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Some results for a kind of Kirchhoff-type problems involving the $p(x)$-biharmonic operator

Mahmoud El Ahmadi[a,*], Anass Lamaizi[a], Gustavo S. A. Costa[b]

[a] *Department of Mathematics, Faculty of Sciences, Mohammed I University, Oujda, 60000, Morocco*
[b] *Departamento de Matemática/CCET, Universidade Federal do Maranhão, 65080-805, São Luís - MA, Brazil*

Abstract. Our focus in this study revolves around investigating a Kirchhoff problem involving the $p(x)$-biharmonic operator. The purpose is to study the existence and multiplicity of weak solutions for our problem without assuming the Ambrosetti-Rabinowitz condition. By using the mountain pass theorem with Cerami condition, we show the existence of non-trivial weak solutions for the considered problem. Furthermore, our second purpose is to determine the precise positive interval of λ for which the problem admits at least two nontrivial solutions. Finally, the existence of infinitely many solutions is proved by employing the fountain theorem.

1. Introduction

Recently, there has been a lot of attention devoted to the study of differential equations and variational problems with variable exponent. Indeed, Some of these equations originate from diverse domains of applied physics and mathematics such like Micro-Electro-Mechanical systems, surface diffusion on solids, flow in Hele-Shaw cells. Additionally, this class of equations can describe the static from the change of beam or the sport of rigid body, there are plenty of authors who have draw attention to that kind of non-linearity furnishes a model to study traveling waves in suspension bridges, for more details, see [1, 8, 18, 30, 31].

In the present article, we mainly study the following Kirchhoff-type problem involving the $p(x)$-biharmonic operator of the form:

$$\begin{cases} m\left(\int_\Omega \frac{|\Delta u|^{p(x)} + d(x)|u|^{p(x)}}{p(x)} dx\right)\left(\Delta^2_{p(x)} u + d(x)|u|^{p(x)-2}u\right) = \lambda f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open domain with smooth boundary $\partial\Omega$, $N \geq 3$, $\lambda > 0$ is a real number, $d \in L^\infty(\Omega)$ such that $\inf_{x\in\Omega} d(x) = d^- > 0$, $m \in C([0,+\infty),\mathbb{R}_+)$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $\Delta^2_{p(x)} u := \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the $p(x)-$ biharmonic operator

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* Corresponding author: Mahmoud El Ahmadi
Email addresses: elahmadi.mahmoud@ump.ac.ma (Mahmoud El Ahmadi), lamaizi.anass@ump.ac.ma (Anass Lamaizi), gsa.costa@ufma.br (Gustavo S. A. Costa)

The differential operator

$$\Delta^2_{p(x)}u := \Delta(|\Delta u|^{p(x)-2}\Delta u)$$

is a natural generalization of the classical $p-$biharmonic operator $\Delta(|\Delta u|^{p-2}\Delta u)$ when $p > 1$ is a real constant. However, the $p(x)$-biharmonic operator has a more complicated nonlinearity than the p-biharmonic operator, due to the fact that $p(x)$-biharmonic operator is not homogeneous. This fact involves certain difficulties, for example, we cannot use the Lagrange Multiplier theorem to solve many problems involving this operator.

Problem (1) is a nonlocal problem due to the presence of the term m, which implies that the equation in (1) is no longer pointwise identities. Note that the problem (1) is similar to the stationary problem introduced by Kirchhof in [22]:

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|dx\right)\frac{\partial^2 u}{\partial x^2} = 0.$$

More precisely, Kirchhoff proposed this model as an extension of D'Alembert's classical wave equation, taking into account the effects of variations in string length during vibration. Furthermore, S. Woinowsky-Krieger in [33] considered the Kirchhoff-type evolution equation:

$$u_{tt} + \Delta^2 u - M(\|\nabla u\|^2)\Delta u = g(x,u),$$

which is a model for the deviation of an extensible beam.

A problem involving the $p(x)$-biharmonic operator was firstly investigated by A. Ayoujil and A.R. El Amrouss in [7], the authors studied the spectrum of the following problem:

$$\begin{cases}\Delta^2_{p(x)}u = \lambda|u|^{p(x)-2}u & \text{in } \Omega,\\ u = \Delta u = 0 & \text{on } \partial\Omega,\end{cases}$$

and they proved the existence of infinitely many eigenvalue sequences and $\sup\Gamma = +\infty$, where Γ is the set of all eigenvalues. Afterwards, various authors studied the existence and multiplicity of solutions for problems of type (1) and a plenty of results have been obtained, see for instance L. Kong [23], A.R. El Amrouss and A. Ourraoui [14], M. Alimohammady and F. Fattahi [4], O. Darhouche [11], G. A. Afrouzi, N. T. Chung and M. Mirzapour [3], R. Ayazoglu, G. Alisoy and I. Ekincioglu [6], K. Kefi [20] and M. Khodabakhshi, S. M. Vaezpour and A. Hadjian [21] and the references therein. In [23], L. Kong considered the problem(1) in the particular case when $m \equiv 1$ and $f(x,u) = a(x)|u|^{\gamma(x)-2}u - c(x)|u|^{\alpha(x)-2}u$, where $a, c, \gamma, \alpha \in C(\overline{\Omega})$ are nonnegative functions. By using variational arguments and the theory of the generalized Lebesgue-Sobolev spaces, the author proved that the problem considered has at least one nontrivial weak solution. Moreover, O. Darhouche [11] studied the existence and multiplicity of weak solutions of problem (1) with $d \equiv 0$ and $\lambda = 1$ under the following compactness condition:

(AR) There exist $K > 0$ and $\theta > l$ such that

$$0 < \theta F(x,t) \le tf(x,t), \quad \text{for all } |t| \ge K \text{ and a.e. } x \in \Omega,$$

where $F(x,t) = \int\limits_0^t f(x,s)ds$ and l is a precise constant.

As we know, the **(AR)**-condition plays a pivotal role in the application of the variational method, which is used extensively to ensure the boundedness of the Palais-Smale sequences and the energy functional has a mountain pass geometry, but in many cases, cannot be satisfied. For example, in the case $p(x) = p$, the function $f(x,t) = |t|^{p-2}t\ln(1+|t|)$ does not satisfy **(AR)**-condition but it satisfies certain weaker conditions. So, a natural question arises, can we ensure the existence and multiplicity of nontrivial solutions without assuming the **(AR)**-condition? Some researchers have answered this question for some problems involving the $p(x)-$biharmonic. L. Li and C. Tang in [26] consider the following condition

$(AR)_0$ There exists a constant $\theta \geq 1$, such that for any $s \in [0,1]$ and $t \in \mathbb{R}$, the inequalities

$$\frac{1}{p^+} f(x, st)st - F(x, st) \leq \frac{\theta}{p^+} f(x,t)t - F(x,t)$$

hold for a.e. $x \in \Omega$, where $F(x,t) = \int_0^t f(x,\tau)d\tau$;

This condition was introduced by L. Jeanjean [19] in the case $p(x) = 2$. Recently, the researchers G. A. Afrouzi, N. T. Chung and M. Mirzapour in [2] consider the following condition:

$(AR)_1$ $f \in C(\mathbb{R}, \mathbb{R})$ and there exist a constant $s_0 \geq 0$ and a decreasing function $\theta(s) \in C\left(\mathbb{R} \backslash (-s_0, s_0), \mathbb{R}\right)$ such that

$$0 < \left(p^+ + \theta(s)\right) F(s) \leq f(s)s, \quad \forall |s| \geq s_0,$$

where $\theta(s) > 0$, $\lim_{|s| \to +\infty} \theta(s)|s| = +\infty$, $\lim_{|s| \to +\infty} \int_{s_0}^{|s|} \frac{\theta(t)}{t} dt = +\infty$ and $F(s) = \int_0^s f(t)dt$;

Then, the goal of this article is to show the existence and multiplicity of nontrivial solutions to (1) without assuming the **(AR)**-condition on nonlinearity f, using the condition more general than $(AR)_0$ and overcome the difficulties generated by the Kirchhoff-type problem.

Before outlining our main results, we list some assumptions imposed on the functions m and f such that:

(H_p) $p \in C(\overline{\Omega})$ and $1 < p^- := \min_{x \in \overline{\Omega}} p(x) \leq p^+ = \max_{x \in \overline{\Omega}} p(x) < p_2^*(x)$ for all $x \in \overline{\Omega}$, where

$$p_2^*(x) := \begin{cases} \frac{Np(x)}{N-2p(x)} & \text{if } p(x) < \frac{N}{2} \\ +\infty & \text{if } p(x) \geq \frac{N}{2}. \end{cases}$$

(m_0) $m \in C([0,+\infty), \mathbb{R}_+)$ and there exists $m^* > 0$ such that $\inf_{t \geq 0} m(t) \geq m^*$.

(m_1) There exists $\mu \in \left[1, \frac{1}{p^+} \min_{x \in \overline{\Omega}} p_2^*(x)\right]$ such that for all $t \in \mathbb{R}_+$, $tm(t) \leq \mu \widehat{m}(t)$,

where $\widehat{m}(t) = \int_0^t m(\tau)d\tau$.

(m_2) m is a decreasing function on $[0,+\infty)$.

(f_1) There exists a function $s \in C(\overline{\Omega})$ which satisfies

$$1 < p^- \leq p^+ < s^- := \min_{x \in \overline{\Omega}} s(x) \leq s^+ := \max_{x \in \overline{\Omega}} s(x) < p_2^*(x) \text{ on } \overline{\Omega},$$

and a positive constant $C > 0$ such that

$$|f(x,t)| \leq C(1 + |t|^{s(x)-1}) \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}.$$

(f_2) $f(x,t) = \circ(|t|^{p^+-1})$ as $t \to 0$ uniformly for a.e. $x \in \Omega$.

(f_3) $\liminf_{|t| \to \infty} \frac{F(x,t)}{|t|^{\mu p^+}} = +\infty$ uniformly a.e $x \in \Omega$, where μ comes from (m_1) above.

(f_4) There are real numbers $\theta_1 \geq p^+$ and $\theta_2 \geq 1$ such that

$$\frac{1}{\theta_1} f(x,t)t - F(x,t) \leq \theta_2 \left[\frac{1}{\theta_1^2} f(x,s)s - F(x,s)\right],$$

a.e $x \in \Omega$ and $\forall$ $(t,s) \in \mathbb{R}^+ \times \mathbb{R}^+$, with $t \leqslant s$.

(f_5) $f(x,-t) = -f(x,t)$ for all $(x,t) \in \Omega \times \mathbb{R}$.

Remark 1.1. *Our assumption (f_4) is more general than hypothesis introduced by L. Jeanjean [19] and L. Li and C. Tang in [26] in the case $p(x) = 2$ and $\theta_1 = p^+ = 2$. In fact, if $f(\cdot,t) \geq 0$ for all $t \geq 0$, we have*

$$\frac{1}{2}f(x,t)t - F(x,t) \leq \theta_2 \left[\frac{1}{(p^+)^2}f(x,s)s - F(x,s)\right] \leq \theta_2 \left[\frac{1}{2}f(x,s)s - F(x,s)\right]$$

a.e $x \in \Omega$ and $\forall\ (t,s) \in \mathbb{R}^+ \times \mathbb{R}^+$, with $t \leqslant s$.

Now, we present the main results of this paper.

Theorem 1.2. *Suppose that $(m_0) - (m_2)$, (H_p) and $(f_1) - (f_4)$ hold. Then for all $\lambda > 0$, problem (1) has at least one nontrivial weak solution in X.*

Theorem 1.3. *Suppose that $(m_0) - (m_2)$, (H_p), (f_1) and $(f_3) - (f_4)$ hold. Then, there exists $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$, problem (1) admits at least two distinct weak solutions in X.*

Theorem 1.4. *Suppose that $(m_0)-(m_2)$, (H_p), (f_1) and $(f_3)-(f_5)$ hold. Then, for all $\lambda > 0$, problem (1) has infinitely many weak solutions $(u_n) \subset X$ such that $I(u_n) \to +\infty$ as $n \to +\infty$.*

2. The functional setting and tools

In this section, we review some necessary definitions and basic properties of the spaces $L^{p(x}(\Omega)$ and $W^{k,p(x)}(\Omega)$ (see [12, 15–17, 24]) and some useful properties of the $p(x)$-biharmonic operator, which we will use later.

Let $p \in C_+(\overline{\Omega}) := \left\{p \in C(\overline{\Omega}) : p^- := \inf_{x\in\overline{\Omega}} p(x) > 1\right\}$, we define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}dx < +\infty\right\},$$

equipped with the Luxemburg norm

$$|u|_{p(x)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}.$$

Proposition 2.1. *(See [17])*

1. *The Lebesgue space $(L^{p(x)}(\Omega), |.|_{p(x)})$ is Banach, separable, uniformly convex, reflexive and its conjugate space is $L^{q(x)}(\Omega)$, where $q(x)$ is conjugate to $p(x)$, i.e., $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$.*
2. *For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have*

$$\left|\int_\Omega uv\, dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right)|u|_{p(x)}|v|_{q(x)} \leq 2|u|_{p(x)}|v|_{q(x)}.$$

On $L^{p(x)}(\Omega)$, we define the modular $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$ as follows

$$\rho(u) = \int_\Omega |u|^{p(x)}dx.$$

The relationship between ρ and $|.|_{p(x)}$ is established by the next result.

Proposition 2.2. *(See [17]) For $u \in L^{p(x)}(\Omega)$ and $(u_n) \subset L^{p(x)}(\Omega)$, we have*

1. $|u|_{p(x)} < 1\,(= 1; > 1) \Longleftrightarrow \rho(u) < 1\,(= 1; > 1)$.
2. *For* $u \neq 0$, $|u|_{p(x)} = \lambda \Longleftrightarrow \rho(\frac{u}{\lambda}) = 1$.
3. $|u|_{p(x)} > 1 \Longrightarrow |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$.
4. $|u|_{p(x)} < 1 \Longrightarrow |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}$.
5. *The following affirmations are equivalent to each other.*
   (a) $\lim_{n\to\infty} |u_n - u|_{p(x)} = 0$.
   (b) $\lim_{n\to\infty} \rho(u_n - u) = 0$.
   (c) $u_n \to u$ *in measure in* Ω *and* $\lim_{n\to\infty} \rho(u_n) = \rho(u)$.
6. $\lim_{n\to\infty} |u_n|_{p(x)} = \infty \Longleftrightarrow \lim_{n\to\infty} \rho(u_n) = \infty$.

Next, for any $k \in \mathbb{N}^*$, as in the case of constant exponent, we can define the variable exponent Sobolev space as

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq k\},$$

where k is an integer, $\alpha = (\alpha_1, ..., \alpha_N)$ is a multi-index, $|\alpha| = \sum_{i=1}^{N} \alpha_i$ and $D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} x_1 ... \partial^{\alpha_N} x_N}$.
The space $W^{k,p(x)}(\Omega)$ endowed with the norm $\|u\|_{k,p(x)} = \sum_{|\alpha|\leq k} |D^\alpha u|_{p(x)}$ is a Banach, separable and reflexive space.

Proposition 2.3. *([17, 28]) For* $p, \gamma \in C_+(\overline{\Omega})$ *such that* $\gamma(x) \leq p_k^*(x)$ *for all* $x \in \overline{\Omega}$*, there is a continuous embedding*

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{\gamma(x)}(\Omega).$$

In addition, if $\gamma(x) < p_k^*(x)$ *for all* $x \in \overline{\Omega}$*, then, the embedding is compact. Hereafter, denote* $p^+ = \max_{x\in\overline{\Omega}} p(x)$, $p^- = \min_{x\in\overline{\Omega}} p(x)$ *and for all* $x \in \overline{\Omega}$,

$$p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N, \end{cases}$$

$$p_k^*(x) := \begin{cases} \frac{Np(x)}{N-kp(x)} & \text{if } p(x) < \frac{N}{k}, \\ +\infty & \text{if } p(x) \geq \frac{N}{k}. \end{cases}$$

We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$. Notice that problem (1) is modeled in the working space $X = W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ equipped with the norm

$$\|u\|_d = \inf\left\{t > 0 : \int_\Omega \left(\left|\frac{\Delta u}{t}\right|^{p(x)} + d(x)\left|\frac{u}{t}\right|^{p(x)}\right) dx \leq 1\right\}.$$

Remark 2.4. *([14])*

1. $(X, \|.\|_d)$ *is a Banach space separable and reflexive space.*
2. *The norms* $\|.\|_{2,p(x)}$, $|\Delta.|_{p(x)}$ *and* $\|.\|_d$ *are equivalent.*
3. *There is a continuous and compact embedding of X into* $L^{\gamma(x)}(\Omega)$ *where* $\gamma(x) < p_2^*(x)$ *for all* $x \in \overline{\Omega}$.

Proposition 2.5. *([9]) Let* $\rho_d(u) = \int_\Omega \left(|\Delta u|^{p(x)} + d(x)|u|^{p(x)}\right) dx$. *For* $u, u_n \in X$*, we have,*

1. $\|u\|_d \leq 1 \Longrightarrow \|u\|_d^{p^+} \leq \rho_d(u) \leq \|u\|_d^{p^-}$.
2. $\|u\|_d \geq 1 \Longrightarrow \|u\|_d^{p^-} \leq \rho_d(u) \leq \|u\|_d^{p^+}$.

3. $\|u_n\|_d \to 0 \Longleftrightarrow \rho_d(u_n) \to 0.$
4. $\|u_n\|_d \to +\infty \Longleftrightarrow \rho_d(u_n) \to +\infty.$

Now, we can define the weak solution of problem (1).

Definition 2.6. *We call $u \in X$ a weak solution of problem* (1) *if*

$$m\left(\int_\Omega \frac{|\Delta u|^{p(x)} + d(x)|u|^{p(x)}}{p(x)}dx\right)\int_\Omega \left(|\Delta u|^{p(x)-2}\Delta u \Delta w + d(x)|u|^{p(x)-2}uw\right)dx$$

$$= \lambda \int_\Omega f(x,u)w dx, \quad \text{for all } w \in X.$$

In order to prove the main results, we introduce the energy functional $I : X \to \mathbb{R}$ of problem (1) as follows $I(u) = J(u) - \lambda\varphi(u)$, where

$$J(u) = \widehat{m}\left(\int_\Omega \frac{|\Delta u|^{p(x)} + d(x)|u|^{p(x)}}{p(x)}dx\right) \quad \text{and} \quad \varphi(u) = \int_\Omega F(x,u)\, dx,$$

with $F(x,t) = \displaystyle\int_0^t f(x,s)ds$ and $\widehat{m}(t) = \displaystyle\int_0^t m(\tau)d\tau$. It is easy to prove that $I \in C^1(X,\mathbb{R})$ and its critical points are solutions to problem (1).

Let the functional $\mathcal{B} : X \to \mathbb{R}$ be defined by

$$\mathcal{B}(u) = \int_\Omega \frac{|\Delta u|^{p(x)} + d(x)|u|^{p(x)}}{p(x)}dx. \tag{2}$$

Proposition 2.7. *([14])*

1. *The functional $\mathcal{B}$ is sequentially weakly lower semi continuous, $\mathcal{B} \in C^1(X,\mathbb{R})$ and it's Fréchet derivative is given by*
$$\langle \mathcal{B}'(u), v\rangle = \int_\Omega \left(|\Delta u|^{p(x)-2}\Delta u\Delta v + d(x)|u|^{p(x)-2}uv\right)dx,$$
for all $u, v \in X$.

2. *The mapping $\mathcal{B}' : X \longrightarrow X^*$ is a strictly monotone, bounded homeomorphism and it is of type (S_+), that is, for every sequence $(u_n) \subset X$ and for every $u \in X$ which satisfy*
$$u_n \rightharpoonup u \text{ weakly in } X \qquad \text{and} \qquad \limsup_{n\to+\infty} \langle \mathcal{B}(u_n), u_n - u\rangle \leq 0,$$
then $u_n \to u$ strongly in X.

3. *φ is a C^1 in $L^{\alpha(x)}(\Omega)$ and φ' is weakly-strongly continuous, i.e. $u_n \rightharpoonup u$ implies $\varphi'(u_n) \longrightarrow \varphi'(u)$.*

Remark 2.8. *Note that $u \in X$ is a weak solution of problem* (1) *if and only if we have*

$$m(\mathcal{B}(u))\langle \mathcal{B}'(u), w\rangle = \lambda \int_\Omega f(x,u)w dx, \quad \text{for all } w \in X.$$

All through this paper, the letters C_i, c_i, $i = 1, 2, 3...$ denote positive constants which may change from line to line.

3. Cerami condition

We now give the definition of the compactness condition of the Cerami which was introduced by G. Cerami in [10].

Definition 3.1. *Let $(X, \|.\|)$ be a real Banach space and $I \in C^1(X, \mathbb{R})$. If any sequence $(u_n) \subset X$ fulfilling*

$$(I(u_n)) \text{ is bounded} \quad \text{and} \quad \|I'(u_n)\|_{X^*}(1 + \|u_n\|) \to 0 \text{ as } n \to +\infty$$

has a strong convergent subsequence in X, then, we say that I satisfies the Cerami condition (we denote $(C)-$ condition) in X.

We are going to show that the functional I fulfills the $(C)-$condition. First, a technical result.

Lemma 3.2. *The following statements hold.*

(i) The conditions (m_0) and (m_1) imply that

$$\widehat{m}(t) \le \widehat{m}(1) \quad \forall\, t \in [0, 1], \qquad \text{and} \qquad \widehat{m}(t) \le \widehat{m}(1)t^{\mu} \quad \forall\, t \in (1, +\infty).$$

Hence,

$$\widehat{m}(t) \le \widehat{m}(1)(1 + t^{\mu}), \quad \forall\, t \in [0, +\infty).$$

(ii) Given the number $\kappa \in [0, 1]$ then, the condition (m_2), implies that $t \mapsto \widehat{m}(t) - \kappa\, m(t)t$ is an increasing function, where $t \ge 0$.

(iii) The condition (f_3) implies that given $D_1 > 0$ there is a positive constant $D_2 = D_2(D_1)$ such that

$$F(x, t) \ge D_1 t^{\mu p^+} - D_2, \qquad \text{for all } (x, t) \in \Omega \times \mathbb{R}.$$

Proof. *To prove (i) note that, if $0 \le t \le 1$, since the function $\widehat{m}(.)$ is strictly increasing on $[0, +\infty)$, then $\widehat{m}(t) \le \widehat{m}(1)$, and if $t \ge 1$. Now, we consider the function $K(t) = \dfrac{\widehat{m}(t)}{t^{\mu}}$. By direct calculation, it is clear that $t \mapsto K(t)$ is strictly decreasing on $[1, +\infty)$, then $\widehat{m}(t) \le \widehat{m}(1)t^{\mu}$.*

To prove the second item we are going to consider $s > t$ and that (m_2) holds, then

$$\begin{aligned}
\widehat{m}(s) - \kappa\, m(s)s &> \int_0^t m(\sigma)d\sigma + \int_t^s m(t)d\sigma - \kappa\, m(s)s \\
&> \widehat{m}(t) + \kappa\, m(t)(s - t) - \kappa\, m(s)s \\
&> \widehat{m}(t) - \kappa\, m(t)t
\end{aligned}$$

Next, the assumption (f_3) means that given $D_1 > 0$ there exists a positive constant t_1 such that

$$F(x, t) > D_1|t|^{\mu p^+} \text{ for all } (x, |t|) \in \Omega \times (t_1, +\infty)\,.$$

Since $F(x, t)$ is continuous at $t \in [-t_1, t_1]$, there is a constant $D_2 > 0$ such that

$$|F(x, t)| \le D_2 \text{ for all } (x, t) \in \Omega \times [-t_1, t_1]\,.$$

Therefore, the result follows. □

Proposition 3.3. *Assuming the assumptions $(m_0) - (m_2)$, (H_p), (f_1) and $(f_3) - (f_4)$ are fulfilled. Then, the functional I fulfills the $(C)-$condition.*

Proof. *Let $(u_n) \subset X$ be a Cerami sequence for I, that is,*

$$(I(u_n)) \text{ is bounded} \quad \text{and} \quad \|I'(u_n)\|_{X^*}(1 + \|u_n\|_d) \to 0 \quad \text{as } n \to +\infty, \tag{3}$$

which implies that

$$\sup_{n\in\mathbb{N}} |I(u_n)| \le M \quad \text{and} \quad \langle I'(u_n), u_n\rangle = \circ_n(1), \tag{4}$$

where $\circ_n(1) \to 0$ as $n \to +\infty$ and M is a positive constant. Let $(u_n) \subset X$ be a bounded sequence satisfying (3). Then there exists $u_0 \in X$ such that up to a subsequence, still denoted by (u_n), we have by Remark 2.4 that

$$\begin{cases} u_n \rightharpoonup u_0 & \text{in } X; \\ u_n \to u_0 & \text{in } L^{\gamma(x)}(\Omega), \text{ for all } \gamma \in C_+(\overline{\Omega}) \text{ with } \gamma(x) < p_2^*(x) \text{ in } \Omega, \\ u_n(x) \to u_0(x) & \text{a.e. } x \in \Omega, \end{cases} \tag{5}$$

Hence, by (f_1), (5), the Hölder inequality and Proposition 2.2, we obtain

$$\begin{aligned} \left|\int_\Omega f(x,u_n)(u_n - u_0)dx\right| &\le C\int_\Omega |u_n - u_0|dx \\ &\quad + C\left(\int_\Omega |u_n|^{s(x)}dx\right)^{\frac{s(x)-1}{s(x)}}\left(\int_\Omega |u_n - u_0|^{s(x)}dx\right)^{\frac{1}{s(x)}} \\ &\to 0, \quad \text{as } n \to +\infty. \end{aligned} \tag{6}$$

On the other hand, using (m_0) and (m_2), we obtain

$$m(\mathcal{B}(u_n)) \in (m^*, m(0)) \quad \forall\, n \in \mathbb{N}, \tag{7}$$

where $\mathcal{B}$ is given in (2). Next, since $u_n \rightharpoonup u_0$, from (3), we have

$$\langle I'(u_n), u_n - u_0\rangle \to 0, \quad \text{as } n \to +\infty.$$

Then, by (6),

$$\begin{aligned} m(\mathcal{B}(u_n))\langle \mathcal{B}'(u_n), u_n - u_0\rangle &= \langle I'(u_n), u_n - u_0\rangle \\ &\quad + \lambda\int_\Omega f(x,u_n)(u_n - u_0)dx \to 0, \quad \text{as } n \to +\infty \end{aligned}$$

Since $\mathcal{B}'$ is a mapping of type (S_+) (see Proposition 2.7) and (7) holds, we can conclude that $u_n \to u_0$ strongly in X.

Next, to complete the demonstration, it remains to show that (u_n) is bounded in X. To this end, assume the contrary that the sequence (u_n) is unbounded in X. Without loss of generality, we may suppose that $\|u_n\|_d > 1$ and $\|u_n\|_d \to +\infty$ as $n \to +\infty$. By (4), (H_p), (m_0), (m_1) and Proposition 2.5, for n large enough, we get

$$\begin{aligned} M &\ge I(u_n) \\ &= \widehat{m}\left(\int_\Omega \frac{|\Delta u_n|^{p(x)} + d(x)|u_n|^{p(x)}}{p(x)}dx\right) - \lambda\int_\Omega F(x,u_n)dx \\ &\ge \frac{m^*}{\mu p^+}\int_\Omega (|\Delta u_n|^{p(x)} + d(x)|u_n|^{p(x)})dx - \lambda\int_\Omega F(x,u_n)dx \\ &\ge \frac{m^*}{\mu p^+}\rho_d(u_n) - \lambda\int_\Omega F(x,u_n)dx \\ &\ge \frac{m^*}{\mu p^+}\|u_n\|_d^{p^-} - \lambda\int_\Omega F(x,u_n)dx. \end{aligned}$$

Since $\|u_n\|_d \to +\infty$ *as* $n \to +\infty$, *we get*

$$\int_\Omega F(x,u_n)dx \geq \frac{m^*}{\lambda\mu p^+}\|u_n\|_d^{p^-} - \frac{M}{\lambda} \to +\infty. \tag{8}$$

On the other hand, using (m_0) *we have that* $\widehat{m}(.)$ *is strictly increasing on* $[0,+\infty)$. *Then, by and Proposition 2.5,*

$$\begin{aligned} I(u_n) &= \widehat{m}\left(\int_\Omega \frac{|\Delta u_n|^{p(x)} + d(x)|u_n|^{p(x)}}{p(x)}dx\right) - \lambda\int_\Omega F(x,u_n)dx \\ &\leq \widehat{m}\left(\int_\Omega (|\Delta u_n|^{p(x)} + d(x)|u_n|^{p(x)})dx\right) - \lambda\int_\Omega F(x,u_n)dx \\ &\leq \widehat{m}(\|u_n\|_d^{p^+}) - \lambda\int_\Omega F(x,u_n)dx. \end{aligned}$$

Thus, we obtain

$$\lambda\int_\Omega F(x,u_n)dx \leq \widehat{m}(\|u_n\|_d^{p^+}) - I(u_n) \leq \widehat{m}(\|u_n\|_d^{p^+}) + M.$$

Since $\widehat{m}(.)$ *is strictly increasing on* $[0,+\infty)$ *and* $\|u_n\|_d \to +\infty$ *as* $n \to +\infty$, *we have*

$$\limsup_{n\to+\infty}\left[\int_\Omega \frac{F(x,u_n)}{\widehat{m}(\|u_n\|_d^{p^+})}dx\right] \leq 1. \tag{9}$$

Let us define the sequence $(\beta_n) \subset X$ *by* $\beta_n = \frac{u_n}{\|u_n\|_d}$ *for any* $n \in \mathbb{N}$. *Then, up to subsequences, for some* $\beta \in X$, *we have*

$$\begin{cases} \beta_n \rightharpoonup \beta & \text{in } X, \\ \beta_n \to \beta & \text{in } L^{\mu p^+}(\Omega), \\ \beta_n \to \beta & \text{in } L^{s(x)}(\Omega), \\ \beta_n(x) \to \beta(x) & \text{a.e. in } \Omega, \end{cases} \tag{10}$$

where $s(.)$ *comes from* (f_1). *Define the set* $\Theta = \{x \in \Omega : \beta(x) \neq 0\}$. *If* $x \in \Theta$, *from* (10), *we obtain*

$$|u_n(x)| = |\beta_n(x)|\|u_n\|_d \to +\infty.$$

Then, by Lemma 3.2 and n large enough, we obtain

$$\frac{F(x,u_n)}{\widehat{m}(\|u_n\|_d^{p^+})} \geq \frac{F(x,u_n)}{\widehat{m}(1)\|u_n\|_d^{\mu p^+}} = \frac{F(x,u_n)}{\widehat{m}(1)|u_n(x)|^{\mu p^+}}|\beta_n(x)|^{\mu p^+} \to +\infty.$$

Hence, if meas$(\Theta) \neq 0$ *(where meas*(Θ) *means the Lebesgue measure of* Θ*), we have, using Fatou's Lemma and* (8),

$$\begin{aligned} +\infty = \int_\Theta \liminf_{n\to+\infty}\left[\frac{F(x,u_n(x))}{\widehat{m}(\|u_n\|_d^{p^+})}\right]dx \quad &\leq \quad \liminf_{n\to+\infty}\left[\int_\Omega \frac{F(x,u_n)}{\widehat{m}(\|u_n\|_d^{p^+})}dx\right] \\ &\leq \quad \limsup_{n\to+\infty}\left[\int_\Omega \frac{F(x,u_n)}{\widehat{m}(\|u_n\|_d^{p^+})}dx\right]. \end{aligned}$$

But, this contradicts (9). *Therefore, we can conclude meas*$(\Theta) = 0$ *and* $\beta(x) = 0$ *almost every* $x \in \Omega$.

As $\xi \mapsto I(\xi u_n)$ *is continuous in* $[0,1]$*, for all* $n \in \mathbb{N}$*, then, there is* $\xi_n \in [0,1]$ *such that*

$$I(\xi_n u_n) := \max_{\xi \in [0,1]} I(\xi u_n). \tag{11}$$

Put $(L_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ *be sequence such that* $L_k > 1$ *for any* k *and* $L_k \to +\infty$ *as* $k \to +\infty$*. Then,*

$$\|L_k\beta_n\|_d = L_k > 1, \quad \forall k, n \in \mathbb{N}.$$

Let k *be fixed. As* $\beta_n \to 0$ *in* $L^{s(x)}(\Omega)$ *and* $\beta_n(x) \to 0$ *a.e.* $x \in \Omega$ *as* $n \to +\infty$*. Using* (f_1) *and Lebesgue's dominated convergence theorem, we get*

$$\int_\Omega F(x, L_k\beta_n)dx \to 0 \quad \text{as } n \to +\infty. \tag{12}$$

Because $\|u_n\|_d \to +\infty$ *as* $n \to +\infty$*, then we have* $\|u_n\|_d > L_k$ *for* n *large enough, which means that* $\frac{L_k}{\|u_n\|_d} \in (0,1)$ *for* n *large enough. Thus, from* $(m_0) - (m_1)$*, Proposition 2.5,* (11) *and* (12) *we obtain*

$$\begin{aligned}
I(\xi_n u_n) &\geq I\left(\frac{L_k}{\|u_n\|_d}u_n\right)\\
&= I(L_k\beta_n)\\
&= \widehat{m}\left(\int_\Omega \frac{|\Delta L_k\beta_n|^{p(x)} + d(x)|L_k\beta_n|^{p(x)}}{p(x)}dx\right) - \lambda\int_\Omega F(x, L_k\beta_n)dx\\
&\geq \frac{m^*}{\mu p^+}\rho_d(L_k\beta_n) - \lambda\int_\Omega F(x, L_k\beta_n)dx\\
&\geq \frac{m^*}{\mu p^+}\|L_k\beta_n\|_d^{p^-} - \lambda\int_\Omega F(x, L_k\beta_n)dx\\
&\geq \frac{m^*}{\mu p^+}L_k^{p^-}
\end{aligned}$$

for n *large enough. Then,*

$$\liminf_{n\to+\infty} I(\xi_n u_n) \geq \frac{m^*}{\mu p^+}L_k^{p^-}, \quad \forall\, k \in \mathbb{N}. \tag{13}$$

Next, we claim that, for n *large enough,* $\xi_n \in (0,1)$*. In fact, by* (13)*, we have* $\xi_n > 0$*, for* n *large enough. On the other hand, if there is* $(\xi_j) \subset (\xi_n)$ *such that* $\xi_j = 1$ *for all* j*. Then, using* (4) *and* (13)*,*

$$\frac{m^*}{\mu p^+}L_k^{p^-} \leq I(u_j) \leq M, \quad \text{for all } j \text{ and } k,$$

which it is a contradiction, because $L_k \to +\infty$*, as* $k \to +\infty$*. Therefore, we can consider, for* n *large enough,*

$$\xi_n \in (0,1) \quad \text{and} \quad \langle I'(\xi_n u_n), \xi_n u_n\rangle = 0.$$

On the other hand, it follows from Remark 2.8, (H_p) *and* (f_4) *that*

$$\begin{aligned}
\frac{1}{\theta_2}I(\xi_n u_n) &= \frac{1}{\theta_2}I(\xi_n u_n) - \frac{1}{\theta_2\theta_1}\langle I'(\xi_n u_n), \xi_n u_n\rangle\\
&\leq \frac{1}{\theta_2}\widehat{m}\left(\int_\Omega \frac{|\Delta(\xi_n u_n)|^{p(x)} + d(x)|\xi_n u_n|^{p(x)}}{p(x)}dx\right)\\
&\quad -\frac{1}{\theta_2\theta_1}m\left(\int_\Omega \frac{|\Delta(\xi_n u_n)|^{p(x)} + d(x)|\xi_n u_n|^{p(x)}}{p(x)}dx\right)\int_\Omega\left(\frac{|\Delta(\xi_n u_n)|^{p(x)} + d(x)|\xi_n u_n|^{p(x)}}{p(x)}\right)dx\\
&\quad +\lambda\int_\Omega\left[\frac{1}{(\theta_1)^2}f(x, u_n)u_n - F(x, u_n)\right]dx.
\end{aligned}$$

Hence, using Lemma 3.2,

$$\frac{1}{\theta_2}I(\xi_n u_n) \le \frac{1}{\theta_2}\Bigg[\widehat{m}\left(\int_\Omega \frac{|\Delta u_n|^{p(x)} + d(x)|u_n|^{p(x)}}{p(x)}dx\right)$$
$$-\frac{1}{\theta_1}m\left(\int_\Omega \frac{|\Delta u_n|^{p(x)} + d(x)|u_n|^{p(x)}}{p(x)}dx\right)\int_\Omega\left(\frac{|\Delta u_n|^{p(x)} + d(x)|u_n|^{p(x)}}{p(x)}\right)dx\Bigg]$$
$$+\lambda\int_\Omega\left[\frac{1}{(\theta_1)^2}f(x,u_n)u_n - F(x,u_n)\right]dx$$
$$\le \widehat{m}\left(\int_\Omega \frac{|\Delta u_n|^{p(x)} + d(x)|u_n|^{p(x)}}{p(x)}dx\right)$$
$$-\frac{1}{(\theta_1)^2}m\left(\int_\Omega \frac{|\Delta u_n|^{p(x)} + d(x)|u_n|^{p(x)}}{p(x)}dx\right)\int_\Omega\left(|\Delta u_n|^{p(x)} + d(x)|u_n|^{p(x)}\right)dx$$
$$+\lambda\int_\Omega\left[\frac{1}{(\theta_1)^2}f(x,u_n)u_n - F(x,u_n)\right]dx$$
$$= I(u_n) - \frac{1}{(\theta_1)^2}\langle I'(u_n), u_n\rangle.$$

Then, letting $n \to +\infty$ we have

$$\lim_{n\to+\infty} I(\xi_n u_n) \le \theta_2 M,$$

which contradicts (13)*. The proof is complete* □

4. Proof of Theorem 1.2

The proof of Theorem 1.2 is based on the application of the mountain pass theorem given by Ambrosetti-Rabinowitz in [5]. By Lemma 3.3, I satisfies the (C)−*condition. According to definition of I, we have I*(0) = 0*. Then, to apply the mountain pass theorem, we are going to show that I has a mountain pass geometry.*

First, we claim that there exist $\alpha,\ R > 0$ such that

$$I(u) \geqslant \alpha, \quad \forall\, u \in X \quad with\ \ \|u\|_d = R. \tag{14}$$

In fact, by $(m_0) - (m_1)$ and Proposition 2.5, for $\|u\|_d < 1$, we obtain

$$\begin{aligned} I(u) &\ge \frac{m^*}{\mu p^+}\rho_d(u) - \lambda\int_\Omega F(x,u)dx \\ &\ge \frac{m^*}{\mu p^+}\|u\|_d^{p^+} - \lambda\int_\Omega F(x,u)dx. \end{aligned} \tag{15}$$

Since $p^+ < p_2^*(x)$ and $s(x) < p_2^*(x)$ for all $x \in \overline{\Omega}$ in view of conditions (m_1) and (f_1), respectively , we have from Remark 2.4 that $X \hookrightarrow L^{p^+}(\Omega)$ and $X \hookrightarrow L^{s(x)}(\Omega)$. So, there exist $c_6, c_7 > 0$ such that

$$|u|_{p^+} \le c_6\|u\|_d \quad and \quad |u|_{s(x)} \le c_7\|u\|_d,\ \ \forall u \in X.$$

From (f_1) and (f_2), given $\varepsilon > 0$ there is a positive constant $C_\varepsilon = C(\varepsilon)$ such that

$$F(x,t) \le \varepsilon|t|^{p^+} + C_\varepsilon|t|^{s(x)}, \quad \forall(x,t) \in \Omega\times\mathbb{R}. \tag{16}$$

Therefore, for $\|u\|_d < 1$ small enough, the relations (15) *and* (16) *yields*

$$\begin{aligned}I(u) &\geqslant \frac{m^*}{\mu p^+}\|u\|_d^{p^+} - \lambda\varepsilon\int_\Omega |u|^{p^+}dx - \lambda C_\varepsilon\int_\Omega |u|^{s(x)}dx\\ &\geqslant \frac{m^*}{\mu p^+}\|u\|_d^{p^+} - \lambda\varepsilon c_6^{p^+}\|u\|_d^{p^+} - \lambda C_\varepsilon c_7^{s^-}\|u\|_d^{s^-}\\ &\geq \left(\frac{m^*}{\mu p^+} - \lambda\varepsilon c_6^{p^+}\right)\|u\|^{p^+} - \lambda C_\varepsilon c_7^{s^-}\|u\|^{s^-}.\end{aligned}$$

Since $s^- > p^+$ and choosing $\varepsilon < \dfrac{m^*}{\lambda c_6^{p^+}\mu p^+}$, we obtain

$$\liminf_{\|u\|_d\to 0^+}\frac{I(u)}{\|u\|_d^{p^+}} \geqslant 0,$$

which implies that (14) *holds.*

Next, we affirm that there exists $\sigma \in X$ with $\|u\|_d > \mu$ such that

$$I(\sigma) < 0. \tag{17}$$

In fact, let $\phi \in X \setminus \{0\}$ such that $\|\phi\| = 1$ and $l > 1$ be large enough, using Lemma 3.2, we obtain

$$\begin{aligned}I(l\phi) &= \widehat{m}\left(\int_\Omega \frac{|\Delta l\phi|^{p(x)} + d(x)|l\phi|^{p(x)}}{p(x)}dx\right) - \lambda\int_\Omega F(x,l\phi)dx\\ &\leq \widehat{m}(1)\left(\int_\Omega \frac{|\Delta l\phi|^{p(x)} + d(x)|l\phi|^{p(x)}}{p(x)}dx\right)^\mu - \lambda\int_\Omega F(x,l\phi)dx\\ &\leq \widehat{m}(1)\left(\frac{l^{p^+}}{p^-}\int_\Omega (|\Delta\phi|^{p(x)} + d(x)|\phi|^{p(x)})dx\right)^\mu - \lambda D_1 l^{\mu p^+}\int_\Omega |\phi|^{\mu p^+}\,dx\\ &\quad + \lambda D_2 meas(\Omega)\\ &\leq l^{\mu p^+}\left(\frac{\widehat{m}(1)}{(p^-)^\mu}l^{p^+(1-\mu)} - \lambda D_1\int_\Omega |\phi|^{\mu p^+}dx\right) + \lambda D_2 meas(\Omega)\end{aligned} \tag{18}$$

where μ comes from (m_1). As

$$\frac{\widehat{m}(1)}{(p^-)^\mu} - \lambda D_1\int_\Omega |\phi|^{\mu p^+}dx < 0,$$

for sufficiently large D_1 , we find

$$I(l\phi) \to -\infty, \quad \text{as } l \to +\infty.$$

Therefore, there exists $t_0 > 1$ and $\sigma = t_0\phi \in X \setminus \overline{B_R(0)}$ such that $I(\sigma) < 0$. The proof of Theorem 1.2 is complete.

5. Proof of Theorem 1.3

In order to prove Theorem 1.3, we shall use the following theorem and prove the technical lemma.

Theorem 5.1. *[27, Theorem 2.6] Let X be a real Banach space, $A, B : X \to \mathbb{R}$ be two continuously Gateaus differentiable functionals such that A is bounded from below and $A(0) = B(0) = 0$. Let $\eta > 0$ be fixed, if for any*

$$\lambda \in \Gamma_0 := \left(0, \frac{\eta}{\sup_{u\in A^{-1}((-\infty,\eta))} B(u)}\right),$$

the functional $I = A - \lambda B$ satisfies the (C)-condition for all $\lambda > 0$ and is unbounded from below. Then, for any $\lambda \in \Gamma_0$, the functional I admits two distinct critical points.

Lemma 5.2. *The following assumptions hols.*

(a_0) *Let hypotheses (m_0) and (m_1) be satisfied. Then, the functional J defined by*

$$J(u) = \widehat{m}\left(\int_\Omega \frac{|\Delta u|^{p(x)} + d(x)|u|^{p(x)}}{p(x)}dx\right)$$

is bounded from below.

(a_1) *Let hypothesis (f_1) be satisfied. Then, there exists $\lambda_0 > 0$ such that*

$$\sup_{u\in J^{-1}((-\infty,1))} \varphi(u) < \frac{1}{\lambda_0},$$

where $\varphi(u) = \int_\Omega F(x,u)dx$.

(a_2) *The functional $I = J - \lambda\varphi$ is unbounded from below, for all $\lambda > 0$.*

Proof. *Using (m_0) and (m_1), we obtain*

$$\begin{aligned} J(u) &= \widehat{m}\left(\int_\Omega \frac{|\Delta u|^{p(x)} + d(x)|u|^{p(x)}}{p(x)}dx\right) \\ &\geq \frac{m^*}{\mu p^+}\int_\Omega (|\Delta u|^{p(x)} + d(x)|u|^{p(x)})dx \\ &\geq \frac{m^*}{\mu p^+}\rho_d(u). \end{aligned}$$

According to Proposition 2.5(4), we deduce that J is coercive. Consequently, J is bounded from below.

To check (a_1) we are going to use (f_1) and Proposition 2.5. Thus, we can find a positive constant c_8 such that

$$\begin{aligned} \varphi(u) &= \int_\Omega F(x,u)dx \\ &\leq C\int_\Omega \left(|u| + \frac{1}{s(x)}|u|^{s(x)}\right)dx \\ &\leq c_8\|u\|_d + c_8\max\{\|u\|_d^{s^+}, \|u\|_d^{s^-}\}. \end{aligned} \tag{19}$$

On the other hand, for all $u \in J^{-1}((-\infty,1))$, according to $(m_0) - (m_1)$ and Proposition 2.5, we find

$$\begin{aligned} \mu p^+ \geq \mu p^+ J(u) = \mu p^+\widehat{m}\left(\int_\Omega \frac{|\Delta u|^{p(x)} + d(x)|u|^{p(x)}}{p(x)}dx\right) &\\ \geq \frac{m^*\mu p^+}{\mu p^+}\int_\Omega (|\Delta u|^{p(x)} + d(x)|u|^{p(x)})dx &\\ \geq m^*\rho_d(u) &\\ \geq m^*\|u_n\|_d^q. & \end{aligned}$$

where $q = p^-$ or $q = p^+$. Hence,

$$\|u\|_d \leq c_9 := \max\left\{\left(\frac{\mu p^+}{m^*}\right)^{\frac{1}{p^-}}, \left(\frac{\mu p^+}{m^*}\right)^{\frac{1}{p^+}}\right\}.$$

In view of (19)*, we have*

$$\sup_{u\in J^{-1}((-\infty,1))} \varphi(u) \le c_8c_9 + c_8 \max\{c_9^{s^+}, c_9^{s^-}\} \tag{20}$$

Let us denote

$$\lambda_0 = c_8c_9 + c_8 \max\{c_9^{s^+}, c_9^{s^-}\}.$$

Then, from (20)*, one yields*

$$\sup_{u\in J^{-1}((-\infty,1))} \varphi(u) \le \frac{1}{\lambda_0} < \frac{1}{\lambda},$$

this finished the proof item (a_1).

Now, we are going to show the proof of the lemma. For this, let $\phi \in X\setminus\{0\}$ *and* $l > 1$ *be large enough. Then, using the same arguments as in* (18)*, we can infer that*

$$I(l\phi) \to -\infty, \quad \text{as } l \to +\infty.$$

Consequently, I is unbounded from below. □

Now, we continue with the proof of Theorem 1.3. Let $A = J$, $B = \varphi$ *and* $\eta = 1$. *In view of the definition of J and* φ, *we have* $J(0) = \varphi(0) = 0$. *According to Lemma 3.3, I satisfies the* (C)*- condition. To apply Theorem 5.1, it suffices to use Lemma 5.2.*

Since all assumptions of Theorem 5.1 are satisfied. Then, for all $\lambda \in (0, \lambda_0) \subset \Gamma_0$, *the problem* (1) *admits at least two distinct weak solutions in X.*

6. Proof of Theorem 1.4

Let X be a real, reflexive, and separable Banach space, then there exist $(f_j)_{j\in\mathbb{N}} \subset X$ *and* $(f_j^*)_{j\in\mathbb{N}} \subset X^*$ *such that*

$$X = \overline{span\{f_j : j = 1, 2, ...\}}, \quad X^* = \overline{span\{f_j^* : j = 1, 2, ...\}},$$

and $\left\langle f_i^*, f_j \right\rangle = 1$ *if* $i = j$, $\left\langle f_i^*, f_j \right\rangle = 0$ *if* $i \ne j$.

We denote $X_j = span\{f_j\}$, $Y_k = \bigoplus_{j=1}^{k} X_j$ *and* $Z_k = \overline{\bigoplus_{j=k}^{+\infty} X_j}$.

Theorem 6.1. *(See [32]) Let* $I \in C^1(X, \mathbb{R})$ *be an even functional and fulfills the* (C)*−condition. For every* $k \in \mathbb{N}$, *there exists* $\gamma_k > \eta_k > 0$ *such that*

(A_1) $b_k := \inf\{I(u) : u \in Z_k, \|u\| = \eta_k\} \to +\infty$ *as* $k \to +\infty$.

(A_2) $c_k := \max\{I(u) : u \in Y_k, \|u\| = \gamma_k\} \le 0$.

Then, I has a sequence of critical values tending to $+\infty$.

Lemma 6.2. *(See [13]) For* $s \in C_+(\overline{\Omega})$ *such that* $s(x) < p_2^*(x)$ *for all* $x \in \overline{\Omega}$. *Then,* $\lim_{k\to+\infty} \delta_k = 0$, *where* $\delta_k = \sup\{|u|_{s(x)} : \|u\| = 1, u \in Z_k\}$.

Lemma 6.3. *(See [13]) For all* $s \in C_+(\overline{\Omega})$ *and* $u \in L^{s(x)}(\Omega)$, *there exists* $y \in \Omega$ *such that*

$$\int_\Omega |u|^{s(x)}\, dx = |u|_{s(x)}^{s(y)}. \tag{21}$$

Now, we continue with the proof of Theorem 1.4. To this end, based on the fountain Theorem 6.1, we will show that the problem (1) possesses infinitely many of solutions with unbounded energy. Evidently, according to (f_5), I is an even functional. By Lemma 3.3, we know that I satisfies the (C)-condition. Then, to prove Theorem 1.4, it only remains to verify the following assertions:

(A_1) $b_k := \inf\{I(u) : u \in Z_k, \|u\|_d = \eta_k\} \to +\infty$ *as* $k \to +\infty$;

(A_2) $c_k := \max\{I(u) : u \in Y_k, \|u\|_d = \gamma_k\} \leq 0.$

To prove (A_1), we are going to take $u \in Z_k$ such that $\|u\|_d = \eta_k > 1$. It follows from (m_0), (m_1), (f_1), Proposition 2.5 and Lemma 6.3 that

$$\begin{aligned} I(u) &= \widehat{m}\left(\int_\Omega \frac{|\Delta u|^{p(x)} + d(x)|u|^{p(x)}}{p(x)}dx\right) - \lambda \int_\Omega F(x,u)dx \\ &\geq \frac{m^*}{\mu p^+}\|u\|_d^{p^-} - \lambda C\int_\Omega |u|\, dx - \lambda C \int_\Omega \frac{|u|^{s(x)}}{s(x)}dx \\ &\geq \frac{m^*}{\mu p^+}\|u\|_d^{p^-} - \lambda c_8\|u\|_d - \frac{\lambda C}{s^-}|u|_{s(x)}^{s(y)}. \end{aligned}$$

On the other hand, by Lemma 6.2,

$$|u|_{s(x)}^{s(y)} \leq \begin{cases} 1 & \text{if } |u|_{s(x)} \leq 1 \\ (\delta_k\|u\|_d)^{s^+} & \text{if } |u|_{s(x)} > 1. \end{cases}$$

Hence,

$$\begin{aligned} I(u) &\geq \frac{m^*}{\mu p^+}\|u\|_d^{p^-} - \lambda c_8\|u\|_d - \frac{\lambda C}{s^-}(\delta_k\|u\|_d)^{s^+} - \frac{\lambda C}{s^-} \\ &\geq \|u\|_d^{p^-}\left[\frac{m^*}{\mu p^+} - \lambda c_8\|u\|_d^{1-p^-} - \frac{\lambda C}{s^-}\delta_k^{s^+}\|u\|_d^{s^+-p^-} - \frac{\lambda C}{s^-}\|u\|_d^{-1}\right]. \end{aligned}$$

Let us fix $\|u\|_d = \eta_k = \delta_k^{-1}$. Then, we obtain

$$I(u) \geq \delta_k^{-p^-}\left[\frac{m^*}{\mu p^+} - \lambda c_8\delta_k^{p^- -1} - \frac{\lambda C}{s^-}\delta_k^{p^+} - \frac{\lambda C}{s^-}\delta_k\right].$$

Since $1 < p^- < s^+$ and $\delta_k \to 0$ as $k \to +\infty$, we conclude that $\eta_k \to +\infty$ as $k \to +\infty$. Finally, $I(u) \to +\infty$ as $k \to +\infty$. Therefore, (A_1) holds.

To prove (A_2), let us suppose by contradiction that (A_2) is not fulfilled for some given k, that is, then there exists a sequence (u_n) in Y_k such that

$$\|u_n\|_d \to +\infty \quad \text{and} \quad I(u_n) \geq 0. \tag{22}$$

Put $v_n = \dfrac{u_n}{\|u_n\|_d}$, then $\|v_n\|_d = 1$. Because Y_k has finite dimension, there exists $v \in Y_k \setminus \{0\}$ such that up to a subsequence,

$$\|v_n - v\|_d \to 0 \quad \text{and} \quad v_n(x) \to v(x) \quad \text{a.e. in } \Omega.$$

If $v(x) \neq 0$, then $|u_n(x)| \to +\infty$. Hence, from (f_3), we have

$$\lim_{n\to+\infty}\frac{F(x,u_n(x))}{\|u_n\|_d^{\mu p^+}} = \lim_{n\to+\infty}\frac{F(x,u_n(x))}{|u_n(x)|^{\mu p^+}}|v_n(x)|^{\mu p^+} = +\infty,$$

for all $x \in \Theta = \{x \in \Omega : v(x) \neq 0\}$. *Then, by Fatou's Lemma,*

$$+\infty = \int_{\Theta} \liminf_{n\to+\infty} \left(\frac{F(x, u_n(x))}{\|u_n\|_d^{\mu p^+}} \right) dx \leq \limsup_{n\to+\infty} \int_{\Theta} \frac{F(x, u_n(x))}{\|u_n\|_d^{\mu p^+}} \, dx. \tag{23}$$

Consequently, by Lemma 3.2, for $\|u_n\|_d > 1$, *we get*

$$\begin{aligned} I(u_n) &= \widehat{m}\left(\int_{\Omega} \frac{|\Delta u_n|^{p(x)} + d(x)|u_n|^{p(x)}}{p(x)} dx \right) - \lambda \int_{\Omega} F(x, u_n) dx \\ &\leq \widehat{m}\left(\frac{1}{p^-} \int_{\Omega} (|\Delta u_n|^{p(x)} + d(x)|u_n|^{p(x)}) dx \right) - \lambda \int_{\Theta} F(x, u_n) dx \\ &\leq \widehat{m}(1) \left(1 + \frac{1}{(p^-)^{\mu}} \left(\int_{\Omega} (|\Delta u_n|^{p(x)} + d(x)|u_n|^{p(x)}) dx \right)^{\mu} \right) - \lambda \int_{\Theta} F(x, u_n) dx \\ &\leq \widehat{m}(1) \left(1 + \frac{\|u_n\|_d^{\mu p^+}}{(p^-)^{\mu}} \right) - \lambda \int_{\Theta} F(x, u_n) dx. \end{aligned}$$

Using (23), *we can deduce that*

$$\limsup_{n\to+\infty} \frac{I(u_n)}{\|u_n\|_d^{\mu p^+}} \leq \limsup_{n\to+\infty} \left[\widehat{m}(1) \|u_n\|_d^{-\mu p^+} + \frac{1}{(p^-)^{\mu}} - \lambda \int_{\Theta} \frac{F(x, u_n)}{\|u_n\|_d^{\mu p^+}} dx \right] = -\infty.$$

which contradiction to (22). *Finally, the assertion* (A_2) *is also valid. This completes the proof.*

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