Filomat 38:25 (2024), 8795–8804
https://doi.org/10.2298/FIL2425795E

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

$P-$Bernstein polynomials

Alper Erdem[a,b]

[a] *Mersin University, Mathematics Department, Mersin / Türkiye*
[b] *Mersin University, Institute of Science, Mersin / Türkiye*

Abstract. In this current study, we introduce a new operator called the $P-$ Bernstein operator derived through the utilization of "P-factorial" (Pell factorial) and "Pellnomial" (Pell binomial). Subsequently, we investigate the fundamental properties of the $P-$Bernstein basis polynomials. Furthermore, we establish some connections between the P-Bernstein Polynomials and Pell Numbers.

1. Background and Brief History

S. Bernstein introduced a polynomial expressed as a sum in order to demonstrate the Weierstrass Approximation Theorem which established the cornerstone of approximation theory in 1912. The Bernstein polynomials find valuable applications in diverse fields including probability theory, numerical analysis and approximation theory. Many generalizations and applications of the Bernstein polynomials have been considered in recent years [6, 13, 16]. A notable instance of these modification is the $q-$analog of the Bernstein polynomials. George M. Phillips introduced the $q-$Bernstein polynomials, utilizing $q-$analysis as outlined in [15]. For details on $q-$ see [18, 19].

With the influence of approximation theory, Bernstein-type polynomials and Bernstein-type operators have recently become a dynamic research area. Ong et al. [13] introduced some probabilistic derivations of the Cheney, Sharma, and Bernstein approximation operators. They established the convergence property of the Bernstein generalization. With this probabilistic approach they ensured the positivity of the approximation operators and made it easier to derive the moments to prove uniform convergence based on the Korovkin Theorem. Özger at al. introduced a new kind of Bernstein–Schurer operators with real parameter α which are stronger than the classical Bernstein operator in [14] . Based upon this new operator, they investigated some shape preserving properties and obtained an approximation formula in terms of Ditzian–Totik uniform modulus of smoothness of first and second order. They give the Voronovskaja-type approximation theorems of the new operators. Srivastava et al. [16] introduced the idea of construction of Stancu-Type Bernstein operators based on Bézier Bases with shape parameter. Then they calculated their moments and established the uniform convergence of the operator and global approximation result by means of Ditzian-Totik modulus of smoothness. They also constructed the bivariate case of Stancu-type λ-Bernstein operators and studied their approximation behaviors.

In 2005, Djordjevic and Srivastava [17] presented the generalized incomplete Fibonacci polynomials and Lucas polynomials, delving into their systematic exploration and analysis. In 2006, They defined

2020 *Mathematics Subject Classification*. Primary 11B39, 11B65, 47A50 ; Secondary 11Y55, 26C05
Keywords. Pell Numbers, Pellnomials, Bernstein Operators, P-Bernstein Polynomials
Received: 02 February 2024; Revised: 27 April 2024; Accepted: 07 May 2024
Communicated by Hari M. Srivastava
Email address: alpererdem@mersin.edu.tr (Alper Erdem)

two separate sequences of numbers, extending the classical Fibonacci numbers, and derived numerous significant combinatorial properties for these generalized number sequences [4]. In 2017, Srivastava, Tuglu, and Cetin [5] introduced new sets of q-Fibonacci and q-Lucas polynomials, thereby offering q-analogues for the incomplete Fibonacci and Lucas numbers, respectively. They provided proofs for various properties within these polynomial families, including recurrence relations, summation formulas, and generating functions.

In 1915, Fontené [7] published a concise one-page document outlining a generalization of binomial coefficients wherein the numbers are exchanged for the elements from a given sequence (a_n) that comprises real or complex numbers. By substituting the Fibonacci sequence (F_n) with (a_n), we can establish the Fibonacci Binomial coefficients, which are commonly referred to as Fibonomial coefficients. Jarden and Motzkin [11] were the trailblazers in study of generalized Fibonomial coefficients. Also Hoggatt [8] studied in Fibonacci numbers and generalized binomial coefficients. From onward, there has been a heightened fascination with the Fibonomial coefficients and some generalizations. One of the modifications is Pellonomial coefficients. As Various aspects of Pell numbers have been extensively studied in the literature, Diskaya and Hamza [3] introduced the Gell numbers which are generalization of Pell numbers and İpek [9, 10] introduced the Pellnomial coefficients and examined their properties.

Inspired by the Fibonacci calculus introduced by E. Krot [12] and related works Diskaya, Erdem and Menken [6] we define the $P-$ Bernstein polynomials then we examine some properties and relations with the Pell numbers.

2. Preliminaries

For the reader's convenience, we give a summary of mathematical notations and foundational concepts. The Pell sequence which is denoted by $\{P_n\}$ is given by the reccurence relation

$$P_n = \begin{cases} 0, & n = 0 \\ 1, & n = 1 \\ 2P_{n-1} + P_{n-2}, & n > 1. \end{cases}$$

It is well known that Pell numbers have an important place among all integer sequences, as they have surprising properties [2] .

Motivated by the Fibonacci calculus [12] the Pell calculus is based on combinatorial interpretation of the Pell numbers. The $P-$factorial definied as $P_n! = P_nP_{n-1}\cdots P_2P_1$ with $P_0! = 1$

The Pellnomial coefficients are definied for $n \geq m \geq 0$ in [10] as

$$\binom{n}{m}_P = \frac{P_n!}{P_{n-m}!P_m!}$$

with $\binom{n}{0}_P = 1$ and $\binom{n}{m}_P = 0$ for $n < m$. It's remarkably peculiar that all Pell coefficients consistently take integer values. This observation can be found in [9] through mathematical induction.

The Pellnomial coefficients exhibit the following characteristics:

- $\binom{n}{r}_P = \binom{n}{n-r}_P$
- $\binom{n}{r}_P\binom{r}{v}_P = \binom{n}{v}_P\binom{n-v}{r-v}_P$
- $\binom{n}{r}_P\binom{n-r}{v}_P = \binom{n}{r+v}_P\binom{r+v}{r}_P$
- $\binom{n}{r}_P = \frac{P_{n-r+1}}{P_r}\binom{n}{r-1}_P \ (r \neq 0)$
- $\binom{n}{r}_P = \frac{P_n}{P_{n-r}}\binom{n-1}{r}_P \ (r \neq n)$
- $\binom{n}{r}_P = P_{r-1}\binom{n-1}{r}_P + P_{n-r}\binom{n-1}{r-1}_P$

The Pellnomial triangle, with its general term denoted as $\binom{n}{k}_P$, showcases alluring properties. By making use of this triangle, the Pellnomial matrix can be derived as displayed below.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 5 & 5 & 1 & 0 & 0 & 0 & \dots \\ 1 & 12 & 30 & 12 & 1 & 0 & 0 & \dots \\ 1 & 29 & 174 & 174 & 29 & 1 & 0 & \dots \\ 1 & 70 & 1015 & 2436 & 1015 & 70 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Let $\{a_n^{(p)}\}$ is the sequence of the numbers which are placed through the $p-$th column in the Pellnomial matrix. A first few columns of the Pellnomial matrix are given below.

$$\begin{aligned} \{a_n^{(1)}\} &= (1,1,1,1,1,\cdots) \\ \{a_n^{(2)}\} &= (0,1,2,5,12,29,70\cdots) \\ \{a_n^{(3)}\} &= (0,0,1,5,30,174,1015\cdots) \\ \{a_n^{(4)}\} &= (0,0,0,1,12,174,2436\cdots) \end{aligned}$$

It is readily apparent that the general term of $\{a_n^{(j)}\}$ can expressed by

$$a_n^{(j)} = \begin{cases} 0, & if \quad n \le j-1 \\ \frac{P_n \cdot P_{n+1} \ \cdots \ P_{n+j-2}}{P_{j-1}!}, & if \quad n > j-1 \end{cases} \tag{1}$$

where $j \ge 2$. Thus, we have the following results.

Proposition 2.1. *The sequence $\{a_n^{(j)}\}$ has a recursive relation as follows:*

$$a_{n+1}^{(j)} = \frac{P_{n+j-1}}{P_n} a_n^{(j)}$$

where $p \ge 1$.

Proof. From (1), we have

$$a_{n+1}^{(j)} = \frac{P_{n+1}P_{n+2}\cdots P_{n+j-2}P_{n+j-1}}{P_{j-1}!} = \frac{P_nP_{n+1}P_{n+2}\cdots P_{n+j-2}P_{n+j-1}}{P_nP_{j-1}!} = \frac{P_{n+j-1}}{P_n} a_n^{(j)}.$$

□

An operator $L(f;t)$ is called linear operator if any the functions $f(x)$ and $g(x)$ which are in its domain, the function $af(x) + bg(x)$ belongs its domain and

$$L(\sigma f + \omega g; t) = \sigma L(f;t) + \omega L(g;t)$$

where σ and ω are constants.

Let $\mathcal{P}$ represent the polynomial algebra over the field $\mathbf{K}$ of characteristic zero. So, the linear operator defined by $\partial_P : \mathcal{P} \to \mathcal{P}$ is the $P-$derivative operator:

$$\partial_P t^m = P_m t^{m-1} \tag{2}$$

for $m \geq 0$.

The binomial theorem for the $P-$analogue is given by

$$(x +_P y)^\sigma = \sum_{0 \leq m \leq \sigma} \binom{\sigma}{m}_P x^{\sigma-m} y^m \tag{3}$$

and the $P-$exponential function e_P^μ is defined by

$$e_P^\mu = \sum_{0 \leq n} \frac{\mu^n}{P_n!} \tag{4}$$

(cf. [12]).

3. $P-$Bernstein Basis Polynomials and Their Properties

Definition 3.1. *Let $\sigma, \omega \in \mathbb{R}$, $r, n \in \mathbb{Z}^+$ where $r \leq n$. Then the $n-$th degree $P-$Bernstein basis polynomials defined by*

$$B_{r,n}^{P[\sigma,\omega]}(t) = \binom{n}{r}_P \frac{(t-\sigma)^r(\omega-t)^{n-r}}{(\omega-\sigma)^n}$$

for $r = 0, 1, \cdots, n$.

If we take $\sigma = 0$ and $\omega = 1$ we have

$$B_{r,n}^{P[0,1]}(t) = \binom{n}{r}_P t^r(1-t)^{n-r} \tag{5}$$

where $r = 0, 1, \cdots, n$. For simplicity we write $B_{r,n}^P$ instead of $B_{r,n}^{P[0,1]}$.

The first few polynomials are detailed and visually presented in the chart below.

- $B_{0,0}^P(t) = 1$
- $B_{0,1}^P(t) = 1 - t$, $B_{1,1}(t) = t$
- $B_{0,2}^P(t) = 1 - 2t + t^2$, $B_{1,2}^P(t) = 2t - 2t^2$, $B_{2,2}^P(t) = t^2$
- $B_{0,3}^P(t) = 1 - 3t + 3t^2 - t^3$, $B_{1,3}^P(t) = 5t - 10t^2 + 5t^3$, $B_{2,3}^P(t) = 5t^2 - 5t^3$, $B_{3,3}^P(t) = t^3$
- $B_{0,4}^P(t) = 1 - 4t + 6t^2 - 4t^3 + t^4$, $B_{1,4}^P(t) = 12t - 36t^2 + 36t^3 - 12t^4$, $B_{2,4}^P(t) = 30t^2 - 60t^3 + 30t^4$,

  $B_{3,4}^P(t) = 12t^3 - 12t^4$, $B_{4,4}^P(t) = t^4$

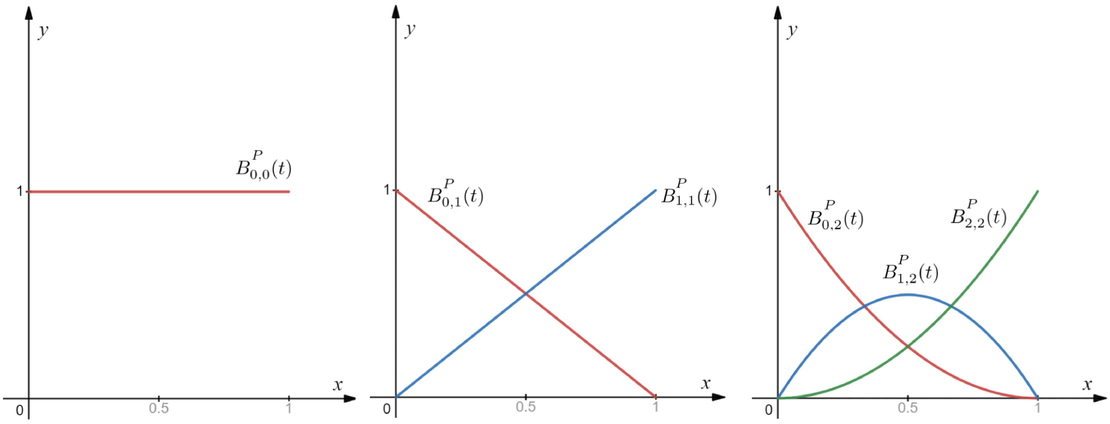

Figure 1: Graphs of $B^P_{r,n}$ for $0 \leq n \leq r \leq 3$

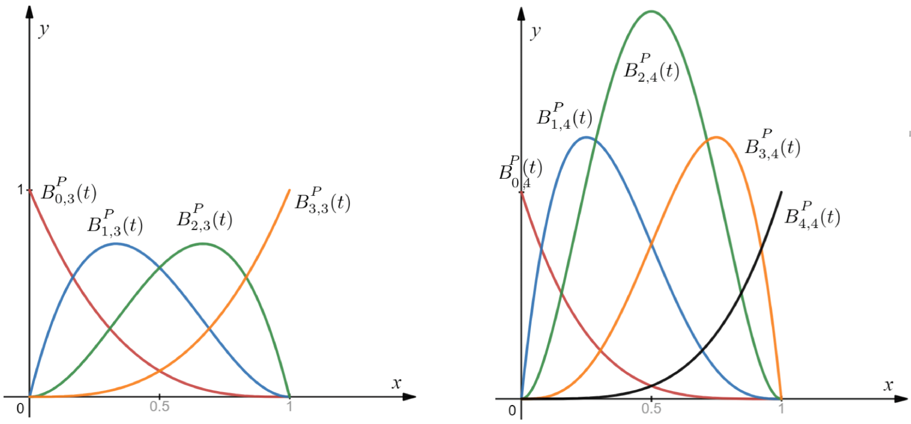

Figure 2: Graphs of $B^P_{r,n}$ for $4 \leq n \leq r \leq 5$

Proposition 3.2. *The P−Bernstein basis polynomials are symmetric.*

Proof. From (5), we have

$$B^P_{r,n}(t) = \binom{n}{r}_P t^r(1-t)^{n-r} = \binom{n}{n-r}_P (1-t)^{n-r}t^r = B^P_{n,n-r}(1-t).$$

□

Proposition 3.3. *The P−Bernstein basis polynomials have the following recursive relation such as*

$$B^P_{r,n}(t) = \frac{P_{n-r+1}}{P_r}\, B^P_{r-1,n}(t)\,\frac{t}{1-t}$$

where $t \in [0, 1)$.

Proof. For the proof, we use (5).

$$\begin{aligned} B^P_{r,n}(t) = \binom{n}{r}_P t^r(1-t)^{n-r} = \frac{P_{n-r+1}}{P_r}\binom{n}{r-1}_P t^r(1-t)^{n-r} &= \frac{P_{n-r+1}}{P_r}\binom{n}{r-1}_P t^{r-1}(1-t)^{n-(r-1)}\frac{t}{1-t} \\ &= \frac{P_{n-r+1}}{P_r}\frac{t}{1-t}B^P_{r-1,n}(t). \end{aligned}$$

□

Theorem 3.4. *Any n-th degree P−Bernstein basis polynomial can be expressed using the power basis which is expressed by* $\{1, t, t^2, \cdots\}$, *i.e.*

$$B^P_{r,n}(t) = \sum_{r\le i\le n} \binom{n}{r}_P \binom{n-r}{i-r} (-1)^{i-r} t^i.$$

Proof. By virtue of the definition of the $P-$Bernstein basis polynomials, we get

$$\begin{aligned} B^P_{r,n}(t) = \binom{n}{r}_P t^r (1-t)^{n-r} = \binom{n}{r}_P t^r \sum_{0\le i\le n-r} (-1)^i \binom{n-r}{i} t^i &= \sum_{0\le i\le n-r} \binom{n}{r}_P \binom{n-r}{i} (-1)^i t^{i+r} \\ &= \sum_{r\le i\le n} \binom{n}{r}_P \binom{n-r}{i-r} (-1)^{i-r} t^i \end{aligned}$$

which is desired. □

Theorem 3.5. *The P−Bernstein basis polynomial of degree n can be written by combining two P−Bernstein basis polynomials with the degree* $n-1$ *such as*

$$B^P_{r,n}(t) = (1-t)\, P_{r-1}\, B^P_{r,n-1}(t) + t\, P_{n-r}\, B^P_{r-1,n-1}(t)$$

where $r = 0, 1, \cdots, n$ *and* $t \in [0, 1]$.

Proof. It follows from (5) and properties of the Pellnomials,

$$\begin{aligned} B^P_{r,n}(t) = \binom{n}{r}_P t^r (1-t)^{n-r} &= \left[P_{r-1} \binom{n-1}{r}_P + P_{n-r} \binom{n-1}{r-1}_P \right] t^r (1-t)^{n-r} \\ &= P_{r-1} \binom{n-1}{r}_P t^r (1-t)^{n-r} + P_{n-r} \binom{n-1}{r-1}_P t^r (1-t)^{n-r} \\ &= (1-t) P_{r-1} \binom{n-1}{r}_P t^r (1-t)^{n-r-1} + t P_{n-r} \binom{n-1}{r-1}_P t^{r-1} (1-t)^{n-r} \\ &= (1-t)\, P_{r-1}\, B^P_{r,n-1}(t) + t\, P_{n-r}\, B^P_{r-1,n-1}(t). \end{aligned}$$

□

Theorem 3.6. *Lower-degree P−Bernstein basis polynomials, those of degree less than n, are representable as linear combinations of n-th degree P−Bernstein basis polynomials as*

$$B^P_{r,n-1}(t) = \frac{P_{n-r}}{P_n} B^P_{r,n}(t) + \frac{P_{r+1}}{P_n} B^P_{r+1,n}(t).$$

Proof. Via a straightforward calculation

$$\frac{P_{n-r}}{P_n} B^P_{r,n}(x) = \frac{P_{n-r}}{P_n} \sum_{0\le r\le n} \binom{n}{r}_P t^r (1-t)^{n-r} = \sum_{0\le r\le n} \binom{n-1}{r}_P t^r (1-t)^{n-r} = (1-t) B^P_{r,n-1}(t)$$

and

$$\frac{P_{r+1}}{P_n} B^P_{r+1,n}(t) = \frac{P_{r+1}}{P_n} \sum_{0\le r\le n} \binom{n}{r+1}_P t^{r+1} (1-t)^{n-r-1} = \sum_{0\le r\le n} \binom{n-1}{r}_P t^{r+2} (1-t)^{n-r-1} = x B^P_{r,n-1}(t)$$

thus

$$B^P_{r,n-1}(t) = \frac{P_{n-r}}{P_n} B^P_{r,n}(t) + \frac{P_{r+1}}{P_n} B^P_{r+1,n}(t)$$

which illustrates a $n-1$ degree $P-$Bernstein basis polynomial regarding the linear combination of $P-$Bernstein basis polynomials with degree n. □

Theorem 3.7. *Polynomials of degree $n-1$ are obtained as the $P-$derivatives of $P-$Bernstein basis polynomials with degree n. Moreover, this derivative can be explicitly expressed as a linear combination of the $P-$Bernstein basis polynomials such as*

$$\partial_P B^P_{r,n}(t) = P_n \left(B^P_{r-1,n-1}(t) - B^P_{r,n-1}(t) \right).$$

Proof. By virtue of the definition of the $P-$derivative (2), we get

$$\begin{aligned} \partial_P B^P_{r,n}(t) = \partial_P \left[\binom{n}{r}_P t^r (1-t)^{n-r} \right] &= \frac{P_n!}{P_{n-r}! P_r!} \left[P_r t^{r-1} (1-t)^{n-r} - P_{n-r} t^r (1-t)^{n-r-1} \right] \\ &= \frac{P_n!}{P_{n-r}! P_r!} P_r t^{r-1} (1-t)^{n-r} - \frac{P_n!}{P_{n-r}! P_r!} P_{n-r} t^r (1-t)^{n-r-1} \\ &= P_n \left(\frac{P_{n-1}!}{P_{n-r}! P_{r-1}!} t^{r-1} (1-t)^{n-r} - \frac{P_{n-1}!}{P_{n-r-1}! P_r!} t^r (1-t)^{n-r-1} \right) \\ &= P_n \left(B^P_{r-1,n-1}(t) - B^P_{r,n-1}(t) \right). \end{aligned}$$

□

Theorem 3.8. *The generating function for the $P-$Bernstein basis polynomials is*

$$\sum_{0 \le r \le n} B^P_{r,n}(t) \mu^r = ((1-x) +_P \mu x)^n.$$

Proof. By using (3), we have

$$((1-t) +_P \mu t)^n = \sum_{0 \le r \le n} \binom{n}{r}_P (\mu t)^r (1-t)^{n-r} \quad = \sum_{0 \le r \le n} \binom{n}{r}_P t^r (1-t)^{n-r} \mu^r = \sum_{0 \le r \le n} B^P_{r,n}(t) \mu^r.$$

□

The next theorem has proved by a similar method which was given in [1].

Theorem 3.9. *The exponential generating function for the $P-$Bernstein polynomials is*

$$\sum_{r \le n} B^P_{r,n}(t) \frac{\mu^n}{P_n!} = \frac{t^r \mu^r}{P_r!} e_P^{(1-t)\mu}.$$

Proof. Using the $P-$exponential function e^t_P (4), we get

$$\begin{aligned} \frac{t^r \mu^r}{P_r!} e_P^{(1-t)_P \mu} = \frac{t^r \mu^r}{P_r!} \sum_{0 \le n} \frac{(1-t)^n \mu^n}{P_n!} = \frac{1}{P_r!} \sum_{0 \le n} \frac{t^r (1-t)^n \mu^{n+r}}{P_n!} &= \sum_{0 \le n} \frac{P_{n+r}!}{P_n! P_r!} \frac{t^r (1-t)^n \mu^{n+r}}{P_{n+r}!} \\ &= \sum_{0 \le n} \binom{n+r}{r} \frac{t^r (1-t)^n \mu^{n+r}}{P_{n+r}!} \\ &= \sum_{r \le n} \binom{n}{r}_P \frac{t^r (1-t)^{n-r} \mu^n}{P_n!} \\ &= \sum_{r \le n} B^P_{r,n}(t) \frac{\mu^n}{P_n!}. \end{aligned}$$

□

4. $P-$Bernstein Polynomials and Their Associations with Pell Numbers

Definition 4.1. *Let $\sigma, \omega \in \mathbb{R}$, $f : [\sigma, \omega] \to \mathbb{R}$ a function, $r, n \in \mathbb{Z}^+$ and $r \leq n$. Then the n-th degree $P-$Bernstein Polynomial with respect to the function f is defined by*

$$\mathfrak{B}_n^{F[\sigma,\omega]}(f;t) = \sum_{0\leq r\leq n} f\left(\frac{P_r}{P_n}\right) B_{r,n}^{F[\sigma,\omega]}(t)$$

for $r = 0, 1, \cdots, n$.

If we take $\sigma = 0$ and $\omega = 1$ then

$$\mathfrak{B}_n^{P[0,1]}(f;t) = \sum_{r=0}^{n} f\left(\frac{P_r}{P_n}\right) B_{r,n}^{P[0,1]}(t)$$

For simplicity we write $\mathfrak{B}_n^P$ instead of $\mathfrak{B}_n^{P[0,1]}$.

The first few $P-$Bernstein polynomials with respect to the functions $f(x) = 1$ and $f(x) = x$ are listed with below.

$\mathfrak{B}_0^P(1;t) = 1$	$\mathfrak{B}_0^P(x;t) = t$
$\mathfrak{B}_1^P(1;t) = 1$	$\mathfrak{B}_1^P(x;t) = t$
$\mathfrak{B}_2^P(1;t) = 1 - t + t^2$	$\mathfrak{B}_2^P(x;t) = t$
$\mathfrak{B}_3^P(1;t) = 1 - t + t^2$	$\mathfrak{B}_3^P(x;t) = t - t^2 + t^3$
$\mathfrak{B}_4^P(1;t) = 1 - t + 3t^2 - 4t^3 + 2t^4$	$\mathfrak{B}_4^P(x;t) = t - t^2 + t^3$
$\mathfrak{B}_5^P(1;t) = 1 + 5t^2 - 10t^3 + 5t^4$	$\mathfrak{B}_5^P(x;t) = t - t^2 + 3t^3 - 4t^4 + 2t^5$
$\vdots \quad \vdots$	$\vdots \quad \vdots$

Proposition 4.2. *The relationship between the $P-$Bernstein Polynomials corresponding to $f(x) = 1$ and $f(x) = x$ is as follows:*

$$\mathfrak{B}_n^P(x;t) = t \cdot \mathfrak{B}_{n-1}^P(1;t).$$

Proof. With smooth calculation, we have

$$\begin{aligned}\mathfrak{B}_n^P(x;t) = \sum_{0\leq r\leq n} \frac{P_r}{P_n}\binom{n}{r}_P t^r(1-t)^{n-r} = \sum_{0\leq r\leq n}\binom{n-1}{r-1}_P t^r(1-t)^{n-r} &= \sum_{-1\leq r\leq n-1}\binom{n-1}{r}_P t^{r+1}(1-t)^{n-1-r} \\ &= t\sum_{0\leq r\leq n-1}\binom{n-1}{r}_P t^r(1-t)^{n-1-r} \\ &= t\cdot \mathfrak{B}_{n-1}^P(1;t).\end{aligned}$$

□

Theorem 4.3. *A notable property of the $P-$Bernstein polynomials with respect to the function $f(x) = 1$ is as follows:*

$$[t^\beta]\mathfrak{B}_n^P(1;t) = \sum_{0\leq j\leq \beta} (-1)^{\beta-j}\binom{n}{j}_P \binom{n-j}{n-\beta} \tag{6}$$

where $[t^\beta]\mathfrak{B}_n^P(1;t)$ denotes the coefficient of t^β in $B_n^P(1;t)$.

Proof. By using the P-Bernstein polynomials to define the following, we get

$$\mathfrak{B}_n^P(1;t) = \sum_{0\le r\le n} \binom{n}{r}_P t^r (1-t)^{n-r} = \sum_{0\le r\le n} \left[\binom{n}{r}_P t^r \sum_{0\le i\le n-r} \left(\binom{n-r}{i} (-1)^{n-r-i} t^{n-r-i} \right) \right]$$
$$= \sum_{0\le r\le n} \left[\binom{n}{r}_P \sum_{0\le i\le n-r} \left(\binom{n-r}{i} (-1)^{n-r-i} t^{n-i} \right) \right].$$

As a consequence, we acquire

$$[t^\beta]\mathfrak{B}_n^P(1;t) = \binom{n}{0}_P \binom{n}{n-\beta} (-1)^\beta + \binom{n}{1}_P \binom{n-1}{n-\beta} (-1)^{\beta-1} + \cdots + \binom{n}{\beta}_P \binom{n-\beta}{n-\beta} (-1)^0 = \sum_{0\le j\le\beta} (-1)^{\beta-j} \binom{n}{j}_P \binom{n-j}{n-\beta}$$

which completes the proof. □

Corollary 4.4. *The $P-$Bernstein polynomials with respect to $f(x) = 1$ is related to Pell numbers in the following manner:*

$$[t]\mathfrak{B}_n^P(1;t) = P_n - n$$

where $[t]\mathfrak{B}_n^P(1;t)$ denotes the coefficients of t in $B_n^P(1;t)$.

Proof. From the equality (6) for $\beta = 1$, we acquire

$$[t]\mathfrak{B}_n^P(1;t) = P_n - n$$

which is desired. □

Theorem 4.5. *For the $P-$Bernstein Polynomials with respect to $f(x) = x^\beta$ where $\beta \ge 2$ has following relation with respect to Pell numbers as*

$$[t]\mathfrak{B}_n^P(x^\beta;t) = \frac{1}{(P_n)^{\beta-1}}$$

where $[t]\mathfrak{B}_n^P(x^\beta;t)$ denotes the coefficients of t in $B_n^P(x^\beta;t)$.

Proof. By virtue of the definition of the $P-$Bernstein polynomials, we get

$$cl\mathfrak{B}_n^P(x^\beta;t) = \sum_{0\le r\le n} \left(\frac{P_r}{P_n}\right)^\beta \binom{n}{r}_P t^r (1-t)^{n-r} = \sum_{0\le r\le n} \left[\left(\frac{P_r}{P_n}\right)^\beta \binom{n}{r}_P t^r \sum_{0\le i\le n-r} \binom{n-r}{i} (-t)^{n-r-i} \right]$$
$$= \sum_{0\le r\le n} \left[\left(\frac{P_r}{P_n}\right)^\beta \binom{n}{r}_P \sum_{0\le i\le n-r} \binom{n-r}{i} (-1)^{n-r-i} t^{n-i} \right]$$
$$= \sum_{0\le r\le n} \left[\left(\frac{P_r}{P_n}\right)^\beta \binom{n}{r}_P \left(\binom{n}{0} (-1)^{n-r} t^n + \binom{n-r}{1} (-1)^{n+r-1} t^{n-1} + \cdots + \binom{n-r}{n-r} t^r \right) \right]$$

From here, we obtain

$$[t]\mathfrak{B}_n^P(x^\beta;t) = \left(\frac{P_0}{P_n}\right)^\beta \binom{n}{0}_P \binom{n}{n-1}_P (-1) + \left(\frac{P_1}{P_n}\right)^\beta \binom{n}{1}_P \binom{n-1}{n-1} = \frac{1}{(P_n)^{\beta-1}}$$

as desired. □

5. Conclusion

The $P-$Bernstein operator, formulated using the concept of "P-factorial" (Pell factorial) and "Pellnomial" (Pell binomial) is given. We delved into the core characteristics of the $P-$Bernstein basis polynomials and explore their essential properties. Additionally, established connections between the P-Bernstein Polynomials and Pell Numbers.

References

[1] M. Acikgoz, S. Araci, *On the generating function for Bernstein polynomials*, American Institute of Physics, In AIP Conference Proceedings Vol. 1281 **1** (2010), 1141–1143.
[2] A.T. Benjamin, J.J. Quinn, *Proofs That Really Count: The Art of Combinatorial Proof*, American Mathematical Soc., 2003.
[3] O. Diskaya, H. Menken, *On The Sequence Of Gell Numbers*, Journal of Universal Mathematics **3** (2020), 77–82.
[4] G. B. Djordjevic, H. M. Srivastava, *Some generalizations of certain sequences associated with the Fibonacci numbers*, J. Indones. Math.Soc. **12** (2006), 99–112.
[5] G. B. Djordjevic, H. M. Srivastava, *Some generalizations of the incomplete Fibonacci and the incomplete Lucas polynomials*, Adv. Stud. Contemp. Math.(Kyungshan) **11** (2005), 11–32.
[6] A. Erdem, O. Diskaya, H. Menken, *On the F−Bernstein Polynomials*, Ukrainian Mathematical Journal, **76** (2024), No. 1, 937–948.
[7] G. Fontené, *Généralisation ďune formule connue*, Nouv. Ann. **(4)** (1915), no. 15, 112.
[8] V. E. Jr. Hoggatt, *Fibonacci numbers and generalized binomial coefficients*, Fibonacci Quart. **5** (1967), 383–400.
[9] A. İpek, *On Pellnomial coefficients and Pell–Catalan numbers*, Acta Et Commentationes Universitatis Tartuensis De Mathematica **26** (2022), no. 2.
[10] A. İpek, *On Triangular Pell and Pell-Lucas Numbers*, Bulletin of International Mathematical Virtual Institute **12** (2022), No. 3.
[11] D. Jarden, T. Motzkin, *The product of sequences with a common linear recursion formula of order 2*, Riveon Lematematika **3** (1949), No. 38, 25–27.
[12] E. Krot, *An Introduction to Finite Fibonomial Calculus*, Cent. Eur. J. Math. **2** (2004), 754–766.
[13] S. H. Ong, C. M. Ng, H. K. Yap, H. M. Srivastava, *Some probabilistic generalizations of the Cheney-Sharma and Bernstein approximation operators*, Axioms **10** (2022), 1–11.
[14] F. Özger, H. M. Srivastava, S. A. Mohiuddine, *Approximation of functions by a new class of generalized Bernstein-Schurer operators*, Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat. (RACSAM) **114** (2020), 1–21.
[15] G.M. Phillips, *A survey of results on the q-Bernstein polynomials*, IMA Journal of Numerical Analysis **30** (2010), no. 1, 277–288.
[16] H. M. Srivastava, F. Özger, S. A. Mohiuddine, *Construction of Stancu-type Bernstein operators based on Bézier bases with shape parameter λ*, Symmetry **11** (2019), 1–22.
[17] H. M. Srivastava, N. Tuglu, M. Cetin, *Some Results on the q-analogues of the Incomplete Fibonacci and Lucas Polynomials*, Miskolc Math. Notes **20** (2019), 511–524.
[18] H. M. Srivastava, *Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformations*, J. Nonlinear Convex Anal. **22** (2021), 1501–1520.
[19] H.M. Srivastava, *Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis*. Iran. J. Sci. Technol. Trans. A. Sci. **44** (2020), 327–344.