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# The core inverse of the sum in a ring with involution

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**Abstract.** We present a necessary and sufficient condition under which the sum of two commuting core invertible elements in a *-ring is core invertible. As applications, we establish various conditions under which a block complex matrix with core invertible subblocks is core invertible.

## 1. Introduction

An involution of a ring $R$ is an anti-automorphism whose square is the identity map 1. Thus an involution of a ring $R$ is an operation $* : R \to R$ such that $(x+y)^* = x^* + y^*$, $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$. A ring $R$ with involution $*$ is called a *-ring.

Let $R$ be a *-ring. An element $a$ in $R$ has group inverse provided that there exists $x \in R$ such that

$$xa^2 = a, ax^2 = x, ax = xa.$$

Such $x$ is unique if it exists, denoted by $a^{\#}$, and called the group inverse of $a$. As is well known, an element $a \in R$ has group inverse if and only if it is strongly regular (i.e., Abelian regular). A square complex matrix $A$ has group inverse if and only if $rank(A) = rank(A^2)$. Group invertibility was extensively studied in ring, matrix and operator theory (see [2, 6, 12, 13, 21]).

An element $a \in R$ has core inverse if there exists some $x \in R$ such that

$$xa^2 = a, ax^2 = x, (ax)^* = ax.$$

If such $x$ exists, it is unique, and denote it by $a^{\textcircled{\#}}$.

Core inverse for complex matrices was firstly introduced by Baksalary and Trenkler in [1]. An element $a \in R$ has $(1,3)$-inverse provided that there exists some $x \in R$ such that $a = axa$ and $(ax)^* = ax$. We denote $x$ by $a^{(1,3)}$. We list several characterizations of core inverse in a *-ring.

**Theorem 1.1.** *(see [6, Theorem 2.8], [6, Theorem 2.14], [7, Theorem 3.4] and [18, Theorem 2.6]). Let $R$ be a *-ring, and let $a \in R$. Then the following are equivalent:*

(1) $a$ has core inverse.

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(2) There exists $x \in R$ such that $axa = a, x = xax, xa^2 = a, ax^2 = x, (ax)^* = ax$.
(3) There exists $x \in R$ such that $axa = a$ and $aR = xR = x^*R$.
(4) There exists some $p^2 = p = p^* \in R$ such that $pa = 0$ and $a + p \in R$ is invertible.
(5) $a \in R$ has group inverse and $Ra = Ra^*a$.
(6) $a \in R$ has group inverse and $a \in R$ has $(1, 3)$-inverse.
In this case, $a^{\#\!\!\!\!\bigcirc} = x = a^{\#}aa^{(1,3)}$.

The core invertibility in a *-ring is attractive. Many authors have studied such problems from many different views, e.g., [1, 3, 6, 8–11, 16, 18, 22].

In [18, Theorem 4.3], Xue, Chen and Zhang proved that $a + b \in R$ has core inverse under the conditions $ab = 0$ and $a^*b = 0$ for two core invertible elements $a$ and $b$ in $R$.

In [21, Theorem 4.1], Zhou et al. considered the core inverse of $a + b$ under the conditions $a^2a^{\#\!\!\!\!\bigcirc}b^{\#\!\!\!\!\bigcirc}b = baa^{\#\!\!\!\!\bigcirc}, ab^{\#\!\!\!\!\bigcirc}b = aa^{\#\!\!\!\!\bigcirc}b$ in a Dedekind-finite ring in which 2 is invertible.

In this paper, we present a new additive result for the core inverse in a ring with involution. We give a necessary and sufficient condition under which the sum of two commuting core invertible elements is core invertible.

Let $C^{n\times n}$ be a *-ring of $n \times n$ complex matrices, with conjugate transpose as the involution. A matrix $A \in C^{n\times n}$ has core inverse $X$ if and only if $AX = P_A$ and $\mathcal{R}(X) \subseteq \mathcal{R}(A)$, where $P_A$ is the projection on the range space $\mathcal{R}(A)$ of $A$ (see [1, Definition 1]). As applications, we establish various conditions under which a block complex matrix with core invertible subblocks is core invertible.

Throughout the paper, all *-rings are associative with an identity. An element $p \in R$ is a projection provided that $p^2 = p = p^*$. Let $a \in R^{\#}$ and $a^{\pi} = 1 - aa^{\#}$. Let $p^2 = p \in R$, and let $x \in R$. We write $x = pxp + px(1 - p) + (1 - p)xp + (1 - p)x(1 - p)$, and induce a Pierce representation given by the matrix $x = \begin{pmatrix} pxp & px(1-p) \\ (1-p)xp & (1-p)x(1-p) \end{pmatrix}_p$. We use $R^{\#}$ and $R^{\#\!\!\!\!\bigcirc}$ to denote the sets of all group and core invertible elements in $R$, respectively. $A^*$ stands for the conjugate transpose $\overline{A}^T$ of the complex matrix $A$.

## 2. The main result

We begin with some elementary results which will be repeatedly used in the next sequel.

**Lemma 2.1.** *(see [5, Corollary 3.4])) Let $a, b \in R^{\#\!\!\!\!\bigcirc}$. If $ab = ba$ and $a^*b = ba^*$, then $a^{\#\!\!\!\!\bigcirc}b = ba^{\#\!\!\!\!\bigcirc}$.*

**Lemma 2.2.** *Let $a \in R^{\#\!\!\!\!\bigcirc}$ and $b \in R$. Then the following are equivalent:*

(1) $(1 - a^{\#\!\!\!\!\bigcirc}a)b = 0$.
(2) $(1 - aa^{\#\!\!\!\!\bigcirc})b = 0$.
(3) $(1 - aa^{\#})b = 0$.

*Proof.* (1) ⇔ (2) See [16, Lemma 2.4].

(1) ⇒ (3) Since $(1 - a^{\#\!\!\!\!\bigcirc}a)b = 0$, we have $b = a^{\#\!\!\!\!\bigcirc}ab = a^{\#}aa^{(1,3)}ab = a^{\#}ab$. Then $(1 - aa^{\#})b = 0$, as required.

(3) ⇒ (1) Since $(1 - aa^{\#})b = 0$, we have $b = aa^{\#}b = a^{\#}aa^{(1,3)}ab = a^{\#\!\!\!\!\bigcirc}ab$; hence, $(1 - a^{\#\!\!\!\!\bigcirc}a)b = 0$. This completes the proof. □

**Lemma 2.3.** *(see [18, Theorem 4.3]) Let $a, b \in R^{\#\!\!\!\!\bigcirc}$. If $ab = a^*b = 0$, then $a + b \in R^{\#\!\!\!\!\bigcirc}$ and $(a + b)^{\#\!\!\!\!\bigcirc} = b^{\pi}a^{\#\!\!\!\!\bigcirc} + b^{\#\!\!\!\!\bigcirc}$.*

**Lemma 2.4.** *(see [5, Theorem 3.5])) Let $a, b \in R^{\#\!\!\!\!\bigcirc}$. If $ab = ba$ and $a^*b = ba^*$, then $ab \in R^{\#\!\!\!\!\bigcirc}$ and $(ab)^{\#\!\!\!\!\bigcirc} = a^{\#\!\!\!\!\bigcirc}b^{\#\!\!\!\!\bigcirc}$.*

We are ready to prove:

**Theorem 2.5.** *Let $a, b \in R^{\#\!\!\!\!\bigcirc}$. If $ab = ba$ and $a^*b = ba^*$, then the following are equivalent:*

(1) $a + b \in R^{\#\!\!\!\!\bigcirc}$ and $a^{\pi}(a + b)^{\#\!\!\!\!\bigcirc}a = 0$.
(2) $1 + a^{\#\!\!\!\!\bigcirc}b \in R^{\#\!\!\!\!\bigcirc}$ and $(1 + a^{\#\!\!\!\!\bigcirc}b)^{\pi}a(1 - aa^{\#\!\!\!\!\bigcirc}) = 0$.

*Proof.* (1) ⇒ (2) Since $ab = ba$ and $a^*b = ba^*$, it follows by Lemma 2.1 that $a^{\text{\textcircled{\#}}}b = ba^{\text{\textcircled{\#}}}$. We observe that

$$\begin{array}{rcl} 1 + a^{\text{\textcircled{\#}}}b & = & (1 - aa^{\text{\textcircled{\#}}}) + (aa^{\text{\textcircled{\#}}} + a^{\text{\textcircled{\#}}}b) \\ & = & (1 - aa^{\text{\textcircled{\#}}}) + (aa^{\text{\textcircled{\#}}} + ba^{\text{\textcircled{\#}}}) \\ & = & (1 - aa^{\text{\textcircled{\#}}}) + (a + b)a^{\text{\textcircled{\#}}} \end{array}$$

Let $p = aa^{\text{\textcircled{\#}}}$. Obviously, $p^\pi(a + b)p = 0$. Then

$$a + b = \begin{pmatrix} p(a + b)p & p(a + b)p^\pi \\ 0 & p^\pi(a + b)p^\pi \end{pmatrix}_p.$$

Since $(1-aa^\#)(a+b)^{\text{\textcircled{\#}}}a = 0$, by using Lemma 2.2, $(1-aa^{\text{\textcircled{\#}}})(a+b)^{\text{\textcircled{\#}}}a = 0$. Then $p^\pi(a+b)^{\text{\textcircled{\#}}}p = [(1-aa^{\text{\textcircled{\#}}})(a+b)^{\text{\textcircled{\#}}}a]a^{\text{\textcircled{\#}}} = 0$. Thus, we have

$$(a + b)^{\text{\textcircled{\#}}} = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}_p.$$

Set $x = a + b, c_1 = p(a + b)p$ and $x_1 = \alpha$. In light of Theorem 1.1, we have

$$x = xx^{\text{\textcircled{\#}}}x, (xx^{\text{\textcircled{\#}}})^* = xx^{\text{\textcircled{\#}}}, x^{\text{\textcircled{\#}}}x^2 = x, x(x^{\text{\textcircled{\#}}})^2 = x^{\text{\textcircled{\#}}}.$$

Hence, $c_1 = c_1x_1c_1, (c_1x_1)^* = c_1x_1, x_1c_1^2 = c_1, c_1x_1^2 = x_1$. Therefore $c_1 = aa^\#(a + b)aa^\# \in R^{\text{\textcircled{\#}}}$ and $[p(a + b)p]^{\text{\textcircled{\#}}} = c_1^{\text{\textcircled{\#}}} = x_1 = \alpha$. Thus, $(a + b)aa^{\text{\textcircled{\#}}} \in R^{\text{\textcircled{\#}}}$. We easily check that

$$\begin{array}{rcl} [(a + b)aa^{\text{\textcircled{\#}}}]a^{\text{\textcircled{\#}}} & = & a^2[a^{\text{\textcircled{\#}}}]^2 + ba[a^{\text{\textcircled{\#}}}]^2 \\ & = & aa^{\text{\textcircled{\#}}} + ba^{\text{\textcircled{\#}}} \\ & = & aa^{\text{\textcircled{\#}}} + b(a^{\text{\textcircled{\#}}}aa^{\text{\textcircled{\#}}}) \\ & = & [a^{\text{\textcircled{\#}}}a^2]a^{\text{\textcircled{\#}}} + a^{\text{\textcircled{\#}}}(baa^{\text{\textcircled{\#}}}) \\ & = & a^{\text{\textcircled{\#}}}[(a + b)aa^{\text{\textcircled{\#}}}]. \end{array}$$

In view of [16, Lemma 2.1], $(a + b)a^{\text{\textcircled{\#}}} = [(a + b)aa^{\text{\textcircled{\#}}}]a^{\text{\textcircled{\#}}} \in R^\#$. Set $y = [(a + b)aa^{\text{\textcircled{\#}}}]^{\text{\textcircled{\#}}}$. Then

$$(a + b)aa^{\text{\textcircled{\#}}} = (a + b)aa^{\text{\textcircled{\#}}}y(a + b)aa^{\text{\textcircled{\#}}}, [(a + b)aa^{\text{\textcircled{\#}}}y]^* = (a + b)aa^{\text{\textcircled{\#}}}y.$$

We verify that

$$\begin{array}{rl} & [(a + b)a^{\text{\textcircled{\#}}}](a^2a^{\text{\textcircled{\#}}}y)[(a + b)a^{\text{\textcircled{\#}}}] \\ = & [(a + b)aa^{\text{\textcircled{\#}}}]y[(a + b)aa^{\text{\textcircled{\#}}}]a^{\text{\textcircled{\#}}} \\ = & [(a + b)aa^{\text{\textcircled{\#}}}]^{\text{\textcircled{\#}}} \\ = & (a + b)a^{\text{\textcircled{\#}}} \end{array}$$

and

$$\begin{array}{rl} & [(a + b)a^{\text{\textcircled{\#}}}(a^2a^{\text{\textcircled{\#}}}y)]^* \\ = & [(a + b)aa^{\text{\textcircled{\#}}}y]^* \\ = & (a + b)aa^{\text{\textcircled{\#}}}y \\ = & (a + b)a^{\text{\textcircled{\#}}}(a^2a^{\text{\textcircled{\#}}}y). \end{array}$$

Therefore $(a + b)a^{\text{\textcircled{\#}}}$ has (1, 3)-inverse $a^2a^{\text{\textcircled{\#}}}y$. By virtue of Theorem 1.1, $(a + b)a^{\text{\textcircled{\#}}} \in R^{\text{\textcircled{\#}}}$. Obviously, we have

$$(1 - aa^{\text{\textcircled{\#}}})(a + b)a^{\text{\textcircled{\#}}} = (1 - aa^{\text{\textcircled{\#}}})^*(a + b)a^{\text{\textcircled{\#}}} = 0.$$

According to Lemma 2.3, $1 + a^{\text{\textcircled{\#}}}b \in R^{\text{\textcircled{\#}}}$.

Since $(a + b)(a + b)^{\text{\textcircled{\#}}}(a + b) = a + b$, we have

$$p(a + b)p^\pi = p(a + b)p\alpha p(a + b)p^\pi + [p(a + b)p\beta + p(a + b)p^\pi\gamma]p^\pi(a + b)p^\pi.$$

Moreover, we have $[(a + b)(a + b)^{\text{\textcircled{\#}}}]^* = (a + b)(a + b)^{\text{\textcircled{\#}}}$, we have

$$p(a + b)p\beta + p(a + b)p^\pi\gamma = 0.$$

Then

$$p(a+b)p^{\pi} = p(a+b)p\alpha p(a+b)p^{\pi},$$

and then $[p(a+b)p]^{\pi}p(a+b)p^{\pi} = [p(a+b)p]^{\pi}[p(a+b)]p\alpha p(a+b)p^{\pi} = 0$. It is easy to verify that

$$\begin{array}{rcl} (a^2a^{\circledR})a^{\circledR} & = & aa^{\circledR} = a^{\circledR}(a^2a^{\circledR}), \\ a^{\circledR}(a^2a^{\circledR})a^{\circledR} & = & a^{\circledR}(aa^{\circledR}) = a^{\circledR}, \\ (a^2a^{\circledR})a^{\circledR}(a^2a^{\circledR}) & = & (aa^{\circledR})(a^2a^{\circledR}) = a^2a^{\circledR}. \end{array}$$

Thus $[a^2a^{\circledR}]^{\#} = a^{\circledR}$. Since $p(a+b)p = a^2a^{\circledR} + baa^{\circledR} = (1+a^{\circledR}b)a^2a^{\circledR}$, we have

$$[p(a+b)p]^{\#} = (1+a^{\circledR}b)^{\#}(a^2a^{\circledR})^{\#} = (1+a^{\circledR}b)^{\#}a^{\circledR}.$$

Hence,

$$\begin{array}{rl} & [p(a+b)p]^{\pi} \\ = & 1-[p(a+b)p][p(a+b)p]^{\#} \\ = & 1-[(1+a^{\circledR}b)a^2a^{\circledR}][(1+a^{\circledR}b)^{\#}a^{\circledR}] \\ = & 1-[(1+a^{\circledR}b)(1+a^{\circledR}b)^{\#}][a^2a^{\circledR}a^{\circledR}] \\ = & 1-(1+a^{\circledR}b)(1+a^{\circledR}b)^{\#} + (1+a^{\circledR}b)(1+a^{\circledR}b)^{\#}(1-aa^{\circledR}) \\ = & (1+a^{\circledR}b)^{\pi} + (1+a^{\circledR}b)(1+a^{\circledR}b)^{\#}(1-aa^{\circledR}). \end{array}$$

Thus we check that

$$\begin{array}{rl} & (1+a^{\circledR}b)^{\pi}a(1-aa^{\circledR}) \\ = & (1+a^{\circledR}b)^{\pi}aa^{\circledR}a(1-aa^{\circledR}) \\ = & [p(a+b)p]^{\pi}aa^{\circledR}(a+b)(1-aa^{\circledR}) \\ = & [p(a+b)p]^{\pi}p(a+b)p^{\pi} \\ = & 0. \end{array}$$

Therefore $(1+a^{\circledR}b)^{\pi}a(1-aa^{\circledR}) = 0$.

(2) ⇒ (1) Let $z = (1+a^{\circledR}b)^{\circledR}$. Then we verify that

$$\begin{array}{rl} & [(1+a^{\circledR}b)a][a^{\circledR}z][(1+a^{\circledR}b)a] \\ = & aa^{\circledR}[(1+a^{\circledR}b)z(1+a^{\circledR}b)]a \\ = & aa^{\circledR}(1+a^{\circledR}b)a \\ = & (1+a^{\circledR}b)a. \end{array}$$

Since $(1+a^{\circledR}b)aa^{\circledR} = aa^{\circledR}(1+a^{\circledR}b)$ and $(aa^{\circledR})^* = aa^{\circledR}$, we have

$$aa^{\circledR}(1+a^{\circledR}b)^* = (1+a^{\circledR}b)^*aa^{\circledR}.$$

In light of Lemma 2.1, we get $aa^{\circledR}z = zaa^{\circledR}$.

Step 1. By the argument above, $a^2a^{\circledR} \in R^{\#}$. In view of Theorem 1.1, $1+a^{\circledR}b \in R^{\#}$. Since

$$\begin{array}{rl} & (1+a^{\circledR}b)a^2a^{\circledR} \\ = & a^2a^{\circledR} + b(a^{\circledR}a^2)a^{\circledR} \\ = & (a+b)aa^{\circledR} \\ = & a^2a^{\circledR} + aa^{\circledR}b \\ = & a^2a^{\circledR}(1+a^{\circledR}b), \end{array}$$

it follows by [16, Lemma 2.1] that $(1+a^{\circledR}b)a^2a^{\circledR} \in R^{\#}$ and

$$\begin{array}{rl} & [(a+b)aa^{\circledR}]^{\pi} \\ = & [(1+a^{\circledR}b)a^2a^{\circledR}]^{\pi} \\ = & 1-(1+a^{\circledR}b)a^2a^{\circledR}(1+a^{\circledR}b)^{\#}a^{\circledR} \\ = & 1-(1+a^{\circledR}b)(1+a^{\circledR}b)^{\#}aa^{\circledR}. \end{array}$$

Step 2. We check that

$$\begin{aligned} & [(1+a^{\#}b)a^2a^{\#}](a^{\#}z) \\ = \; & [(1+a^{\#}b)z](aa^{\#}). \end{aligned}$$

Hence,

$$\begin{aligned} & [(1+a^{\#}b)a^2a^{\#}(a^{\#}z)]^* \\ = \; & (aa^{\#})^*[(1+a^{\#}b)z]^* \\ = \; & (aa^{\#})[(1+a^{\#}b)z] \\ = \; & [(1+a^{\#}b)z](aa^{\#}) \\ = \; & [(1+a^{\#}b)a^2a^{\#}](a^{\#}z). \end{aligned}$$

So $(1+a^{\#}b)a^2a^{\#}$ has a $(1,3)$ inverse $a^{\#}z$.

Accordingly, $(a+b)aa^{\#} = (1+a^{\#}b)a^2a^{\#} \in R^{\#}$. Let $p = aa^{\#}$. Then $p^{\pi}bp = (1-aa^{\#})baa^{\#} = (1-aa^{\#})aba^{\#} = 0$. Similarly, $pbp^{\pi} = 0$. So we get

$$a = \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix}_p, b = \begin{pmatrix} b_1 & 0 \\ 0 & b_4 \end{pmatrix}_p.$$

Hence

$$a+b = \begin{pmatrix} a_1+b_1 & a_2 \\ 0 & b_4 \end{pmatrix}_p.$$

Here $a_1+b_1 = a(a^{\#}a^2)a^{\#} + aa^{\#}baa^{\#} = a^2a^{\#} + b(aa^{\#})^2 = (a+b)aa^{\#}$, $b_4 = p^{\pi}(a+b)p^{\pi} = bp^{\pi}$. Since $bp^{\pi} = p^{\pi}b$, $b^*p^{\pi} = (p^{\pi}b)^* = (bp^{\pi})^* = p^{\pi}b^*$. In light of Lemma 2.4, $b_4 = bp^{\pi} \in R^{\#}$ and $b_4^{\#} = b^{\#}p^{\pi}$.

Let

$$x = \begin{pmatrix} (a_1+b_1)^{\#} & -(a_1+b_1)^{\#}a_2b_4^{\#} \\ 0 & b_4^{\#} \end{pmatrix}_p.$$

Since $(1+a^{\#}b)^{\pi}a(1-aa^{\#}) = 0$, we verify that

$$\begin{aligned} & a_2 - (a_1+b_1)(a_1+b_1)^{\#}a_2 \\ = \; & [1-(a_1+b_1)(a_1+b_1)^{\#}]aa^{\#}a(1-aa^{\#}) \\ = \; & [1-(1+a^{\#}b)(1+a^{\#}b)^{\#}aa^{\#}]aa^{\#}a(1-aa^{\#}) \\ = \; & (1+a^{\#}b)^{\pi}a(1-aa^{\#}) \\ = \; & 0. \end{aligned}$$

That is, $(a_1+b_1)^{\pi}a_2 = 0$. In view of [12, Theorem 2.1], $a+b \in R^{\#}$ and

$$(a+b)^{\#} = \begin{pmatrix} (a_1+b_1)^{\#} & * \\ 0 & (b_4)^{\#} \end{pmatrix}_p.$$

Then we we have

$$\begin{aligned} & (a+b)x \\ = \; & \begin{pmatrix} a_1+b_1 & a_2 \\ 0 & b_4 \end{pmatrix}_p \begin{pmatrix} (a_1+b_1)^{\#} & -(a_1+b_1)^{\#}a_2b_4^{\#} \\ 0 & b_4^{\#} \end{pmatrix}_p \\ = \; & \begin{pmatrix} (a_1+b_1)(a_1+b_1)^{\#} & 0 \\ 0 & b_4b_4^{\#} \end{pmatrix}_p. \end{aligned}$$

Hence $[(a+b)x]^* = (a+b)x$. We further verify that

$$\begin{aligned} & (a+b)x(a+b) \\ = \; & \begin{pmatrix} (a_1+b_1)(a_1+b_1)^{\#} & 0 \\ 0 & b_4b_4^{\#} \end{pmatrix}_p \begin{pmatrix} a_1+b_1 & a_2 \\ 0 & b_4 \end{pmatrix}_p \\ = \; & a+b. \end{aligned}$$

Thus $a+b \in R^{(1,3)}$. According to Theorem 1.1, $a+b$ has core inverse.

Moreover, we have

$$\begin{array}{rl} & (a+b)^{\circledast} \\ = & (a+b)^{\#}(a+b)x \\ = & \begin{pmatrix} (a_1+b_1)^{\#} & * \\ 0 & (b_4)^{\#} \end{pmatrix}_p \begin{pmatrix} (a_1+b_1)(a_1+b_1)^{\circledast} & 0 \\ 0 & b_4b_4^{\circledast} \end{pmatrix}_p \\ = & \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}_p . \end{array}$$

We infer that $p^{\pi}(a+b)^{\circledast}a = p^{\pi}(a+b)^{\circledast}pa = 0$. In light of Lemma 2.2, $a^{\pi}(a+b)^{\circledast}a = 0$, as asserted. □

An element $a \in R$ has dual core inverse if there exists some $x \in R$ such that

$$a^2x = a, x^2a = x, (xa)^* = xa.$$

If such $x$ exists, it is unique, and denote it by $a_{\circledast}$ (see [7]).

**Corollary 2.6.** *Let $a, b \in R_{\circledast}$. If $ab = ba$ and $a^*b = ba^*$, then the following are equivalent:*

(1) $a + b \in R_{\circledast}$ and $a(a+b)_{\circledast}a^{\pi} = 0$.
(2) $1 + a_{\circledast}b \in R_{\circledast}$ and $(1 - aa_{\circledast})a(1 + a_{\circledast}b)^{\pi} = 0$.

*Proof.* Since $x \in R$ has dual core if and only if $x^* \in R$ has core inverse and $x_{\circledast} = (x^*)^{\circledast}$. In view of Lemma 2.1, we have $a_{\circledast}b = ba_{\circledast}$. Therefore we complete the proof by Theorem 2.5. □

Recall that $a \in R$ is EP, if there exists $x \in R$ such that $xa^2 = a, ax = xa, (ax)^* = ax$. Evidently, $a \in R$ is EP if and only if $a \in R^{\circledast}$ and $a^{\circledast} = a^{\#}$ if and only if $a \in R^{\circledast}$ and $(aa^{\#})^* = aa^{\#}$ if and only if $a \in R^{\circledast} \bigcap R_{\circledast}$ and $a^{\circledast} = a_{\circledast}$ (see [14, 15, 17]). We now derive

**Corollary 2.7.** *Let $a, b \in R$ be EP. If $ab = ba$ and $a^*b = ba^*$, then the following are equivalent:*

(1) $a + b \in R$ is EP.
(2) $1 + a^{\#}b \in R$ is EP.

*Proof.* (1) ⇒ (2) Since $a + b \in R$ is EP, $a + b \in R^{\circledast}$ and $(a+b)^{\circledast} = (a+b)^{\#}$. As $a \in R$ is EP, $a^{\circledast} = a^{\#}$. Clearly, $a(a+b) = (a+b)a$, and so $a^{\#}(a+b) = (a+b)a^{\#}$. Write $1 + a^{\#}b = a_1 + a_2$, where $a_1 = 1 - aa^{\circledast}$ and $a_2 = (a+b)a^{\circledast}$ In view of Lemma 2.1, we have $a_1, a_2 \in R^{\circledast}$. Obviously, $a_1a_2 = a_2a_1 = a_1^*a_2 = 0$. In light of [18, Theorem 4.3], $a_1 + a_2 \in R^{\circledast}$. By virtue of [2, Theorem 2.1], we have $(1 + a^{\#}b)^{\#} = a_1^{\#} + a_2^{\#}$. Hence,

$$\begin{array}{rcl} (1+a^{\#}b)(1+a^{\#}b)^{\#} & = & (a_1+a_2)(a_1^{\#}+a_2^{\#}) \\ & = & (a_1+a_2)(a_1^{\#}+a_2^{\#}) \\ & = & a_1a_1^{\#} + (a_1a_2)(a_2^{\#})^2 + (a_2a_1)(a_1^{\#})^2 + a_2a_2^{\#} \\ & = & a_1a_1^{\#} + a_2a_2^{\#} \\ & = & (1 - aa^{\circledast}) + (a+b)(a+b)^{\circledast}aa^{\circledast}, \end{array}$$

and then $[(1+a^{\#}b)(1+a^{\#}b)^{\#}]^* = (1+a^{\#}b)(1+a^{\#}b)^{\#}$. Therefore $1 + a^{\#}b \in R$ is EP, as required.

(2) ⇒ (1) Since $a \in R$ is EP, $a \in R^{\circledast}$ and $a^{\circledast} = a^{\#}$. Then $(1 + a^{\circledast}b)^{\pi}a(1 - aa^{\circledast}) = 0$. In light of Theorem 2.5, $a + b \in R^{\circledast}$. One easily checks that

$$a + b = a(1 + a^{\#}b) + (1 - aa^{\#})b.$$

By hypothesis, we see that $a(1 + a^{\#}b), (1 - aa^{\#})b \in R^{\#}$ and $a(1 + a^{\#}b)(1 - aa^{\#})b = (1 - aa^{\#})ba(1 + a^{\#}b) = 0$. According to [2, Theorem 2.1], we have

$$(a+b)^{\#} = a^{\#}(1 + a^{\#}b)^{\#} + (1 - aa^{\#})b^{\#}.$$

Hence,

$$\begin{array}{rcl} (a+b)(a+b)^{\#} & = & aa^{\#}(1+a^{\#}b)(1+a^{\#}b)^{\#} + (1-aa^{\#})bb^{\#} \\ & = & aa^{\circledast}(1+a^{\#}b)(1+a^{\#}b)^{\circledast} + (1-aa^{\circledast})bb^{\circledast}. \end{array}$$

Then $[(a+b)(a+b)^{\#}]^* = (a+b)(a+b)^{\#}$, thus yielding the result. □

## 3. Applications

Let $A, B, C, D \in \mathbb{C}^{n\times n}$ have core inverses and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. The aim of this section is to present the core invertibility of the block complex matrix $M$ by using the core invertibility of its subblocks.

**Lemma 3.1.** *If $B(CB)^{\pi} = 0$ and $C(BC)^{\pi} = 0$, then $\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$ has core inverse. In this case,*

$$Q^{\textcircled{\#}} = \begin{pmatrix} 0 & (BC)^{\#}BCC^{\textcircled{\#}} \\ (CB)^{\#}CBB^{\textcircled{\#}} & 0 \end{pmatrix}.$$

*Proof.* Let $Q = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$. Then $(CB)(CB)^D = (CB)^D(CB), (CB)^D = (CB)^D(CB)(CB)^D$. Since $B(CB)^{\pi} = 0$, we have $CB(CB)^{\pi} = 0$. Hence $CB$ has group inverse. Likewise, $BC$ has group inverse. One directly checks that $Q^{\#} = \begin{pmatrix} 0 & B(CB)^{\#} \\ C(BC)^{\#} & 0 \end{pmatrix}$. Moreover, we verify that

$$\begin{array}{ll} & Q\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}\begin{pmatrix} B^{\textcircled{\#}} & 0 \\ 0 & C^{\textcircled{\#}} \end{pmatrix}Q \\ = & \begin{pmatrix} 0 & BB^{\textcircled{\#}}B \\ CC^{\textcircled{\#}}C & 0 \end{pmatrix} \\ = & Q; \\ & (Q\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}\begin{pmatrix} B^{\textcircled{\#}} & 0 \\ 0 & C^{\textcircled{\#}} \end{pmatrix})^* \\ = & \begin{pmatrix} BB^{\textcircled{\#}} & 0 \\ 0 & CC^{\textcircled{\#}} \end{pmatrix}^* \\ = & Q\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}\begin{pmatrix} B^{\textcircled{\#}} & 0 \\ 0 & C^{\textcircled{\#}} \end{pmatrix}. \end{array}$$

This implies that $Q$ has $(1,3)$-inverse. In light of [16, Lemma 2.1], $Q$ has core inverse. In this case,

$$\begin{array}{rcl} Q^{\textcircled{\#}} & = & Q^{\#}QQ^{(1,3)} \\ & = & \begin{pmatrix} 0 & B(CB)^{\#} \\ C(BC)^{\#} & 0 \end{pmatrix}\begin{pmatrix} BB^{\textcircled{\#}} & 0 \\ 0 & CC^{\textcircled{\#}} \end{pmatrix} \\ & = & \begin{pmatrix} 0 & B(CB)^{\#}CC^{\textcircled{\#}} \\ C(BC)^{\#}BB^{\textcircled{\#}} & 0 \end{pmatrix} \\ & = & \begin{pmatrix} 0 & (BC)^{\#}BCC^{\textcircled{\#}} \\ (CB)^{\#}CBB^{\textcircled{\#}} & 0 \end{pmatrix}, \end{array}$$

as asserted. □

We are now ready to prove:

**Theorem 3.2.** *If $AB = BD, DC = CA, A^*B = BD^*, D^*C = CA^*, B(CB)^{\pi} = 0$ and $C(BC)^{\pi} = 0$ and $A^{\textcircled{\#}}BD^{\textcircled{\#}}C$ is nilpotent, then $M$ has core inverse.*

*Proof.* Write $M = P + Q$, where

$$P = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, Q = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

Since $A$ and $D$ have core inverses, so has $P$, and that

$$P^{\textcircled{\#}} = \begin{pmatrix} A^{\textcircled{\#}} & 0 \\ 0 & D^{\textcircled{\#}} \end{pmatrix}.$$

In view of Lemma 3.1, $Q$ has core inverse. We easily check that

$$PQ = \begin{pmatrix} 0 & AB \\ DC & 0 \end{pmatrix} = \begin{pmatrix} 0 & BD \\ CA & 0 \end{pmatrix} = QP.$$

Likewise, we verify that $P^*Q = QP^*$. Moreover, we check that

$$I_{2n} + P^{\circledast}Q = \begin{pmatrix} I_n & A^{\circledast}B \\ D^{\circledast}C & I_n \end{pmatrix}.$$

It is easy to verify that

$$\begin{pmatrix} I_n & A^{\circledast}B \\ D^{\circledast}C & I_n \end{pmatrix} = \begin{pmatrix} I_n - A^{\circledast}BD^{\circledast}C & A^{\circledast}B \\ 0 & I_n \end{pmatrix}\begin{pmatrix} I_n & 0 \\ D^{\circledast}C & I_n \end{pmatrix}.$$

Since $A^{\circledast}BD^{\circledast}C$ is nilpotent, we see that $I_n - A^{\circledast}BD^{\circledast}C$ is invertible, and so $\begin{pmatrix} I_n & A^{\circledast}B \\ D^{\circledast}C & I_n \end{pmatrix}$ is invertible. This implies that $I_{2n} + P^{\circledast}Q$ has core inverse. Additionally, $(I_{2n} + P^{\circledast}Q)^{\pi} = 0$. According to Theorem 2.5, $M$ has core inverse, as asserted. □

**Theorem 3.3.** *If $AB = BD, DC = CA, B^*A = DB^*, C^*D = AC^*, B(CB)^{\pi} = 0$ and $C(BC)^{\pi} = 0$ and $A^{\circledast}BD^{\circledast}C$ is nilpotent, then $M$ has core inverse.*

*Proof.* Write $M = P + Q$, where

$$P = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, Q = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

Then we check that

$$\begin{aligned} Q^*P &= \begin{pmatrix} 0 & C^* \\ B^* & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \\ &= \begin{pmatrix} 0 & C^*D \\ B^*A & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & AC^* \\ DB^* & 0 \end{pmatrix} \\ &= \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} 0 & C^* \\ B^* & 0 \end{pmatrix} \\ &= PQ^*. \end{aligned}$$

Similarly, $QP = PQ$. Further, we verify that

$$\begin{aligned} I_{2n} + Q^{\circledast}P &= I_{2n} + \begin{pmatrix} 0 & (BC)^{\#}BCC^{\circledast} \\ (CB)^{\#}CBB^{\circledast} & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \\ &= \begin{pmatrix} I_n & (BC)^{\#}BCC^{\circledast}D \\ (CB)^{\#}CBB^{\circledast}A & I_n \end{pmatrix}. \end{aligned}$$

Since $A^{\circledast}BD^{\circledast}C$ is nilpotent, we prove that $I_{2n} + Q^{\circledast}P$ is invertible; hence, it has core inverse. Additionally, $(I_{2n} + Q^{\circledast}P)^{\pi} = 0$. In light of Theorem 2.5, $M$ has core inverse, as required. □

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