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Contact screen generic lightlike submanifolds of indefinite Kenmotsu manifold

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Abstract. In this paper, we study contact screen generic lightlike (CSGL) submanifolds, totally umbilical CSGL submanifolds and minimal CSGL submanifolds of indefinite Kenmotsu manifolds. We investigate the necessary and sufficient (n & s) conditions for the induced connection on a CSGL submanifold to be a metric connection, for integrability & parallelism of some associated distributions, and for some distributions to be totally geodesic foliations. We also discuss about non-parallel distributions and more than one n & s conditions for a CSGL submanifold to be mixed geodesic. We further study some properties satisfied by proper totally umbilical CSGL submanifolds and the n & s conditions for minimality of an associated distribution & also of a CSGL submanifold. At last, we construct an example of a CSGL submanifold of an indefinite Kenmotsu manifold.

1. Introduction

K. L. Duggal introduced the geometry of lightlike submanifolds in 1996 along with A. Bejancu [5] and later in 2010, he along with B. Sahin wrote another book on it [10]. As the tangent and normal bundles have non-trivial intersection in lightlike submanifolds, many researchers used this theory widely in their works such as [2], [4], [6], [7], [8], [9], [13], [14], [16].

K. Yano and M. Kon introduced the notion of generic submanifolds as the generalization of CR-submanifolds in 1980 [17]. Generic submanifold is the most general case of submanifolds because CR-submanifolds include holomorphic, as well as totally real submanifolds as subspaces. Also, screen CR-lightlike submanifold has invariant and anti-invariant lightlike submanifolds as its particular cases. Hence, generic lightlike submanifolds must include CR-lightlike submanifolds. Now, R. Gupta and S. Ahamad introduced the notion of slant lightlike submanifolds of indefinite Kenmotsu manifolds in 2011 [11]. Then, one year later in 2012, K. L. Duggal and D. H. Jin introduced the concept of generic lightlike submanifolds of an indefinite Sasakian manifold [7]. In 2015, D. H. Jin and J. W. Lee further studied generic lightlike submanifolds of an indefinite Kahler manifold [16] but yet, this concept did not contain proper screen CR-lightlike submanifolds. Hence, later in 2019, screen generic lightlike submanifold was introduced by

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B. Dogan et al. [4]. In 2020, R. S. Gupta modified that concept in the context of contact geometry and introduced a general notion of screen generic lightlike submanifolds of an indefinite Sasakian manifold with the structure vector field tangent to the submanifold [12].

Motivated by the works mentioned above, in this paper we have studied contact screen generic lightlike (CSGL) submanifolds, totally umbilical CSGL submanifolds and minimal CSGL submanifolds of indefinite Kenmotsu manifolds. This paper consists of five sections. After introduction and preliminaries sections, in the third section, we have investigated the necessary and sufficient (n & s) conditions for the induced connection on a CSGL submanifold to be a metric connection, for integrability & parallelism of some associated distributions, and for some distributions to be totally geodesic foliations. We have also discussed about non-parallel distributions and more than one n & s conditions for a CSGL submanifold to be mixed geodesic. In the fourth and fifth sections respectively, we have further studied some properties satisfied by proper totally umbilical CSGL submanifolds and the n & s conditions for minimality of an associated distribution & also of a CSGL submanifold. At last, we have constructed an example of a CSGL submanifold of an indefinite Kenmotsu manifold.

2. Preliminaries

Definition 2.1. Let $\bar{M}$ be an odd dimensional differentiable manifold equipped with a metric structure $(\phi, \xi, \eta, \bar{g})$ consisting of a (1,1) tensor field ϕ, a vector field ξ, a 1-form η and a semi-Riemannian metric $\bar{g}$ satisfying the following relations−

$$\phi^2 X = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \eta \circ \phi = 0,\ \phi\xi = 0, \tag{1}$$

$$\bar{g}(\phi X, \phi Y) = \bar{g}(X, Y) - \eta(X)\eta(Y), \tag{2}$$

$$\bar{g}(\phi X, Y) = -\bar{g}(X, \phi Y), \tag{3}$$

$$\eta(X) = \bar{g}(X, \xi)\ \ \forall\ X, Y \in \chi(\bar{M}), \tag{4}$$

then $\bar{M}$ is called *indefinite almost contact metric manifold* [3].

Definition 2.2. An indefinite almost contact metric manifold $\bar{M}(\phi, \xi, \eta, \bar{g})$ is called *indefinite Kenmotsu manifold* [15] if $\forall\ X, Y \in \chi(\bar{M})$,

$$\bar{\nabla}_X \xi = X - \eta(X)\xi, \tag{5}$$

$$(\bar{\nabla}_X \phi)Y = \bar{g}(\phi X, Y)\xi - \eta(Y)\phi X, \tag{6}$$

where $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$.

Here, without loss of generality, the structure vector field ξ is assumed to be spacelike i.e. $\bar{g}(\xi, \xi) = 1$.

Definition 2.3. A submanifold (M^m, g) immersed in a proper semi-Riemannian manifold $(\bar{M}^{m+n}, \bar{g})$ is called *lightlike submanifold* [5] if the metric g induced from $\bar{g}$ is degenerate and the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ is of rank r such that $1 \leq r \leq m$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM i.e.,

$$TM = Rad(TM) \oplus_{orth} S(TM). \tag{7}$$

$S(TM)$ is a non-degenerate distribution which is generally not unique because of the degenerate metric g [10].

Let us consider a screen transversal vector bundle $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $Rad(TM)$ in $TM^{\perp}$ i.e.,

$$TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}).$$

Since for any local basis $\{\xi_i\}$ of $Rad(TM)$, there exists a local frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $S(TM)^{\perp}$ such that $\bar{g}(\xi_i, N_j) = \delta_{ij}$ and $\bar{g}(N_i, N_j) = 0$, it follows that there exists a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$ [16]. Let $tr(TM)$ be the complementary (not orthogonal) vector bundle to TM in $T\bar{M}$. Now we have the following decompositions−

$$T\bar{M}|_M = TM \oplus tr(TM),$$

$$tr(TM) = S(TM^{\perp}) \oplus_{orth} ltr(TM), \tag{8}$$

$$T\bar{M}|_M = S(TM) \oplus_{orth} [Rad(TM) \oplus ltr(TM)] \oplus_{orth} S(TM^{\perp}). \tag{9}$$

A submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\bar{M}$ is called
(i) *r-lightlike* if $r < min\{m, n\}$,
(ii) *co-isotropic* if $r = n < m,\ S(TM^{\perp}) = \{0\}$,
(iii) *isotropic* if $r = m < n,\ S(TM) = \{0\}$,
(iv) *totally lightlike* if $r = m = n,\ S(TM) = \{0\} = S(TM^{\perp})$.

Let M be a lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ and $\nabla,\ \bar{\nabla}$ be the Levi-Civita connections on $M,\ \bar{M}$ respectively. The Gauss and Weingarten formulae are given by−

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \ \ \forall\ X, Y \in \Gamma(TM), \tag{10}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^t_X W \ \ \forall\ X \in \Gamma(TM),\ W \in \Gamma(tr(TM)), \tag{11}$$

where $\nabla_X Y, A_W X \in \Gamma(TM)$ and $h(X, Y), \nabla^t_X W \in \Gamma(tr(TM))$. Here h is a symmetric bilinear form on $\Gamma(TM)$ with values in $\Gamma(tr(TM))$ known as the *second fundamental form*, A is a linear operator on TM known as the *shape operator* and ∇^t is a linear connection on $tr(TM)$ known as the *transversal linear connection* on M.

Now, the equations (10) and (11) further reduce to−

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y) \ \ \forall\ X, Y \in \Gamma(TM), \tag{12}$$

$$\bar{\nabla}_X W = -A_W X + D^l(X, W) + D^s(X, W) \ \ \forall\ X \in \Gamma(TM),\ W \in \Gamma(tr(TM)),$$

where $h^l(X, Y) = L(h(X, Y))$, $h^s(X, Y) = S(h(X, Y))$, $D^l(X, W) = L(\nabla^t_X W)$, $D^s(X, W) = S(\nabla^t_X W)$ and $L,\ S$ are the projection morphisms of $tr(TM)$ on $ltr(TM),\ S(TM^{\perp})$ respectively. h^l and h^s are called the *lightlike second fundamental form* and the *screen second fundamental form* of M respectively.

In particular, we have

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N) \ \ \forall\ N \in \Gamma(ltr(TM)), \tag{13}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^s_X V + D^l(X, V) \ \ \forall\ V \in \Gamma(S(TM^{\perp})), \tag{14}$$

where ∇^l and ∇^s are linear connections on $ltr(TM)$ and $S(TM^{\perp})$ called the *lightlike transversal connection* and the *screen transversal connection* on M respectively.

Again, from (12)-(14) we get

$$\bar{g}(h^s(X, Y), V) + \bar{g}(Y, D^l(X, V)) = g(A_V X, Y), \tag{15}$$

$$\bar{g}(D^s(X, N), V) = \bar{g}(N, A_V X). \tag{16}$$

Let $\bar{P}$ be the projection morphism of TM on $S(TM)$, then we have $\forall\ X, Y \in \Gamma(TM)$, $Z \in \Gamma(Rad(TM))$,

$$\nabla_X \bar{P}Y = \nabla^*_X \bar{P}Y + h^*(X, \bar{P}Y), \tag{17}$$

$$\nabla_X Z = -A^*_Z X + \nabla^{*t}_X Z, \tag{18}$$

where h^* is the local second fundamental form on $S(TM)$ and A^* is the shape operator of $Rad(TM)$, $\nabla^*_X \bar{P}Y, A^*_Z X \in \Gamma(S(TM))$ and $h^*(X, \bar{P}Y), \nabla^{*t}_X Z \in \Gamma(Rad(TM))$. Here ∇^* and ∇^{*t} are induced connections on $S(TM)$ and $Rad(TM)$ respectively.

From (17) and (18) we have

$$\bar{g}(h^l(X, \bar{P}Y), Z) = g(A^*_Z X, \bar{P}Y), \tag{19}$$

$$\bar{g}(h^*(X, \bar{P}Y), N) = g(A_N X, \bar{P}Y), \tag{20}$$

$$\bar{g}(h^l(X, Z), Z) = 0,\ A^*_Z Z = 0. \tag{21}$$

Although the induced connection ∇ on M is not a metric connection, ∇^* and ∇^{*t} are metric connections on $S(TM)$ and $Rad(TM)$ respectively. As $\bar{\nabla}$ is a metric connection on $\bar{M}$, from (12) we get $\forall\ X', Y', Z' \in \Gamma(TM)$,

$$(\nabla_{X'} g)(Y', Z') = \bar{g}(h^l(X', Y'), Z') + \bar{g}(h^l(X', Z'), Y'). \tag{22}$$

Definition 2.4. A lightlike submanifold M of an indefinite Kenmotsu manifold $\bar{M}$, with the structure vector field ξ tangent to M, is called *totally umbilical* [6] if there exists a smooth transversal vector field $H \in \Gamma(tr(TM))$ on M, which is called the *transversal curvature vector field* of M, such that $\forall\ Z, W \in \Gamma(TM)$,

$$h(Z, W) = g(Z, W)H. \tag{23}$$

From (12), (14) and (23), we easily conclude that M is totally umbilical if and only if on each coordinate neighbourhood, there exist smooth vector fields $H^l \in \Gamma(ltr(TM))$, $H^s \in \Gamma(S(TM^\perp))$, such that $\forall\ V \in \Gamma(S(TM^\perp))$,

$$h^l(Z, W) = g(Z, W)H^l,\ D^l(Z, V) = 0, \tag{24}$$

$$h^s(Z, W) = g(Z, W)H^s. \tag{25}$$

Definition 2.5. A lightlike submanifold M of an indefinite Kenmotsu manifold $\bar{M}$, with the structure vector field ξ tangent to M, is called *contact totally umbilical lightlike submanifold* [18] if for a vector field α transversal to M, $\forall\ Z, W \in \Gamma(TM)$,

$$h(Z, W) = [g(Z, W) - \eta(Z)\eta(W)]\alpha + \eta(Z)h(W, \xi) + \eta(W)h(Z, \xi). \tag{26}$$

If $\alpha = 0$, then M is called *contact totally geodesic lightlike submanifold*.

Now, equating components from both sides of (26) belonging to $ltr(TM)$ and $S(TM^\perp)$ respectively, we have [10]

$$h^l(Z, W) = [g(Z, W) - \eta(Z)\eta(W)]\alpha_l + \eta(Z)h^l(W, \xi) + \eta(W)h^l(Z, \xi), \tag{27}$$

$$h^s(Z, W) = [g(Z, W) - \eta(Z)\eta(W)]\alpha_s + \eta(Z)h^s(W, \xi) + \eta(W)h^s(Z, \xi), \tag{28}$$

where $\alpha_l \in \Gamma(ltr(TM))$, $\alpha_s \in \Gamma(S(TM^\perp))$.

Definition 2.6. A lightlike submanifold M of an indefinite Kenmotsu manifold $\bar{M}$, with the structure vector field ξ tangent to M, is called *minimal* [1] if
(i) $h^s = 0$ on $Rad(TM)$,

(ii) *trace*$(h) = 0$ with respect to g restricted to $S(TM)$.

Definition 2.7. An r-lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$ of an indefinite Kenmotsu manifold $(\bar{M}, \bar{g})$, with the structure vector field ξ tangent to M, is called *contact screen generic lightlike (CSGL) submanifold* [12] if the following conditions are satisfied−

(i) $Rad(TM)$ is invariant with respect to ϕ i.e.,

$$\phi(Rad(TM)) = Rad(TM), \tag{29}$$

(ii) there exists a subbundle D_0 of $S(TM)$ such that

$$D_0 = \phi(S(TM)) \cap S(TM), \tag{30}$$

where D_0 is a non-degenerate distribution on M.

From Definition 2.7 we get

$$S(TM) = D_0 \oplus D' \oplus_{orth} < \xi >, \tag{31}$$

where D' is a complementary non-degenerate distribution to D_0 in $S(TM)$ such that

$$\phi(D') \not\subseteq S(TM),\ \phi(D') \not\subseteq S(TM^{\perp}).$$

Let P_0, P_1 and P' be the projection morphisms on D_0, $Rad(TM)$ and D' respectively, then we have $\forall\ X \in \Gamma(TM)$,

$$X = P_0X + P_1X + P'X + \eta(X)\xi \tag{32}$$

$$\Rightarrow X = PX + P'X + \eta(X)\xi, \tag{33}$$

where

$$D = D_0 \oplus_{orth} Rad(TM), \tag{34}$$

so that

$$TM = D \oplus D' \oplus_{orth} < \xi >, \tag{35}$$

D is invariant i.e. $\phi(D) = D$ and $PX \in \Gamma(D)$, $P'X \in \Gamma(D')$.

From (29) we have

$$\phi X = TX + \omega X, \tag{36}$$

where TX and ωX are the tangential and transversal parts of ϕX respectively. Also, it is clear that $\phi(D') \neq D'$.

Again, $\forall\ Y \in \Gamma(D')$,

$$\phi Y = TY + \omega Y, \tag{37}$$

where $TY \in \Gamma(D')$ and $\omega Y \in \Gamma(S(TM^{\perp}))$.

Similarly, $\forall\ W \in \Gamma(tr(TM))$,

$$\phi W = BW + CW, \tag{38}$$

where BW and CW are the tangential and transversal parts of ϕW respectively.

A lightlike submanifold M of an indefinite Kenmotsu manifold $\bar{M}$ is called *proper CSGL submanifold* if $D_0 \neq \{0\}$, $D' \neq \{0\}$ and then, from Definition 2.7 we have−

(A) $dim(Rad(TM)) = 2s \geq 2$ (by condition (i)),
(B) $dim(D_0) = 2a \geq 2$ (by condition (ii)),
(C) $dim(D') = 2p \geq 2$ so that $dim(M) \geq 7$ and $dim(\bar{M}) \geq 11$,
(D) any proper 7-dimensional CSGL submanifold must be 2-lightlike,
(E) $index(\bar{M}) \geq 2$ (by condition (i), since $\bar{M}$ is an indefinite Kenmotsu manifold).

Proposition 2.1. [12] *A contact SCR-lightlike submanifold M of an indefinite Kenmotsu manifold $\bar{M}$ is a CSGL submanifold such that the distribution D' is totally anti-invariant i.e.,*

$$S(TM^{\perp}) = \omega D' \oplus \mu, \tag{39}$$

where μ is a non-degenerate invariant distribution ($\phi(\mu) = \mu$).

Definition 2.8. An r-lightlike submanifold M of an indefinite Kenmotsu manifold $\bar{M}$ is called *generic r-lightlike submanifold* [16] if there exists a screen distribution $S(TM)$ of M such that

$$\phi(S(TM^{\perp})) \subset S(TM). \tag{40}$$

Proposition 2.2. [12] *A generic r-lightlike submanifold M of an indefinite Kenmotsu manifold $\bar{M}$ is a screen generic lightlike submanifold with $\mu = \{0\}$.*

Proposition 2.3. [12] *Any CSGL submanifold M of an indefinite Kenmotsu manifold $\bar{M}$ is an invariant lightlike submanifold if $D' = \{0\}$.*

Definition 2.9. [12] A CSGL submanifold M of an indefinite Kenmotsu manifold $\bar{M}$ is called *D-geodesic* if

$$h(X, Y) = 0 \ \ \forall X, Y \in \Gamma(D), \tag{41}$$

which implies that M is D-geodesic if

$$h^l(X, Y) = 0 = h^s(X, Y) \ \ \forall X, Y \in \Gamma(D). \tag{42}$$

Again, M is called *mixed geodesic* if

$$h(X, Y) = 0 \ \ \forall X \in \Gamma(D), \ Y \in \Gamma(D' \oplus_{orth} < \xi >). \tag{43}$$

3. CSGL Submanifolds

In this section, we investigate the necessary and sufficient (n & s) conditions for the induced connection on a CSGL submanifold M of an indefinite Kenmotsu manifold $\bar{M}$ to be a metric connection, for integrability & parallelism of some associated distributions, and for some distributions to be totally geodesic foliations. We also discuss about non-parallel distributions and more than one n & s conditions for M to be mixed geodesic.

Theorem 3.1. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a CSGL submanifold of an indefinite Kenmotsu manifold $(\bar{M}, \bar{g})$ with the structure vector field ξ tangent to M, then the induced connection ∇ on M is a metric connection if and only if $\forall$ $X, Y \in \Gamma(Rad(TM))$, $U \in \Gamma(S(TM))$,*

$$\bar{g}(h^l(X, \phi Y), \omega U) + \bar{g}(h^s(X, \phi Y), \omega U) = g(X, Y)\eta(U). \tag{44}$$

Proof. From (6) we have $\forall\ X, Y \in \Gamma(Rad(TM))$,

$$(\bar{\nabla}_X\phi)Y = \bar{\nabla}_X\phi Y - \phi(\bar{\nabla}_X Y) = \bar{g}(\phi X, Y)\xi \ \ (as\ \eta(Y) = 0),$$

on which applying ϕ and then using (1) we get

$$\bar{\nabla}_X Y = -\phi(\bar{\nabla}_X\phi Y) - g(X, Y)\xi. \tag{45}$$

Using (12), (18), (36), (38) in (45) we obtain

$$\nabla_X Y + h(X, Y) = TA^*_{\phi Y}X + \omega A^*_{\phi Y}X - B\nabla^{*t}_X\phi Y - C\nabla^{*t}_X\phi Y - Bh^l(X, \phi Y) - Ch^l(X, \phi Y) - Bh^s(X, \phi Y) - Ch^s(X, \phi Y) - g(X, Y)\xi. \tag{46}$$

Equating the tangential parts from both sides of (46) we get

$$\nabla_X Y = TA^*_{\phi Y}X - B\nabla^{*t}_X\phi Y - Bh^l(X, \phi Y) - Bh^s(X, \phi Y) - g(X, Y)\xi. \tag{47}$$

Now, we know that ∇ is a metric connection if and only if $Rad(TM)$ is a parallel distribution i.e., $g(\nabla_X Y, U) = 0\ \ \forall\ U \in \Gamma(S(TM))$.

From (47), on applying (3) and (36), we have $\forall\ U \in \Gamma(S(TM))$,

$$g(\nabla_X Y, U) = \bar{g}(h^l(X, \phi Y), \omega U) + \bar{g}(h^s(X, \phi Y), \omega U) - g(X, Y)\eta(U),$$

which implies that $g(\nabla_X Y, U) = 0$ if and only if (44) holds. This completes the proof.

Theorem 3.2. *Let $(M, g, S(TM), S(TM^\perp))$ be a CSGL submanifold of an indefinite Kenmotsu manifold $(\bar{M}, \bar{g})$ with the structure vector field ξ tangent to M, then*

(i) the distribution D_0 is integrable if and only if $\forall\ X, Y \in \Gamma(D_0),\ Z \in \Gamma(Rad(TM)),\ N \in \Gamma(ltr(TM)),\ V \in \Gamma(S(TM^\perp))$,

$$g(\nabla^*_X\phi Y - \nabla^*_Y\phi X, TZ) + g(Bh^*(X, \phi Y) - Bh^*(Y, \phi X), TZ) = 0, \tag{48}$$

$$\bar{g}(h^*(X, \phi Y) - h^*(Y, \phi X), \phi N) = 0, \tag{49}$$

$$g(\nabla^*_X\phi Y - \nabla^*_Y\phi X, BV) + \bar{g}(h^*(X, \phi Y) - h^*(Y, \phi X), CV) = 0; \tag{50}$$

(ii) the distribution D' is integrable if and only if (48), (49), (50) hold $\forall\ X, Y \in \Gamma(D')$;

(iii) the distribution D is integrable if and only if $\forall\ X, Y \in \Gamma(D),\ V \in \Gamma(S(TM^\perp))$,

$$g(\nabla_X TY - \nabla_Y TX, BV) + \bar{g}(h(X, TY) - h(Y, TX), CV) = 0. \tag{51}$$

Proof. (i) $\forall\ X, Y \in \Gamma(D_0)$, using (5) in the following equation

$$\bar{g}([X, Y], \xi) = \bar{g}(\bar{\nabla}_X Y, \xi) - \bar{g}(\bar{\nabla}_Y X, \xi) = -\bar{g}(Y, \bar{\nabla}_X\xi) + \bar{g}(X, \bar{\nabla}_Y\xi),$$

we have

$$\bar{g}([X, Y], \xi) = 0. \tag{52}$$

Now, $\forall\ Z \in \Gamma(Rad(TM))$, using (2) we have

$$\bar{g}([X, Y], Z) = \bar{g}(\phi\bar{\nabla}_X Y - \phi\bar{\nabla}_Y X, \phi Z).$$

Applying (1), (3), (6), (17), (36) and (38) on the above equation we obtain

$$\bar{g}([X,Y],Z) = g(\nabla^*_X\phi Y - \nabla^*_Y\phi X, TZ) + g(Bh^*(X,\phi Y) - Bh^*(Y,\phi X), TZ). \tag{53}$$

Again, $\forall\ N \in \Gamma(ltr(TM))$, using (2) we have

$$\bar{g}([X,Y],N) = \bar{g}(\phi\bar{\nabla}_X Y - \phi\bar{\nabla}_Y X, \phi N),$$

in which using (1), (3), (6) and (17) we get

$$\bar{g}([X,Y],N) = \bar{g}(h^*(X,\phi Y) - h^*(Y,\phi X), \phi N). \tag{54}$$

Also, $\forall\ V \in \Gamma(S(TM^\perp))$, using (2) we have

$$\bar{g}([X,Y],V) = \bar{g}(\phi\bar{\nabla}_X Y - \phi\bar{\nabla}_Y X, \phi V),$$

on which applying (1), (3), (6), (17) and (38) we get

$$\bar{g}([X,Y],V) = g(\nabla^*_X\phi Y - \nabla^*_Y\phi X, BV) + \bar{g}(h^*(X,\phi Y) - h^*(Y,\phi X), CV). \tag{55}$$

From (52)−(55), we conclude that $\forall\ X,Y \in \Gamma(D_0)$, $[X,Y] \in \Gamma(D_0)$ if and only if $\forall\ Z \in \Gamma(Rad(TM))$, $N \in \Gamma(ltr(TM))$, $V \in \Gamma(S(TM^\perp))$, equations (48), (49) and (50) hold.

(ii) The proof is similar as of (i).

(iii) $\forall\ X,Y \in \Gamma(D)$, similarly as (52) we have

$$\bar{g}([X,Y],\xi) = 0. \tag{56}$$

Now, $\forall\ V \in \Gamma(S(TM^\perp))$, using (2) we have

$$\bar{g}([X,Y],V) = \bar{g}(\phi\bar{\nabla}_X Y - \phi\bar{\nabla}_Y X, \phi V).$$

Using (1), (6), (10) and (38) in the above equation we obtain

$$\bar{g}([X,Y],V) = g(\nabla_X TY - \nabla_Y TX, BV) + \bar{g}(h(X,TY) - h(Y,TX), CV). \tag{57}$$

From (56) and (57), we conclude that $\forall\ X,Y \in \Gamma(D)$, $[X,Y] \in \Gamma(D)$ if and only if $\forall\ V \in \Gamma(S(TM^\perp))$, equation (51) holds.

Theorem 3.3. *Let $(M, g, S(TM), S(TM^\perp))$ be a CSGL submanifold of an indefinite Kenmotsu manifold $(\bar{M}, \bar{g})$ with the structure vector field ξ tangent to M, then*

(i) the distribution D_0 is not parallel,

(ii) the distribution D' is not parallel,

(iii) the distribution D is not parallel.

Proof. (i) Let $X,Y \in \Gamma(D_0)$, then using (5) in the following equation

$$g(\nabla_X Y, \xi) = -\bar{g}(Y, \bar{\nabla}_X \xi),$$

we get $g(\nabla_X Y, \xi) = -\bar{g}(X,Y) \neq 0$ since D_0 is non-degenerate. Hence, D_0 is not parallel.

(ii) The proof is similar as of (i).

(iii) Let $X, Y \in \Gamma(D) = \Gamma(D_0 \oplus_{orth} Rad(TM))$, then using (5) in the following equation

$$g(\nabla_X Y, \xi) = -\bar{g}(Y, \bar{\nabla}_X \xi),$$

we get $g(\nabla_X Y, \xi) = -\bar{g}(X, Y) \neq 0$ since D_0 is non-degenerate. Hence, D is not parallel.

Theorem 3.4. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a CSGL submanifold of an indefinite Kenmotsu manifold $(\bar{M}, \bar{g})$ with the structure vector field ξ tangent to M, then*

(i) the distribution $D_0 \oplus_{orth} <\xi>$ is parallel if and only if $\forall\, X \in \Gamma(TM),\ Y \in \Gamma(D_0 \oplus_{orth} <\xi>)$,

$$\nabla^*_X TY - A_{\omega Y} X \in \Gamma(D_0 \oplus_{orth} <\xi>),$$

$$h^*(X, TY) + \nabla^t_X \omega Y = 0;$$

(ii) the distribution $D' \oplus_{orth} <\xi>$ is parallel if and only if $\forall\, X \in \Gamma(TM),\ Y \in \Gamma(D' \oplus_{orth} <\xi>)$,

$$\nabla^*_X TY - A_{\omega Y} X \in \Gamma(D' \oplus_{orth} <\xi>),$$

$$h^*(X, TY) + \nabla^t_X \omega Y = 0;$$

(iii) the distribution $D \oplus_{orth} <\xi>$ is parallel if and only if $\forall\, X \in \Gamma(TM),\ Y \in \Gamma(D \oplus_{orth} <\xi>)$, $\bar{\nabla}_X TY$ has no component in $\phi(S(TM^{\perp}))$.

Proof. (i) Let $X \in \Gamma(TM),\ Y \in \Gamma(D_0 \oplus_{orth} <\xi>)$.

Now, for $Z \in \Gamma(Rad(TM))$, using (2) we have

$$g(\nabla_X Y, Z) = \bar{g}(\bar{\nabla}_X Y, \phi Z),$$

which leads to the following equation with the help of (1), (6), (11), (17) and (36)–

$$g(\nabla_X Y, Z) = g(\nabla^*_X TY - A_{\omega Y} X, TZ)$$

$$\Rightarrow g(\nabla_X Y, Z) = 0 \iff g(\nabla^*_X TY - A_{\omega Y} X, TZ) = 0. \tag{58}$$

Again, for $N \in \Gamma(ltr(TM))$, using (2) we have

$$g(\nabla_X Y, N) = \bar{g}(\bar{\nabla}_X Y, \phi N).$$

Applying (1), (6), (11), (17), (36) and (38) on the above equation we get

$$g(\nabla_X Y, N) = g(\nabla^*_X TY - A_{\omega Y} X, BN) + \bar{g}(h^*(X, TY) + \nabla^t_X \omega Y, CN)$$

$$\Rightarrow g(\nabla_X Y, N) = 0 \ \textit{if and only if}$$

$$g(\nabla^*_X TY - A_{\omega Y} X, BN) = 0, \tag{59}$$

$$\bar{g}(h^*(X, TY) + \nabla^t_X \omega Y, CN) = 0. \tag{60}$$

Also, for $V \in \Gamma(S(TM^{\perp}))$, using (2) we have

$$g(\nabla_X Y, V) = \bar{g}(\bar{\nabla}_X Y, \phi V),$$

in which using (1), (6), (11), (17), (36) and (38) we get

$$g(\nabla_X Y, V) = g(\nabla^*_X TY - A_{\omega Y} X, BV) + \bar{g}(h^*(X, TY) + \nabla^t_X \omega Y, CV)$$

$$\Rightarrow g(\nabla_X Y, V) = 0 \ \ if\ and\ only\ if$$

$$g(\nabla^*_X TY - A_{\omega Y}X, BV) = 0, \tag{61}$$

$$\bar{g}(h^*(X, TY) + \nabla^t_X \omega Y, CV) = 0. \tag{62}$$

The distribution $D_0 \oplus_{orth} < \xi >$ is parallel if and only if $\forall\ X \in \Gamma(TM),\ Y \in \Gamma(D_0 \oplus_{orth} < \xi >)$, $\nabla_X Y \in \Gamma(D_0 \oplus_{orth} < \xi >)$.

Now, combining (58), (59), (61) and then (60), (62) respectively, we have, $\nabla_X Y \in \Gamma(D_0 \oplus_{orth} < \xi >)$ if and only if

$$g(\nabla^*_X TY - A_{\omega Y}X, \phi U) = 0 \ \ \forall\ U \in \Gamma([Rad(TM) \oplus ltr(TM)] \oplus_{orth} S(TM^\perp))$$

$$\iff \nabla^*_X TY - A_{\omega Y}X \in \Gamma(D_0 \oplus_{orth} < \xi >),$$

$$and\ \ \bar{g}(h^*(X, TY) + \nabla^t_X \omega Y, \phi W) = 0 \ \ \forall\ W \in \Gamma(ltr(TM) \oplus_{orth} S(TM^\perp)) = \Gamma(tr(TM))$$

$$\iff h^*(X, TY) + \nabla^t_X \omega Y = 0.$$

(ii) The proof is similar as of (i).

(iii) Let $X \in \Gamma(TM),\ Y \in \Gamma(D \oplus_{orth} < \xi >)$. For $V \in \Gamma(S(TM^\perp))$, using (2) we have

$$g(\nabla_X Y, V) = \bar{g}(\bar{\nabla}_X Y, \phi V),$$

which leads to the following equation by the help of (1), (6), (10) and (36)−

$$g(\nabla_X Y, V) = \bar{g}(\bar{\nabla}_X TY, \phi V). \tag{63}$$

Now, the distribution $D \oplus_{orth} < \xi >$ is parallel if and only if $\forall\ X \in \Gamma(TM),\ Y \in \Gamma(D \oplus_{orth} < \xi >)$, $\nabla_X Y \in \Gamma(D \oplus_{orth} < \xi >)$.

Therefore, from (63), we get, the distribution $D \oplus_{orth} < \xi >$ is parallel if and only if $\bar{\nabla}_X TY$ has no component in $\phi(S(TM^\perp))$.

Theorem 3.5. *Let $(M, g, S(TM), S(TM^\perp))$ be a CSGL submanifold of an indefinite Kenmotsu manifold $(\bar{M}, \bar{g})$ with the structure vector field ξ tangent to M, then the distribution $D_0 \oplus_{orth} < \xi >$ is a totally geodesic foliation in $\bar{M}$ if and only if M is $D_0 \oplus_{orth} < \xi >$ −geodesic and $D_0 \oplus_{orth} < \xi >$ is parallel with respect to ∇ on M.*

Proof. $D_0 \oplus_{orth} < \xi >$ is a totally geodesic foliation in $\bar{M}$ if and only if $\forall\ X, Y \in \Gamma(D_0 \oplus_{orth} < \xi >)$, $\bar{\nabla}_X Y \in \Gamma(D_0 \oplus_{orth} < \xi >)$ i.e., $\bar{g}(\bar{\nabla}_X Y, Z) = \bar{g}(\bar{\nabla}_X Y, N) = \bar{g}(\bar{\nabla}_X Y, V) = 0 \ \ \forall\ Z \in \Gamma(Rad(TM)),\ N \in \Gamma(ltr(TM)),\ V \in \Gamma(S(TM^\perp))$.

Now, from (12), we have $\forall\ X, Y \in \Gamma(D_0 \oplus_{orth} < \xi >),\ N \in \Gamma(ltr(TM)),\ V \in \Gamma(S(TM^\perp))$,

$$\bar{g}(\bar{\nabla}_X Y, N) = \bar{g}(h^l(X, Y), N),$$

$$\bar{g}(\bar{\nabla}_X Y, V) = \bar{g}(h^s(X, Y), V).$$

Hence, if $D_0 \oplus_{orth} < \xi >$ is a totally geodesic foliation in $\bar{M}$, then $\bar{\nabla}_X Y \in \Gamma(D_0 \oplus_{orth} < \xi >)$ and thus, from the above two equations, we get $h^l(X, Y) = 0 = h^s(X, Y) \Rightarrow M$ is $D_0 \oplus_{orth} < \xi >$ −geodesic and from (12), $\nabla_X Y = \bar{\nabla}_X Y \in \Gamma(D_0 \oplus_{orth} < \xi >)$ so that $D_0 \oplus_{orth} < \xi >$ is parallel with respect to ∇ on M.

Conversely, if M is $D_0 \oplus_{orth} < \xi >$ −geodesic, then $h^l(X, Y) = 0 = h^s(X, Y)$ and hence, from (12), $\bar{\nabla}_X Y = \nabla_X Y \in \Gamma(TM)$. As $D_0 \oplus_{orth} < \xi >$ is parallel with respect to ∇ on M, $\bar{\nabla}_X Y = \nabla_X Y \in \Gamma(D_0 \oplus_{orth} < \xi >)$

$\Rightarrow D_0 \oplus_{orth} < \xi >$ is a totally geodesic foliation in $\bar{M}$.

Theorem 3.6. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a CSGL submanifold of an indefinite Kenmotsu manifold $(\bar{M}, \bar{g})$ with the structure vector field ξ tangent to M, then the distribution $D' \oplus_{orth} < \xi >$ is a totally geodesic foliation in $\bar{M}$ if and only if M is $D' \oplus_{orth} < \xi > -$geodesic and $D' \oplus_{orth} < \xi >$ is parallel with respect to ∇ on M.*

Proof. The proof is similar as of Theorem 3.5.

Theorem 3.7. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a CSGL submanifold of an indefinite Kenmotsu manifold $(\bar{M}, \bar{g})$ with the structure vector field ξ tangent to M, then the distribution $D \oplus_{orth} < \xi >$ is a totally geodesic foliation in $\bar{M}$ if and only if $h^s = 0$ on $D \oplus_{orth} < \xi >$.*

Proof. $D \oplus_{orth} < \xi >$ is a totally geodesic foliation in $\bar{M}$ if and only if $\forall\ X, Y \in \Gamma(D \oplus_{orth} < \xi >)$, $\bar{\nabla}_X Y \in \Gamma(D \oplus_{orth} < \xi >)$ i.e., $\bar{g}(\bar{\nabla}_X Y, V) = 0\ \ \forall\ V \in \Gamma(S(TM^{\perp}))$.

Now, from (12), we have $\forall\ X, Y \in \Gamma(D \oplus_{orth} < \xi >),\ V \in \Gamma(S(TM^{\perp}))$,

$$\bar{g}(\bar{\nabla}_X Y, V) = \bar{g}(\nabla_X Y + h^l(X, Y) + h^s(X, Y), V) = \bar{g}(h^s(X, Y), V)$$

$$\Rightarrow \bar{g}(\bar{\nabla}_X Y, V) = 0 \iff h^s(X, Y) = 0.$$

Hence, the proof is completed.

Theorem 3.8. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a CSGL submanifold of an indefinite Kenmotsu manifold $(\bar{M}, \bar{g})$ with the structure vector field ξ tangent to M. If M is mixed geodesic, then $\forall\ X \in \Gamma(D),\ Y \in \Gamma(D' \oplus_{orth} < \xi >),\ Z \in \Gamma(Rad(TM)),\ V \in \Gamma(S(TM^{\perp}))$,*

$$(i)\ g((\nabla_X T)Y, Z) = g(A_{\omega Y} X - \eta(Y)\phi X, Z),\ D^l(X, \omega Y) = -h^l(X, TY), \tag{64}$$

$$(ii)\ g(A_{\omega Y} X - \nabla_X TY, BV) = \bar{g}(\nabla^s_X \omega Y + h^s(X, TY), CV). \tag{65}$$

Proof. Let M be mixed geodesic, then $\forall X \in \Gamma(D),\ Y \in \Gamma(D' \oplus_{orth} < \xi >)$,

$$h(X, Y) = 0 \Rightarrow \bar{g}(h(X, Y), Z) = 0 = \bar{g}(h(X, Y), V)$$

$$\Rightarrow \bar{g}(h^l(X, Y), Z) = 0\ \ \forall\ Z \in \Gamma(Rad(TM)), \tag{66}$$

$$\bar{g}(h^s(X, Y), V) = 0\ \ \forall\ V \in \Gamma(S(TM^{\perp})). \tag{67}$$

(i) We have, on using (12) and (66), $\forall\ X \in \Gamma(D),\ Y \in \Gamma(D' \oplus_{orth} < \xi >),\ Z \in \Gamma(Rad(TM))$,

$$\bar{g}(\bar{\nabla}_X Y, Z) = \bar{g}(\nabla_X Y, Z). \tag{68}$$

Replacing Z by ϕZ in (68) we have

$$\bar{g}(\bar{\nabla}_X Y, \phi Z) = \bar{g}(\nabla_X Y, \phi Z),$$

on which applying (6), (12), (14), (37) to the left side and (3), (36) to the right side we obtain

$$g(\nabla_X TY - A_{\omega Y} X, Z) + \eta(Y) g(\phi X, Z) + \bar{g}(D^l(X, \omega Y), Z) + \bar{g}(h^l(X, TY), Z) = g(T(\nabla_X Y), Z). \tag{69}$$

Comparing the tangential and transversal parts of (69) from both sides, we get respectively

$$g(\nabla_X TY - A_{\omega Y} X, Z) + \eta(Y) g(\phi X, Z) = g(T(\nabla_X Y), Z)$$

$$\Rightarrow g(\nabla_X TY - A_{\omega Y} X + \eta(Y)\phi X, Z) = g(T(\nabla_X Y), Z)$$

$$\Rightarrow g((\nabla_X T)Y, Z) = g(A_{\omega Y}X - \eta(Y)\phi X, Z),$$

and

$$D^l(X, \omega Y) + h^l(X, TY) = 0$$

$$\Rightarrow D^l(X, \omega Y) = -h^l(X, TY).$$

(ii) By the help of the equations (12), (14), (37) and (38) we have

$$\bar{g}(\bar{\nabla}_X \phi Y, \phi V) = g(\nabla_X TY - A_{\omega Y}X, BV) + \bar{g}(\nabla^s_X \omega Y + h^s(X, TY), CV). \quad (70)$$

Now, using (12) and (67) we have

$$\bar{g}(\bar{\nabla}_X Y, V) = \bar{g}(h^s(X, Y), V) = 0. \quad (71)$$

Again, using (2) we have

$$\bar{g}(\bar{\nabla}_X Y, V) = \bar{g}(\phi(\bar{\nabla}_X Y), \phi V).$$

Using (1), (2) and (6) in the above equation we obtain

$$\bar{g}(\bar{\nabla}_X Y, V) = \bar{g}(\bar{\nabla}_X \phi Y, \phi V). \quad (72)$$

Equations (71) and (72) imply

$$\bar{g}(\bar{\nabla}_X \phi Y, \phi V) = 0. \quad (73)$$

Equations (70) and (73) lead to

$$g(\nabla_X TY - A_{\omega Y}X, BV) + \bar{g}(\nabla^s_X \omega Y + h^s(X, TY), CV) = 0$$

$$\Rightarrow g(A_{\omega Y}X - \nabla_X TY, BV) = \bar{g}(\nabla^s_X \omega Y + h^s(X, TY), CV).$$

Theorem 3.9. *Let $(M, g, S(TM), S(TM^\perp))$ be a CSGL submanifold of an indefinite Kenmotsu manifold $(\bar{M}, \bar{g})$ with the structure vector field ξ tangent to M, then M is mixed geodesic if and only if $\forall\ X \in \Gamma(D),\ Y \in \Gamma(D' \oplus_{orth} < \xi >)$,*

$$D^l(X, \omega Y) = -h^l(X, TY), \quad (74)$$

$$\omega(A_{\omega Y}X - \nabla_X TY) = C(h^s(X, TY) + \nabla^s_X \omega Y). \quad (75)$$

Proof. Let $X \in \Gamma(D),\ Y \in \Gamma(D' \oplus_{orth} < \xi >)$, then from (1) we have

$$\phi^2 Y = -Y + \eta(Y)\xi$$

$$\Rightarrow \phi(\phi Y) = -Y + \eta(Y)\xi,$$

on which applying (37) we get

$$\phi(TY + \omega Y) = -Y + \eta(Y)\xi.$$

Now, differentiating the above equation with respect to X i.e., operating with $\bar{\nabla}_X$ on both sides we obtain

$$(\bar{\nabla}_X \phi)\phi Y + \phi(\bar{\nabla}_X TY) + \phi(\bar{\nabla}_X \omega Y) = -\bar{\nabla}_X Y - \bar{g}(Y, \bar{\nabla}_X \xi)\xi + \eta(Y)\bar{\nabla}_X \xi,$$

in which using (5), (6), (10), (12), (14), (36) and (38) we obtain

$$[T(\nabla_X TY) + \omega(\nabla_X TY) + Bh^l(X, TY) + Ch^l(X, TY) + Bh^s(X, TY) + Ch^s(X, TY)]$$

$$+[-TA_{\omega Y}X - \omega A_{\omega Y}X + B\nabla^s_X\omega Y + C\nabla^s_X\omega Y + BD^l(X,\omega Y) + CD^l(X,\omega Y)] + 2\bar{g}(X,Y)\xi - \eta(X)\eta(Y)\xi$$
$$= -\nabla_X Y - h(X,Y) + \eta(Y)X. \tag{76}$$

Equating the transversal parts from both sides of (76), we have

$$h(X,Y) = [\omega(A_{\omega Y}X - \nabla_X TY) - C(h^s(X,TY) + \nabla^s_X\omega Y)] - C(h^l(X,TY) + D^l(X,\omega Y)). \tag{77}$$

Now, M is mixed geodesic if and only if $h(X,Y) = 0\ \ \forall\ X \in \Gamma(D),\ Y \in \Gamma(D'\oplus_{orth} < \xi >)$. Hence, from (77), we have, M is mixed geodesic if and only if the equations (74) and (75) hold.

Theorem 3.10. *Let $(M, g, S(TM), S(TM^\perp))$ be a CSGL submanifold of an indefinite Kenmotsu manifold $(\bar{M}, \bar{g})$ with the structure vector field ξ tangent to M. If M is mixed geodesic, then $\forall\ X \in \Gamma(D),\ Y \in \Gamma(D'\oplus_{orth} < \xi >)$,*

$$(\nabla_X T)Y = A_{\omega Y}X + g(TX,Y)\xi - \eta(Y)TX, \tag{78}$$

$$\omega\nabla_X Y = h^s(X,TY) + \nabla^s_X\omega Y. \tag{79}$$

Proof. As M is mixed geodesic, we have, $\forall\ X \in \Gamma(D),\ Y \in \Gamma(D'\oplus_{orth} < \xi >)$,

$$h(X,Y) = 0. \tag{80}$$

From (37) we have

$$\phi Y = TY + \omega Y,$$

which gives us, on differentiating both sides with respect to X,

$$(\bar{\nabla}_X\phi)Y + \phi(\bar{\nabla}_X Y) = \bar{\nabla}_X TY + \bar{\nabla}_X\omega Y.$$

Now, using (6), (10), (12), (14), (36), (38) and (80) in the above equation we obtain

$$\bar{g}(\phi X,Y)\xi - \eta(Y)\phi X + T\nabla_X Y + \omega\nabla_X Y = \nabla_X TY + h^l(X,TY) + h^s(X,TY) - A_{\omega Y}X + \nabla^s_X\omega Y + D^l(X,\omega Y),$$

on which applying (74) we have

$$\bar{g}(\phi X,Y)\xi - \eta(Y)\phi X + T\nabla_X Y + \omega\nabla_X Y = \nabla_X TY + h^s(X,TY) - A_{\omega Y}X + \nabla^s_X\omega Y. \tag{81}$$

Again, comparing the tangential and transversal parts from both sides of (81), we have respectively

$$g(TX,Y)\xi - \eta(Y)TX = (\nabla_X T)Y - A_{\omega Y}X\ \ (using\ (36))$$
$$\Rightarrow (\nabla_X T)Y = A_{\omega Y}X + g(TX,Y)\xi - \eta(Y)TX,$$
$$\omega\nabla_X Y = h^s(X,TY) + \nabla^s_X\omega Y.$$

4. Totally Umbilical CSGL Submanifolds

In this section, we study some properties satisfied by a proper totally umbilical CSGL submanifold M of an indefinite Kenmotsu manifold $\bar{M}$.

Theorem 4.1. *Let $(M, g, S(TM), S(TM^\perp))$ be a proper totally umbilical CSGL submanifold of an indefinite Kenmotsu manifold $(\bar{M}, \bar{g})$ with the structure vector field ξ tangent to M, then $\alpha_s \notin \Gamma(\mu)$.*

Proof. Let $X, Y \in \Gamma(TM)$, then from (36) we have

$$\phi Y = TY + \omega Y.$$

Now, differentiating the above equation with respect to X, we get

$$(\bar{\nabla}_X\phi)Y + \phi(\bar{\nabla}_X Y) = \bar{\nabla}_X TY + \bar{\nabla}_X \omega Y.$$

Applying (6), (12), (14), (36) and (38) on the above equation we obtain

$$g(TX,Y)\xi - \eta(Y)TX - \eta(Y)\omega X + T(\nabla_X Y) + \omega(\nabla_X Y) + Bh^l(X,Y) + Ch^l(X,Y) + Bh^s(X,Y) + Ch^s(X,Y)$$

$$= \nabla_X TY + h^l(X,TY) + h^s(X,TY) - A_{\omega Y}X + \nabla^s_X \omega Y + D^l(X,\omega Y).$$

Comparing the tangential and transversal parts of the above equation, we get respectively

$$g(TX,Y)\xi - \eta(Y)TX + T(\nabla_X Y) + Bh^l(X,Y) + Bh^s(X,Y) = \nabla_X TY - A_{\omega Y}X, \tag{82}$$

$$-\eta(Y)\omega X + \omega(\nabla_X Y) + Ch^l(X,Y) + Ch^s(X,Y) = h^l(X,TY) + h^s(X,TY) + \nabla^s_X \omega Y + D^l(X,\omega Y). \tag{83}$$

Again, from (27) and (28), we have respectively

$$Ch^l(X,Y) = [g(X,Y) - \eta(X)\eta(Y)]C\alpha_l + \eta(X)Ch^l(Y,\xi) + \eta(Y)Ch^l(X,\xi), \tag{84}$$

$$Ch^s(X,Y) = [g(X,Y) - \eta(X)\eta(Y)]C\alpha_s + \eta(X)Ch^s(Y,\xi) + \eta(Y)Ch^s(X,\xi). \tag{85}$$

Adding (84), (85) and then using (83) to replace the value obtained in the left hand side of the resultant equation, we get

$$\eta(Y)\omega X - \omega(\nabla_X Y) + h^l(X,TY) + h^s(X,TY) + \nabla^s_X \omega Y + D^l(X,\omega Y)$$

$$= [g(X,Y) - \eta(X)\eta(Y)]C(\alpha_l + \alpha_s) + \eta(X)C[h^l(Y,\xi) + h^s(Y,\xi)] + \eta(Y)C[h^l(X,\xi) + h^s(X,\xi)]. \tag{86}$$

Let $X, Y \in \Gamma(D)$, then $\phi X, \phi Y \in \Gamma(\phi(D)) = \Gamma(D) \Rightarrow \phi X = TX$, $\phi Y = TY$ and $\omega X = 0 = \omega Y$. Also, $\eta(X) = 0 = \eta(Y)$. Hence, from (86) we obtain

$$-\omega(\nabla_X Y) + h^l(X,TY) + h^s(X,TY) = g(X,Y)C(\alpha_l + \alpha_s). \tag{87}$$

Equating the $S(TM^\perp)$-components from both sides of (87), we have

$$-\omega(\nabla_X Y) + h^s(X,TY) = g(X,Y)C\alpha_s. \tag{88}$$

Replacing X by ϕX, Y by ϕY in (88) and then using (2) and $\eta(X) = 0 = \eta(Y)$, we get

$$-\omega(\nabla_{\phi X}\phi Y) + h^s(\phi X, \phi^2 Y) = g(X,Y)C\alpha_s. \tag{89}$$

Again, from (28) and with the help of (1), (2) we have

$$h^s(\phi X, \phi^2 Y) = g(X, \phi Y)\alpha_s. \tag{90}$$

Now, applying (90) on (89) we obtain

$$-\omega(\nabla_{\phi X}\phi Y) + g(X,\phi Y)\alpha_s = g(X,Y)C\alpha_s. \tag{91}$$

Putting $X = \phi Y$ in (91) and using the fact that $g(Y,\phi Y) = -g(\phi Y, Y) \Rightarrow g(Y,\phi Y) = 0$, we get

$$g(\phi Y, \phi Y)\alpha_s = \omega(\nabla_{\phi^2 Y}\phi Y),$$

which gives, on replacing ϕY by Y,

$$g(Y,Y)\alpha_s = \omega(\nabla_{\phi Y} Y). \tag{92}$$

Let $Y \in \Gamma(D_0)$, then (92) gives $\alpha_s \notin \Gamma(\mu)$ since D_0 is a non-degenerate distribution.

Theorem 4.2. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a proper totally umbilical CSGL submanifold of an indefinite Kenmotsu manifold $(\bar{M}, \bar{g})$ with the structure vector field ξ tangent to M, then the induced connection ∇ is a metric connection on $D \oplus_{orth} < \xi >$.*

Proof. Equating $ltr(TM)$-components from (87), we have $\forall\ X, Y \in \Gamma(D)$,

$$h^l(X, TY) = g(X, Y)C\alpha_l. \tag{93}$$

Replacing X by ϕX, Y by ϕY and then using (2), (27) and $\eta(X) = 0 = \eta(Y)$, we get

$$g(X, \phi Y)\alpha_l = g(X, Y)C\alpha_l. \tag{94}$$

Now, interchanging X, Y and then applying (3), we obtain

$$-g(X, \phi Y) = g(X, Y)C\alpha_l. \tag{95}$$

Subtracting (95) from (94) we get

$$2g(X, \phi Y)\alpha_l = 0. \tag{96}$$

Putting $X = \phi Y$ in (96) we have

$$2g(\phi Y, \phi Y)\alpha_l = 0$$

$$\Rightarrow \alpha_l = 0 \tag{97}$$

since $D = D_0 \oplus_{orth} < \xi >$ and D_0 is non-degenerate.

Again, applying (97) and $\eta(X) = 0 = \eta(Y)$ on (27), we get $\forall\ X, Y \in \Gamma(D)$,

$$h^l(X, Y) = 0. \tag{98}$$

By (5) and (12) we have

$$\nabla_X \xi + h^l(X, \xi) + h^s(X, \xi) = X - \eta(X)\xi,$$

which gives, on equating the tangential and transversal parts from both sides respectively

$$\nabla_X \xi = X - \eta(X)\xi, \tag{99}$$

$$h^l(X, \xi) = 0, \tag{100}$$

$$h^s(X, \xi) = 0. \tag{101}$$

Combining (98) and (100) we get

$$h^l = 0 \ \ on \ \ D \oplus_{orth} < \xi >$$

$$\Rightarrow \nabla g = 0 \ \ on \ \ D \oplus_{orth} < \xi > \quad (by\ (22))$$

$$\Rightarrow \nabla \ \ is\ a\ metric\ connection\ on \ \ D \oplus_{orth} < \xi > .$$

5. Minimal CSGL Submanifolds

In this section, we find the necessary and sufficient conditions for minimality of the distribution $D_0 \oplus_{orth} < \xi >$ associated to a CSGL submanifold M of an indefinite Kenmotsu manifold $\bar{M}$ and also of M itself.

Theorem 5.1. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a CSGL submanifold of an indefinite Kenmotsu manifold $(\bar{M}, \bar{g})$ with the structure vector field ξ tangent to M, then the distribution $D_0 \oplus_{orth} < \xi >$ is minimal if and only if $\nabla_X X + \nabla_{\phi X}\phi X \in \Gamma(D_0 \oplus_{orth} < \xi >)$ $\forall X \in \Gamma(D_0 \oplus_{orth} < \xi >)$.*

Proof. From the description of D_0, it is clear that $D_0 \oplus_{orth} < \xi >$ is minimal if and only if $h^s = 0$ on $D_0 \oplus_{orth} < \xi >$. Now, let $X \in \Gamma(D_0 \oplus_{orth} < \xi >)$.

For $V \in \Gamma(S(TM^{\perp}))$, by the help of the equations (3), (6), (12) and (15), we obtain

$$g(\nabla_X X, \phi V) = -g(A_V X, \phi X) - \bar{g}(h^s(X, X), \phi V). \tag{102}$$

Also, using (1), (2), (6), (12), (15) and $\eta(X) = 0 = \eta(V)$, we get

$$g(\nabla_{\phi X}\phi X, \phi V) = g(A_V \phi X, X) - \bar{g}(h^s(\phi X, \phi X), \phi V)$$

$$\Rightarrow g(\nabla_{\phi X}\phi X, \phi V) = g(\phi X, A_V X) - \bar{g}(h^s(\phi X, \phi X), \phi V) \tag{103}$$

since A is symmetric on $S(TM^{\perp})$.

Addition of (102) and (103) gives

$$g(\nabla_X X + \nabla_{\phi X}\phi X, \phi V) = -\bar{g}(h^s(X, X) + h^s(\phi X, \phi X), \phi V),$$

which implies that $h^s = 0$ on $D_0 \oplus_{orth} < \xi > \iff \nabla_X X + \nabla_{\phi X}\phi X \in \Gamma(D_0 \oplus_{orth} < \xi >)$.

Theorem 5.2. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a CSGL submanifold of an indefinite Kenmotsu manifold $(\bar{M}, \bar{g})$ with the structure vector field ξ tangent to M, then M is minimal if and only if $h^s|_{Rad(TM)} = 0$ and $trace(A^*_{Z_k})|_{S(TM)} = 0$, $trace(A_{V_p})|_{S(TM)} = 0$, $Z_k \in \Gamma(Rad(TM))$, $V_p \in \Gamma(S(TM^{\perp}))$.*

Proof. Putting $X = \xi$ in (5) and then using (1) in the right side and (10) in the left side, we have

$$\nabla_\xi \xi + h(\xi, \xi) = 0$$

$$\Rightarrow h(\xi, \xi) = 0. \tag{104}$$

Now, let us consider a quasi orthonormal frame $\{Z_1, ..., Z_{2r}, e_1, ..., e_{m-2r-1}, \xi, N_1, ..., N_{2r}, V_1, ..., V_{n-2r}\}$ such that $\{e_i\}_1^{2a}$ are tangent to D_0 and $\{e_j\}_{2a+1}^{m-2r-1}$ are tangent to D' with signatures $\{\epsilon_i\}_1^{m-2r-1}$, $Z_k \in \Gamma(Rad(TM))$, $N_k \in \Gamma(ltr(TM))$, $V_p \in \Gamma(S(TM^{\perp}))$. Then we have

$$trace(h)|_{S(TM)} = trace(h)|_{D_0} + trace(h)|_{D'} \quad (by\ (104))$$

$$= \sum_{i=1}^{2a} \epsilon_i [h^l(e_i, e_i) + h^s(e_i, e_i)] + \sum_{j=2a+1}^{m-2r-1} \epsilon_j [h^l(e_j, e_j) + h^s(e_j, e_j)]$$

$$= \sum_{i=1}^{2a} \epsilon_i \Big[\frac{1}{2r}\sum_{k=1}^{2r} \bar{g}(h^l(e_i, e_i), Z_k)N_k + \frac{1}{n-2r}\sum_{p=1}^{n-2r} \bar{g}(h^s(e_i, e_i), V_p)V_p\Big]$$

$$+ \sum_{j=2a+1}^{m-2r-1} \epsilon_j \Big[\frac{1}{2r}\sum_{k=1}^{2r} \bar{g}(h^l(e_j, e_j), Z_k)N_k + \frac{1}{n-2r}\sum_{p=1}^{n-2r} \bar{g}(h^s(e_j, e_j), V_p)V_p\Big]. \tag{105}$$

Again, from (15) and (19), we have respectively

$$\bar{g}(h^l(e_i,e_i),Z_k)N_k = g(A^*_{Z_k}e_i,e_i)N_k, \quad (106)$$

$$\bar{g}(h^s(e_j,e_j),V_p)V_p = g(A_{V_p}e_j,e_j)V_p. \quad (107)$$

Applying (106) and (107) on (105), we obtain

$$trace(h)|_{S(TM)} = \sum_{k=1}^{2r} trace(A^*_{Z_k})|_{D_0\oplus D'} + \sum_{p=1}^{n-2r} trace(A_{V_p})|_{D_0\oplus D'}$$

$$\Rightarrow trace(h)|_{S(TM)} = 0 \iff trace(A^*_{Z_k})|_{S(TM)} = 0 = trace(A_{V_p})|_{S(TM)}, \quad (108)$$

Using Definition 2.6, we conclude that, M is minimal if and only if (108) holds and $h^s|_{Rad(TM)} = 0$.

Example

Let us consider the 13-dimensional manifold $\bar{M} = \{(x^1,...,x^{13}) \in \mathbb{R}^{13}_6 : x^{13} \neq 0\}$, where $(x^1,...,x^{13})$ are the standard coordinates in $\mathbb{R}^{13}_6$. Then $\bar{M}$ forms an indefinite Kenmotsu manifold together with the indefinite almost contact metric structure $(\phi,\xi,\eta,\bar{g})$ such that $\bar{g}$ is the semi-Riemannian metric with signature $(+,+,+,+,+,+,-,-,-,-,-,-,+)$ defined by

$$\bar{g}(e_i,e_i) = 1 \text{ for } i = 1,2,3,4,5,6,13 \ \ and \ \ \bar{g}(e_i,e_i) = -1 \ \ for \ \ i = 7,8,9,10,11,12,$$

$$\bar{g}(e_i,e_j) = 0 \ \ \forall i \neq j, \ \ i,j = 1,...,13,$$

where $\{e_i\}_{i=1}^{13}$ are linearly independent vector fields at each point of $T\bar{M}$ given by

$$e_i = x^{13}\frac{\partial}{\partial x^i} \ \ for \ \ i = 1,2,3,4,5,6 \ \ and \ \ e_i = -x^{13}\frac{\partial}{\partial x^i} \ \ for \ \ i = 7,8,9,10,11,12,13;$$

$$\phi e_1 = -e_2,\ \phi e_2 = e_1,\ \phi e_3 = -e_4,\ \phi e_4 = e_3,\ \phi e_5 = -e_6,\ \phi e_6 = e_5,$$

$$\phi e_7 = -e_8,\ \phi e_8 = e_7,\ \phi e_9 = -e_{10},\ \phi e_{10} = e_9,\ \phi e_{11} = -e_{12},\ \phi e_{12} = e_{11},\ \phi e_{13} = 0;$$

$$\xi = e_{13} = -x^{13}\frac{\partial}{\partial x^{13}},\ \eta = -\frac{1}{x^{13}}dx^{13}.$$

Now, the map given by

$$x(u_1,u_2,u_3,u_4,u_5,u_6,u_7) = (u_1,u_2,u_3,u_4,u_5,u_6,0,0,u_1,u_2,u_4,u_3,u_7)$$

defines a 7-dimensional submanifold M of $\bar{M}$, where

$$E_1 = e_1 + e_9,\ E_2 = e_2 + e_{10},\ E_3 = e_5,\ E_4 = e_6,\ E_5 = e_3 + e_{12},\ E_6 = e_4 + e_{11},\ E_7 = \xi$$

form a local orthogonal basis of $TM = Rad(TM)\oplus_{orth}S(TM) = D\oplus D'\oplus_{orth} <\xi>$ such that $D = Rad(TM)\oplus_{orth}D_0$ and $S(TM) = D_0 \oplus D'\oplus_{orth} <\xi>$ with $Rad(TM) = <E_1,E_2>$, $D_0 = <E_3,E_4>$, $D' = <E_5,E_6>$ so that $D = <E_1,E_2,E_3,E_4>$ and $S(TM) = <E_3,E_4,E_5,E_6,E_7>$.

Again, $tr(TM) = S(TM^{\perp}) \oplus_{orth} ltr(TM)$ such that $ltr(TM) = <N_1,N_2>$ and $S(TM^{\perp}) = <V_1,V_2,V_3,V_4>$, where

$$N_1 = e_1,\ N_2 = e_2$$

so that $\bar{g}(E_1,N_1) = 1 = \bar{g}(E_2,N_2),\ \bar{g}(E_2,N_1) = 0 = \bar{g}(E_1,N_2),\ \bar{g}(N_1,N_2) = 0$, and

$$V_1 = e_{11},\ V_2 = e_{12},\ V_3 = e_7,\ V_4 = e_8$$

such that $\omega D' = <V_1,V_2>$ and $\mu = <V_3,V_4>$ satisfying $\phi(\mu) = \mu$ since $\phi V_3 = -V_4,\ \phi V_4 = V_3$.

Now, $\phi E_1 = -E_2,\ \phi E_2 = E_1,\ \phi E_3 = -E_4,\ \phi E_4 = E_3,\ \phi E_5 = -e_4 + e_{11},\ \phi E_6 = e_3 - e_{12},\ \phi E_7 = \phi\xi = 0$ so that $\phi(Rad(TM)) = Rad(TM)$, $D_0 = \phi(S(TM)) \cap S(TM)$, $\phi(D) = D$ and $\phi(D') \not\subseteq S(TM)$, $\phi(D') \not\subseteq S(TM^{\perp})$.

Therefore, M is a CSGL submanifold of $\bar{M}$.

Conclusion

The primary difference between the theory of lightlike submanifolds and the classical theory of Riemannian or semi-Riemannian submanifolds arises due to the fact that, in the first case, a part of the normal vector bundle lies in the tangent bundle of the submanifold such that the intersection of the tangent bundle and the normal bundle is called the radical or lightlike or null distribution, whereas, in the second case, that intersection is null. Hence we can see that, the lightlike or null cone of a semi-Euclidean space is a typical example of lightlike submanifold of a semi-Riemannian manifold. This unique property of lightlike submanifolds has made it an interesting topic for the researchers since its conceptualization and the author is no exception. In this paper, contact screen generic lightlike (CSGL) submanifolds of indefinite Kenmotsu manifold has been studied as a next step in the study of such submanifolds which are recently introduced in the context of indefinite Sasakian manifolds by R. S. Gupta. In addition, an example of such submanifold has been constructed at the end. Therefore, the extensive applications of the topic of this paper (discussed below) makes it an active field for researchers of Physics as well as of Differential Geometry.

Geometry of lightlike submanifolds is used in Mathematical Physics, in particular, in general theory of relativity since lightlike submanifolds produce models of different types of horizons for e.g. event horizons, Cauchy horizons, Kruskal's horizons. Lightlike hypersurfaces are also studied in the theory of electromagnetism, radiation fields, Killing horizons, asymptotically flat spacetimes. Lightlike submanifolds appear as smooth parts of event horizons of the Kruskal and Keer black holes.

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