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The optimization problems of a quadratic Hermitian matrix-valued function with the constraint of matrix equations

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Abstract. Let $\Omega = \left\{X \in \mathbb{C}^{n\times p} \,\middle|\, AX = B, XH = K, \text{ and } AA^\dagger B = B, KH^\dagger H = K, AK = BH\right\}$, and let $f(X) = (XC + D)M(XC + D)^* - G$ be a given quadratic Hermitian matrix-valued function. In this paper, we first establish a series of closed-form formulas for calculating the extremal ranks and inertias of $f(X)$ subject to $X \in \Omega$ by applying the generalized inverses of matrices. Further, we present the solvability conditions for $X \in \Omega$ to satisfy the matrix equality $(XC + D)M(XC + D)^* = G$ and matrix inequalities $(XC + D)M(XC + D)^* > G$ $(\geq G, < G, \leq G)$ to hold, respectively. In addition, we provide closed-form solutions to two Löwner partial ordering optimization problems on $f(X)$ subject to $X \in \Omega$.

1. Introduction

Throughout the paper, $\mathbb{C}^{m\times n}$ and $\mathbb{C}_{\rm H}^{m}$ stand for the sets of all $m \times n$ complex matrices and all $m \times m$ complex Hermitian matrices, respectively. The notations $A^\dagger, A^*, \mathscr{R}(A)$ and $r(A)$ denote the Moore-Penrose inverse, the conjugate transpose, the column space and the rank of a matrix $A \in \mathbb{C}^{m\times n}$, respectively. Further, the matrices $F_A = I_n - A^\dagger A$ and $E_A = I_m - AA^\dagger$ are the orthogonal projectors induced by matrix A. For two matrices $B, C \in \mathbb{C}_{\rm H}^{m}$, if $B - C$ is positive (nonnegative) definite, we denote it by $B > C$ $(B \geqslant C)$. In addition, $i_-(N)$ and $i_+(N)$ denote the negative and positive inertia of $N \in \mathbb{C}_{\rm H}^{m}$.

We know that the rank and inertia of matrices are important tools in matrix theory and applications, and many problems are closely related to the ranks and inertias of some matrix expressions under constraint conditions. The extremal ranks and inertias of an Hermitian matrix have many applications in control theory [1–3], statistics [8, 15, 17] and so on.

The following pair of linear matrix equations

$$AX = B, \; XH = K \tag{1}$$

have been extensively studied by many authors. In 1971, Rao and Mitra [9] showed that Eq.(1) has a common solution if and only if

$$AA^\dagger B = B, \; KH^\dagger H = K, \; AK = BH, \tag{2}$$

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and the general common solution of Eq.(1) is

$$X = A^{\dagger}B + F_A K H^{\dagger} + F_A V E_H, \tag{3}$$

where $V \in \mathbb{C}^{n\times p}$ is an arbitrary matrix. Further, Mitra [6] obtained a common solution with the minimum possible rank for Eq.(1) by using the generalized inverses of matrices. Liu [5] gave the extremal ranks of the common least-square solutions to Eq.(1). Wang et al. [18] presented the extremal ranks and inertias of the common bisymmetric nonnegative definite solution of Eq.(1). Zhang and Wang [20] investigated the (P, Q)-symmetric and (P, Q)-skewsymmetric maximal and minimal rank solutions to Eq.(1).

The well-known quadratic Hermitian matrix-valued function (QHMF)

$$f(X) = (XC + D)M(XC + D)^* - G, \tag{4}$$

where $C \in \mathbb{C}^{p\times m}, D \in \mathbb{C}^{n\times m}, M \in \mathbb{C}^{m}_{\mathrm{H}}, G \in \mathbb{C}^{n}_{\mathrm{H}}$ are given matrices and $X \in \mathbb{C}^{n\times p}$ is an unknown matrix, have been considered by many scholars. For example, Tian [14] derived a group of basic problems on ranks, inertias, equalities, inequalities, maximization and minimization in the Löwner partial ordering of QHMF.(4) subject to a consistent linear matrix equation $XA = B$ by pure algebraic operations of matrices. Tian [12], Tian and Jiang [4, 16] established the explicit formulas for calculating the extremal ranks and inertias of matrix function $XPX^* - Q^*$ subject to the general solution, the least-squares solutions and the weighted least-squares solutions of the matrix equation $AX = B$, respectively. Wang [19] considered the extreme inertias and ranks of a new quasi-quadratic Hermitian structure $XX^* - P$ subject to a consistent system of Eq.(1). Motivated by the works mentioned above, we will discuss the optimization problems on ranks and inertias of QHMF.(4) subject to Eq.(1).

It follows from the relations of (2) and (3) that

$$\Omega = \left\{ X \in \mathbb{C}^{n\times p} :\ AX = B, XH = K,\ and\ AA^{\dagger}B = B, KH^{\dagger}H = K, AK = BH \right\}$$

is a nonempty set, and

$$\Omega = \left\{ X \in \mathbb{C}^{n\times p} :\ X = A^{\dagger}B + F_A K H^{\dagger} + F_A V E_H, \forall\, V \in \mathbb{C}^{n\times p} \right\}.$$

In this paper, the following problems are considered:

Problem 1. Let $f(X)$ be given by (4). Find explicit algebraic formulas for calculating the following global extremum ranks and inertias:

$$\max_{X\in\Omega} r[f(X)], \quad \min_{X\in\Omega} r[f(X)], \quad \max_{X\in\Omega} i_{\pm}[f(X)], \quad \min_{X\in\Omega} i_{\pm}[f(X)].$$

Problem 2. Given $f(X)$ in (4), find necessary and sufficient conditions for the following constrained matrix equation and four constrained matrix inequalities

$$(XC + D)M(XC + D)^* = G \ \text{ s.t. }\ X \in \Omega,$$

$$(XC + D)M(XC + D)^* > G \ (\geqslant G,\ < G,\ \leqslant G) \ \text{ s.t. }\ X \in \Omega$$

to hold, respectively.

Problem 3. Given $f(X)$ in (4), find $\hat{X},\ \tilde{X} \in \Omega$ such that

$$f(X) \leqslant f(\hat{X}),\ \ f(X) \geqslant f(\tilde{X}), \tag{5}$$

as well as find necessary and sufficient conditions for (5) to hold, and find the explicit analytical expressions of $\hat{X}$ and $\tilde{X}$.

We organize this paper as follows. In Section 2, we introduce some known results on matrix ranks and inertias. In Sections 3 and 4, by means of the generalized inverses of matrices, we derive explicit algebraic formulas and the sufficient and necessary conditions for the existence of solutions to Problems 1-3.

2. Preliminaries

We present some known results, which will be used as effective tools for solving the previous problems.

Lemma 2.1. *[10, 11] Let $A \in \mathbb{C}_{\mathrm{H}}^{m}, E \in \mathbb{C}_{\mathrm{H}}^{n}, D \in \mathbb{C}^{m\times n}$, and $F \in \mathbb{C}^{m\times m}$ be nonsingular. Then,*

$$i_{\pm}(FAF^{*}) = i_{\pm}(A),$$

$$i_{\pm}(\zeta A) = \begin{cases} i_{\pm}(A) & if\ \zeta > 0 \\ i_{\mp}(A) & if\ \zeta < 0, \end{cases}$$

$$i_{\pm}\begin{bmatrix} A & 0 \\ 0 & E \end{bmatrix} = i_{\pm}(A) + i_{\pm}(E),$$

$$i_{+}\begin{bmatrix} 0 & D \\ D^{*} & 0 \end{bmatrix} = i_{-}\begin{bmatrix} 0 & D \\ D^{*} & 0 \end{bmatrix} = r(D).$$

Lemma 2.2. *[10, 11] Let $\mathcal{T}$ and $\mathcal{H}$ be two matrix subsets in $\mathbb{C}^{p\times q}$ and $\mathbb{C}_{\mathrm{H}}^{p}$, respectively. Then,*
(a) *For $p = q$, $\mathcal{T}$ exists a nonsingular matrix iff* $\max_{X\in\mathcal{T}} r(X) = p$.
(b) *For $p = q$, all $X \in \mathcal{T}$ are nonsingular iff* $\min_{X\in\mathcal{T}} r(X) = p$.
(c) $0 \in \mathcal{T}$ *iff* $\min_{X\in\mathcal{T}} r(X) = 0$.
(d) $\mathcal{T} = \{0\}$ *iff* $\max_{X\in\mathcal{T}} r(X) = 0$.
(e) *$\mathcal{H}$ exists a matrix $X < 0$ ($X > 0$) iff* $\max_{X\in\mathcal{H}} i_{-}(X) = p$ ($\max_{X\in\mathcal{H}} i_{+}(X) = p$).
(f) *All $X \in \mathcal{H}$ satisfy $X < 0$ ($X > 0$) iff* $\min_{X\in\mathcal{H}} i_{-}(X) = p$ ($\min_{X\in\mathcal{H}} i_{+}(X) = p$).
(g) *$\mathcal{H}$ exists a matrix $X \leqslant 0$ ($X \geqslant 0$) iff* $\min_{X\in\mathcal{H}} i_{+}(X) = 0$ ($\min_{X\in\mathcal{H}} i_{-}(X) = 0$).
(h) *All $X \in \mathcal{H}$ satisfy $X \leqslant 0$ ($X \geqslant 0$) iff* $\max_{X\in\mathcal{H}} i_{+}(X) = 0$ ($\max_{X\in\mathcal{H}} i_{-}(X) = 0$).

Lemma 2.3. *[7] Let $A \in \mathbb{C}^{m\times n}, C \in \mathbb{C}^{m\times k}$, and $B \in \mathbb{C}^{l\times n}$ be given. Then,*

$$r(E_{A}C) + r(A) = r(E_{C}A) + r(C) = r[A,\ C],$$

$$r(BF_{A}) + r(A) = r(AF_{B}) + r(B) = r\begin{bmatrix} A \\ B \end{bmatrix},$$

$$r(E_{C}AF_{B}) + r(B) + r(C) = r\begin{bmatrix} A & C \\ B & 0 \end{bmatrix}.$$

In particular,

$$r[A,\ C] = r(A) \Leftrightarrow \mathscr{R}(C) \subseteq \mathscr{R}(A) \Leftrightarrow AA^{\dagger}C = C \Leftrightarrow E_{A}C = 0,$$

$$r\begin{bmatrix} A \\ B \end{bmatrix} = r(A) \Leftrightarrow \mathscr{R}(B^{*}) \subseteq \mathscr{R}(A^{*}) \Leftrightarrow BA^{\dagger}A = B \Leftrightarrow BF_{A} = 0.$$

Lemma 2.4. *[10] Let $A \in \mathbb{C}_{\mathrm{H}}^{m}, B \in \mathbb{C}^{m\times n}, G \in \mathbb{C}_{\mathrm{H}}^{n}$, and denote*

$$K_{1} = \begin{bmatrix} A & B \\ B^{*} & 0 \end{bmatrix},\ K_{2} = \begin{bmatrix} A & B \\ B^{*} & G \end{bmatrix}.$$

Then, the ranks and inertias of K_1 and K_2 can be expanded as

$$i_{\pm}(K_{1}) = r(B) + i_{\pm}(E_{B}AE_{B}),\ i_{\pm}(K_{2}) = i_{\pm}(A) + i_{\pm}\begin{bmatrix} 0 & E_{A}B \\ B^{*}E_{A} & G - B^{*}A^{\dagger}B \end{bmatrix}, \tag{6}$$

$$r(K_{1}) = 2r(B) + r(E_{B}AE_{B}),\ r(K_{2}) = r(A) + r\begin{bmatrix} 0 & E_{A}B \\ B^{*}E_{A} & G - B^{*}A^{\dagger}B \end{bmatrix}.$$

In particular,

$$i_{+}\begin{bmatrix} A & B \\ B^{*} & 0 \end{bmatrix} = 0 \Leftrightarrow A \leqslant 0 \text{ and } B = 0,\ i_{-}\begin{bmatrix} A & B \\ B^{*} & 0 \end{bmatrix} = 0 \Leftrightarrow A \geqslant 0 \text{ and } B = 0. \tag{7}$$

Some useful expansion formulas derived from (6) *are*

$$r\begin{bmatrix} A & BF_Q \\ C & 0 \end{bmatrix} = r\begin{bmatrix} A & B \\ C & 0 \\ 0 & Q \end{bmatrix} - r(Q),$$

$$r\begin{bmatrix} A & B \\ E_PC & 0 \end{bmatrix} = r\begin{bmatrix} A & B & 0 \\ C & 0 & P \end{bmatrix} - r(P),\ r\begin{bmatrix} A & BF_Q \\ E_PC & 0 \end{bmatrix} = r\begin{bmatrix} A & B & 0 \\ C & 0 & P \\ 0 & Q & 0 \end{bmatrix} - r(P) - r(Q),$$

$$i_\pm\begin{bmatrix} A & BF_P \\ F_PB^* & 0 \end{bmatrix} = i_\pm\begin{bmatrix} A & B & 0 \\ B^* & 0 & P^* \\ 0 & P & 0 \end{bmatrix} - r(P),\ i_\pm\begin{bmatrix} E_QAE_Q & E_QB \\ B^*E_Q & G \end{bmatrix} = i_\pm\begin{bmatrix} A & B & Q \\ B^* & G & 0 \\ Q^* & 0 & 0 \end{bmatrix} - r(Q).$$

We will use them to simplify the ranks and inertias of block matrices.

Lemma 2.5. *[13] Let* $A \in \mathbb{C}^m_{\mathrm{H}}, C \in \mathbb{C}^n_{\mathrm{H}}, B \in \mathbb{C}^{m\times n}$ *and* $D \in \mathbb{C}^{n\times p}$ *be given, and* $X \in \mathbb{C}^{p\times m}$ *be a variable matrix. Also, let*

$$\varphi(X) = DXAX^*D^* + DXB + B^*X^*D^* + C,\ E_1 = [C,\ B^*,\ D],$$
$$E_2 = \begin{bmatrix} C & D \\ D^* & 0 \end{bmatrix},\ E_3 = \begin{bmatrix} C & D & B^* \\ D^* & 0 & 0 \end{bmatrix},\ E_4 = \begin{bmatrix} C & B^* \\ B & A \end{bmatrix},\ E_5 = \begin{bmatrix} C & B^* & D \\ B & A & 0 \end{bmatrix}.$$

Then, we have:

$$\max_{X\in\mathbb{C}^{p\times m}} r[\varphi(X)] = \min\{r(E_1),\ r(E_2),\ r(E_4)\},$$
$$\min_{X\in\mathbb{C}^{p\times m}} r[\varphi(X)] = \max\{t_1,\ t_2,\ t_3,\ t_4\},$$
$$\max_{X\in\mathbb{C}^{p\times m}} i_\pm[\varphi(X)] = \min\{i_\pm(E_2),\ i_\pm(E_4)\},$$
$$\min_{X\in\mathbb{C}^{p\times m}} i_\pm[\varphi(X)] = \max\{r(E_1) - r(E_3) + i_\pm(E_2),\ r(E_1) - r(E_5) + i_\pm(E_4)\},$$

where

$$t_1 = 2r(E_1) + r(E_2) - 2r(E_3),\ t_3 = 2r(E_1) - r(E_3) - r(E_5) + i_+(E_2) + i_-(E_4),$$
$$t_2 = 2r(E_1) + r(E_4) - 2r(E_5),\ t_4 = 2r(E_1) - r(E_3) - r(E_5) + i_-(E_2) + i_+(E_4).$$

3. The solutions of Problems 1 and 2

Let

$$N_1 = \begin{bmatrix} (AD+BC)MD^* - AG & (AD+BC)MC^* \\ -K^* & H^* \end{bmatrix},$$
$$N_2 = \begin{bmatrix} (AD+BC)MD^*A^* - AGA^* & (AD+BC)MC^* \\ -K^*A^* & H^* \end{bmatrix}, \tag{8}$$
$$N_3 = \begin{bmatrix} DMD^* - G & DMC^* & -K \\ CMD^* & CMC^* & H \\ -K^* & H^* & 0 \end{bmatrix},\ N_4 = \begin{bmatrix} ADMD^* - AG & ADMC^* & -AK \\ CMD^* & CMC^* & H \\ -K^* & H^* & 0 \end{bmatrix}.$$

Now we give the first theorem of this paper.

Theorem 3.1. *Let* $f(X)$ *be as given by* (4), *and let* $N_1,\ N_2,\ N_3$ *and* N_4 *be the matrices of* (8). *Then,*
(a) *The global maximum rank of* $f(X)$ *subject to* $X \in \Omega$ *is*

$$\max_{X\in\Omega} r[f(X)] = \min\{n + r(N_1) - r(A) - r(H),\ r(N_3) - 2r(H),$$
$$2n + r[(AD+BC)M(AD+BC)^* - AGA^*] - 2r(A)\}. \tag{9}$$

(b) *The global minimum rank of $f(X)$ subject to $X \in \Omega$ is*

$$\min_{X\in\Omega} r[f(X)] = \max\{v_1,\ v_2,\ v_3,\ v_4\}, \tag{10}$$

where

$$\begin{aligned}
v_1 &= 2r(N_1) + r[(AD+BC)M(AD+BC)^* - AGA^*] - 2r(N_2),\\
v_2 &= 2r(N_1) + r(N_3) - 2r(N_4),\\
v_3 &= 2r(N_1) - r(N_2) - r(N_4) + i_+[(AD+BC)M(AD+BC)^* - AGA^*] + i_-(N_3),\\
v_4 &= 2r(N_1) - r(N_2) - r(N_4) + i_-[(AD+BC)M(AD+BC)^* - AGA^*] + i_+(N_3).
\end{aligned}$$

(c) *The global maximum partial inertia of $f(X)$ subject to $X \in \Omega$ is*

$$\max_{X\in\Omega} i_\pm[f(X)] = \min\{n - r(A) + i_\pm[(AD+BC)M(AD+BC)^* - AGA^*],\ i_\pm(N_3) - r(H)\}. \tag{11}$$

(d) *The global minimum partial inertia of $f(X)$ subject to $X \in \Omega$ is*

$$\min_{X\in\Omega} i_\pm[f(X)] = \max\left\{r(N_1) - r(N_4) + i_\pm(N_3),\ r(N_1) - r(N_2) + i_\pm\left[(AD+BC)M(AD+BC)^* - AGA^*\right]\right\}. \tag{12}$$

Proof. By substituting (3) into $f(X)$ leads to

$$\begin{aligned}
f(X) &= (XC+D)M(XC+D)^* - G\\
&= [(A^\dagger B + F_AKH^\dagger + F_AVE_H)C + D]M[(A^\dagger B + F_AKH^\dagger + F_AVE_H)C + D]^* - G\\
&= (F_AVE_HC + L)M(F_AVE_HC + L)^* - G\\
&= F_AVE_HCMC^*E_HV^*F_A + F_AVE_HCML^* + LMC^*E_HV^*F_A + LML^* - G,
\end{aligned} \tag{13}$$

where $L = A^\dagger BC + F_AKH^\dagger C + D$. Applying Lemma 2.5 to $f(X)$ in (13), we obtain

$$\max_{X\in\Omega} r[f(X)] = \min\{r(M_1),\ r(M_2),\ r(M_4)\}, \tag{14}$$

$$\min_{X\in\Omega} r[f(X)] = \max\{v_1,\ v_2,\ v_3,\ v_4\}, \tag{15}$$

$$\max_{X\in\Omega} i_\pm[f(X)] = \min\{i_\pm(M_2),\ i_\pm(M_4)\}, \tag{16}$$

$$\min_{X\in\Omega} i_\pm[f(X)] = \max\{r(M_1) - r(M_3) + i_\pm(M_2),\ r(M_1) - r(M_5) + i_\pm(M_4)\}, \tag{17}$$

where

$$\begin{aligned}
M_1 &= [LML^* - G,\ LMC^*E_H,\ F_A],\\
M_2 &= \begin{bmatrix} LML^* - G & F_A \\ F_A & 0 \end{bmatrix},\ M_3 = \begin{bmatrix} LML^* - G & LMC^*E_H & F_A \\ F_A & 0 & 0 \end{bmatrix},\\
M_4 &= \begin{bmatrix} LML^* - G & LMC^*E_H \\ E_HCML^* & E_HCMC^*E_H \end{bmatrix},\ M_5 = \begin{bmatrix} LML^* - G & LMC^*E_H & F_A \\ E_HCML^* & E_HCMC^*E_H & 0 \end{bmatrix},
\end{aligned}$$

and

$$v_1 = 2r(M_1) + r(M_2) - 2r(M_3), \tag{18}$$

$$v_2 = 2r(M_1) + r(M_4) - 2r(M_5), \tag{19}$$

$$v_3 = 2r(M_1) - r(M_3) - r(M_5) + i_+(M_2) + i_-(M_4), \tag{20}$$

$$v_4 = 2r(M_1) - r(M_3) - r(M_5) + i_-(M_2) + i_+(M_4). \tag{21}$$

Applying Lemmas 2.1,2.3 and 2.4, and simplifying by $AA^{\dagger}B = B, KH^{\dagger}H = K$ and $AK = BH$, congruence matrix operations and elementary matrix operations, we can find that

$$\begin{aligned}
r(M_1) &= r(F_A) + r[A^{\dagger}A(LML^* - G), A^{\dagger}ALMC^*E_H] \\
&= n - r(A) + r[ALML^* - AG, ALMC^*E_H] \\
&= n - r(A) - r(H) + r\begin{bmatrix} ALML^* - AG & ALMC^* \\ 0 & H^* \end{bmatrix} \\
&= n - r(A) - r(H) + r\begin{bmatrix} (AD + BC)M(A^{\dagger}BC + F_AKH^{\dagger}C + D)^* - AG & (AD + BC)MC^* \\ 0 & H^* \end{bmatrix} \\
&= n - r(A) - r(H) + r\begin{bmatrix} (AD + BC)MD^* - AG & (AD + BC)MC^* \\ -K^* & H^* \end{bmatrix} \\
&= n + r(N_1) - r(A) - r(H),
\end{aligned} \tag{22}$$

$$\begin{aligned}
i_{\pm}(M_2) &= r(F_A) + i_{\pm}[A^{\dagger}A(LML^* - G)A^{\dagger}A] \\
&= n - r(A) + i_{\pm}[ALML^*A^* - AGA^*] \\
&= n - r(A) + i_{\pm}[(AD + BC)M(AD + BC)^* - AGA^*],
\end{aligned} \tag{23}$$

$$r(M_2) = 2n + r[(AD + BC)M(AD + BC)^* - AGA^*] - 2r(A), \tag{24}$$

$$\begin{aligned}
r(M_3) &= 2r(F_A) + r[ALML^*A^* - AGA^*,\ ALMC^*E_H] \\
&= 2n - 2r(A) - r(H) + r\begin{bmatrix} (AD + BC)M(AD + BC)^* - AGA^* & (AD + BC)MC^* \\ 0 & H^* \end{bmatrix} \\
&= 2n - 2r(A) - r(H) + r\begin{bmatrix} (AD + BC)MC^*B^* + (AD + BC)MD^*A^* - AGA^* & (AD + BC)MC^* \\ 0 & H^* \end{bmatrix} \\
&= 2n - 2r(A) - r(H) + r\begin{bmatrix} (AD + BC)MD^*A^* - AGA^* & (AD + BC)MC^* \\ -K^*A^* & H^* \end{bmatrix} \\
&= 2n + r(N_2) - 2r(A) - r(H),
\end{aligned} \tag{25}$$

$$\begin{aligned}
i_{\pm}(M_4) &= i_{\pm}\begin{bmatrix} LML^* - G & LMC^* & 0 \\ CML^* & CMC^* & H \\ 0 & H^* & 0 \end{bmatrix} - r(H) \\
&= i_{\pm}\begin{bmatrix} (A^{\dagger}BC + F_AKH^{\dagger}C + D)M(A^{\dagger}BC + F_AKH^{\dagger}C + D)^* - G & (A^{\dagger}BC + F_AKH^{\dagger}C + D)MC^* & 0 \\ CM(A^{\dagger}BC + F_AKH^{\dagger}C + D)^* & CMC^* & H \\ 0 & H^* & 0 \end{bmatrix} \\
&\quad - r(H) \\
&= i_{\pm}\begin{bmatrix} DMD^* - G & DMC^* & -K \\ CMD^* & CMC^* & H \\ -K^* & H & 0 \end{bmatrix} - r(H) \\
&= i_{\pm}(N_3) - r(H),
\end{aligned} \tag{26}$$

$$r(M_4) = r(N_3) - 2r(H), \tag{27}$$

$$\begin{aligned}
r(M_5) &= r(F_A) + r\begin{bmatrix} ALML^* - AG & ALMC^*E_H \\ E_HCML^* & E_HCMC^*E_H \end{bmatrix} \\
&= n - r(A) - 2r(H) + r\begin{bmatrix} (AD + BC)M(A^{\dagger}BC + F_AKH^{\dagger}C + D)^* - AG & (AD + BC)MC^* & 0 \\ CM(A^{\dagger}BC + F_AKH^{\dagger}C + D)^* & CMC^* & H \\ 0 & H^* & 0 \end{bmatrix} \\
&= n - r(A) - 2r(H) + r\begin{bmatrix} ADM(A^{\dagger}BC + F_AKH^{\dagger}C + D)^* - AG & ADMC^* & -BH \\ CM(A^{\dagger}BC + F_AKH^{\dagger}C + D)^* & CMC^* & H \\ 0 & H^* & 0 \end{bmatrix}
\end{aligned} \tag{28}$$

$$= n - r(A) - 2r(H) + r\begin{bmatrix} ADMD^* - AG & ADMC^* & -AK \\ CMD^* & CMC^* & H \\ -K^* & H^* & 0 \end{bmatrix}$$
$$= n + r(N_4) - r(A) - 2r(H).$$

Substituting (22)–(28) into (14)–(21), and then we obtain (9)–(12). □

Using Lemma 2.2 to Theorem 3.1, we obtain the following consequences.

Corollary 3.2. *Let $f(X)$ and v_1, v_2, v_3, v_4 be as given by* (4) *and* (10)*, respectively, and let N_1, N_2, N_3 and N_4 be the matrices of* (8)*. Then, the following results hold.*
(a) *There exists an $X \in \Omega$ such that $f(X)$ is nonsingular iff*

$$r(N_1) \geqslant r(A) + r(H), \;\; r[(AD+BC)M(AD+BC)^* - AGA^*] \geqslant 2r(A) - n \;\; and \;\; r(N_3) \geqslant n + 2r(H).$$

(b) *$f(X)$ is nonsingular for all $X \in \Omega$ iff one of $v_i = n, i = 1, \dots, 4$ holds.*
(c) *There exists an $X \in \Omega$ such that $f(X) = 0$ iff*

$$(AD+BC)M(AD+BC)^* = AGA^*, \;\; \mathscr{R}\begin{bmatrix} (AD+BC)MD^* - AG \\ -K^* \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} (AD+BC)MC^* \\ H^* \end{bmatrix},$$
$$2r\begin{bmatrix} (AD+BC)MC^* \\ H^* \end{bmatrix} + r(N_3) - 2r(N_4) \leqslant 0, \;\; r\begin{bmatrix} (AD+BC)MC^* \\ H^* \end{bmatrix} - r(N_4) + i_+(N_3) \leqslant 0,$$
$$r\begin{bmatrix} (AD+BC)MC^* \\ H^* \end{bmatrix} - r(N_4) + i_-(N_3) \leqslant 0.$$

(d) *All $X \in \Omega$ satisfy $f(X) = 0$ iff*

$$r(A) = n \;\; and \;\; (AD+BC)M(AD+BC)^* = AGA^*, \;\; or \;\; M = G = 0.$$

(e) *There exists an $X \in \Omega$ such that $f(X) > 0$ iff*

$$i_+[(AD+BC)M(AD+BC)^* - AGA^*] \geqslant r(A) \;\; and \;\; i_+(N_3) \geqslant n + r(H).$$

(f) *All $X \in \Omega$ satisfy $f(X) > 0$ iff*

$$r(N_1) + i_+[(AD+BC)M(AD+BC)^* - AGA^*] = n + r(N_2) \;\; or \;\; r(N_1) + i_+(N_3) = n + r(N_4).$$

(g) *There exists an $X \in \Omega$ such that $f(X) < 0$ iff*

$$i_-[(AD+BC)M(AD+BC)^* - AGA^*] \geqslant r(A) \;\; and \;\; i_-(N_3) \geqslant n + r(H).$$

(h) *All $X \in \Omega$ satisfy $f(X) < 0$ iff*

$$r(N_1) + i_-[(AD+BC)M(AD+BC)^* - AGA^*] = n + r(N_2) \;\; or \;\; r(N_1) + i_-(N_3) = n + r(N_4).$$

(i) *There exists an $X \in \Omega$ such that $f(X) \geqslant 0$ iff*

$$(AD+BC)M(AD+BC)^* \geqslant AGA^* \;\; and \;\; r(N_2) = r(N_1) \leqslant r(N_4) - i_-(N_3).$$

(j) *All $X \in \Omega$ satisfy $f(X) \geqslant 0$ iff*

$$r(A) = n \;\; and \;\; (AD+BC)M(AD+BC)^* \geqslant AGA^*, \;\; or \;\; r(H) = i_-(N_3).$$

(k) *There exists an $X \in \Omega$ such that $f(X) \leqslant 0$ iff*

$$(AD+BC)M(AD+BC)^* \leqslant AGA^* \;\; and \;\; r(N_2) = r(N_1) \leqslant r(N_4) - i_+(N_3).$$

(l) *All $X \in \Omega$ satisfy $f(X) \leqslant 0$ iff*

$$r(A) = n \;\; and \;\; (AD+BC)M(AD+BC)^* \leqslant AGA^*, \;\; or \;\; r(H) = i_+(N_3).$$

Corollary 3.3. *Let $f(X)$ and N_3 be as given by* (4) *and* (8)*, respectively, with $\mathscr{R}[LML^* - G,\ LMC^*E_H] \subseteq \mathscr{R}(F_A)$. Write $N_5 = \begin{bmatrix} CMC^* & CMD^* & H \\ H^* & -K^* & 0 \end{bmatrix}$. Then,*

$$\begin{aligned}
\max_{X\in\Omega} r[f(X)] &= \min\{n - r(A),\ r(N_3) - 2r(H)\}, \\
\min_{X\in\Omega} r[f(X)] &= \max\{0,\ 2r(H) + r(N_3) - 2r(N_5), \\
&\qquad r(H) - r(N_5) + i_+(N_3),\ r(H) - r(N_5) + i_-(N_3)\}, \\
\max_{X\in\Omega} i_\pm[f(X)] &= \min\{n - r(A),\ i_\pm(N_3) - r(H)\}, \\
\min_{X\in\Omega} i_\pm[f(X)] &= \max\{0,\ r(H) - r(N_5) + i_\pm(N_3)\}.
\end{aligned}$$

Hence, the following hold

(*a*) *There exists an $X \in \Omega$ such that $f(X)$ is nonsingular iff $A = 0$ and $r(N_3) \geqslant n + 2r(H)$.*

(*b*) *$f(X)$ is nonsingular for all $X \in \Omega$ iff $2r(H) + r(N_3) = n + 2r(N_5)$ or $r(H) + i_+(N_3) = n + r(N_5)$ or $r(H) + i_-(N_3) = n + r(N_5)$.*

(*c*) *There exists an $X \in \Omega$ such that $f(X) = 0$ iff $r(H) + i_+(N_3) \leqslant r(N_5)$ and $r(H) + i_-(N_3) \leqslant r(N_5)$.*

(*d*) *All $X \in \Omega$ satisfy $f(X) = 0$ iff $r(A) = n$ or $r(N_3) = 2r(H)$.*

(*e*) *There exists an $X \in \Omega$ such that $f(X) > 0$ ($f(X) < 0$) iff $A = 0$ and $i_+(N_3) \geqslant n + r(H)$ $\big(A = 0$ and $i_-(N_3) \geqslant n + r(H)\big)$.*

(*f*) *All $X \in \Omega$ satisfy $f(X) > 0$ ($f(X) < 0$) iff $r(H) + i_+(N_3) = n + r(N_5)$ $\big(r(H) + i_-(N_3) = n + r(N_5)\big)$.*

(*g*) *There exists an $X \in \Omega$ such that $f(X) \geqslant 0$ ($f(X) \leqslant 0$) iff $r(H) + i_-(N_3) \leqslant r(N_5)$ $\big(r(H) + i_+(N_3) \leqslant r(N_5)\big)$.*

(*h*) *All $X \in \Omega$ satisfy $f(X) \geqslant 0$ ($f(X) \leqslant 0$) iff $i_-(N_3) = r(H)$ or $r(A) = n$ $\big(i_+(N_3) = r(H)$ or $r(A) = n\big)$.*

Corollary 3.4. *Let $f(X)$ be as given by* (4) *with $C = I,\ D = 0,\ M > 0$ and $G > 0$. Also denote*

$$N_6 = \begin{bmatrix} A & B \\ K^*G^{-1} & H^*M^{-1} \end{bmatrix},\ N_7 = \begin{bmatrix} AGA^* & B \\ K^*A^* & H^*M^{-1} \end{bmatrix},\ N_8 = \begin{bmatrix} A & 0 \\ K^*G^{-1} & K^*G^{-1}K - H^*M^{-1}H \end{bmatrix}.$$

Then,

$$\begin{aligned}
\max_{X\in\Omega} r(XMX^* - G) &= \min\{m + n + r(K^*G^{-1}K - H^*M^{-1}H) - 2r(H), \\
&\qquad n + r(N_6) - r(A) - r(H),\ 2n + r(BMB^* - AGA^*) - 2r(A)\}, \\
\min_{X\in\Omega} r(XMX^* - G) &= \max\{\bar{v}_1,\ \bar{v}_2,\ \bar{v}_3,\ \bar{v}_4\},
\end{aligned}$$

where

$$\begin{aligned}
\bar{v}_1 &= 2r(N_6) + r(BMB^* - AGA^*) - 2r(N_7), \\
\bar{v}_2 &= n - m + 2r(N_6) + r(K^*G^{-1}K - H^*M^{-1}H) - 2r(N_8), \\
\bar{v}_3 &= n - m + 2r(N_6) - r(N_7) - r(N_8) + i_+(BMB^* - AGA^*) + i_-(K^*G^{-1}K - H^*M^{-1}H), \\
\bar{v}_4 &= 2r(N_6) - r(N_7) - r(N_8) + i_-(BMB^* - AGA^*) + i_+(K^*G^{-1}K - H^*M^{-1}H),
\end{aligned}$$

$$\max_{X\in\Omega} i_+(XMX^* - G) = \min\{n - r(A) + i_+(BMB^* - AGA^*),\ m - r(H) + i_+(K^*G^{-1}K - H^*M^{-1}H)\},$$
$$\max_{X\in\Omega} i_-(XMX^* - G) = \min\{n - r(A) + i_-(BMB^* - AGA^*),\ n - r(H) + i_-(K^*G^{-1}K - H^*M^{-1}H)\},$$
$$\min_{X\in\Omega} i_+(XMX^* - G) = \max\{r(N_6) - r(N_7) + i_+(BMB^* - AGA^*),\ r(N_6) - r(N_8) + i_+(K^*G^{-1}K - H^*M^{-1}H)\},$$
$$\min_{X\in\Omega} i_-(XMX^* - G) = \max\{r(N_6) - r(N_7) + i_-(BMB^* - AGA^*),\ n - m + r(N_6) - r(N_8) + i_-(K^*G^{-1}K - H^*M^{-1}H)\}.$$

Under the condition $\mathscr{R}[K^*G^{-1},\ H^*M^{-1}] \subseteq \mathscr{R}[A,\ B]$, the following equalities

$$\max_{X\in\Omega} r(XMX^* - G) = \min\{m + n + r(K^*G^{-1}K - H^*M^{-1}H) - 2r(H), n - r(H),\ 2n + r(BMB^* - AGA^*) - 2r(A)\},$$
$$\min_{X\in\Omega} r(XMX^* - G) = \max\{\tilde{v}_1,\ \tilde{v}_2,\ \tilde{v}_3,\ \tilde{v}_4\},$$

where

$$\tilde{v}_1 = 2r(A) + r(BMB^* - AGA^*) - 2r(N_7),\ \tilde{v}_2 = n - m - r(K^*G^{-1}K - H^*M^{-1}H),$$
$$\tilde{v}_3 = n - m + r(A) - r(N_7) + i_+(BMB^* - AGA^*) - i_+(K^*G^{-1}K - H^*M^{-1}H),$$
$$\tilde{v}_4 = r(A) - r(N_7) + i_-(BMB^* - AGA^*) - i_-(K^*G^{-1}K - H^*M^{-1}H),$$

$$\max_{X\in\Omega} i_+(XMX^* - G) = \min\{n - r(A) + i_+(BMB^* - AGA^*),\ m - r(H) + i_+(K^*G^{-1}K - H^*M^{-1}H)\},$$
$$\max_{X\in\Omega} i_-(XMX^* - G) = \min\{n - r(A) + i_-(BMB^* - AGA^*),\ n - r(H) + i_-(K^*G^{-1}K - H^*M^{-1}H)\},$$
$$\min_{X\in\Omega} i_+(XMX^* - G) = \max\{r(A) - r(N_7) + i_+(BMB^* - AGA^*),\ -i_-(K^*G^{-1}K - H^*M^{-1}H)\},$$
$$\min_{X\in\Omega} i_-(XMX^* - G) = \max\{r(A) - r(N_7) + i_-(BMB^* - AGA^*),\ n - m - i_+(K^*G^{-1}K - H^*M^{-1}H)\}$$

hold. In consequence,

(*a*) *There exists an $X \in \Omega$ such that $XMX^* - G$ is nonsingular iff $H = 0$, $r(BMB^* - AGA^*) \geqslant 2r(A) - n$ and $r(K^*G^{-1}K - H^*M^{-1}H) \geqslant 2r(H) - m$.*

(*b*) *$XMX^* - G$ is nonsingular for all $X \in \Omega$ iff $\tilde{v}_i = n, i = 1, \ldots, 4$ holds.*

(*c*) *There exists an $X \in \Omega$ such that $XMX^* = G$ iff $BMB^* = AGA^*$, $r(A) = r(B)$, $K^*G^{-1}K = H^*M^{-1}H$ and $n \leqslant m$.*

(*d*) *All $X \in \Omega$ satisfy $XMX^* = G$ iff $r(H) = n$, or $r(A) = n$ and $BMB^* = AGA^*$, or $2r(H) = m + n$ and $K^*G^{-1}K = H^*M^{-1}H$.*

(*e*) *There exists an $X \in \Omega$ such that $XMX^* > G$ iff $i_+(BMB^* - AGA^*) \geqslant r(A)$ and $m + i_+(K^*G^{-1}K - H^*M^{-1}H) \geqslant n + r(H)$.*

(*f*) *All $X \in \Omega$ satisfy $XMX^* > G$ iff $r(A) + i_+(BMB^* - AGA^*) = n + r(N_7)$.*

(*g*) *There exists an $X \in \Omega$ such that $XMX^* < G$ iff $i_-(BMB^* - AGA^*) \geqslant r(A)$ and $i_-(K^*G^{-1}K - H^*M^{-1}H) \geqslant r(H)$.*

(*h*) *All $X \in \Omega$ satisfy $XMX^* < G$ iff $r(A) + i_-(BMB^* - AGA^*) = n + r(N_7)$.*

(*i*) *There exists an $X \in \Omega$ such that $XMX^* \geqslant G$ iff $r(A) + i_-(BMB^* - AGA^*) \leqslant r(N_7)$ and $n - m \leqslant i_+(K^*G^{-1}K - H^*M^{-1}H)$.*

(*j*) *All $X \in \Omega$ satisfy $XMX^* \geqslant G$ iff $r(A) = n$ and $BMB^* \geqslant AGA^*$, or $r(H) = n$ and $K^*G^{-1}K \geqslant H^*M^{-1}H$.*

(*k*) *There exists an $X \in \Omega$ such that $XMX^* \leqslant G$ iff $r(A) + i_+(BMB^* - AGA^*) \leqslant r(N_7)$.*

(*l*) *All $X \in \Omega$ satisfy $XMX^* \leqslant G$ iff $BMB^* \leqslant AGA^*$ and $r(A) = n$, or $K^*G^{-1}K \leqslant H^*M^{-1}H$ and $r(H) = m$.*

Corollary 3.5. *Let $f(X)$ be as given by (4) with $C = I,\ D = 0,\ M = I_m$ and $G = I_n$. Also write*

$$\hat{N}_6 = \begin{bmatrix} A & B \\ K^* & H^* \end{bmatrix},\ \hat{N}_7 = \begin{bmatrix} AA^* & B \\ K^*A^* & H^* \end{bmatrix},\ \hat{N}_8 = \begin{bmatrix} A & 0 \\ K^* & K^*K - H^*H \end{bmatrix}.$$

Then,

$$\max_{X\in\Omega} r(XX^* - I_n) = \min\{m + n + r(K^*K - H^*H) - 2r(H),\ n + r(\hat{N}_6) - r(A) - r(H),\ 2n + r(BB^* - AA^*) - 2r(A)\},$$

$$\min_{X\in\Omega} r(XX^* - I_n) = \max\{\hat{v}_1,\ \hat{v}_2,\ \hat{v}_3,\ \hat{v}_4\},$$

where

$$\hat{v}_1 = 2r(\hat{N}_6) + r(BB^* - AA^*) - 2r(\hat{N}_7),\ \ \hat{v}_2 = n - m + 2r(\hat{N}_6) + r(K^*K - H^*H) - 2r(\hat{N}_8),$$
$$\hat{v}_3 = n - m + 2r(\hat{N}_6) - r(\hat{N}_7) - r(\hat{N}_8) + i_+(BB^* - AA^*) + i_-(K^*K - H^*H),$$
$$\hat{v}_4 = 2r(\hat{N}_6) - r(\hat{N}_7) - r(\hat{N}_8) + i_-(BB^* - AA^*) + i_+(K^*K - H^*H),$$

$$\max_{X\in\Omega} i_+(XX^* - I_n) = \min\{n - r(A) + i_+(BB^* - AA^*),\ m - r(H) + i_+(K^*K - H^*H)\},$$
$$\max_{X\in\Omega} i_-(XX^* - I_n) = \min\{n - r(A) + i_-(BB^* - AA^*),\ n - r(H) + i_-(K^*K - H^*H)\},$$
$$\min_{X\in\Omega} i_+(XX^* - I_n) = \max\{r(\hat{N}_6) - r(\hat{N}_7) + i_+(BB^* - AA^*),\ r(\hat{N}_6) - r(\hat{N}_8) + i_+(K^*K - H^*H)\},$$
$$\min_{X\in\Omega} i_-(XX^* - I_n) = \max\{r(\hat{N}_6) - r(\hat{N}_7) + i_-(BB^* - AA^*),\ n - m + r(\hat{N}_6) - r(\hat{N}_8) + i_-(K^*K - H^*H)\}.$$

Hence, the following hold.

(*a*) *There exists an $X \in \Omega$ such that $XX^* - I_n$ is nonsingular iff $r(K^*K - H^*H) \geqslant 2r(H) - m,\ r(BB^* - AA^*) \geqslant 2r(A) - n$ and $r(\hat{N}_6) \geqslant r(A) + r(H)$.*

(*b*) *$XX^* - I_n$ is nonsingular for all $X \in \Omega$ iff one of $\hat{v}_i = n, i = 1, \ldots, 4$ holds.*

(*c*) *There exists an $X \in \Omega$ such that $XX^* = I_n$ iff*

$$BB^* = AA^*,\ K^*K = H^*H,\ \mathscr{R}\begin{bmatrix} A \\ K^* \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} B \\ H^* \end{bmatrix},\ \ n - m + 2r\begin{bmatrix} B \\ H^* \end{bmatrix} - 2r(\hat{N}_8) \leqslant 0,$$
$$n - m + r\begin{bmatrix} B \\ H^* \end{bmatrix} - r(\hat{N}_8) \leqslant 0,\ \ r\begin{bmatrix} B \\ H^* \end{bmatrix} - r(\hat{N}_8) \leqslant 0.$$

(*d*) *All $X \in \Omega$ satisfy $XX^* = I_n$ iff $n + r(\hat{N}_6) = r(A) + r(H)$, or $r(A) = n$ and $BB^* = AA^*$, or $2r(H) = m + n$ and $K^*K = H^*H$.*

(*e*) *There exists an $X \in \Omega$ such that $XX^* > I_n$ iff $i_+(BB^* - AA^*) \geqslant r(A)$ and $m + i_+(K^*K - H^*H) \geqslant n + r(H)$.*

(*f*) *All $X \in \Omega$ satisfy $XX^* > I_n$ if iff $r(\hat{N}_6) + i_+(BB^* - AA^*) = n + r(\hat{N}_7)$ or $r(\hat{N}_6) + i_+(K^*K - H^*H) = n + r(\hat{N}_8)$.*

(*g*) *There exists an $X \in \Omega$ such that $XX^* < I_n$ iff $i_-(BB^* - AA^*) \geqslant r(A)$ and $i_-(K^*K - H^*H) \geqslant r(H)$.*

(*h*) *All $X \in \Omega$ satisfy $XX^* < I_n$ iff $r(\hat{N}_6) + i_-(BB^* - AA^*) = n + r(\hat{N}_7)$ or $r(\hat{N}_6) + i_-(K^*K - H^*H) = m + r(\hat{N}_8)$.*

(*i*) *There exists an $X \in \Omega$ such that $XX^* \geqslant I_n$ iff $BB^* \geqslant AA^*,\ K^*K \geqslant H^*H$ and $r(\hat{N}_7) = r(\hat{N}_6) \leqslant m - n + r(\hat{N}_8)$.*

(*j*) *All $X \in \Omega$ satisfy $XX^* \geqslant I_n$ iff $r(A) = n$ and $BB^* \geqslant AA^*$, or $r(H) = n$ and $K^*K \geqslant H^*H$.*

(*k*) *There exists an $X \in \Omega$ such that $XX^* \leqslant I_n$ iff $BB^* \leqslant AA^*,\ K^*K \leqslant H^*H$ and $r(\hat{N}_7) = r(\hat{N}_6) \leqslant r(\hat{N}_8)$.*

(l) *All $X \in \Omega$ satisfy $XX^* \leqslant I_n$ iff $BB^* \leqslant AA^*$ and $r(A) = n$, or $K^*K \leqslant H^*H$ and $r(H) = m$.*

(m) *Under the condition $\mathscr{R}[K^*,\ H^*] \subseteq \mathscr{R}[A,\ B]$, the following equalities*

$$\max_{X\in\Omega} r(XX^* - I_n) = \min\{m + n + r(K^*K - H^*H) - 2r(H),\ n - r(H),\ 2n + r(BB^* - AA^*) - 2r(A)\},$$
$$\min_{X\in\Omega} r(XX^* - I_n) = \max\left\{\breve{v}_1,\ \breve{v}_2,\ \breve{v}_3,\ \breve{v}_4\right\},$$

where

$$\breve{v}_1 = 2r(A) + r(BB^* - AA^*) - 2r(\hat{N}_7),\ \breve{v}_2 = n - m - r(K^*K - H^*H),$$
$$\breve{v}_3 = n - m + r(A) - r(\hat{N}_7) + i_+(BB^* - AA^*) - i_+(K^*K - H^*H),$$
$$\breve{v}_4 = r(A) - r(\hat{N}_7) + i_-(BB^* - AA^*) - i_-(K^*K - H^*H),$$

$$\max_{X\in\Omega} i_+(XX^* - I_n) = \min\left\{n - r(A) + i_+(BB^* - AA^*),\ m - r(H) + i_+(K^*K - H^*H)\right\},$$
$$\max_{X\in\Omega} i_-(XX^* - I_n) = \min\left\{n - r(A) + i_-(BB^* - AA^*),\ n - r(H) + i_-(K^*K - H^*H)\right\},$$
$$\min_{X\in\Omega} i_+(XX^* - I_n) = \max\{r(A) - r(\hat{N}_7) + i_+(BB^* - AA^*),\ -i_-(K^*K - H^*H)\},$$
$$\min_{X\in\Omega} i_-(XX^* - I_n) = \max\{r(A) - r(\hat{N}_7) + i_-(BB^* - AA^*),\ n - m - i_+(K^*K - H^*H)\}$$

hold. In consequence,

(i) *There exists an $X \in \Omega$ such that $XX^* - I_n$ is nonsingular iff $H = 0$, $r(BB^* - AA^*) \geqslant 2r(A) - n$ and $r(K^*K - H^*H) \geqslant 2r(H) - m$.*

(ii) *$XX^* - I_n$ is nonsingular for all $X \in \Omega$ iff one of $\breve{v}_i = n, i = 1, \ldots, 4$ holds.*

(iii) *There exists an $X \in \Omega$ such that $XX^* = I_n$ iff $BB^* = AA^*$, $r(A) = r(B)$, $K^*K = H^*H$ and $n \leqslant m$.*

(iv) *All $X \in \Omega$ satisfy $XX^* = I_n$ iff $r(H) = n$, or $r(A) = n$ and $BB^* = AA^*$, or $2r(H) = m + n$ and $K^*K = H^*H$.*

(v) *There exists an $X \in \Omega$ such that $XX^* > I_n$ iff $i_+(BB^* - AA^*) \geqslant r(A)$ and $m + i_+(K^*K - H^*H) \geqslant n + r(H)$.*

(vi) *All $X \in \Omega$ satisfy $XX^* > I_n$ iff $r(A) + i_+(BB^* - AA^*) = n + r(\hat{N}_7)$.*

(vii) *There exists an $X \in \Omega$ such that $XX^* < I_n$ iff $i_-(BB^* - AA^*) \geqslant r(A)$ and $i_-(K^*K - H^*H) \geqslant r(H)$.*

($viii$) *All $X \in \Omega$ satisfy $XX^* < I_n$ iff $r(A) + i_-(BB^* - AA^*) = n + r(\hat{N}_7)$.*

(ix) *There exists an $X \in \Omega$ such that $XX^* \geqslant I_n$ iff $r(A) + i_-(BB^* - AA^*) \leqslant r(\hat{N}_7)$ and $n - m \leqslant i_+(K^*K - H^*H)$.*

(x) *All $X \in \Omega$ satisfy $XX^* \geqslant I_n$ iff $r(A) = n$ and $BB^* \geqslant AA^*$, or $r(H) = n$ and $K^*K \geqslant H^*H$.*

(xi) *There exists an $X \in \Omega$ such that $XX^* \leqslant I_n$ iff $r(A) + i_+(BB^* - AA^*) \leqslant r(\hat{N}_7)$.*

(xii) *All $X \in \Omega$ satisfy $XX^* \leqslant I_n$ iff $BB^* \leqslant AA^*$ and $r(A) = n$, or $K^*K \leqslant H^*H$ and $r(H) = m$.*

4. The solution of Problem 3

According to Problem 3, we can derive the following theorem.

Theorem 4.1. *Let $f(X)$ be as given by* (4). *Then, the following results hold.*
(a) *There exists an $\hat{X} \in \Omega$ such that $f(X) \leqslant f(\hat{X})$ for all $X \in \Omega$ holds iff*

$$E_H CMC^* E_H \leqslant 0,\ \mathscr{R}\begin{bmatrix} K^* \\ -CMD^* \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} H^* & 0 \\ CMC^* & H \end{bmatrix} \text{ and } (AD+BC)MC^* = (AD+BC)MC^*HH^{\dagger}.$$

In this case, the global maximizer $\hat{X}$ is determined by the consistent matrix equations

$$\begin{cases} A\hat{X} = B \\ \hat{X}[H,\ CMC^*E_H] = [K, -DMC^*E_H], \end{cases}$$

and the general expression of the global maximizer is given by

$$\arg\max_{\geqslant}\{f(X) \mid X \in \Omega\} = A^{\dagger}B + F_A[K, -DMC^*E_H][H, CMC^*E_H]^{\dagger} + F_A V\Big(I_P - [H, CMC^*E_H][H, CMC^*E_H]^{\dagger}\Big), \quad (29)$$

where $V \in \mathbb{C}^{n\times p}$ is an arbitrary matrix.
(b) *There exists an $\tilde{X} \in \Omega$ such that $f(X) \geqslant f(\tilde{X})$ for all $X \in \Omega$ holds iff*

$$E_H CMC^* E_H \geqslant 0,\ \mathscr{R}\begin{bmatrix} K^* \\ -CMD^* \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} H^* & 0 \\ CMC^* & H \end{bmatrix} \text{ and } (AD+BC)MC^* = (AD+BC)MC^*HH^{\dagger}.$$

In this case, the global minimizer $\tilde{X}$ is determined by the consistent matrix equations

$$\begin{cases} A\tilde{X} = B \\ \tilde{X}[H,\ CMC^*E_H] = [K, -DMC^*E_H], \end{cases}$$

and the general expression of the global minimizer is given by

$$\arg\max_{\leqslant}\{f(X) \mid X \in \Omega\} = A^{\dagger}B + F_A[K, -DMC^*E_H][H, CMC^*E_H]^{\dagger} + F_A V\Big(I_P - [H, CMC^*E_H][H, CMC^*E_H]^{\dagger}\Big),$$

where $V \in \mathbb{C}^{n\times p}$ is an arbitrary matrix.

Proof. Let

$$\Psi(X) = f(X) - f(\hat{X}) = (XC+D)M(XC+D)^* - (\hat{X}C+D)M(\hat{X}C+D)^*,$$
$$\Phi(X) = f(X) - f(\tilde{X}) = (XC+D)M(XC+D)^* - (\tilde{X}C+D)M(\tilde{X}C+D)^*.$$

Then, (5) are equivalent to finding $\hat{X},\ \tilde{X} \in \Omega$ such that

$$\Psi(X) \leqslant 0 \quad \text{holds for all} \quad X \in \Omega, \quad (30)$$
$$\Phi(X) \geqslant 0 \quad \text{holds for all} \quad X \in \Omega. \quad (31)$$

From Corollary 3.2(l), (30) are equivalent to finding $\hat{X} \in \Omega$ such that

$$i_+\begin{bmatrix} DMD^* - (\hat{X}C+D)M(\hat{X}C+D)^* & DMC^* & -K \\ CMD^* & CMC^* & H \\ -K^* & H^* & 0 \end{bmatrix} = r(H). \quad (32)$$

From Corollary 3.2(j), (31) are equivalent to finding $\tilde{X} \in \Omega$ such that

$$i_-\begin{bmatrix} DMD^* - (\tilde{X}C+D)M(\tilde{X}C+D)^* & DMC^* & -K \\ CMD^* & CMC^* & H \\ -K^* & H^* & 0 \end{bmatrix} = r(H). \quad (33)$$

It follows from (32), (33) and (6) that

$$\begin{aligned}
& i_+\begin{bmatrix} DMD^* - (\hat{X}C+D)M(\hat{X}C+D)^* & DMC^* & -K \\ CMD^* & CMC^* & H \\ -K^* & H^* & 0 \end{bmatrix} \\
=& i_+\begin{bmatrix} 0 & DMC^* + \hat{X}CMC^* & 0 \\ CMD^* + CMC^*\hat{X}^* & CMC^* & H \\ 0 & H^* & 0 \end{bmatrix} \\
=& r(H) + i_+\begin{bmatrix} 0 & (DMC^* + \hat{X}CMC^*)E_H \\ E_H(CMD^* + CMC^*\hat{X}^*) & E_HCMC^*E_H \end{bmatrix} \geqslant r(H),
\end{aligned} \tag{34}$$

$$\begin{aligned}
& i_-\begin{bmatrix} DMD^* - (\tilde{X}C+D)M(\tilde{X}C+D)^* & DMC^* & -K \\ CMD^* & CMC^* & H \\ -K^* & H^* & 0 \end{bmatrix} \\
=& i_-\begin{bmatrix} 0 & DMC^* + \tilde{X}CMC^* & 0 \\ CMD^* + CMC^*\tilde{X}^* & CMC^* & H \\ 0 & H^* & 0 \end{bmatrix} \\
=& r(H) + i_-\begin{bmatrix} 0 & (DMC^* + \tilde{X}CMC^*)E_H \\ E_H(CMD^* + CMC^*\tilde{X}^*) & E_HCMC^*E_H \end{bmatrix} \geqslant r(H).
\end{aligned} \tag{35}$$

Combining (32) and (34) leads to

$$i_+\begin{bmatrix} 0 & (DMC^* + \hat{X}CMC^*)E_H \\ E_H(CMD^* + CMC^*\hat{X}^*) & E_HCMC^*E_H \end{bmatrix} = 0. \tag{36}$$

Combining (33) and (35) leads to

$$i_-\begin{bmatrix} 0 & (DMC^* + \tilde{X}CMC^*)E_H \\ E_H(CMD^* + CMC^*\tilde{X}^*) & E_HCMC^*E_H \end{bmatrix} = 0. \tag{37}$$

Applying (7) to (36) results in

$$(DMC^* + \hat{X}CMC^*)E_H = 0 \ \text{ and } \ E_HCMC^*E_H \leqslant 0. \tag{38}$$

Applying (7) to (37) results in

$$(DMC^* + \tilde{X}CMC^*)E_H = 0 \ \text{ and } \ E_HCMC^*E_H \geqslant 0. \tag{39}$$

Thus, (32) and (33) are equivalent to

$$E_HCMC^*E_H \leqslant 0 \ \text{ and } \ \begin{cases} A\hat{X} = B \\ \hat{X}[H,\ CMC^*E_H] = [K, -DMC^*E_H], \end{cases} \tag{40}$$

$$E_HCMC^*E_H \geqslant 0 \ \text{ and } \ \begin{cases} A\tilde{X} = B \\ \tilde{X}[H,\ CMC^*E_H] = [K, -DMC^*E_H], \end{cases} \tag{41}$$

Using the relations of (2), the matrix equations in (40) have a solution for $\hat{X}$ iff

$$r\begin{bmatrix} H^* & K^* \\ E_HCMC^* & -E_HCMD^* \end{bmatrix} = r\begin{bmatrix} H^* \\ E_HCMC^* \end{bmatrix}, \text{ i.e., } r\begin{bmatrix} H^* & K^* & 0 \\ CMC^* & -CMD^* & H \end{bmatrix} = r\begin{bmatrix} H^* & 0 \\ CMC^* & H \end{bmatrix}.$$

In which case, the general solution of (40) can be expressed as (29). Result (b) can be shown similarly. □

5. Conclusions

In this paper, by applying the generalized inverses of matrices and elementary matrix operations, we established some explicit formulas for calculating the global maximum and minimum ranks and inertias of QHMF.(4) with the constraint of Eq.(1). As applications of these inertia and rank formulas, we derived a variety of necessary and sufficient conditions for some quadratic matrix equations and inequalities to hold. In particular, we obtained the analytical solutions to two classical optimization problems on QHMF.(4) subject to $X \in \Omega$ in the Löwner partial ordering.

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