)\le N+1}\left|\hat{\mu}_{N+t}\left(\xi+\frac{\rho^t}{5p}\tau^*(I)\right)\right|^2\le 1,$$

where τ is a maximal mapping with respect to $\rho=5p$ and $t,N\in\mathbb{N}$.

Proof. For any distinct elements $I,J\in\Sigma_5^\tau$ and $1\le l(I),l(J)\le N+1$, let k be the first index in which I,J disagree, then

$$\begin{aligned}\frac{\rho^t}{5p}\tau^*(I)-\frac{\rho^t}{5p}\tau^*(J)&=\frac{\rho^t}{5p}(5p)^{k-1}((\tau(I|_k)-\tau(J|_k))+5pz_0)\\&\in M^{k+t-1}\mathcal{Z}(m_{\mathcal{D}})\subseteq\mathcal{Z}(\hat{\mu}_{N+t})\end{aligned}$$

for some integer z_0. This means that $\frac{\rho^t}{5p}\tau^*(\Omega_{N+1})$ is an orthogonal set of μ_{N+t}, where $\Omega_{N+1}=\{I\in\Sigma_5^\tau: l(I)\le N+1\}$. The result follows. □

Proof of Proposition 4.5. Assume for contradiction's sake that $\mu_{M,\mathcal{D}}$ is a spectral measure. Let Λ be a spectrum of $\mu_{M,\mathcal{D}}$ with $0\in\Lambda$, and also Λ is a maximal orthogonal set of $\mu_{M,\mathcal{D}}$. Then, by Proposition 4.4, there exists a maximal mapping τ with respect to $\mu_{M,\mathcal{D}}$, such that

$$\Lambda=\frac{\rho^t}{5p}\left\{\sum_{j=1}^{\infty}(5p)^{j-1}\tau(I|_j)\,:\,I\in\Sigma_5^*\text{ and }\tau(I|_j)=0\text{ for sufficient large }j\right\},\tag{30}$$

for $t\in\mathbb{N}$.

Fix $\xi=\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix}\in\left(-\frac{1}{2},\frac{1}{2}\right)^2$ and an integer N such that $N>\beta^{-1}$, where β is the number obtained in Lemma 4.7. Let $\mathcal{I}_n=\{I\in\Sigma_5^\tau:\,l(I)\le n^N\}$ and $\mathcal{I}_{n,n+1}=\{I\in\Sigma_5^\tau:\,n^N<l(I)\le(n+1)^N\}$. For any $I\in\mathcal{I}_{n,n+1}$, we have

$$\begin{aligned}\left|\pi_j\left(\rho^{-(n+1)^N-t}\left(\xi+\frac{\rho^t}{5p}\tau^*(I)\right)\right)\right|&\ge\left(\frac{q}{5p}\right)^{(n+1)^N}\rho^{-t}\left(\rho^t(5p)^{l(I)-3}-\frac{1}{2}\right)\\&\ge\left(\frac{q}{5p}\right)^{(n+1)^N}\rho^{-t}\left(\rho^t(5p)^{n^N-3}-\frac{1}{2}\right)\\&\ge\left(\frac{q}{5p}\right)^{(n+1)^N}\frac{(5p)^{n^N-3}}{2}\\&\ge\frac{1}{2(5p)^3}\left(\frac{q^n}{(5p)^{N^2}}\right)^{n^{N-1}}>1,\end{aligned}\tag{31}$$

where $\pi_j(E)$ is the set of j-th coordinate for $E \subseteq \mathbb{R}^2$ and $j \in \{1, 2\}$.

Therefore, by Lemma 4.7 and (31), we have

$$\left|\hat{\mu}_{M,\mathcal{D}}\left(\rho^{-(n+1)^N-t}\left(\xi+\frac{\rho^t}{5p}\tau^*(I)\right)\right)\right| \leq \left(\ln\left|\rho^{-(n+1)^N-t}\pi_j\left(\xi+\frac{\rho^t}{5p}\tau^*(I)\right)\right|\right)^{-\beta} \leq \left(n^N\left(\ln q-\frac{N^2\ln(5p)}{n}-\frac{3\ln 5p}{n^N}-\frac{\ln 2}{n^N}\right)\right)^{-\beta}. \tag{32}$$

Let $C = \ln q - \frac{N^2\ln(5p)}{n_0} - \frac{3\ln 5p}{n_0^N} - \frac{\ln 2}{n_0^N}$, then we have $\left|\hat{\mu}_{M,\mathcal{D}}\left(\rho^{-(n+1)^N-t}\left(\xi+\frac{\rho^t}{5p}\tau^*(I)\right)\right)\right| \leq (Cn^N)^{-\beta}$ for sufficient large n_0.

Now, let n_0 be the number which satisfy (31) and (32) whenever $n \geq n_0$. Denote

$$Q_n(\xi) = \sum_{I\in\mathcal{I}_n}\left|\hat{\mu}_{M,\mathcal{D}}\left(\xi+\frac{\rho^t}{5p}\tau^*(I)\right)\right|^2.$$

Then by (32) and Proposition 4.4, we obtain

$$\begin{aligned} Q_{n+1}(\xi) &= Q_n(\xi)+\sum_{I\in\mathcal{I}_{n,n+1}}\left|\hat{\mu}_{M,\mathcal{D}}\left(\xi+\frac{\rho^t}{5p}\tau^*(I)\right)\right|^2 \\ &= Q_n(\xi)+\sum_{I\in\mathcal{I}_{n,n+1}}\left|\hat{\mu}_{(n+1)^N+t}\left(\xi+\frac{\rho^t}{5p}\tau^*(I)\right)\right|^2\cdot\left|\hat{\mu}_{M,\mathcal{D}}\left(\rho^{-(n+1)^N-t}\left(\xi+\frac{\rho^t}{5p}\tau^*(I)\right)\right)\right|^2 \\ &\leq Q_n(\xi)+(Cn^N)^{-2\beta}\sum_{I\in\mathcal{I}_{n,n+1}}\left|\hat{\mu}_{(n+1)^N+t}\left(\xi+\frac{\rho^t}{5p}\tau^*(I)\right)\right|^2 \\ &\leq Q_n(\xi)+(Cn^N)^{-2\beta}\left(1-\sum_{I\in\mathcal{I}_n}\left|\hat{\mu}_{(n+1)^N+t}\left(\xi+\frac{\rho^t}{5p}\tau^*(I)\right)\right|^2\right) \\ &\leq Q_n(\xi)+(Cn^N)^{-2\beta}(1-Q_n(\xi)). \end{aligned}$$

For any $n \geq n_0$, this implies

$$1-Q_{n+1}(\xi)\geq(1-C^{-2\beta}n^{-2\beta N})(1-Q_n(\xi))\geq\cdots\geq\prod_{k=n_0}^{n}(1-C^{-2\beta}k^{-2\beta N})(1-Q_{n_0^N}(\xi)).$$

Taking $n\to\infty$, we obtain

$$1-Q_\Lambda(\xi)\geq\prod_{k=n_0}^{\infty}(1-C^{-2\beta}k^{-2\beta N})(1-Q_{n_0^N}(\xi)),$$

and $\prod_{k=n_0}^{\infty}(1-C^{-2\beta}k^{-2\beta N})\neq 0$. This means that $Q_\Lambda(\xi)\neq 1$ for $\xi\in\left(0,\frac{1}{2}\right)^2$, so by Theorem 2.1, we have $Q_\Lambda(\xi)\neq 1$ for any $\xi\in\mathbb{R}$. That is, $\mu_{M,\mathcal{D}}$ is not a spectral measure when $\rho=\frac{5p}{q}$, $\gcd(5p,q)=1$ and $q\neq 1$. This is a contradiction. Therefore, the proof is completed. □

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