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A note on additive formulas for the Drazin inverse of matrices and block representations

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Abstract. In this paper, we investigate the additive properties of the Drazin inverse for complex matrices. We derive new additive formulas for the Drazin inverse, which generalize some previous results on the subject. Furthermore, we give a new representation for the Drazin inverse of a block matrix, which extends some known representations.

1. Introduction

Throughout this paper, we use $\mathbb{C}^{n\times n}$ to denote the set of all $n \times n$ complex matrices. For $A \in \mathbb{C}^{n\times n}$, by $\mathcal{R}(A)$, $\mathcal{N}(A)$ and $\text{rank}(A)$, we denote the range, the null space and the rank of a matrix A, respectively. The index of a matrix A, denoted by $\text{ind}(A)$, is the smallest nonnegative integer k, such that $\text{rank}(A^{k+1}) = \text{rank}(A^k)$. For every matrix $A \in \mathbb{C}^{n\times n}$, such that $\text{ind}(A) = k$, there exists the unique matrix $A^d \in \mathbb{C}^{n\times n}$, which satisfies following relations:

$$A^{k+1}A^d = A^k,\ A^dAA^d = A^d,\ AA^d = A^dA.$$

The matrix A^d is called the Drazin inverse of A [1]. By $A^\pi = I - AA^d$, we denote the projection on $\mathcal{N}(A^k)$ along $\mathcal{R}(A^k)$. Also, we suppose that $A^0 = I$, where I is the identity matrix of an appropriate size. Moreover, if the lower limit of a sum is greater than its upper limit, we define the sum to be 0.

Suppose $P, Q \in \mathbb{C}^{n\times n}$. In 1958, Drazin [2] investigated additive properties of the Drazin inverse (in the concept of associative rings and semigroups) and proved that $(P+Q)^d = P^d + Q^d$ holds when $PQ = QP = 0$. In 2001, Hartwig, Wang and Wei reopened this problem and offered the formula for $(P+Q)^d$, which is valid when $PQ = 0$ [3]. Since then, this topic attracts a great attention and many authors have studied this problem, which still remains open (we refer the reader to see the review [4] on this subject). Some of the conditions, under which is obtained a formula for $(P+Q)^d$ are as follows:

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(i) $P^2Q = 0$ and $Q^2 = 0$ [5, Theorem 2.2];

(ii) $P^2Q = 0$ and $Q^2P = 0$ [6, Theorem 3.1];

(iii) $P^2QP = 0$, $P^3Q = 0$ and $Q^2 = 0$ [7, Theorem 3.3]; (a)

(iv) $PQP^2 = 0$, $PQ^2P = 0$, $PQ^3 = 0$, $QP^3 = 0$, $QPQ^2 = 0$ and $P^2Q^2 = 0$ [8, Theorem 3.2];

(v) $PQP^2 = 0$, $PQ^2 = 0$ and $QP^3 = 0$ [9, Theorem 3.1].

In this paper, we give the explicit formula for $(P + Q)^d$ under conditions $PQP^2 = 0$, $PQ^2P = 0$, $PQ^3 = 0$ and $QP^3 = 0$ (Theorem 2.1). Also, we derive its symmetrical formulation, that is the explicit formula for $(P + Q)^d$ under conditions $P^2QP = 0$, $PQ^2P = 0$, $Q^3P = 0$ and $P^3Q = 0$ (Theorem 2.3). Note that these conditions generalizes the conditions (i)–(v) from the list (a).

Consider the following 2×2 complex block matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{1.1}$$

where A and D are square matrices, not necessarily of the same size. The problem of finding the Drazin inverse of M was opened in 1979, by Campbell and Meyer [10]. Since then, many authors have studied this problem and offered some formulas for M^d, when blocks of matrix M satisfy some certain conditions (see [4] for a development of this subject). In some papers on this topic, authors considered the block matrix of the form (1.1), for which generalized Schur complement $S = D - CA^dB$ is equal to zero. Some of the conditions, under which is derived the formula for M^d, are given in the following list:

(i) $CA^\pi = 0$, $A^\pi B = 0$ and $S = 0$ [11];

(ii) $ABC = 0$ and $S = 0$ [5, Theorem 3.6];

(iii) $ABCA^\pi = 0$, $A^\pi ABC = 0$ and $S = 0$ [6, Theorem 4.1]; (b)

(iv) $A^2BCA^\pi A = 0$, $A^2BCA^\pi B = 0$, $A^\pi ABC = 0$ and $S = 0$ [8, Theorem 4.1];

(v) $A^dBC = 0$, $CAA^\pi BC = 0$, $A^2A^\pi B = 0$ and $S = 0$ [9, Theorem 3.4].

In this paper, in Theorem 2.4, as an application of our new additive result, we derive the formula for M^d, when conditions $A^dBCA^\pi A = 0$, $A^dBCA^\pi B = 0$, $A^\pi A^2BC = 0$, $CA^\pi ABC = 0$ and $S = 0$ are satisfied. We remark that these conditions are weaker than conditions (i)–(v) from the list (b).

Before we give our results, we state the following auxiliary lemmas.

Lemma 1.1. [1] *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times m}$. *Then* $(AB)^d = A((BA)^2)^dB$.

Lemma 1.2. [3, Theorem 2.1] *Let* $P, Q \in \mathbb{C}^{n\times n}$ *be such that* $\operatorname{ind}(P) = r$ *and* $\operatorname{ind}(Q) = s$. *If* $PQ = 0$ *then*

$$(P + Q)^d = \sum_{i=0}^{s-1} Q^\pi Q^i (P^d)^{i+1} + \sum_{i=0}^{r-1} (Q^d)^{i+1} P^i P^\pi.$$

Lemma 1.3. [12, Theorem 2.1] *Let* $P, Q \in \mathbb{C}^{n\times n}$ *be such that* $\operatorname{ind}(P) = r$, $\operatorname{ind}(Q) = s$. *If* $PQP = 0$ *and* $PQ^2 = 0$, *then*

$$(P + Q)^d = Y_1 + Y_2 + \left(Y_1(P^d)^2 + (Q^d)^2Y_2 - Q^d(P^d)^2 - (Q^d)^2P^d\right)PQ,$$

where Y_1 *and* Y_2 *are defined as follows*

$$Y_1 = \sum_{i=0}^{s-1} Q^\pi Q^i (P^d)^{i+1}, \quad Y_2 = \sum_{i=0}^{r-1} (Q^d)^{i+1} P^i P^\pi.$$

Lemma 1.4. [13, 14] *Let M_1 and M_2 be block matrices of a form:*

$$M_1 = \begin{bmatrix} A & 0 \\ C & B \end{bmatrix}, \quad M_2 = \begin{bmatrix} B & C \\ 0 & A \end{bmatrix},$$

where A and B are square matrices, with $\operatorname{ind}(A) = k$, $\operatorname{ind}(B) = l$. *Then* $\max\{k, l\} \le \operatorname{ind}(M_i) \le k + l$, *for* $i \in \{1, 2\}$, *and*

$$M_1^d = \begin{bmatrix} A^d & 0 \\ X & B^d \end{bmatrix}, \quad M_2^d = \begin{bmatrix} B^d & X \\ 0 & A^d \end{bmatrix},$$

where

$$X = X(B, C, A) = \sum_{i=0}^{l-1} (B^d)^{i+2} C A^i A^\pi + \sum_{i=0}^{k-1} B^\pi B^i C (A^d)^{i+2} - B^d C A^d.$$

Lemma 1.5. [15, Theorem 3.2] *Let M be matrix of a form* (1.1) *such that* $BCA = 0$, $ABD = 0$ *and* $CBD = 0$. *Then*

$$M^d = \begin{bmatrix} A\Omega + B(F_1 + F_2) & \begin{matrix} \Omega B + BD(F_1\Omega + (D^d)^2 F_2)B \\ +B(D^d)^2 - BD^d(CA + DC)\Omega^2 B \end{matrix} \\ C\Omega + D(F_1 + F_2) & D^d + (F_1 + F_2)B \end{bmatrix},$$

where

$$\Omega = (A^2 + BC)^d = \sum_{i=0}^{\nu_4 - 1} (A^d)^{2i+2} (BC)^i (BC)^\pi + \sum_{i=0}^{\nu_1 - 1} A^\pi A^{2i} ((BC)^d)^{i+1},$$

$$F_1 = \sum_{i=0}^{\nu_2 - 1} D^\pi D^{2i} (CA + DC) \Omega^{i+2},$$

$$F_2 = \sum_{i=0}^{\nu_3 - 1} (D^d)^{2i+4} (CA + DC)(A^2 + BC)^i (BC)^\pi - \sum_{i=0}^{\nu_3} (D^d)^{2i+2} (CA + DC) A^{2i} \Omega,$$

$\nu_1 = \operatorname{ind}(A^2)$, $\nu_2 = \operatorname{ind}(D^2)$, $\nu_3 = \operatorname{ind}(A^2 + BC)$ *and* $\nu_4 = \operatorname{ind}(BC)$.

Lemma 1.6. [11] *Let M be matrix of a form* (1.1), *such that* $S = 0$. *If* $A^\pi B = 0$ *and* $CA^\pi = 0$, *then*

$$M^d = \begin{bmatrix} I \\ CA^d \end{bmatrix} \left((AW)^d \right)^2 A \begin{bmatrix} I & A^d B \end{bmatrix},$$

where $W = AA^d + A^d BCA^d$.

2. Results

In 2017, Yang et al. offered a formula for $(P + Q)^d$, which is valid when conditions $PQP^2 = 0$, $PQ^2P = 0$, $PQ^3 = 0$, $QP^3 = 0$, $QPQ^2 = 0$ and $P^2Q^2 = 0$ hold [8, Theorem 3.2]. In the following theorem we prove that conditions $QPQ^2 = 0$ and $P^2Q^2 = 0$ from the previously mentioned result are superfluous for finding the explicit formula for $(P + Q)^d$. Namely, we derive the formula for $(P + Q)^d$, which is valid when conditions $PQP^2 = 0$, $PQ^2P = 0$, $PQ^3 = 0$ and $QP^3 = 0$ are satisfied.

Theorem 2.1. *Let $P, Q \in \mathbb{C}^{n\times n}$. If $PQP^2 = 0$, $PQ^2P = 0$, $PQ^3 = 0$ and $QP^3 = 0$, then*

$$\begin{aligned}(P+Q)^d =& \sum_{i=0}^{r-1}\left(P^\pi P^{2i+1} + Q^\pi Q^{2i+1}\right)\left(\left((PQ)^d\right)^{i+1} + \left((QP)^d\right)^{i+1}\left(I + (QP)^dQ^2\right)\right)\\ &+ \sum_{i=0}^{\lceil\frac{r_2-1}{2}\rceil-1} Q^\pi Q^{2i+1}\left(QP\left((PQ)^d\right)^{i+2} + P^2\left((QP)^d\right)^{i+2}\left(I+(QP)^dQ^2\right)\right)\\ &+ \sum_{i=0}^{s-1}\left((P^d)^{2i+1} + (Q^d)^{2i+1}\right)\left((PQ)^i(PQ)^\pi + (QP)^i(QP)^\pi\right)\\ &+ \sum_{i=0}^{s-1}(Q^d)^{2i+3}\left(QP(PQ)^i(PQ)^\pi + P^2(QP)^i(QP)^\pi\right)\\ &+ \sum_{i=0}^{s_2-1}\left((P^d)^{2i+3} + (Q^d)^{2i+5}P^2 + (Q^d)^{2i+3}\right)(QP)^i(QP)^\pi Q^2\\ &- P^d - 2Q^d - (Q^d)^2P - Q^dP^2(QP)^d - Q^dQP(PQ)^d - Q^dP(PQ)^dQ\\ &- P^d(QP)^dQ^2 - Q^d(QP)^dQ^2 - (Q^d)^3P^2(QP)^dQ^2,\end{aligned} \tag{2.1}$$

where $r_1 = \text{ind}(P)$, $r_2 = \text{ind}(Q)$, $s_1 = \text{ind}(PQ)$, $s_2 = \text{ind}(QP)$, $r = \max\left\{\left\lceil\frac{r_1-1}{2}\right\rceil, \left\lceil\frac{r_2-1}{2}\right\rceil\right\}$ *and* $s = \max\{s_1, s_2\}$.

Proof. Using Lemma 1.1, we have that

$$\begin{aligned}(P+Q)^d &= \left(\begin{bmatrix} P & Q\end{bmatrix}\begin{bmatrix} I \\ I\end{bmatrix}\right)^d = \begin{bmatrix} P & Q\end{bmatrix}\left(\left(\begin{bmatrix} I \\ I\end{bmatrix}\begin{bmatrix} P & Q\end{bmatrix}\right)^2\right)^d\begin{bmatrix} I \\ I\end{bmatrix}\\ &= \begin{bmatrix} P & Q\end{bmatrix}\begin{bmatrix} P^2+QP & Q^2+PQ \\ P^2+QP & Q^2+PQ\end{bmatrix}^d\begin{bmatrix} I \\ I\end{bmatrix}.\end{aligned}$$

If we denote by $E = \begin{bmatrix} QP & PQ \\ QP & PQ\end{bmatrix}$ and $F = \begin{bmatrix} P^2 & Q^2 \\ P^2 & Q^2\end{bmatrix}$, we have

$$(P+Q)^d = \begin{bmatrix} P & Q\end{bmatrix}(E+F)^d\begin{bmatrix} I \\ I\end{bmatrix}. \tag{2.2}$$

By the hypothesis of the theorem, we get that $EFE = 0$ and $EF^2 = 0$. Hence, matrices E and F satisfy the conditions of Lemma 1.3 and therefore

$$(E+F)^d = Z_1 + Z_2 + \left(Z_1(E^d)^2 + (F^d)^2Z_2 - F^d(E^d)^2 - (F^d)^2E^d\right)EF, \tag{2.3}$$

where

$$Z_1 = \sum_{i=0}^{t_2-1}F^\pi F^i(E^d)^{i+1}, \quad Z_2 = \sum_{i=0}^{t_1-1}(F^d)^{i+1}E^iE^\pi, \quad t_1 = \text{ind}(E) \quad \text{and} \quad t_2 = \text{ind}(F). \tag{2.4}$$

Thus, we should find the expressions of E^i, F^i, $(E^d)^i$ and $(F^d)^i$, for every $i \in \mathbb{N}$, and also E^π and F^π. Using the induction by i and by the hypothesis of the theorem, we get:

$$E^i = \begin{bmatrix} (QP)^i & (PQ+QP)(PQ)^{i-1} \\ (QP)^i & (PQ+QP)(PQ)^{i-1}\end{bmatrix}, \quad \text{for every } i \geq 2, \tag{2.5}$$

$$F^i = \begin{bmatrix} (P^2+Q^2)^{i-1}P^2 & (P^2+Q^2)^{i-1}Q^2 \\ (P^2+Q^2)^{i-1}P^2 & (P^2+Q^2)^{i-1}Q^2 \end{bmatrix}, \quad \text{for every } i \geq 1. \tag{2.6}$$

In order to find the expression for E^d, denote by $E_1 = \begin{bmatrix} 0 & PQ \\ 0 & PQ \end{bmatrix}$ and $E_2 = \begin{bmatrix} QP & 0 \\ QP & 0 \end{bmatrix}$. Obviously, $E = E_1 + E_2$. Moreover, since $PQ^2P = 0$, we have that $E_1E_2 = 0$. Thus, matrices E_1 and E_2 satisfy the conditions of Lemma 1.2 and after applying this lemma we get:

$$E^d = \sum_{i=0}^{m_2-1} E_2^\pi E_2^i (E_1^d)^{i+1} + \sum_{i=0}^{m_1-1} (E_2^d)^{i+1} E_1^i E_1^\pi, \tag{2.7}$$

where $m_1 = \text{ind}(E_1)$ and $m_2 = \text{ind}(E_2)$. One can easily check that:

$$E_1^i = \begin{bmatrix} 0 & (PQ)^i \\ 0 & (PQ)^i \end{bmatrix} \quad \text{and} \quad E_2^i = \begin{bmatrix} (QP)^i & 0 \\ (QP)^i & 0 \end{bmatrix}, \quad \text{for every } i \geq 1.$$

Also, using Lemma 1.4, we get:

$$(E_1^d)^i = \begin{bmatrix} 0 & ((PQ)^d)^i \\ 0 & ((PQ)^d)^i \end{bmatrix} \quad \text{and} \quad (E_2^d)^i = \begin{bmatrix} ((QP)^d)^i & 0 \\ ((QP)^d)^i & 0 \end{bmatrix}, \quad \text{for every } i \geq 1.$$

Moreover, we obtain:

$$E_1^\pi = \begin{bmatrix} I & -PQ(PQ)^d \\ 0 & (PQ)^\pi \end{bmatrix} \quad \text{and} \quad E_2^\pi = \begin{bmatrix} (QP)^\pi & 0 \\ -QP(QP)^d & I \end{bmatrix}.$$

Using the above expressions, we have that

$$\sum_{i=0}^{m_2-1} E_2^\pi E_2^i (E_1^d)^{i+1} = \begin{bmatrix} 0 & \sum_{i=0}^{s_2-1} (QP)^\pi (QP)^i ((PQ)^d)^{i+1} \\ 0 & \sum_{i=0}^{s_2-1} (QP)^\pi (QP)^i ((PQ)^d)^{i+1} \end{bmatrix}.$$

Since $PQP^2 = 0$, we have that $(QP)^i(PQ)^d = 0$, for every $i \geq 2$. Therefore, we obtain

$$\sum_{i=0}^{m_2-1} E_2^\pi E_2^i (E_1^d)^{i+1} = \begin{bmatrix} 0 & (PQ+QP)((PQ)^d)^2 \\ 0 & (PQ+QP)((PQ)^d)^2 \end{bmatrix}. \tag{2.8}$$

Further, we have

$$\sum_{i=0}^{m_1-1} (E_2^d)^{i+1} E_1^i E_1^\pi = \begin{bmatrix} (QP)^d & \sum_{i=0}^{s_1-1} ((QP)^d)^{i+1} (PQ)^i (PQ)^\pi - (QP)^d \\ (QP)^d & \sum_{i=0}^{s_1-1} ((QP)^d)^{i+1} (PQ)^i (PQ)^\pi - (QP)^d \end{bmatrix}.$$

From $PQP^2 = 0$, it follows that $(QP)^d PQ = 0$ and thereby

$$\sum_{i=0}^{m_1-1} (E_2^d)^{i+1} E_1^i E_1^\pi = \begin{bmatrix} (QP)^d & 0 \\ (QP)^d & 0 \end{bmatrix}. \tag{2.9}$$

Substituting (2.8) and (2.9) into (2.7), we get

$$E^d = \begin{bmatrix} (QP)^d & (PQ+QP)((PQ)^d)^2 \\ (QP)^d & (PQ+QP)((PQ)^d)^2 \end{bmatrix}.$$

Using the induction by i, we obtain

$$(E^d)^i = \begin{bmatrix} ((QP)^d)^i & (PQ+QP)((PQ)^d)^{i+1} \\ ((QP)^d)^i & (PQ+QP)((PQ)^d)^{i+1} \end{bmatrix}, \quad \text{for} \quad i \geq 1. \tag{2.10}$$

Moreover, we get

$$E^\pi = \begin{bmatrix} (QP)^\pi & -(PQ+QP)(PQ)^d \\ -QP(QP)^d & I-(PQ+QP)(PQ)^d \end{bmatrix}. \tag{2.11}$$

In order to find the expression for F^d, notice that matrix F satisfies the conditions of Lemma 1.5. Hence,

$$F^d = \begin{bmatrix} P^2\Omega_1 + Q^2(G_1+G_2) & \begin{matrix} \Omega_1 Q^2 + Q^4(G_1\Omega_1 + (Q^d)^4 G_2)Q^2 \\ +(Q^d)^2 - QQ^d(P^4+Q^2P^2){\Omega_1}^2 Q^2 \end{matrix} \\ P^2\Omega_1 + Q^2(G_1+G_2) & (Q^d)^2 + (G_1+G_2)Q^2 \end{bmatrix}, \tag{2.12}$$

where

$$\begin{aligned}
\Omega_1 &= \sum_{i=0}^{\nu_4-1} (P^d)^{4(i+1)}(Q^2P^2)^i(Q^2P^2)^\pi + \sum_{i=0}^{\nu_1-1} P^\pi P^{4i}((Q^2P^2)^d)^{i+1}, \\
G_1 &= \sum_{i=0}^{\nu_2-1} Q^\pi Q^{4i}(P^4+Q^2P^2)\Omega_1^{i+2}, \\
G_2 &= \sum_{i=0}^{\nu_3-1} (Q^d)^{4(i+2)}(P^4+Q^2P^2)(P^4+Q^2P^2)^i(Q^2P^2)^\pi - \sum_{i=0}^{\nu_3} (Q^d)^{4(i+1)}(P^4+Q^2P^2)P^{4i}\Omega_1,
\end{aligned}$$

$\nu_1 = \text{ind}(P^4)$, $\nu_2 = \text{ind}(Q^4)$, $\nu_3 = \text{ind}(P^4+Q^2P^2)$ and $\nu_4 = \text{ind}(Q^2P^2)$. Since $PQ^2P = 0$, we get $P^d(Q^2P^2)^i = 0$, for $i \geq 1$. Moreover, since $(Q^2P^2)^2 = 0$, we have $(Q^2P^2)^d = 0$ and $(Q^2P^2)^\pi = I$. Thus,

$$\Omega_1 = (P^d)^4. \tag{2.13}$$

Further, since $QP^3 = 0$ and $PQ^2P = 0$ we obtain:

$$G_1 = (P^d)^4 \quad \text{and} \quad G_2 = (Q^d)^6 P^2. \tag{2.14}$$

After substituting (2.14) and (2.13) into (2.12), we get

$$F^d = \begin{bmatrix} ((P^d)^4 + (Q^d)^4)P^2 & ((P^d)^4 + (Q^d)^4)Q^2 + (Q^d)^6P^2Q^2 \\ ((P^d)^4 + (Q^d)^4)P^2 & ((P^d)^4 + (Q^d)^4)Q^2 + (Q^d)^6P^2Q^2 \end{bmatrix}.$$

Using the induction by i, we obtain

$$(F^d)^i = \begin{bmatrix} ((P^d)^{2(i+1)} + (Q^d)^{2(i+1)})P^2 & ((P^d)^{2(i+1)} + (Q^d)^{2(i+1)})Q^2 + (Q^d)^{2(i+2)}P^2Q^2 \\ ((P^d)^{2(i+1)} + (Q^d)^{2(i+1)})P^2 & ((P^d)^{2(i+1)} + (Q^d)^{2(i+1)})Q^2 + (Q^d)^{2(i+2)}P^2Q^2 \end{bmatrix}, \tag{2.15}$$

for every $i \geq 1$. Moreover, we get

$$F^\pi = \begin{bmatrix} I - PP^d - (Q^d)^2P^2 & -QQ^d - (P^d)^2Q^2 - (Q^d)^4P^2Q^2 \\ -PP^d - (Q^d)^2P^2 & I - QQ^d - (P^d)^2Q^2 - (Q^d)^4P^2Q^2 \end{bmatrix}. \tag{2.16}$$

In order to find the expressions for Z_1 and Z_2 from (2.4), we note the following. Using the hypothesis of the theorem, we get:

$$\begin{aligned}(PQ+QP)^i &= (PQ+QP)(PQ)^{i-1}+(QP)^i, \quad \text{for} \quad i\geq 2,\\ (P^2+Q^2)^i &= P^{2i}+P^{2i-2}Q^2+Q^{2i-4}P^2Q^2+Q^{2i-2}P^2+Q^{2i}, \quad \text{for} \quad i\geq 3.\end{aligned} \tag{2.17}$$

Moreover, since $PQ^2P=0$, using Lemma 1.2 we obtain

$$(PQ+QP)^d = (PQ+QP)((PQ)^d)^2+(QP)^d.$$

Further, since $PQ^2P=0$ and $PQ^3=0$, using Lemma 1.3 we get

$$(P^2+Q^2)^d = \left((P^d)^4+(Q^d)^4\right)(P^2+Q^2)+(Q^d)^6P^2Q^2. \tag{2.18}$$

Using the induction by i, we obtain:

$$\begin{aligned}\left((PQ+QP)^d\right)^i &= (PQ+QP)((PQ)^d)^{i+1}+((QP)^d)^i, \quad \text{for} \quad i\geq 1,\\ \left((P^2+Q^2)^d\right)^i &= \left((P^d)^{2i+2}+(Q^d)^{2i+2}\right)(P^2+Q^2)+(Q^d)^{2i+4}P^2Q^2, \quad \text{for} \quad i\geq 1.\end{aligned} \tag{2.19}$$

In addition, we have:

$$\begin{aligned}(PQ+QP)^\pi &= I-\left((PQ+QP)(PQ)^d+QP(QP)^d\right),\\ (P^2+Q^2)^\pi &= I-(PP^d+QQ^d+(Q^d)^2P^2+(P^d)^2Q^2+(Q^d)^4P^2Q^2).\end{aligned} \tag{2.20}$$

Also,

$$\begin{aligned}(PQ+QP)(PQ+QP)^\pi &= PQ-(PQ+QP)PQ(PQ)^d+QP(QP)^\pi,\\ (PQ+QP)^i(PQ+QP)^\pi &= (PQ+QP)(PQ)^{i-1}(PQ)^\pi+(QP)^i(QP)^\pi, \quad \text{for} \quad i\geq 2.\end{aligned} \tag{2.21}$$

Now, we can determine the expressions for Z_1 and Z_2. Using (2.6), (2.15),(2.16) and (2.20), we get:

$$F^\pi E^d = \begin{bmatrix} (P^2+Q^2)^\pi(QP)^d & (P^2+Q^2)^\pi(PQ+QP)((PQ)^d)^2 \\ (P^2+Q^2)^\pi(QP)^d & (P^2+Q^2)^\pi(PQ+QP)((PQ)^d)^2 \end{bmatrix},$$

$$F^\pi F^i(E^d)^{(i+1)} = \begin{bmatrix} (P^2+Q^2)^\pi(P^2+Q^2)^i((QP)^d)^{i+1} & (P^2+Q^2)^\pi(P^2+Q^2)^i(PQ+QP)((PQ)^d)^{i+2} \\ (P^2+Q^2)^\pi(P^2+Q^2)^i((QP)^d)^{i+1} & (P^2+Q^2)^\pi(P^2+Q^2)^i(PQ+QP)((PQ)^d)^{i+2} \end{bmatrix},$$

for $i\geq 1$. Hence,

$$Z_1 = \begin{bmatrix} \sum\limits_{i=0}^{\mu_1-1}(P^2+Q^2)^\pi(P^2+Q^2)^i((QP)^d)^{i+1} & \sum\limits_{i=0}^{\mu_1-1}(P^2+Q^2)^\pi(P^2+Q^2)^i(PQ+QP)((PQ)^d)^{i+2} \\ \sum\limits_{i=0}^{\mu_1-1}(P^2+Q^2)^\pi(P^2+Q^2)^i((QP)^d)^{i+1} & \sum\limits_{i=0}^{\mu_1-1}(P^2+Q^2)^\pi(P^2+Q^2)^i(PQ+QP)((PQ)^d)^{i+2} \end{bmatrix},$$

where $\mu_1=\text{ind}(P^2+Q^2)$.

Now, we need to determine Z_2. By (2.11) and (2.15), we have

$$F^dE^\pi = \begin{bmatrix} \begin{matrix}((P^d)^4+(Q^d)^4)P^2 \\ -((P^d)^4+(Q^d)^4)(P^2+Q^2)QP(QP)^d\end{matrix} & \begin{matrix}((P^d)^4+(Q^d)^4)Q^2+(Q^d)^6P^2Q^2 \\ -(((P^d)^4+(Q^d)^4)(P^2+Q^2)+(Q^d)^6P^2Q^2)(PQ+QP)(PQ)^d\end{matrix} \\ \\ \begin{matrix}((P^d)^4+(Q^d)^4)P^2 \\ -((P^d)^4+(Q^d)^4)(P^2+Q^2)QP(QP)^d\end{matrix} & \begin{matrix}((P^d)^4+(Q^d)^4)Q^2+(Q^d)^6P^2Q^2 \\ -(((P^d)^4+(Q^d)^4)(P^2+Q^2)+(Q^d)^6P^2Q^2)(PQ+QP)(PQ)^d\end{matrix} \end{bmatrix},$$

Using (2.18), we get that the entry (1.1) (and also (2.1)), for matrix F^dE^π, is $(P^2+Q^2)^d(QP)^\pi-((P^d)^4+(Q^d)^4)Q^2-(Q^d)^6P^2Q^2$. Further, by (2.20), we have that $-(PQ+QP)(PQ)^d=(PQ+QP)^\pi-(QP)^\pi$. Thereby, the entry (1.2) (and also (2.2)), for matrix F^dE^π, is $((P^d)^4+(Q^d)^4)Q^2+(Q^d)^6P^2Q^2+(P^2+Q^2)^d(PQ+QP)^\pi-(P^2+Q^2)^d(QP)^\pi$. Hence,

$$F^dE^\pi=\begin{bmatrix} \begin{matrix}(P^2+Q^2)^d(QP)^\pi\\ -((P^d)^4+(Q^d)^4)Q^2-(Q^d)^6P^2Q^2\end{matrix} & \begin{matrix}((P^d)^4+(Q^d)^4)Q^2+(Q^d)^6P^2Q^2\\ +(P^2+Q^2)^d(PQ+QP)^\pi-(P^2+Q^2)^d(QP)^\pi\end{matrix} \\ \\ \begin{matrix}(P^2+Q^2)^d(QP)^\pi\\ -((P^d)^4+(Q^d)^4)Q^2-(Q^d)^6P^2Q^2\end{matrix} & \begin{matrix}((P^d)^4+(Q^d)^4)Q^2+(Q^d)^6P^2Q^2\\ +(P^2+Q^2)^d(PQ+QP)^\pi-(P^2+Q^2)^d(QP)^\pi\end{matrix}\end{bmatrix}.$$

Further, by (2.11), (2.15) and (2.19), we have

$$\begin{aligned}(F^d)^2EE^\pi &= (F^d)^2\begin{bmatrix} QP(QP)^\pi & PQ-(PQ+QP)PQ(PQ)^d\\ QP(QP)^\pi & PQ-(PQ+QP)PQ(PQ)^d\end{bmatrix}\\ &= \begin{bmatrix} ((P^2+Q^2)^d)^2QP(QP)^\pi & ((P^2+Q^2)^d)^2(PQ-(PQ+QP)PQ(PQ)^d)\\ ((P^2+Q^2)^d)^2QP(QP)^\pi & ((P^2+Q^2)^d)^2(PQ-(PQ+QP)PQ(PQ)^d)\end{bmatrix}.\end{aligned}$$

By (2.21), we have that $PQ-(PQ+QP)PQ(PQ)^d=(PQ+QP)(PQ+QP)^\pi-QP(QP)^\pi$. Thereby

$$(F^d)^2EE^\pi=\begin{bmatrix} ((P^2+Q^2)^d)^2QP(QP)^\pi & ((P^2+Q^2)^d)^2(PQ+QP)(PQ+QP)^\pi-((P^2+Q^2)^d)^2QP(QP)^\pi\\ ((P^2+Q^2)^d)^2QP(QP)^\pi & ((P^2+Q^2)^d)^2(PQ+QP)(PQ+QP)^\pi-((P^2+Q^2)^d)^2QP(QP)^\pi\end{bmatrix}.$$

Moreover, by (2.5), (2.11) and (2.15), for $i\geq 2$ we have

$$\begin{aligned}(F^d)^{i+1}E^iE^\pi &= (F^d)^{i+1}\begin{bmatrix} (QP)^i(QP)^\pi & (PQ+QP)(PQ)^{i-1}(PQ)^\pi\\ (QP)^i(QP)^\pi & (PQ+QP)(PQ)^{i-1}(PQ)^\pi\end{bmatrix}\\ &= \begin{bmatrix} ((P^2+Q^2)^d)^{i+1}(QP)^i(QP)^\pi & ((P^2+Q^2)^d)^{i+1}(PQ+QP)(PQ)^{i-1}(PQ)^\pi\\ ((P^2+Q^2)^d)^{i+1}(QP)^i(QP)^\pi & ((P^2+Q^2)^d)^{i+1}(PQ+QP)(PQ)^{i-1}(PQ)^\pi\end{bmatrix}.\end{aligned}$$

Using (2.21), we have $(PQ+QP)(PQ)^{i-1}(PQ)^\pi=(PQ+QP)^i(PQ+QP)^\pi-(QP)^i(QP)^\pi$ and thereby

$$(F^d)^{i+1}E^iE^\pi=\begin{bmatrix} ((P^2+Q^2)^d)^{i+1}(QP)^i(QP)^\pi & \begin{matrix}((P^2+Q^2)^d)^{i+1}(PQ+QP)^i(PQ+QP)^\pi\\ -((P^2+Q^2)^d)^{i+1}(QP)^i(QP)^\pi\end{matrix}\\ \\ ((P^2+Q^2)^d)^{i+1}(QP)^i(QP)^\pi & \begin{matrix}((P^2+Q^2)^d)^{i+1}(PQ+QP)^i(PQ+QP)^\pi\\ -((P^2+Q^2)^d)^{i+1}(QP)^i(QP)^\pi\end{matrix}\end{bmatrix},$$

for $i\geq 2$. Therefore, we get

$$Z_2 = \begin{bmatrix} \begin{array}{l} \sum_{i=0}^{s_2-1}((P^2+Q^2)^d)^{i+1}(QP)^i(QP)^{\pi} \\ -((P^d)^4+(Q^d)^4)Q^2-(Q^d)^6P^2Q^2 \end{array} & \begin{array}{l} ((P^d)^4+(Q^d)^4)Q^2+(Q^d)^6P^2Q^2 \\ -\sum_{i=0}^{s_2-1}((P^2+Q^2)^d)^{i+1}(QP)^i(QP)^{\pi} \\ +\sum_{i=0}^{\mu_2-1}((P^2+Q^2)^d)^{i+1}(PQ+QP)^i(PQ+QP)^{\pi} \end{array} \\ \begin{array}{l} \sum_{i=0}^{s_2-1}((P^2+Q^2)^d)^{i+1}(QP)^i(QP)^{\pi} \\ -((P^d)^4+(Q^d)^4)Q^2-(Q^d)^6P^2Q^2 \end{array} & \begin{array}{l} ((P^d)^4+(Q^d)^4)Q^2+(Q^d)^6P^2Q^2 \\ -\sum_{i=0}^{s_2-1}((P^2+Q^2)^d)^{i+1}(QP)^i(QP)^{\pi} \\ +\sum_{i=0}^{\mu_2-1}((P^2+Q^2)^d)^{i+1}(PQ+QP)^i(PQ+QP)^{\pi} \end{array} \end{bmatrix},$$

where $\mu_2 = \mathrm{ind}(PQ+QP)$. In order to find the expression for $(E+F)^d$, given in (2.3), we also need to determine Z_1E^dF, $(F^d)^2Z_2EF$, F^dE^dF and $(F^d)^2E^dEF$. We obtain:

$$Z_1E^dF = \begin{bmatrix} 0 & \sum_{i=0}^{\mu_1-1}(P^2+Q^2)^{\pi}(P^2+Q^2)^i((QP)^d)^{i+2}Q^2 \\ 0 & \sum_{i=0}^{\mu_1-1}(P^2+Q^2)^{\pi}(P^2+Q^2)^i((QP)^d)^{i+2}Q^2 \end{bmatrix},$$

$$(F^d)^2Z_2EF = \begin{bmatrix} 0 & \sum_{i=0}^{\mu_2-1}((P^2+Q^2)^d)^{i+3}(PQ+QP)^i(PQ+QP)^{\pi}QPQ^2 \\ 0 & \sum_{i=0}^{\mu_2-1}((P^2+Q^2)^d)^{i+3}(PQ+QP)^i(PQ+QP)^{\pi}QPQ^2 \end{bmatrix},$$

$$F^dE^dF = \begin{bmatrix} 0 & (P^2+Q^2)^d(QP)^dQ^2 \\ 0 & (P^2+Q^2)^d(QP)^dQ^2 \end{bmatrix}$$

and

$$(F^d)^2E^dEF = \begin{bmatrix} 0 & ((P^2+Q^2)^d)^2(QP)^dQPQ^2 \\ 0 & ((P^2+Q^2)^d)^2(QP)^dQPQ^2 \end{bmatrix}.$$

Substituting the above equalities in (2.3), we get:

$$(E+F)^d = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_1 & \alpha_2 \end{bmatrix}, \tag{2.22}$$

where

$$\begin{aligned} \alpha_1 = &\sum_{i=0}^{\mu_1-1}(P^2+Q^2)^{\pi}(P^2+Q^2)^i((QP)^d)^{i+1} \\ &+\sum_{i=0}^{s_2-1}((P^2+Q^2)^d)^{i+1}(QP)^i(QP)^{\pi} \\ &-((P^d)^4+(Q^d)^4)Q^2-(Q^d)^6P^2Q^2 \end{aligned}$$

and

$$
\begin{aligned}
\alpha_2 = &\sum_{i=0}^{\mu_1-1}(P^2+Q^2)^\pi(P^2+Q^2)^i(PQ+QP)((PQ)^d)^{i+2}\\
&+\sum_{i=0}^{\mu_2-1}((P^2+Q^2)^d)^{i+1}(PQ+QP)^i(PQ+QP)^\pi\\
&+((P^d)^4+(Q^d)^4)Q^2+(Q^d)^6P^2Q^2\\
&-\sum_{i=0}^{s_2-1}((P^2+Q^2)^d)^{i+1}(QP)^i(QP)^\pi\\
&+\sum_{i=0}^{\mu_1-1}(P^2+Q^2)^\pi(P^2+Q^2)^i((QP)^d)^{i+2}Q^2\\
&+\sum_{i=0}^{\mu_2-1}((P^2+Q^2)^d)^{i+3}(PQ+QP)^i(PQ+QP)^\pi QPQ^2\\
&-(P^2+Q^2)^d(QP)^dQ^2-((P^2+Q^2)^d)^2(QP)^dQPQ^2.
\end{aligned}
$$

Now, after substituting (2.22) in (2.2), we obtain

$$
\begin{aligned}
(P+Q)^d = (P+Q)\Bigg(&\sum_{i=0}^{\mu_1-1}(P^2+Q^2)^\pi(P^2+Q^2)^i((PQ+QP)^d)^{i+1}\\
&+\sum_{i=0}^{\mu_2-1}((P^2+Q^2)^d)^{i+1}(PQ+QP)^i(PQ+QP)^\pi\\
&+\sum_{i=0}^{\mu_1-1}(P^2+Q^2)^\pi(P^2+Q^2)^i((QP)^d)^{i+2}Q^2\\
&+\sum_{i=0}^{\mu_2-1}((P^2+Q^2)^d)^{i+3}(PQ+QP)^i(PQ+QP)^\pi QPQ^2\\
&-(P^2+Q^2)^d(QP)^dQ^2-((P^2+Q^2)^d)^2(QP)^dQPQ^2\Bigg).
\end{aligned}
\tag{2.23}
$$

Using (2.17), (2.19), (2.20), (2.21) and after some computation, we get:

$$
\begin{aligned}
(P+Q)&\sum_{i=0}^{\mu_1-1}(P^2+Q^2)^\pi(P^2+Q^2)^i((PQ+QP)^d)^{i+1} =\\
&=\sum_{i=0}^{r-1}\left(P^\pi P^{2i+1}+Q^\pi Q^{2i+1}\right)\left(\left((PQ)^d\right)^{i+1}+\left((QP)^d\right)^{i+1}\right)\\
&+\sum_{i=0}^{r_2-1}Q^\pi Q^{2i+1}P^2((QP)^d)^{i+2}+\sum_{i=0}^{r_2-1}Q^\pi Q^{2i+2}P((PQ)^d)^{i+2}\\
&-Q^dP^2(QP)^d,
\end{aligned}
$$

$$(P+Q)\sum_{i=0}^{\mu_2-1}((P^2+Q^2)^d)^{i+1}(PQ+QP)^i(PQ+QP)^\pi =$$
$$= \sum_{i=0}^{s-1}\left((P^d)^{2i+1}+(Q^d)^{2i+1}\right)\left((PQ)^i(PQ)^\pi+(QP)^i(QP)^\pi\right)$$
$$+\sum_{i=0}^{s_2-1}(Q^d)^{2i+3}P^2(QP)^i(QP)^\pi+\sum_{i=0}^{s_1-1}(Q^d)^{2i+2}P(PQ)^i(PQ)^\pi$$
$$+(P^d)^3Q^2+(Q^d)^5P^2Q^2-QQ^dP(PQ)^d-(Q^d)^2P-Q^d-P^d,$$

$$(P+Q)\sum_{i=0}^{\mu_1-1}(P^2+Q^2)^\pi(P^2+Q^2)^i((QP)^d)^{i+2}Q^2 =$$
$$= \sum_{i=0}^{r-1}\left(P^\pi P^{2i+1}+Q^\pi Q^{2i+1}\right)\left((QP)^d\right)^{i+2}Q^2$$
$$+\sum_{i=0}^{r_2-1}Q^\pi Q^{2i+1}P^2((QP)^d)^{i+3}Q^2-Q^dP(PQ)^dQ$$

$$(P+Q)\sum_{i=0}^{\mu_2-1}((P^2+Q^2)^d)^{i+3}(PQ+QP)^i(PQ+QP)^\pi QPQ^2$$
$$= \sum_{i=0}^{s_2-1}\left((P^d)^{2i+5}+(Q^d)^{2i+7}P^2+(Q^d)^{2i+5}\right)(QP)^{i+1}(QP)^\pi Q^2,$$

$$(P+Q)(P^2+Q^2)^d(QP)^dQ^2 = P^d(QP)^dQ^2+(Q^d)^3P^2(QP)^dQ^2+Q^d(QP)^dQ^2$$

and

$$(P+Q)((P^2+Q^2)^d)^2(QP)^dQPQ^2 = (P^d)^3QP(QP)^dQ^2+(Q^d)^5P^2QP(QP)^dQ^2+(Q^d)^3QP(QP)^dQ^2.$$

Substituting the previously obtained expressions into (2.23), we get that the additive formula (2.1) is valid.
□

In the following example, we analyze two matrices P and Q, which do not satisfy the conditions of [8, Theorem 3.2], but which satisfy the conditions of the previously proved theorem.

Example 2.2. Let $P, Q \in \mathbb{C}^{5\times 5}$ be such that:

$$P = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We have that $PQP^2 = 0$, $PQ^2P = 0$, $PQ^3 = 0$ and $QP^3 = 0$. Meanwhile, $QPQ^2 \neq 0$ and $P^2Q^2 \neq 0$, so we can not apply the formula for $(P+Q)^d$ from [8, Theorem 3.2]. However, we have that the conditions of Theorem 2.1 are satisfied and therefore we can apply the additive formula (2.1). In order to determine the expression for $(P+Q)^d$, we have the following. We get that $\text{ind}(P) = 3$, $\text{ind}(Q) = 3$ and:

$$P^d = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad Q^d = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Moreover, we obtain $\text{ind}(PQ) = 2$, $\text{ind}(QP) = 2$ and:

$$(PQ)^d = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (QP)^d = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

After applying the formula (2.1) and after some computation, we get that

$$(P+Q)^d = \begin{bmatrix} 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Now we give a symmetrical formulation of Theorem 2.1. We note that this result extends the result from [8, Theorem 3.1], where the formula for $(P+Q)^d$ is given under conditions $P^2QP = 0$, $PQ^2P = 0$, $Q^3P = 0$, $P^3Q = 0$, $Q^2PQ = 0$ and $Q^2P^2 = 0$.

Theorem 2.3. *Let $P, Q \in \mathbb{C}^{n\times n}$. If $P^2QP = 0$, $PQ^2P = 0$, $Q^3P = 0$ and $P^3Q = 0$, then*

$$\begin{aligned}
(P+Q)^d =& \sum_{i=0}^{r-1} \left(\left((QP)^d\right)^{i+1} + \left(I + Q^2(PQ)^d\right)\left((PQ)^d\right)^{i+1} \right) \left(P^{2i+1}P^\pi + Q^{2i+1}Q^\pi\right) \\
&+ \sum_{i=0}^{\lceil \frac{r_2-1}{2} \rceil - 1} \left(\left((QP)^d\right)^{i+2} PQ + \left(I + Q^2(PQ)^d\right)\left((PQ)^d\right)^{i+2} P^2 \right) Q^{2i+1}Q^\pi \\
&+ \sum_{i=0}^{s-1} \left((QP)^\pi (QP)^i + (PQ)^\pi (PQ)^i\right)\left((P^d)^{2i+1} + (Q^d)^{2i+1}\right) \\
&+ \sum_{i=0}^{s-1} \left((QP)^\pi (QP)^i PQ + (PQ)^\pi (PQ)^i P^2\right)(Q^d)^{2i+3} \\
&+ Q^2 \sum_{i=0}^{s_1-1} (PQ)^\pi (PQ)^i \left((P^d)^{2i+3} + P^2(Q^d)^{2i+5} + (Q^d)^{2i+3}\right) \\
&- P^d - 2Q^d - P(Q^d)^2 - (PQ)^d P^2 Q^d - (QP)^d PQQ^d - Q(QP)^d PQ^d \\
&- Q^2(PQ)^d P^d - Q^2(PQ)^d Q^d - Q^2(PQ)^d P^2 (Q^d)^3,
\end{aligned} \tag{2.24}$$

where $r_1 = \text{ind}(P)$, $r_2 = \text{ind}(Q)$, $s_1 = \text{ind}(PQ)$, $s_2 = \text{ind}(QP)$, $r = \max\left\{\left\lceil \frac{r_1-1}{2} \right\rceil, \left\lceil \frac{r_2-1}{2} \right\rceil\right\}$ and $s = \max\{s_1, s_2\}$.

In the following theorem we give a new representation for a block matrix M of a form (1.1), as an application of Theorem 2.3.

Theorem 2.4. *Let M be a complex block matrix of a form* (1.1), *such that $S = 0$. If $A^dBCA^\pi A = 0$, $A^dBCA^\pi B = 0$, $A^\pi A^2BC = 0$ and $CA^\pi ABC = 0$, then*

$$\begin{aligned}
M^d = &\begin{bmatrix} (A^\pi BC)^\pi - AA^\pi B((CA^\pi B)^d)^2CA^\pi A & -A^\pi BC(A^\pi BC)^dA^dB - AA^\pi B(CA^\pi B)^d \\ -(CA^\pi B)^dCA^\pi A & (CA^\pi B)^\pi \end{bmatrix} P^d \\
&+ \begin{bmatrix} A^\pi A^2 - AA^\pi B(CA^\pi B)^dCA^\pi A & AA^\pi B(CA^\pi B)^\pi \\ 0 & 0 \end{bmatrix} (P^d)^3 \\
&+ \sum_{i=0}^{t-1} \begin{bmatrix} ((A^\pi BC)^d)^{i+1} + A^\pi AB((CA^\pi B)^d)^{i+3}CA^\pi A & ((A^\pi BC)^d)^{i+1}A^dB + AA^\pi B((CA^\pi B)^d)^{i+2} \\ ((CA^\pi B)^d)^{i+2}CA^\pi A & ((CA^\pi B)^d)^{i+1} \end{bmatrix} \cdot \\
&\quad \cdot \left(P^{2i+1}P^\pi + \begin{bmatrix} A^\pi A^{2i+1} & A^\pi A^{2i}B \\ 0 & 0 \end{bmatrix} \right) \\
&+ \sum_{i=1}^{v} \begin{bmatrix} (A^\pi BC)^\pi (A^\pi BC)^i & (A^\pi BC)^\pi (A^\pi BC)^iA^dB \\ (CA^\pi B)^\pi (CA^\pi B)^{i-1}CA^\pi A & (CA^\pi B)^\pi (CA^\pi B)^i \end{bmatrix} (P^d)^{2i+1} \\
&+ \sum_{i=1}^{v_2} \begin{bmatrix} AA^\pi B(CA^\pi B)^\pi (CA^\pi B)^{i-1}CA^\pi A & AA^\pi B(CA^\pi B)^\pi (CA^\pi B)^i \\ 0 & 0 \end{bmatrix} (P^d)^{2i+3},
\end{aligned}$$

where

$$\begin{aligned}
&P = \begin{bmatrix} A^2A^d & AA^dB \\ C & CA^dB \end{bmatrix}, \\
&(P^d)^i = (P_1^d)^i \left(I + P_1^d \begin{bmatrix} 0 & 0 \\ CA^\pi & 0 \end{bmatrix} \right), \\
&(P_1^d)^i = \begin{bmatrix} I \\ CA^d \end{bmatrix} ((AW)^d)^{i+1} A \begin{bmatrix} I & A^dB \end{bmatrix}, \; W = AA^d + A^dBCA^d,
\end{aligned}$$

for every $i \in \mathbb{N}$ and for $r_1 = \operatorname{ind}(P)$, $t_1 = \operatorname{ind}(A)$, $t = \max\left\{ \left\lceil \frac{r_1 - 1}{2} \right\rceil, \left\lceil \frac{t_1}{2} \right\rceil \right\}$, $v_1 = \operatorname{ind}(A^\pi BC)$ and $v_2 = \operatorname{ind}(CA^\pi B)$, $v = \max\{v_1, v_2\}$.

Proof. Let the assumptions of the theorem hold. If we denote by

$$P = \begin{bmatrix} A^2A^d & AA^dB \\ C & CA^dB \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} AA^\pi & A^\pi B \\ 0 & 0 \end{bmatrix},$$

we have that $M = P + Q$. Moreover, we have that $P^2Q = 0$, $PQ^2P = 0$ and $Q^3P = 0$. Therefore, we can apply Theorem 2.3. In order to find the expression for M^d, we need to determine P^d, Q^d, $(PQ)^d$ and $(QP)^d$. If we use notation:

$$P_1 = \begin{bmatrix} A^2A^d & AA^dB \\ CAA^d & CA^dB \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 0 & 0 \\ CA^\pi & 0 \end{bmatrix},$$

we have that $P = P_1 + P_2$. Since $P_2P_1 = 0$ and $P_2^2 = 0$, we can apply Lemma 1.2 and we get $P^d = P_1^d + (P_1^d)^2P_2$. By induction, we get

$$(P^d)^i = (P_1^d)^i + (P_1^d)^{i+1}P_2, \quad \text{for } i \geq 1. \tag{2.25}$$

In order to determine P_1^d, notice that matrix P_1 satisfies the conditions of Lemma 1.6 (we have $(A^2A^d)^d = A^d$) and after applying this lemma, we get $P_1^d = \begin{bmatrix} I \\ CA^d \end{bmatrix}((AW)^d)^2A\begin{bmatrix} I & A^dB \end{bmatrix}$, where $W = AA^d + A^dBCA^d$. Moreover, since $(AW)^dAA^d = (AW)^d$ and by induction, we derive

$$(P_1^d)^i = \begin{bmatrix} I \\ CA^d \end{bmatrix}((AW)^d)^{i+1}A\begin{bmatrix} I & A^dB \end{bmatrix}, \text{ for every } i \in \mathbb{N}.$$

Further, we have that

$$Q^i = \begin{bmatrix} A^\pi A^i & A^\pi A^{i-1}B \\ 0 & 0 \end{bmatrix}, \text{ for every } i \in \mathbb{N}. \tag{2.26}$$

Thereby, matrix Q is $(t_1 + 1)$-nilpotent, where $t_1 = \operatorname{ind}(A)$. Therefore, $Q^d = 0$ and $Q^\pi = I$. Furthermore, by induction we obtain:

$$(PQ)^i = \begin{bmatrix} 0 & 0 \\ (CA^\pi B)^{i-1}CA^\pi A & (CA^\pi B)^i \end{bmatrix} \quad \text{and} \quad (QP)^i = \begin{bmatrix} (A^\pi BC)^i & (A^\pi BC)^iA^dB \\ 0 & 0 \end{bmatrix} \tag{2.27}$$

for every $i \in \mathbb{N}$. Moreover, after applying Lemma 1.4, we get:

$$\begin{aligned} ((PQ)^d)^i &= \begin{bmatrix} 0 & 0 \\ ((CA^\pi B)^d)^{i+1}CA^\pi A & ((CA^\pi B)^d)^i \end{bmatrix}, \\ ((QP)^d)^i &= \begin{bmatrix} ((A^\pi BC)^d)^i & ((A^\pi BC)^d)^iA^dB \\ 0 & 0 \end{bmatrix}, \end{aligned} \tag{2.28}$$

for every $i \in \mathbb{N}$. It remains to apply the expressions (2.25), (2.26), (2.27) and (2.28) into the following formula

$$\begin{aligned} (P+Q)^d = &\sum_{i=0}^{r-1}\left(\left((QP)^d\right)^{i+1} + \left(I + Q^2(PQ)^d\right)\left((PQ)^d\right)^{i+1}\right)\left(P^{2i+1}P^\pi + Q^{2i+1}\right) \\ &+ \sum_{i=0}^{s-1}\left((QP)^\pi(QP)^i + (PQ)^\pi(PQ)^i\right)(P^d)^{2i+1} + Q^2\sum_{i=0}^{s_1-1}(PQ)^\pi(PQ)^i(P^d)^{2i+3} - P^d - Q^2(PQ)^dP^d, \end{aligned}$$

where $r_1 = \operatorname{ind}(P)$, $r_2 = \operatorname{ind}(Q)$, $s_1 = \operatorname{ind}(PQ)$, $s_2 = \operatorname{ind}(QP)$, $r = \max\left\{\left\lceil \frac{r_1-1}{2} \right\rceil, \left\lceil \frac{r_2-1}{2} \right\rceil\right\}$ and $s = \max\{s_1, s_2\}$. After some computation, we get that the statement of the theorem is true.
□

Now we give the following example, to illustrate Theorem 2.4.

Example 2.5. Let $M \in \mathbb{C}^{7\times 7}$, $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We have that $S = D - CA^dB = 0$. Also, we have that $A^\pi ABC \neq 0$ and $A^dBC \neq 0$. Thereby, we can not apply formulas for M^d given under conditions (i)–(v) from the list (b). However, since $A^dBCA^\pi A = 0$, $A^dBCA^\pi B = 0$, $A^\pi A^2BC = 0$ and $CA^\pi ABC = 0$, we can apply Theorem 2.4. We have that $\operatorname{ind}(A) = 3$ and

$$A^d = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Moreover, we get that ind(AW) = 1, ind(P) = 2 and

$$(AW)^d = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad P^d = \begin{bmatrix} \frac{1}{4} & \frac{1}{8} & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{8} & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

After applying Theorem 2.4, we get

$$M^d = \begin{bmatrix} \frac{1}{4} & \frac{1}{8} & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{4} & \frac{17}{8} & 0 & 1 & \frac{1}{4} & 1 & \frac{1}{4} \\ -\frac{1}{2} & \frac{3}{4} & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{8} & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ -\frac{1}{4} & \frac{11}{8} & 0 & 1 & -\frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

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