Filomat 38:25 (2024), 8805–8812
https://doi.org/10.2298/FIL2425805Y

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Existence and stability results for fractional Langevin equation in complex domain

Ashish Yadav[a,*], Trilok Mathur[a], Shivi Agarwal[a], Babli Yadav[a]

[a] *Department of Mathematics, Birla Institute of Technology and Science Pilani, Pilani Campus,* 333031, *India*

Abstract. One generalization of the Langevin equation for systems in fractal media replaces the ordinary derivative with a fractional derivative, providing a more accurate description of dynamics in complex environments. In this paper, we examine a boundary value problem using a Langevin equation with two real fractional orders. The operators are taken in Srivastava-Owa sense in the unit disk. The existence of solutions in a Banach space is established using the contraction mapping concept and Krasnoselskii's fixed point theorem. Moreover, Ulam Hyers stability for the fractional Langevin equation(FLE) is introduced in this study.

1. Introduction

Fractional derivatives are significant tool for understanding the memory and hereditary properties of many materials and processes. Particularly, Fractional differential equations (FDEs) occur naturally in a variety of disciplines, including electrical circuitry, biology, the phenomena of blood flow, polymer rheology, geophysics, aerodynamics, nonlinear seismic oscillation, etc. Srivastava *et al*. [26] studied the FDE in the sense of Hadamard fractional derivative involving p-Laplacian operator with three point boundary conditions in porous medium and in turbulent flow model. They have used some well known fixed point theorem for obtaining the results and also discussed the stability. For deeper investigation on FDEs, it is advised to view [4, 5, 11, 14, 18, 19, 21, 22, 24, 29].

The Langevin equations have been widely used to describe stochastic problems in physics, chemistry and electrical engineering. For instance, when the random fluctuation force is supposed to be white noise (or not white noise), the Langevin equation (or generalised Langevin equation) accurately describes Brownian motion. The Langevin equation with two fractional orders is one such generalisation, which includes fractal and memory features with a dissipative memory kernel. In 2006, K. Sau Fa [23] discussed FLE with Riemann-Liouville fractional time derivative which modifies the classical Newtonian force, nonlocal dissipative force, and long-time correlation. Bashir Ahmad *et al*. [1] studied a nonlinear

2020 *Mathematics Subject Classification*. Primary 34A08; Secondary 26A33, 35R11.
Keywords. Fractional Langevin equation, Analytic function, Banach fixed point theorem, Krasnoselskii's Lemma, Unit disk.
Received: 01 March 2024; Revised: 29 April 2024; Accepted: 07 May 2024
Communicated by Hari M. Srivastava
The first author gratefully acknowledges Birla Institute of Technology and Science (BITS) Pilani, Pilani Campus, 333031, India; for providing the financial support. The fourth author is grateful to CSIR-UGC for financial assistance with (1006/(CSIR-UGC NET JUNE 2018)).
* Corresponding author: Ashish Yadav
Email addresses: as20111995rao@gmail.com (Ashish Yadav), tmathur@pilani.bits-pilani.ac.in (Trilok Mathur), shivi@pilani.bits-pilani.ac.in (Shivi Agarwal), p20200046@pilani.bits-pilani.ac.in (Babli Yadav)

Langevin equation based on generalized Liouville-Caputo-type fractional differential operator of different order. They considered nonlocal boundary conditions involving generalized integral operator and showed the existence-uniqueness results using fixed point theorems. For a thorough examination of FLE, see [2, 9, 17, 20].

Till now FLE have been studied in real variables. But to the best of our knowledge, there are no attempts on FLE in complex variable. Using complex plane which is analogous to $\mathbb{R}^2$, FLE can be used more precisely to describe the Brownian motion of particles.
Motivated by the work in [3, 15, 16], we study the following boundary value problem of FLE in complex domain with two real fractional orders :

$$\begin{aligned} D_s^{\eta}(D_s^{\varrho}+\lambda)x(s) &= u(s,x(s)),\ s\in\mathbb{U},\ 0\le\varrho,\eta<1,\ 0<\varrho+\eta<1,\\ x(0)&=\mu_1,\ x(1)=\mu_2, \end{aligned} \tag{1}$$

where $\mathbb{U} := \{s \in \mathbb{C} : |s| < 1\}$ is the unit disk. $D_s^{\varrho}, D_s^{\eta}$ are the Srivastava-Owa fractional derivative operators of order real ϱ and η respectively. λ, μ_1, μ_2 are complex constants. Both $u : \mathbb{U}\times\mathcal{B} \to \mathcal{B}$, $x : \mathbb{U} \to \mathcal{B}$ are analytic functions. $\mathcal{B}$ is the space of analytic bounded functions on the unit disk. Our objective is to demonstrate the existence and stability of solutions of considered FLE(1).

2. Preliminaries

Srivastava-Owa [27] defined fractional derivative and integral operators in 1989 as:

Definition 2.1. The fractional derivative of order $0 \le \eta < 1$ of a complex valued function $x(s)$ is

$$D_s^{\eta}x(s) = \frac{1}{\Gamma(1-\eta)}\frac{d}{ds}\int_0^s \frac{x(\theta)-x(0)}{(s-\theta)^{\eta}}\,d\theta,$$

where the function $x(s)$ is assumed to be analytic in a simply connected region of the complex plane $\mathbb{C}$ which contains the origin, and when $(s-\theta) > 0$, the requirement that $\log(s-\theta)$ be real, eliminates the multiplicity of $(s-\theta)^{-\eta}$.

Definition 2.2. The fractional integral of order $\eta > 0$ of a complex valued function $x(s)$ is

$$I_s^{\eta}x(s) = \frac{1}{\Gamma(\eta)}\int_0^s x(\theta)(s-\theta)^{\eta-1}\,d\theta,$$

where the function $x(s)$ is assumed to be analytic in a simply connected region of the complex plane $\mathbb{C}$ which contains the origin, and when $(s-\theta) > 0$, the requirement that $\log(s-\theta)$ be real, eliminates the multiplicity of $(s-\theta)^{\eta-1}$.

Hypergeometric Function [25]: Hypergeometric function defined by Carl Friedrich Gauss in 1812, as

$$F(a,b;q;s) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(q)_n}\frac{s^n}{n!},\quad |s|<1,$$

$$(a,b,q\in\mathbb{C};\ \ q\notin\mathbb{Z}_0^- = \{0,-1,-2,\ldots\}),$$

where $(l)_n$ represents the Pochhammer symbol by

$$(l)_n = \frac{\Gamma(l+n)}{\Gamma(l)} = \begin{cases} 1 & \text{for } n=0,\\ l(l+1)\ldots(l+n-1) & \text{for } n\in\mathbb{N}. \end{cases}$$

Krasnoselskii Lemma [12]: Assume E is a Banach space and D is a non empty subset of E which is bounded, convex and closed. Let T_1, T_2 be the two operators on D such that

(1) if $h, k \in D$, $T_1 h + T_2 k \in D$.
(2) T_1 is compact and continuous mapping.
(3) T_2 is a contraction map.
Then, there exists a $w \in D$ such that $w = T_1 w + T_2 w$.

3. Existence and Uniqueness Results

Assume $(\mathcal{B}, \|.\|)$ be a complex Banach space of analytic functions defined on unit disk with sup norm

$$\|x\| := \sup_{s \in \mathbb{U}} |x(s)|.$$

Theorem 3.1. *(Existence) let $u : \mathbb{U} \times \mathcal{B} \to \mathcal{B}$ be a analytic function satisfying the following conditions*

(i) $\|u(s, x) - u(s, y)\| \leq N\|x - y\|,\ N > 0,\ s \in \mathbb{U},\ x, y \in \mathcal{B}$,
(ii) *there exists* $M > 0$ *such that* $\|u(s, x)\| \leq M,\ \forall s \in \mathbb{U},\ x \in \mathcal{B}$,
(iii) $\frac{N\, B(\varrho, \eta+1)}{\Gamma(\varrho)\Gamma(\eta+1)} + \frac{|\lambda|}{\Gamma(\varrho+1)} < 1$.

Then, there exists a function $x : \mathbb{U} \to \mathcal{B}$ solving the FLE(1).

Proof. We construct an operator $P : \mathcal{B} \to \mathcal{B}$ by

$$\begin{aligned}(Px)(s) = &\int_0^s \frac{(s-\zeta)^{\varrho-1}}{\Gamma(\varrho)}\left(\int_0^\zeta \frac{(\zeta-\theta)^{\eta-1}}{\Gamma(\eta)} u(\theta, x(\theta))d\theta - \lambda x(\zeta)\right)d\zeta \\ &- s^\varrho\left[\int_0^1 \frac{(1-\zeta)^{\varrho-1}}{\Gamma(\varrho)}\left(\int_0^\zeta \frac{(\zeta-\theta)^{\eta-1}}{\Gamma(\eta)} u(\theta, x(\theta))d\theta - \lambda x(\zeta)\right)d\zeta\right] \\ &+ (\mu_2 - \mu_1)s^\varrho + \mu_1.\end{aligned} \tag{2}$$

To show P is a bounded operator on D where the set $D = \{x \in \mathcal{B} : \|x\| \leq \tau, \tau > 0\} \subset \mathcal{B}$.
Let us choose

$$\tau \geq \frac{\frac{2M\, B(\varrho,\eta+1)}{\Gamma(\varrho)\Gamma(\eta+1)} + |\mu_2| + 2|\mu_1|}{1 - \frac{2N\, B(\varrho,\eta+1)}{\Gamma(\varrho)\Gamma(\eta+1)} - \frac{2|\lambda|}{\Gamma(\eta+1)}} \quad \text{with} \quad 1 > \frac{2N\, B(\varrho, \eta+1)}{\Gamma(\varrho)\Gamma(\eta+1)} + \frac{2|\lambda|}{\Gamma(\eta+1)}.$$

Now

$$\begin{aligned}|(Px)(s)| \leq &\int_0^s \left|\frac{(s-\zeta)^{\varrho-1}}{\Gamma(\varrho)}\right|\left(\int_0^\zeta \left|\frac{(\zeta-\theta)^{\eta-1}}{\Gamma(\eta)}\right|\left|\Big(u(\theta, x(\theta)) + u(\theta, 0) - u(\theta, 0)\Big)\right|d\theta + \left|\lambda x(\zeta)\right|\right)d\zeta \\ &+ \left[\int_0^1 \left|\frac{(1-\zeta)^{\varrho-1}}{\Gamma(\varrho)}\right|\left(\int_0^\zeta \left|\frac{(\zeta-\theta)^{\eta-1}}{\Gamma(\eta)}\right|\left|\Big(u(\theta, x(\theta)) + u(\theta, 0) - u(\theta, 0)\Big)\right|d\theta\right.\right. \\ &+ \left.\left.\left|\lambda x(\zeta)\right|\right)d\zeta\right] + |\mu_2| + 2|\mu_1| \\ \implies |(Px)(s)| \leq &\frac{2(N\tau + M)\, B(\varrho, \eta+1)}{\Gamma(\varrho)\Gamma(\eta+1)} + \frac{2|\lambda|\tau}{\Gamma(\varrho+1)} + |\mu_2| + 2|\mu_1| \leq \tau,\end{aligned}$$

hence P is bounded operator.
Now consider two operators $T_1 : D \to D$ and $T_2 : D \to D$ as follows:

$$(T_1 x)(s) = \int_0^s \frac{(s-\zeta)^{\varrho-1}}{\Gamma(\varrho)}\left(\int_0^\zeta \frac{(\zeta-\theta)^{\eta-1}}{\Gamma(\eta)} u(\theta, x(\theta))d\theta - \lambda x(\zeta)\right)d\zeta.$$

$$(T_2x)(s) = -s^{\varrho}\Big[\int_0^1 \frac{(1-\zeta)^{\varrho-1}}{\Gamma(\varrho)}\Big(\int_0^{\zeta} \frac{(\zeta-\theta)^{\eta-1}}{\Gamma(\eta)} u(\theta, x(\theta))d\theta - \lambda x(\zeta)\Big)d\zeta\Big]$$
$$+ (\mu_2 - \mu_1)s^{\varrho} + \mu_1.$$

It is obvious that set D is bounded, convex and closed.
Now if $x, y \in D$, then

$$\|(T_1x)(s) + (T_2y)(s)\| \leq \frac{2(N\tau + M)\, B(\varrho, \eta+1)}{\Gamma(\varrho)\Gamma(\eta+1)} + \frac{2|\lambda|\tau}{\Gamma(\varrho+1)} + |\mu_2| + 2|\mu_1|,$$

which yields

$$\|(T_1x)(s) + (T_2y)(s)\| \leq \tau.$$

So $(T_1x)(s) + (T_2y)(s) \in D$.
The continuity of g implies the continuity of the operator T_1.
The operator T_1 is uniformly bounded on D as

$$|(T_1x)(s)| \leq \frac{2(N\tau + M)\, B(\varrho, \eta+1)}{\Gamma(\varrho)\Gamma(\eta+1)} + \frac{2|\lambda|\tau}{\Gamma(\varrho+1)}.$$

Next we prove that T_1 is compact operator. For this, we will prove the equicontinuity of the operator T_1. For any $s_1, s_2 \in \mathbb{U}, s_1 \neq s_2$, and for all $x \in D$, we obtain

$$|(T_1x)(s_1) - (T_1x)(s_2)| \leq \frac{2MB(\varrho, \eta+1)}{\Gamma(\varrho)\Gamma(\eta+1)} + \frac{2|\lambda|\tau}{\Gamma(\varrho+1)},$$

which is independent from x. This shows the equicontinuity of the operator T_1. Hence using Arzela Ascoli theorem [8] , T_1 is compact on $\bar{D}$. Hence T_1 is compact on D.
Further, we prove that T_2 is a contraction map.

$$\|(T_2x)(s) - (T_2y)(s)\| \leq \Big(\frac{N\, B(\varrho, \eta+1)}{\Gamma(\varrho)\Gamma(\eta+1)} + \frac{|\lambda|}{\Gamma(\varrho+1)}\Big)\|x - y\| < \|x - y\|.$$

So, T_2 is contraction mapping.
Using Krasnoselskii lemma, there exists $x : \mathbb{U} \to \mathcal{B}$ such that $(Px)(s) = (T_1x)(s) + (T_2x)(s) = x(s)$. Hence, the FLE (1) has atleast a solution. □

Theorem 3.2. *(Uniqueness) Let the conditions* (i) - (ii) *hold with*

$$\frac{2N\, B(\varrho, \eta+1)}{\Gamma(\varrho)\Gamma(\eta+1)} + \frac{2|\lambda|}{\Gamma(\varrho+1)} < 1. \tag{3}$$

Then, there exists a unique solution $x : \mathbb{U} \to \mathcal{B}$ *of the FLE* (1).

Proof. We show the uniqueness of the fixed point of operator P defined by eq(2).

$$\|(Px)(s) - (Py)(s)\| \leq \int_0^s \Big|\frac{(w-\zeta)^{\varrho-1}}{\Gamma(\varrho)}\Big|\Big(\int_0^{\zeta}\Big|\frac{(\zeta-\theta)^{\eta-1}}{\Gamma(\eta)}\Big|\|u(\theta, x(\theta)) - u(\theta, y(\theta))\|d\theta + |\lambda|\|x - y\|\Big)d\zeta$$
$$+ \int_0^1 \Big|\frac{(1-\zeta)^{\varrho-1}}{\Gamma(\varrho)}\Big|\Big(\int_0^{\zeta}\Big|\frac{(\zeta-\theta)^{\eta-1}}{\Gamma(\eta)}\Big|\|u(\theta, x(\theta)) - u(\theta, y(\theta))\|d\theta + |\lambda|\|x - y\|\Big)d\zeta$$
$$\leq \Big(\frac{2N\, B(\varrho, \eta+1)}{\Gamma(\varrho)\Gamma(\eta+1)} + \frac{2|\lambda|}{\Gamma(\varrho+1)}\Big)\|x - y\| < \|x - y\|.$$

Thus P is a contraction map. Hence by applying Banach fixed point theorem, P has a unique fixed point which leads to the solution of the FLE (1). □

4. Hyers - Ulam Stability

Due to the fact that fractional derivatives are nonlocal and have weakly singular kernels, the analysis of the stability of FDE is more difficult than that of a classical differential equation. An overview of the stability of FDEs was provided by Li and Zhang [13]. Ulam initially established the Ulam stability in 1940 and Hyers [10] later extended it in 1941. Ulam stability and data dependence for FDE with Caputo derivative has been posed by Wang *et al* [28]. In 2023, Chen *et al*. [7] performed the existence and stability of the bifurcating solution in Chemotaxis model, which bifurcated from the steady state, by using the Crandall-Rabinowitz local bifurcation theory.

In this section, we are going to discuss stability in Hyers-Ulam sense of FLE in the complex domain with two point boundary conditions on the unit disk.
Let $\epsilon > 0$. Consider the FLE(1) and the inequality

$$|D_s^{\eta}(D_s^{\varrho} + \lambda)x(s) - u(s, x(s))| < \epsilon, \qquad s \in \mathbb{U}. \tag{4}$$

Definition 4.1 [6] FLE(1) is Hyers Ulam stable if there exists a real number $c > 0$ such that for each $\epsilon > 0$ and for each analytic solution $\psi : \mathbb{U} \to \mathcal{B}$ of the inequality (4), there exists an analytic solution $x : \mathbb{U} \to \mathcal{B}$ of the FLE(1) with $\|\psi - x\| \leq c\epsilon$.

Definition 4.2 [6] An analytic function $\psi : \mathbb{U} \to \mathcal{B}$ is a solution of the inequality (4) if there exists an analytic function $f : \mathbb{U} \to \mathcal{B}$ such that $|f(s)| \leq \epsilon$ and

$$D_s^{\eta}(D_s^{\varrho} + \lambda)\psi(s) = u(s, \psi(s)) + f(s), \qquad s \in \mathbb{U}.$$

Definition 4.3 [6] FLE(1) is generalized Hyers Ulam stable if there exist an analytic function $v : \mathbb{R}_+ \to \mathbb{R}_+$ such that for each analytic solution $\psi : \mathbb{U} \to \mathcal{B}$ of the inequality (4), there exists an analytic solution $x : \mathbb{U} \to \mathcal{B}$ of the FLE(1) with $\|\psi - x\| \leq v(\epsilon)$.

Theorem 4.1. *Under the assumption of Theorem* 3.2, *FLE*(1) *is stable in Hyers -Ulam sense on the unit disk* $\mathbb{U}$.

Proof. Let $\psi : \mathbb{U} \to \mathcal{B}$ be analytic solution of inequality (4) and $x : \mathbb{U} \to \mathcal{B}$ be the unique solution of FLE(1), then

$$\begin{aligned}\|\psi(s) - x(s)\| \leq & \int_0^s \left|\frac{(s-\zeta)^{\varrho-1}}{\Gamma(\varrho)}\right| \left(\int_0^{\zeta} \left|\frac{(\zeta-\theta)^{\eta-1}}{\Gamma(\eta)}\right| \left(\left| u(\theta, \psi(\theta)) - u(\theta, x(\theta)) + f(\theta) \right| \right) d\theta + |\lambda| |\psi(\zeta) - x(\zeta)| \right) d\zeta \\ & + \left[\int_0^1 \left|\frac{(1-\zeta)^{\varrho-1}}{\Gamma(\varrho)}\right| \left(\int_0^{\zeta} \left|\frac{(\zeta-\theta)^{\eta-1}}{\Gamma(\eta)}\right| \left(\left| u(\theta, \psi(\theta)) - u(\theta, x(\theta)) + f(\theta) \right| \right) d\theta + |\lambda| |\psi(\zeta) - x(\zeta)| \right) d\zeta \right] \\ \|\psi(s) - x(s)\| \leq & \frac{2N\|\psi - x\|\, B(\varrho, \eta+1)}{\Gamma(\varrho)\Gamma(\eta+1)} + 2\epsilon \frac{B(\varrho, \eta+1)}{\Gamma(\varrho)\Gamma(\eta+1)} + \frac{2|\lambda| \|\psi - x\|}{\Gamma(\varrho+1)}.\end{aligned}$$

Hence, there exist a constant $c > 0$ achieving

$$\|\psi - x\| \leq c\, \epsilon, \ \text{ where } \ c = \frac{2\frac{B(\varrho,\eta+1)}{\Gamma(\varrho)\Gamma(\eta+1)}}{1 - \frac{2N\, B(\varrho,\eta+1)}{\Gamma(\varrho)\Gamma(\eta+1)} - \frac{2|\lambda|}{\Gamma(\varrho+1)}}.$$

Hence FLE(1) is Ulam-Hyers stable. By putting $v(\epsilon) = c\epsilon$, it is generalized stable in Hyers-Ulam sense. □

Theorem 4.2. *[25] Let* $0 < \eta < 1$, *and* $x(s)$ *be the univalent function, Then*

$$|D_s^{\eta} x(s)| \leq \frac{r^{1-\eta}}{\Gamma(1-\eta)} \int_0^1 \frac{1 + r\alpha}{(1-\alpha)^{\eta}(1-r\alpha)^3} d\alpha,$$
$$(|s| = r < 1, s \in \mathbb{U}).$$

Theorem 4.3. *Let $0 < \eta < 1, 0 < \varrho < 1$ with $|\lambda| < \frac{\Gamma(\varrho+1)}{2}$. If the FLE (1) has an univalent solution, then It is generalized stable in Hyers -Ulam sense on the unit disk $\mathbb{U}$.*

Proof. By applying the Theorem 4.2 , we get

$$|D_s^{\eta}(D_s^{\varrho}+\lambda)x(s)| \leq \frac{r^{1-(\varrho+\eta)}}{\Gamma(1-(\varrho+\eta))}\int_0^1 \frac{1+r\alpha}{(1-\alpha)^{\varrho+\eta}(1-r\alpha)^3}d\alpha + \frac{|\lambda|\, r^{1-\eta}}{\Gamma(1-\eta)}\int_0^1 \frac{1+r\alpha}{(1-\alpha)^{\eta}(1-r\alpha)^3}d\alpha, \ \ (|s|=r<1),$$

Using the integral representation [25],

$$(rF(2,1,2-\Lambda:r))' = (1-\Lambda)\int_0^1 \frac{1+r\alpha}{(1-\alpha)^{\Lambda}(1-r\alpha)^3}d\alpha, \ \text{ where } \ 0<\Lambda<1.$$

We obtain

$$|D_s^{\eta}(D_s^{\varrho}+\lambda)x(s)| \leq \frac{r^{1-(\varrho+\eta)}}{\Gamma(2-(\varrho+\eta))}(rF(2,1,2-(\varrho+\eta):r))' + \frac{|\lambda|\, r^{1-\eta}}{\Gamma(2-\eta)}(rF(2,1,2-\eta:r))',$$

where F is hypergeometric function.
Let $|g| \leq M < \infty$, thus we obtain

$$|D_s^{\eta}(D_s^{\varrho}+\lambda)x(s)-u(s,x(s))| \leq \frac{r^{1-(\varrho+\eta)}}{\Gamma(2-(\varrho+\eta))}(rF(2,1,2-(\varrho+\eta):r))' + \frac{|\lambda|\, r^{1-\eta}}{\Gamma(2-\eta)}(rF(2,1,2-\eta:r))' + M = \epsilon.$$

To show that FLE(1) is generalized stable in Ulam-Hyers sense. Now,

$$|x(s)-\psi(s)| \leq \frac{4\frac{B(\varrho,\eta+1)}{\Gamma(\varrho)\Gamma(\eta+1)}M}{1-\frac{2|\lambda|}{\Gamma(\varrho+1)}},$$

$$|x(s)-\psi(s)| \leq \frac{4\frac{B(\varrho,\eta+1)}{\Gamma(\varrho)\Gamma(\eta+1)}\left(\epsilon - \frac{r^{1-(\varrho+\eta)}}{\Gamma(2-(\varrho+\eta))}(rF(2,1,2-(\varrho+\eta):r))' - \frac{|\lambda|\, r^{1-\eta}}{\Gamma(2-\eta)}(rF(2,1,2-\eta:r))'\right)}{1-\frac{2|\lambda|}{\Gamma(\varrho+1)}} = v(\epsilon),$$

for sufficiently small r. Hence FLE(1) is generalized Ulam-Hyers stable. □

5. Examples

Example 5.1. Consider FLE in complex domain

$$D_s^{\frac{1}{3}}(D_s^{\frac{1}{4}} + \frac{1}{15+3\iota})x(s) = \frac{1}{(5+7\iota)^{1/4}}\frac{1}{(5+7\iota)^{1/3}}x(s) + \frac{1}{(15+3\iota)}\frac{1}{(5+7\iota)^{1/3}}x(s), \ s\in\mathbb{U}, \tag{5}$$

$$x(0)=1, \ \ x(1)=e^{\frac{1}{5+7\iota}}. \tag{6}$$

where $x(s), u(s,x(s))$ are analytic functions. $x(s) = e^{\frac{s}{5+7\iota}}$ is a solution of above problem.
Now, $\forall x, y \in \mathcal{B}$,

$$\begin{aligned}\|u(s,x(s))-u(s,y(s))\| &\leq \left|\frac{1}{(5+7\iota)^{1/4}}\frac{1}{(5+7\iota)^{1/3}} + \frac{1}{(15+3\iota)}\frac{1}{(5+7\iota)^{1/3}}\right||x-y| \\ &\leq N\|x-y\|, \quad \text{where } N = 0.317\ldots\end{aligned}$$

thus condition (i) is satisfied with $N = 0.317\ldots$. Moreover

$$2\,\frac{(0.317\ldots)B(\frac{1}{4},\frac{4}{3})}{\Gamma(\frac{1}{4})\Gamma(\frac{4}{3})} + \frac{2|\frac{1}{15+3\iota}|}{\Gamma(\frac{5}{4})} = 0.8559\ldots < 1.$$

Hence by Theorem 3.2, there exists a unique solution of the problem(5) satisfying boundary conditions (6). Also, it is Ulam- Hyers stable.

Example 5.2. Consider the following FLE in complex domain

$$D_s^{\frac{2}{3}}(D_s^{\frac{1}{6}} + \frac{1}{100+5\iota})x(s) = \frac{|x|}{35(1+|x|)}, \; s \in \mathbb{U}, \tag{7}$$

$$x(0) = \mu_1, \;\; x(1) = \mu_2. \tag{8}$$

where $x(s), u(s, x(s))$ are analytic functions,
Now, $\forall x, y \in \mathcal{B}$,

$$\|u(s, x(s)) - u(s, y(s))\| = \|\frac{|x|}{35(1+|x|)} - \frac{|y|}{35(1+|y|)}\| \quad \leq \frac{1}{35}\|x-y\|,$$

thus condition (i) is satisfied with $N = \frac{1}{35}$. Moreover

$$\frac{2B(\frac{1}{6}, \frac{5}{3})}{35\Gamma(\frac{1}{6})\Gamma(\frac{5}{3})} + \frac{2|\frac{1}{100+5\iota}|}{\Gamma(\frac{7}{6})} = 0.0822792436\ldots\ldots < 1.$$

Hence using Theorem 3.2, the considered problem (7) has a unique solution satisfying boundary conditions (8). Also the problem (7) is Ulam-Hyers stable.

6. Conclusion

In this paper, we established the conditions for the existence and uniqueness of the solutions of FLE(1) in complex domain using fixed point theory. We employed Srivastava-Owa fractional operator in the unit disk for this problem. In the section 4, we have presented some results dealing with the Hyers-Ulam stability in terms of Hypergeometric function using the univalency of the solution for the considered FLE(1). Examples are provided to verify the obtained results.

Conflicts of Interest

The authors declare that they have no conflict of interest.

References

[1] B. Ahmad, M. Alghanmi, A. Alsaedi, H. M. Srivastava, and S. K. Ntouyas. The Langevin equation in terms of generalized Liouville–Caputo derivatives with nonlocal boundary conditions involving a generalized fractional integral. *Mathematics*, 7(6):533, 2019.

[2] B. Ahmad and P. Eloe. A nonlocal boundary value problem for a nonlinear fractional differential equation with two indices. *Comm. Appl. Nonlinear Anal.*, 17(3):69–80, 2010.

[3] B. Ahmad, J. J. Nieto, A. Alsaedi, and M. El-Shahed. A study of nonlinear Langevin equation involving two fractional orders in different intervals. *Nonlinear Anal. Real World Appl.*, 13(2):599–606, 2012.

[4] B. Ahmad and S. K. Ntouyas. Existence of solutions for fractional differential inclusions with four-point nonlocal Riemann-Liouville type integral boundary conditions. *Filomat*, 27(6):1027–1036, 2013.

[5] S. Arora, T. Mathur, and K. Tiwari. A fractional-order model to study the dynamics of the spread of crime. *J. Comput. Appl. Math.*, 426:115102, 2023.

[6] M. Benchohra and S. Bouriah. Existence and stability results for nonlinear boundary value problem for implicit differential equations of fractional order. *Moro. J. Pure Appl. Anal.*, 1:1–16, 2015.

[7] M. Chen and H. M. Srivastava. Existence and stability of bifurcating solution of a chemotaxis model. *Proc. Amer. Math. Soc.*, 151(11):4735–4749, 2023.

[8] R. F. Curtain and A. J. Pritchard. *Functional analysis in modern applied mathematics*, volume 132. Academic Press, 1977.

[9] K. S. Fa. Generalized Langevin equation with fractional derivative and long-time correlation function. *Phys. Rev. E*, 73(6):061104, 2006.

[10] D. H. Hyers. On the stability of the linear functional equation. *Proc. Nat. Acad. Sci. U.S.A.*, 27:222–224, 1941.

[11] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo. *Theory and applications of fractional differential equations*, volume 204. Elsevier, Amsterdam, 2006.

[12] M. A. Krasnosel'skiĭ. Two remarks on the method of successive approximations. *Uspehi Mat. Nauk.*, 10(1):123–127, 1955.

[13] C. Li and F. Zhang. A survey on the stability of fractional differential equations: Dedicated to Prof. YS Chen on the occasion of his 80th birthday. *The Euro. Phys. J. Spec. Topics*, 193(1):27–47, 2011.

[14] S. Liang and J. Zhang. Positive solutions for boundary value problems of nonlinear fractional differential equation. *Nonlinear Anal.*, 71(11):5545–5550, 2009.

[15] S. C. Lim, M. Li, and L. P. Teo. Langevin equation with two fractional orders. *Phys. Lett. A*, 372(42):6309–6320, 2008.

[16] S. C. Lim and L. P. Teo. The fractional oscillator process with two indices. *J. Phys. A*, 42(6):065208, 2009.

[17] E. Lutz. Fractional Langevin equation. In *Fractional dynamics*, pages 285–305. World Sci. Publ., 2012.

[18] T. Mathur, S. Agarwal, S. Goyal, and K. S. Pritam. Analytical solutions of some fractional diffusion boundary value problems. In *Frac. Order Syst. Appl. Engi.*, pages 37–50. Elsevier, 2023.

[19] K. S. Miller and B. Ross. *An introduction to the fractional calculus and fractional differential equations*. John Wiley & Sons, Inc., New York, 1993.

[20] S. Picozzi and B. J. West. Fractional Langevin model of memory in financial markets. *Phys. Rev. E*, 66(4):046118, 2002.

[21] I. Podlubny. *Fractional differential equations*, volume 198. Academic Press, Inc., San Diego, CA, 1999.

[22] K. S. Pritam, Sugandha, T. Mathur, and S. Agarwal. Underlying dynamics of crime transmission with memory. *Chaos Solitons Fractals*, 146:110838, 2021.

[23] K. Sau Fa. Fractional Langevin equation and Riemann-Liouville fractional derivative. *The Euro. Phys. Journal E*, 24:139–143, 2007.

[24] H. M. Srivastava, M. Darus, and R. W. Ibrahim. Classes of analytic functions with fractional powers defined by means of a certain linear operator. *Integral Transforms Spec. Funct.*, 22(1):17–28, 2011.

[25] H. M. Srivastava, Y. Ling, and G. Bao. Some distortion inequalities associated with the fractional derivatives of analytic and univalent functions. *J. Inequal. Pure Appl. Math.*, 2(2), 2001.

[26] H. M. Srivastava, A. K. Nain, R. K. Vats, and P. Das. A theoretical study of the fractional-order p-Laplacian nonlinear Hadamard type turbulent flow models having the Ulam-Hyers stability. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. (RACSAM)*, 117(4):160, 1– 19, 2023.

[27] H. M. Srivastava and S. Owa. *Univalent functions, fractional calculus, and their applications*. Chichester: Ellis Horwood; New York; Toronto: Halsted Press, 1989.

[28] J. Wang, L. Lv, and Y. Zhou. Ulam stability and data dependence for fractional differential equations with Caputo derivative. *Electron. J. Qual. Theory Differ. Equ.*, 2011(63):1–10, 2011.

[29] A. Yadav, T. Mathur, S. Agarwal, and B. Yadav. Fractional boundary value problem in complex domain. *J. Math. Anal. Appl.*, 526(1):127178, 2023.