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# The composition of some approximation operators of exponential type

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**Abstract.** Recently, Gupta, López-Pellicer, and Srivastava [5] studied the convergence of approximation operators of exponential type connected to the function $x^{3/2}$. In the present article, we consider a new operator obtained by the composition of two exponential type integral operators connected to $x^2$ and $x^3$. The new operator is based on the modified Bessel function of the second kind. We obtain the moments, present an estimate for the rate of convergence and establish the complete asymptotic expansion of the new operator.

**Dedicated to the 80th anniversary of the birthday of Professor Manuel López-Pellicer**

## 1. Introduction and main results

In the present article, we consider two exponential-type integral operators and study the operator obtained by their composition. The exponential-type integral operators given by

$$(L_n f)(x) = \int_0^\infty \phi_n(x,t) f(t) dt$$

are such that their kernels $\phi_n(x,t)$ satisfy the differential equation $\dfrac{\partial}{\partial x}\phi_n(x,t) = \dfrac{n(t-x)}{p(x)}\phi_n(x,t)$.

Very recently, V. Gupta, M. López-Pellicer, and H. M. Srivastava [5] studied the convergence of approximation operators of exponential type connected to the function $p(x) = x^{3/2}$. Here we deal with the two well-known exponential-type operators, namely Post–Widder operators $P_n$ and the Ismail–May operators $Q_n$, the values of $p(x)$ are $x^2$ and $x^3$, respectively. The Post–Widder operator [7, (3.9)] is defined by

$$(P_n f)(t) = \frac{n^n}{t^n}\frac{1}{\Gamma(n)}\int_0^\infty e^{-nu/t} u^{n-1} f(u)\, du.$$

The exponential type operator $Q_n$ introduced by Ismail and May [7, Eq. (3.11)] is given by

$$(Q_n f)(x) = \frac{n^{1/2} e^{n/x}}{\sqrt{2\pi}} \int_0^\infty t^{-3/2} \exp\left(-\frac{n}{2t} - \frac{nt}{2x^2}\right) f(t)\, dt.$$

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The composition of the operators $Q_m$ and $P_n$ provides a new integral operator $C_{m,n} = Q_m \circ P_n$. Obviously, $C_{m,n}$ is a positive linear operator. Since $P_n$ and $Q_n$ preserve constant functions, so its composition $C_{m,n}$ does. We present a concise form of $C_{m,n}$ as an integral operator by expressing the kernel in terms of certain special functions.

**Theorem 1.1.** *The operator $C_{m,n}$ can be written in the form*

$$(C_{m,n}f)(x) = \int_0^\infty \phi_{m,n}(x,u)\, f(u)\, du,$$

*with the kernel function*

$$\phi_{m,n}(x,u) = \frac{n^n m^{1/2} e^{m/x}}{\Gamma(n)\, x^{n+1/2}} \sqrt{\frac{2}{\pi}} \left(1 + 2\frac{n}{m}u\right)^{-n/2-1/4} K_{n+1/2}\left(\frac{m}{x}\sqrt{1 + 2\frac{n}{m}u}\right) u^{n-1},$$

*where $K_\nu$ denotes the modified Bessel function of the second kind.*

*Proof.* We have

$$(C_{m,n}f)(x) = \frac{n^n m^{1/2} e^{m/x}}{\Gamma(n)\sqrt{2\pi}} \int_0^\infty \left(\int_0^\infty t^{-n-\frac{3}{2}} \exp\left(-\frac{m}{2t} - \frac{nu}{t} - \frac{mt}{2x^2}\right) dt\right) u^{n-1} f(u)\, du.$$

Applying the integral representation

$$K_\nu(az) = \frac{z^\nu}{2} \int_0^\infty t^{-\nu-1} \exp\left(-\frac{a}{2}\left(t + \frac{z^2}{t}\right)\right) dt,$$

of the modified Bessel function of the second kind [8, page 39] (cf. [4, entry 8.432, formula 7, p. 917]), with $a = mx^{-2}$ and $z = x\sqrt{1 + 2\frac{n}{m}u}$, we get

$$(C_{m,n}f)(x) = \frac{n^n m^{1/2} e^{m/x}}{\Gamma(n)\, x^{n+1/2}} \sqrt{\frac{2}{\pi}} \int_0^\infty \left(1 + 2\frac{n}{m}u\right)^{-n/2-1/4} K_{n+1/2}\left(\frac{m}{x}\sqrt{1 + 2\frac{n}{m}u}\right) u^{n-1} f(u)\, du.$$

This completes the proof of theorem. □

The asymptotic relation $K_\nu(z) \sim \sqrt{\pi/(2z)}e^{-z}$ as $z \to +\infty$ ([4, entry 8.451, formula 5, p. 962], cf. [10, p. 80, Eq. (12)]), shows that the operator $C_{m,n}$ is well-defined, for all locally integrable functions on $[0, +\infty)$ satisfying the growth condition $f(u) = O\left(e^{u^\beta}\right)$ as $u \to +\infty$, for some constant $\beta < 1/2$. In the special case when $m = n$, we get the approximation operator

$$(C_{n,n}f)(x) = \int_0^\infty \phi_{n,n}(x,u)\, f(u)\, du \tag{1}$$

with the kernel function

$$\phi_{n,n}(x,u) = \sqrt{\frac{2}{\pi}} \frac{e^{n/x}}{\Gamma(n)} \left(\frac{n}{x}\right)^{n+1/2} (1 + 2u)^{-n/2-1/4} K_{n+1/2}\left(\frac{n}{x}\sqrt{1 + 2u}\right) u^{n-1}.$$

In this paper we study the operator $C_{n,n}$.

Firstly, we present an estimate of the rate of convergence in terms of the classical modulus of continuity. Let $C_b[0, +\infty)$ be the space of all real-valued, continuous and bounded functions defined on the interval $[0, +\infty)$.

**Theorem 1.2.** *Let $x > 0$. For $f \in C_b[0, +\infty)$, the rate of convergence can be estimated by*

$$\left|(C_{n,n}f)(x) - f(x)\right| \le \left(1 + x\sqrt{1 + 2x}\right) \omega\left(f, \frac{1}{\sqrt{n}}\right). \tag{2}$$

Throughout the paper, let $e_r$ denote the monomials, given by $e_r(x) = x^r$ $(r = 0, 1, 2, \ldots)$. Furthermore, define $\psi_x = e_1 - xe_0$, for $x \in \mathbb{R}$. The proof of Theorem 1.2 is based on a classical estimate using the second central moment (see [2, Theorem 5.1.2]).

**Lemma 1.3.** *Let $I$ be a real interval and $E$ be a sublattice of $C(I)$ containing $C_b(I)$ as well as the functions $e_1$ and $e_2$. If $L : E \to C[0, +\infty)$, then, for $f \in C_b[0, +\infty)$,*

$$\left|(Lf)(x) - f(x)\right| \quad \leq \quad |(Le_0)(x) - 1| \left|f(x)\right| + \left((Le_0)(x) + \frac{1}{\delta}\sqrt{\left(L\psi_x^2\right)(x)}\sqrt{(Le_0)(x)}\right)\omega(f, \delta).$$

*Proof of Theorem 1.2.* By Lemma 1.3, $C_{n,n}e_0 = e_0$ and the estimate (8), for $f \in C_b[0, +\infty)$ and $\delta > 0$,

$$\left|(C_{n,n}f)(x) - f(x)\right| \leq \left(1 + \frac{1}{\delta}\sqrt{\left(C_{n,n}\psi_x^2\right)(x)}\right)\omega(f, \delta) \leq \left(1 + \frac{x}{\delta}\sqrt{\frac{1+2x}{n}}\right)\omega(f, \delta).$$

Choosing $\delta = 1/\sqrt{n}$ we obtain (2). □

Now we deal with the asymptotic properties of the operators $C_{n,n}$. For $q \in \mathbb{N}$ and $x \in (0, \infty)$, let $K[q; x]$ be the class of all locally integrable functions $f$ of polynomial growth $f(t) = O(t^q)$ as $t \to +\infty$ which are $q$ times differentiable at $x$. The following theorem presents as our main result the complete asymptotic expansion for the operators $C_{n,n}$. The corresponding result for the operator $Q_n$ can be found in [1].

**Theorem 1.4.** *Let $q \in \mathbb{N}$ and $x \in (0, \infty)$. For each function $f \in K[2q; x]$, the operators $C_{n,n}$ possess the asymptotic expansion*

$$(C_{n,n}f)(x) = f(x) + \sum_{k=1}^{q} \frac{c_k(f, x)}{n^k} + o\left(n^{-q}\right) \qquad (n \to \infty),$$

*with the coefficients* $c_k(f, x) = \sum_{s=2}^{2k} \frac{f^{(s)}(x)}{s!} \sum_{j=0}^{k} \frac{x^{s+j}}{2^j j!} T(s, j, k-j)$, *where*

$$T(s, j, \ell) = \sum_{r=1}^{s} (-1)^{s-r} \binom{s}{r} \left[\begin{matrix} r \\ r-\ell \end{matrix}\right] (r-1+j)^{\underline{2j}}. \tag{3}$$

Here and in the following $\left[\begin{matrix} r \\ \ell \end{matrix}\right]$ denote the (signless) Stirling numbers of the first kind. We follow the convention that $\left[\begin{matrix} r \\ \ell \end{matrix}\right] = 0$ if $\ell < 0$ or if $\ell > r$. A definition and several properties of the Stirling numbers can be found in Section 3. For the convenience of the reader we explicitly present the initial part of the expansion:

$$\begin{aligned}(C_{n,n}f)(x) &= f(x) + \frac{x^2(1+x)}{2n} f^{(2)}(x) \\ &+ \frac{12x^3 f^{(2)}(x) + 4x^3\left(3x^2 + 6x + 2\right) f^{(3)}(x) + 3x^4(1+x)^2 f^{(4)}(x)}{24n^2} + O\left(n^{-3}\right) \qquad (n \to \infty).\end{aligned}$$

In the special case $q = 1$, we obtain the following Voronovskaja-type result.

**Corollary 1.5.** *Let $x \in (0, \infty)$. For each function $f \in K[2; x]$, the operators $C_{n,n}$ satisfy the asymptotic relation*

$$\lim_{n\to\infty} n\left[(C_{n,n}f)(x) - f(x)\right] = \frac{x^2(1+x)}{2} f''(x).$$

This limit is not unexpected. It is well known that exponential operators $L_n$ have Voronovskaja-type limit $\lim_{n\to\infty} n\left[(L_nf)(x) - f(x)\right] = p(x) f''(x)/2$. Noting that $C_{n,n} = Q_n \circ P_n$ it holds

$$\lim_{n\to\infty} n\left[(C_{n,n}f)(x) - f(x)\right] = \lim_{n\to\infty} n\left[(P_nf)(x) - f(x)\right] + \lim_{n\to\infty} n\left[(Q_nf)(x) - f(x)\right] = \frac{x^2}{2} f''(x) + \frac{x^3}{2} f''(x).$$

## 2. The moments

Firstly, we study the moments of the operators $C_{n,n}$. Throughout the paper, let $e_r$ denote the monomials, given by $e_r(x) = x^r$ $(r = 0, 1, 2, \ldots)$. Furthermore, define $\psi_x = e_1 - xe_0$, for $x \in \mathbb{R}$. The following integral formula is essential for the evaluation of the moments of the operator $C_{n,n}$.

**Lemma 2.1.** *For real $a > 0$ and complex $\nu$ with $\operatorname{Re}(\nu) > -1$, it holds*

$$\int_0^\infty K_\nu\left(a\sqrt{2u+1}\right)(2u+1)^{-\nu/2}u^\mu du = \frac{\Gamma(\mu+1)}{a^{\mu+1}}K_{\nu-\mu-1}(a).$$

*Proof.* For real $a, z > 0$ and complex $\nu$ with $\operatorname{Re}(\nu) > -1$, Sonine's formula [10, p. 147, Eq. (6)] states that

$$\int_0^\infty K_\nu\left(a\sqrt{t^2+z^2}\right)\left(t^2+z^2\right)^{-\nu/2}t^{2\mu+1}dt = \frac{2^\mu\Gamma(\mu+1)}{z^{\nu-\mu-1}a^{\mu+1}}K_{\nu-\mu-1}(az).$$

The change of variable $t^2 = 2u$ and choosing $z = 1$ leads to the desired formula. □

A direct consequence is the following representation of the moments of the operators $C_{n,n}$.

**Lemma 2.2.** *The moments of the operators $C_{n,n}$ are given by*

$$(C_{n,n}e_r)(x) = \sqrt{\frac{2}{\pi}}\frac{\Gamma(n+r)}{\Gamma(n)}\left(\frac{x}{n}\right)^{r-1/2}e^{n/x}K_{1/2-r}\left(\frac{n}{x}\right) \qquad (x > 0, r = 0, 1, 2, \cdots).$$

*Proof.* By definition, we have

$$(C_{n,n}e_r)(x) = \sqrt{\frac{2}{\pi}}\frac{e^{n/x}}{\Gamma(n)}\left(\frac{n}{x}\right)^{n+1/2}\int_0^\infty (1+2u)^{-n/2-1/4}K_{n+1/2}\left(\frac{n}{x}\sqrt{1+2u}\right)u^{n-1+r}du,$$

such that the desired representation follows from Lemma 2.1. □

For $\nu = n + 1/2$ $(n = 0, 1, 2, \ldots)$, the modified Bessel function of the second kind has the explicit representation ([10, p. 80, Eq. (12)] or [4, entry 8.468, p. 967])

$$K_{n+1/2}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}\sum_{k=0}^{n}\frac{(n+k)!}{k!(n-k)!}(2z)^{-k} \qquad (z > 0),$$

in particular, $K_{1/2}(z) = \sqrt{\pi/(2z)}e^{-z}$. Using the symmetry $K_{-\nu}(z) = K_\nu(z)$, we obtain $C_{n,n}e_0 = e_0$ and

$$(C_{n,n}e_{r+1})(x) = \frac{\Gamma(n+r+1)}{\Gamma(n)}\sum_{k=0}^{r}\frac{(2r-k)!}{2^{r-k}k!(r-k)!}\left(\frac{x}{n}\right)^{2r+1-k}. \tag{4}$$

Hence, for $r \geq 1$, the moment $C_{n,n}e_r$ is a polynomial of degree $2r - 1$, being a multiple of $e_r$.

## 3. The asymptotic expansion

We make use of the Stirling numbers and certain of their properties. Recall that, the quantities $\begin{bmatrix} m \\ j \end{bmatrix}$ denote the (signless) Stirling numbers of the first kind defined by $z^{\underline{m}} = \sum_{j=0}^{m}(-1)^{m-j}\begin{bmatrix} m \\ j \end{bmatrix}z^j$, $(m = 0, 1, 2, \cdots)$, where $z^{\underline{0}} = 1$, $z^{\underline{m}} = z(z-1)\cdots(z-m+1)$, $m \in \mathbb{N}$, are the falling factorials. Using $(-z)^{\underline{m}} = (-1)^m(z+m-1)^{\underline{m}}$ we obtain the relations

$$(z+m-1)^{\underline{m}} = \sum_{j=0}^{m}\begin{bmatrix} m \\ j \end{bmatrix}z^j \qquad (m = 0, 1, 2, \cdots). \tag{5}$$

We recall some known facts about Stirling numbers which will be useful in the sequel. The Stirling numbers of the first kind possess the representation

$$\left[\begin{matrix} r \\ r-\ell \end{matrix}\right] = (-1)^{\ell} \sum_{j=\ell}^{2\ell} s_2\left(j, j-\ell\right) \binom{r}{j}, \tag{6}$$

for $0 \le \ell \le r$, (see [3, page 226–227, Ex. 16]). The coefficients $s_2\left(j, j-\ell\right)$, called associated Stirling numbers of the first kind, are independent of $r$.

**Lemma 3.1.** *The moments of the operators $C_{n,n}$ are given by $C_{n,n}e_0 = e_0$ and, for $r \ge 1$, by*

$$\left(C_{n,n}e_r\right)(x) = \sum_{k=0}^{2r-1} \frac{1}{n^k} \sum_{j+\ell=k} \left[\begin{matrix} r \\ r-\ell \end{matrix}\right] \frac{(r-1+j)^{\underline{2j}}}{2^j j!} x^{r+j}$$

*Proof.* We have $\Gamma(n+r)/\Gamma(n) = (n+r-1)^{\underline{r}} = \sum_{\ell=0}^{r} \left[\begin{matrix} r \\ r-\ell \end{matrix}\right] n^{r-\ell}$. By Eq. (4), we obtain, for $r \ge 1$,

$$\left(C_{n,n}e_r\right)(x) = \sum_{\ell=0}^{r} \left[\begin{matrix} r \\ r-\ell \end{matrix}\right] n^{r-\ell} \sum_{j=0}^{r-1} \frac{(r-1+j)!}{2^j j!\,(r-1-j)!} \left(\frac{x}{n}\right)^{r+j} = \sum_{k=0}^{2r-1} \frac{1}{n^k} \sum_{j+\ell=k} \left[\begin{matrix} r \\ r-\ell \end{matrix}\right] \frac{(r-1+j)^{\underline{2j}}}{2^j j!} x^{r+j}$$

with the convention that $\left[\begin{matrix} r \\ r-\ell \end{matrix}\right] = 0$ if $\ell > r$. □

**Lemma 3.2.** *The central moments of the operators $C_{n,n}$ are given by $C_{n,n}\psi_x^0 = e_0$ and, for $s \ge 1$, by*

$$\left(C_{n,n}\psi_x^s\right)(x) = \sum_{k=1}^{2s-1} \frac{1}{n^k} \sum_{j+\ell=k} \frac{x^{s+j}}{2^j j!} T(s,j,\ell),$$

*where $T(s,j,\ell)$ is as defined in* (3). *For each $x > 0$ and $s = 0, 1, 2, \ldots$, the central moments satisfy the relation*

$$\left(C_{n,n}\psi_x^s\right)(x) = O\left(n^{-\lfloor (s+1)/2 \rfloor}\right) \qquad (n \to \infty). \tag{7}$$

*Proof.* Plainly, $C_{n,n}\psi_x^0 = C_{n,n}e_0 = e_0$. Let $s \ge 1$. Since $\left(C_{n,n}e_r\right)(x) = x^r + O\left(n^{-1}\right)$ as $n \to \infty$ we have

$$\left(C_{n,n}\psi_x^s\right)(x) = \sum_{r=0}^{s} \binom{s}{r} (-x)^{s-r} \left(C_{n,n}e_r\right)(x) = \sum_{k=1}^{2s-1} \frac{1}{n^k} \sum_{j+\ell=k} \frac{x^{s+j}}{2^j j!} T(s,j,\ell)$$

with $T(s,j,\ell)$ as defined in Eq. (3). To prove the second part (3), we show that $T(s,j,\ell) = 0$ if $1 \le j+\ell < s/2$. Using the representation (6) and taking advantage of the binomial identity $\binom{s}{r}\binom{r}{j} = \binom{s}{j}\binom{s-j}{r-j}$ we obtain

$$T(s,j,\ell) = (-1)^{\ell} \sum_{i=\ell}^{2\ell} s_2(i, i-\ell) \binom{s}{i} \sum_{r=i}^{s} (-1)^{s-r} \binom{s-i}{r-i} (r-1+j)^{\underline{2j}}.$$

The inner sum is equal to

$$\sum_{r=0}^{s-i} (-1)^{s-i-r} \binom{s-i}{r} (r+i-1+j)^{\underline{2j}} = 0$$

if $2j < s-i$. This is the case, for $\ell \le i \le 2\ell$, if $2j < s-2\ell$, i.e., $j+\ell < s/2$. □

For the convenience of the reader we list some instances:

$$\begin{aligned}
\left(C_{n,n}\psi_x^0\right)(x) &= 1, \qquad \left(C_{n,n}\psi_x^1\right)(x) = 0, \qquad \left(C_{n,n}\psi_x^2\right)(x) = \frac{x^2(1+x)}{n} + \frac{x^3}{n^2}, \\
\left(C_{n,n}\psi_x^3\right)(x) &= \frac{x^3\left(3x^2+6x+2\right)}{n^2} + \frac{3x^4(3x+2)}{n^3} + \frac{6x^5}{n^4}, \\
\left(C_{n,n}\psi_x^4\right)(x) &= \frac{3x^4(1+x)^2}{n^2} + \frac{3x^4\left(5x^3+18x^2+14x+2\right)}{n^3} + \frac{3x^5\left(30x^2+47x+12\right)}{n^4} + \frac{15x^6(11x+6)}{n^5} + \frac{90x^7}{n^6}.
\end{aligned}$$

A direct consequence is the following estimate of the second central moment:

$$\left|\left(C_{n,n}\psi_x^2\right)(x)\right| \le \frac{x^2+2x^3}{n}. \tag{8}$$

*Proof of Theorem 1.4.* In order to derive Theorem 1.4, a general approximation theorem due to Sikkema will be applied. Let $q \in \mathbb{N}$ and $x \in (0,\infty)$. By Sikkema's theorem [9, Theorem 3], we have, for each function $f \in K[2q;x]$,

$$(C_{n,n}f)(x) = \sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!}\left(C_{n,n}\psi_x^s\right)(x) + o\left(n^{-q}\right) \qquad (n \to \infty).$$

By Lemma 3.1 and observing that $\left(C_{n,n}\psi_x^1\right)(x) = 0$, it follows that

$$(C_{n,n}f)(x) = f(x) + \sum_{k=1}^{q} \frac{1}{n^k} \sum_{s=2}^{2q} \frac{f^{(s)}(x)}{s!} \sum_{j+\ell=k} \frac{x^{s+j}}{2^j j!} T(s,j,\ell) + o\left(n^{-q}\right)$$

as $n \to \infty$. The final form follows by Eq. (7). □

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