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Generalized Sumudu transform and tempered ξ-Caputo fractional derivative

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Abstract. This paper applied the generalized Sumudu transform to the tempered ξ-Hilfer fractional integral and the tempered ξ-Caputo fractional derivative. Our findings are utilized to address non homogeneous linear fractional differential equations in an initial value problem involving the tempered ξ-Caputo fractional derivative of an order ς for $n-1<\varsigma<n\in\mathbb{N}$. An example is provided for $0<\varsigma<1$.

1. Introduction

In the early 90s, Watugala [8] introduced a new integral transform called the Sumudu transform, characterized by its simple formulation and various useful properties. He applied it to solve ordinary differential equations and engineering control problems [7]. Subsequently, Weerakoon [21] extended it to partial differential equations. The Sumudu transform has not really been accepted or used by the mathematical world. Thanks to the inversion theorems of this new integral transform, Weerakoon [22] defends this transform against definitions that perceive no difference between Sumudu and Laplace. In 2006, Belgacem et al [2] showed that it was the theoretical dual of the Laplace transform and should therefore compete with it in problem solving. The Sumudu transform is now classified as one of the most widely used integral transforms for solving differential equations due to its conservation of the unit of measurement. For example in [18] this transform was applied to fractional differential equations, while in [1], it was combined with the homotopy pertibation method to solve fractional gas dynamics equation. Additionally, in [11] nonlinear fractional partial differential equations systems was solved through a hybrid homotopy perturbation sumudu transform method, and it was also utilized to construct solutions of local-fractional PDEs [12].

In a recent study [6], researchers utilized a modified version of the Sumudu transform to handle differential equations containing Riemann-Liouville and Caputo type fractional derivatives arising from the ξ-Hilfer fractional integral. Furthermore, in [16] the generalized Laplace transform [3] was applied to derive solutions for fractional differential equations with a new tempered fractional derivative of ξ-Caputo as defined in [15]. This derivative serves as a link between the tempered Caputo derivative [5] and the

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fractional ξ-Caputo derivative [9, 19], and offering greater flexibility in real-world applications, including diffusion processes [10, 13, 17, 23, 24]. (See also the included references). Inspired by the above papers, we employ the generalized Sumudu trnsform in this work to obtain the solutions discussed in [16].

The paper is structured as follows. We present preliminary results in the next section. Section 3 is devoted to the derivation under the generalized Sumudu transform of representations of the tempered ξ-Hilfer fractional integral and the tempered ξ-Caputo fractional derivative. The final section focuses on Cauchy problems for differential equations.

2. Preliminaries

Let's recall some fundamental definitions and notations of generalized Sumudu transform and fractinal calculus that are used in this work.
Throughout the paper, we concede functions in the set $\widetilde{A}$ defined by:

$$\widetilde{A} = \left\{\phi(t)/\exists B, \tau_1, \tau_2, \ \ such \ \ that \mid \phi(t) \mid < Be^{\frac{|t|}{\tau_j}}, \ \ if \ \ t \in (-1)^j \times [0, +\infty)\right\}.$$

Definition 2.1. *The function ϕ is said to be $\xi(t)$-exponential order if positive constants $N, \tilde{c}$, and $\tilde{T}$ exist such that :*

$$|\phi(t)| \leq Ne^{\tilde{c}\xi(t)}, \ \ for \ \ t \geq \tilde{T}.$$

Definition 2.2. *[6] Let $\phi \in C[\tilde{a}, +\infty)$ be of $\xi(t)$-exponential order, and $\xi \in C^1[\tilde{a}, +\infty)$ such that for all $t \geq \tilde{a}$ $\xi'(t) > 0$. The generalized Sumudu transform of ϕ is given by:*

$$\mathbb{S}_\xi\{\phi(t)\}(v) = \frac{1}{v}\int_{\tilde{a}}^{+\infty} e^{-\frac{\xi(t)-\xi(\tilde{a})}{v}}\phi(t)\xi'(t)dt, \tag{1}$$

for all $v > \tilde{c}$.

Remark 2.3. *Let $\phi \in C[\tilde{a}, +\infty)$ be of $\xi(t)$-exponential order for some increasing $\xi \in C[\tilde{a}, +\infty)$, and $N, \tilde{c} \geq 0, \tilde{T} \geq \tilde{a}$ be such that $|\phi(t)| \leq N\mathrm{e}^{\tilde{c}\xi(t)}$ for all $t \geq \tilde{T}$. Then the continuity of ϕ gives the existence of $N_1 \geq 0$ such that $\max_{t\in[\tilde{a},\tilde{T}]} |\phi(t)| \leq N_1$. So taking $\widetilde{N} = \max\left\{N, N_1\mathrm{e}^{-\tilde{c}\xi(\tilde{a})}\right\}$, the increasing property of ξ implies $|\phi(t)| \leq \widetilde{N}\mathrm{e}^{\tilde{c}\xi(t)}$ for all $t \geq \tilde{a}$. Therefore, whenever ξ is increasing, we can take $\tilde{T} = \tilde{a}$.*

Furthermore, when ξ is increasing, we can always assume that $\tilde{c} > 0$. Indeed, if $\xi(t) \geq 0$ for all $t \geq \tilde{a}$, this is obvious. If $\xi(\tilde{a}) < 0$, we have

$$|\phi(t)| \leq N\mathrm{e}^{\tilde{c}\xi(t)} \leq N\mathrm{e}^{\tilde{c}\xi(t)}\mathrm{e}^{\varepsilon(\xi(t)-\xi(\tilde{a}))} = \left(N\mathrm{e}^{-\varepsilon\xi(\tilde{a})}\right)\mathrm{e}^{(\tilde{c}+\varepsilon)\xi(t)},$$

for all $t \geq \tilde{a}$, where $\varepsilon > 0$ is arbitrary fixed.

The relation between the classical and generalized Sumudu transform is represented in the following theorem.

Theorem 2.4. *[6] Considering $\phi, \xi : [\tilde{a}, +\infty) \to \mathbb{R}$ as two functions where ξ is continuous and $\xi'(t) > 0$ on $[\tilde{a}, +\infty)$, and assuming the existence of $\mathbb{S}_\xi\{\phi(t)\}$. Then*

$$\mathbb{S}_\xi\{\phi(t)\}(v) = \mathbb{S}\left\{\phi\left(\xi^{-1}(t + \xi(\tilde{a}))\right)\right\}(v), \tag{2}$$

where $\mathbb{S}\{\phi\}$ is the classical Sumudu transform of ϕ.

Lemma 2.5. *Let ϕ, h be two continuous functions on $[\tilde{a}, +\infty)$, and of $\xi(t)$-exponential order. Then the generalized Sumudu transform has the following properties:*

1. *$\mathbb{S}_\xi\left\{\left(\phi *_\xi h\right)(t)\right\}(v) = v\mathbb{S}_\xi\{\phi(t)\}\mathbb{S}_\xi\{h(t)\}(v)$, where*

$$(\phi *_\xi h)(t) = \int_{\tilde{a}}^{t} \phi(\tau)h\left(\xi^{-1}(\xi(t)+\xi(\tilde{a})-\xi(\tau))\right)\xi'(\tau)d\tau,$$

is the generalized convolution [4].

2. *If $\varsigma > 0, \mu \geq 0$, then*

$$\mathbb{S}_\xi\left\{[\xi(t)-\xi(\tilde{a})]^{\varsigma-1}\mathrm{e}^{-\mu(\xi(t)-\xi(\tilde{a}))}\right\}(v) = \frac{v^{\varsigma-1}}{(\mu v+1)^{\varsigma}}\Gamma(\varsigma),$$

for $\frac{1}{v} > -\mu$, where $\Gamma(z) = \int_0^{+\infty} t^{z-1}\mathrm{e}^{-t}dt$ is the well-known Gamma function.

Proof. For statement 1,

$$\begin{aligned}\mathbb{S}_\xi\{\phi\}\mathbb{S}_\xi\{h\}(v) &= \frac{1}{v}\int_{\tilde{a}}^{+\infty} e^{-\frac{(\xi(t)-\xi(\tilde{a}))}{v}}\phi(t)\xi'(t)dt\frac{1}{v}\int_{a}^{+\infty} e^{-\frac{(\xi(s)-\xi(\tilde{a}))}{v}}h(s)\xi'(s)ds\\ &= \frac{1}{v^2}\int_{\tilde{a}}^{+\infty}\int_{\tilde{a}}^{+\infty} e^{-\frac{(\xi(t)+\xi(s)-2\xi(\tilde{a}))}{v}}\phi(t)h(s)\xi'(t)\xi'(s)dtds\end{aligned}$$

Now, by choosing τ such that $\xi(\tau) = \xi(t)+\xi(s)-\xi(\tilde{a})$, we get

$$\begin{aligned}\mathbb{S}_\xi\{\phi\}\mathbb{S}_\xi\{h\}(v) &= \frac{1}{v^2}\int_{0}^{+\infty}\int_{s}^{+\infty} e^{-\frac{(\xi(\tau)-\xi(\tilde{a}))}{v}}\phi\left(\xi^{-1}(\xi(\tau)-\xi(s)+\xi(\tilde{a}))\right)\\ &\times h(s)\xi'(\tau)\xi'(s)d\tau ds\\ &= \frac{1}{v^2}\int_{\tilde{a}}^{+\infty} e^{-\frac{(\xi(\tau)-\xi(\tilde{a}))}{v}}\left[\int_{\tilde{a}}^{\tau}\phi\left(\xi^{-1}(\xi(\tau)-\xi(s)+\xi(\tilde{a}))\right)\right.\\ &\times h(s)\xi'(s)ds]\,\xi'(\tau)d\tau\\ \mathbb{S}_\xi\{\phi\}\mathbb{S}_\xi\{h\}(v) &= \frac{1}{v}\mathbb{S}_\xi\left\{\phi *_\xi h\right\}(v).\end{aligned}$$

For the verification of statement 2, we put $q = \xi(t)-\xi(\tilde{a})$ to deduce

$$\begin{aligned}&\mathbb{S}_\xi\left\{[\xi(t)-\xi(\tilde{a})]^{\varsigma-1}\mathrm{e}^{-\mu(\xi(t)-\xi(\tilde{a}))}\right\}(v)\\ &= \frac{1}{v}\int_{\tilde{a}}^{+\infty}[\xi(t)-\xi(\tilde{a})]^{\varsigma-1}\mathrm{e}^{-(\frac{1}{v}+\mu)[\xi(t)-\xi(\tilde{a})]}\xi'(t)dt\\ &= \frac{1}{v}\int_{0}^{+\infty} q^{\varsigma-1}\mathrm{e}^{-(\frac{1}{v}+\mu)q}dq\\ &= \frac{1}{v(\frac{1}{v}+\mu)^{\varsigma}}\int_0^{+\infty}\tilde{q}^{\varsigma-1}\mathrm{e}^{-\tilde{q}}d\tilde{q} = \frac{v^{\varsigma-1}}{(\mu v+1)^{\varsigma}}\Gamma(\varsigma),\end{aligned}$$

where $\frac{1}{v}+\mu > 0$ was needed when we changed $\tilde{q} = (\frac{1}{v}+\mu)q$. □

We will also require the following result from [6, Corollary 1].

Lemma 2.6. *Let $\xi \in C^{n-1}[\tilde{a},+\infty)$ such that for all $t \geq \tilde{a}$, $\xi'(t) > 0$. And let $\phi \in C_\xi^{n-1}[\tilde{a},+\infty)$ be such that*

$$\phi_\xi^{[k]}(t) = \left(\frac{1}{\xi'(t)}\frac{d}{dt}\right)^k\phi(t),$$

is of $\xi(t)$-exponential order for each $k = 1,2,\ldots,n-1$, and $\phi_\xi^{[n]}$ a continuous function on the interval $[\tilde{a},+\infty)$. Then the generalized Sumudu transform of $\phi_\xi^{[n]}$ exists and

$$\mathbb{S}_\xi\left\{\phi_\xi^{[n]}(t)\right\}(v) = v^{-n}\mathbb{S}_\xi\{\phi(t)\}(v) - \sum_{k=0}^{n-1} v^{k-n}\phi_\xi^{[k]}(b).$$

Throughout the remainder of the paper, we shall represent $u_{\xi}^{[0]}(t) = u(t)$.

Definition 2.7. *[16] Consider $\varsigma > 0, \mu \geq 0$, $u(t)$ is a real continuous function on $[\tilde{a}, \tilde{b}]$ and $\xi \in C^1[\tilde{a}, \tilde{b}]$ such that $\xi(t) > 0$. Then the tempered ξ-Hilfer fractional integral of order ς is defined as follows:*

$$I_{\tilde{a}}^{\varsigma,\mu,\xi} u(t) = \frac{1}{\Gamma(\varsigma)} \int_{\tilde{a}}^{t} (\xi(t) - \xi(\tau))^{\varsigma-1} e^{-\mu(\xi(t)-\xi(\tau))} \xi'(\tau) u(\tau) d\tau, \tag{3}$$

for $t \in [\tilde{a}, \tilde{b}]$.

Definition 2.8. *[16] Let $\xi \in C^n[\tilde{a}, \tilde{b}]$ be such that $\xi'(t) > 0$. For $n-1 < \varsigma < n, n \in \mathbb{N}, \mu \geq 0$, the tempered ξ-Caputo fractional derivative of order ς is defined by*

$$^{C}D_{\tilde{a}}^{\varsigma,\mu,\xi} u(t) = \frac{e^{-\mu\xi(t)}}{\Gamma(n-\varsigma)} \int_{\tilde{a}}^{t} (\xi(t) - \xi(\tau))^{n-\varsigma-1} \xi'(\tau) u_{\mu,\xi}^{[n]}(\tau) d\tau, \tag{4}$$

for $t \in [\tilde{a}, \tilde{b}]$, where

$$u_{\mu,\xi}^{[n]}(t) = \left[\frac{1}{\xi'(t)} \frac{d}{dt}\right]^n \left(e^{\mu\xi(t)} u(t)\right).$$

Using ξ-convolution, we can express (3) *by:*

$$I_{\tilde{a}}^{\varsigma,\mu,\xi} u(t) = \frac{1}{\Gamma(\varsigma)} \left([\xi(\cdot) - \xi(\tilde{a})]^{\varsigma-1} e^{-\mu(\xi(\cdot)-\xi(\tilde{a}))} *_{\xi} u\right)(t). \tag{5}$$

Moreover,

$$u_{\mu,\xi}^{[n]}(t) = \left(e^{\mu\xi(\cdot)} u(\cdot)\right)_{\xi}^{[n]}(t). \tag{6}$$

Remark 2.9. *For $\mu = 0$ we get:*

$$I_{\tilde{a}}^{\varsigma,0,\xi} u(t) = I_{\tilde{a}}^{\varsigma,\xi} u(t), \quad {}^{C}D_{\tilde{a}}^{\varsigma,0,\xi} u(t) = {}^{C}D_{\tilde{a}}^{\varsigma,\xi} u(t).$$

where $I_{\tilde{a}}^{\varsigma,\xi}$ is the ξ-Hilfer fractional integral [4], and ${}^{C}D_{\tilde{a}}^{\varsigma,\xi}$ is the ξ-Caputo fractional derivative [19].

3. Generalized Sumudu transform and tempered ξ-fractional calculus

This section presents several proprties of the generalised Sumudu transform that are applicable to the tempered ξ-fractional calculus.

Lemma 3.1. *Let $\varsigma > 0, \mu \geq 0, \xi \in C^1[\tilde{a}, +\infty)$ an increasing function, and $u \in C[\tilde{a}, +\infty)$. Then*

$$\mathbb{S}_{\xi} \left\{I_{\tilde{a}}^{\varsigma,\mu,\xi} u(t)\right\}(v) = \left(\frac{v}{\mu v + 1}\right)^{\varsigma} \mathbb{S}_{\xi}\{u(t)\}(v), \tag{7}$$

for $\mu v > -1$ such that the right-hand side exists.

Proof. In order to simplify the notation, we set $U_{\xi}(s) = \mathbb{S}_{\xi}\{u(t)\}(s)$. Lemma 2.5, and identity (5) allow us to get

$$\begin{aligned} \mathbb{S}_{\xi} \left\{I_{\tilde{a}}^{\varsigma,\mu,\xi} u(t)\right\}(v) &= \frac{1}{\Gamma(\varsigma)} \mathbb{S}_{\xi} \left\{\left([\xi(\cdot) - \xi(\tilde{a})]^{\varsigma-1} e^{-\mu(\xi(\cdot)-\xi(\tilde{a}))} *_{\xi} u\right)(t)\right\}(v) \\ &= \frac{v}{\Gamma(\varsigma)} \mathbb{S}_{\xi} \left\{[\xi(t) - \xi(\tilde{a})]^{\varsigma-1} e^{-\mu(\xi(t)-\xi(\tilde{a}))}\right\}(v) U_{\xi}(v) = \left(\frac{v}{\mu v + 1}\right)^{\varsigma} U_{\xi}(v). \end{aligned}$$

□

Lemma 3.2. *Let $\xi \in C^n[\tilde{a},+\infty)$ be an increasing function and $n-1<\varsigma<n$ for some $n\in\mathbb{N},\mu\geq 0$. Let $u\in C^n[\tilde{a},+\infty)$ be such that $u^{[k]}_{\mu,\xi}$ is of $\xi(t)$-exponential order for each $k=1,2,\ldots,n-1$, and $u^{[n]}_{\mu,\xi}\in C[\tilde{a},+\infty)$. Then*

$$\mathbb{S}_\xi\left\{{}^C D^{\varsigma,\mu,\xi}_{\tilde{a}}u(t)\right\}(v)=\left(\frac{1+\mu v}{v}\right)^{\varsigma}\mathbb{S}_\xi\{u(t)\}(v)-\mathrm{e}^{-\mu\xi(\tilde{a})}\sum_{k=0}^{n-1}\frac{(1+\mu v)^{\varsigma-k-1}}{v^{\varsigma-k}}u^{[k]}_{\mu,\xi}(\tilde{a}).$$

Proof. Considering the relations

$$ {}^C D^{\varsigma,\mu,\xi}_{\tilde{a}}u(t)=\mathrm{e}^{-\mu\xi(t)}\,{}^C D^{\varsigma,\xi}_{\tilde{a}}\left(\mathrm{e}^{\mu\xi(t)}u(t)\right),\quad I^{\varsigma,\mu,\xi}_{\tilde{a}}u(t)=\mathrm{e}^{-\mu\xi(t)}I^{\varsigma,\xi}_{\tilde{a}}\left(\mathrm{e}^{\mu\xi(t)}u(t)\right), \tag{8}$$

using the formula from [15]

$$ {}^C D^{\varsigma,\xi}_{\tilde{a}}u(t)=I^{n-\varsigma,\xi}_{\tilde{a}}\left[\left(\frac{1}{\xi'(t)}\frac{d}{dt}\right)^n u(t)\right]=I^{n-\varsigma,\xi}_{\tilde{a}}\left(x^{[n]}_{\xi}(t)\right),$$

using relation (6) and [19], we derive

$$\begin{aligned}{}^C D^{\varsigma,\mu,\xi}_{\tilde{a}}u(t)&=\mathrm{e}^{-\mu\xi(t)}\,{}^C D^{\varsigma,\xi}_{\tilde{a}}\left(\mathrm{e}^{\mu\xi(t)}u(t)\right)=\mathrm{e}^{-\mu\xi(t)}I^{n-\varsigma,\xi}_{\tilde{a}}\left(\left[\mathrm{e}^{\mu\xi(\cdot)}u(\cdot)\right]^{[n]}_{\xi}(t)\right)\\&=\mathrm{e}^{-\mu\xi(t)}I^{n-\varsigma,\xi}_{\tilde{a}}\left(u^{[n]}_{\mu,\xi}(t)\right)=I^{n-\varsigma,\mu,\xi}_{\tilde{a}}\left(\mathrm{e}^{-\mu\xi(t)}u^{[n]}_{\mu,\xi}(t)\right).\end{aligned}$$

Next, by Lemma 3.1, we obtain

$$\begin{aligned}\mathbb{S}_\xi\left\{{}^C D^{\varsigma,\mu,\xi}_{\tilde{a}}u(t)\right\}(v)&=\mathbb{S}_\xi\left\{I^{n-\varsigma,\mu,\xi}_{\tilde{a}}\left(\mathrm{e}^{-\mu\xi(t)}u^{[n]}_{\mu,\xi}(t)\right)\right\}(v)\\&=\left(\frac{v}{\mu v+1}\right)^{n-\varsigma}\mathbb{S}_\xi\left\{\mathrm{e}^{-\mu\xi(t)}u^{[n]}_{\mu,\xi}(t)\right\}(v).\end{aligned}$$

After that, we use Theorem 2.4 and the translation property [14]:

$$\mathbb{S}\left\{\mathrm{e}^{\mu t}\phi(t)\right\}(v)=\frac{1}{1-\mu v}\mathbb{S}\{\phi(t)\}\left(\frac{v}{1-\mu v}\right).$$

So, we have

$$\begin{aligned}\mathbb{S}_\xi\left\{{}^C D^{\varsigma,\mu,\xi}_{\tilde{a}}u(t)\right\}(v)&=\left(\frac{v}{1+\mu v}\right)^{n-\varsigma}\mathbb{S}\left\{\mathrm{e}^{-\mu(t+\xi(\tilde{a}))}u^{[n]}_{\mu,\xi}\left(\xi^{-1}(t+\xi(\tilde{a}))\right)\right\}(v)\\&=\left(\frac{v}{1+\mu v}\right)^{n-\varsigma}\frac{\mathrm{e}^{-\mu\xi(\tilde{a})}}{1+\mu v}\mathbb{S}\left\{u^{[n]}_{\mu,\xi}\left(\xi^{-1}(t+\xi(\tilde{a}))\right)\right\}\left(\frac{v}{1+\mu v}\right)\\&=\frac{v^{n-\varsigma}\mathrm{e}^{-\mu\xi(\tilde{a})}}{(1+\mu v)^{n-\varsigma+1}}\mathbb{S}_\xi\left\{u^{[n]}_{\mu,\xi}(t)\right\}\left(\frac{v}{1+\mu v}\right)\\&=\frac{v^{n-\varsigma}\mathrm{e}^{-\mu\xi(\tilde{a})}}{(1+\mu v)^{n-\varsigma+1}}\mathbb{S}_\xi\left\{\left(\mathrm{e}^{\mu\xi(\cdot)}u(\cdot)\right)^{[n]}_{\xi}(t)\right\}\left(\frac{v}{1+\mu v}\right),\end{aligned}$$

we complete the proof by employing Lemma 2.6

$$\begin{aligned}
\mathbb{S}_{\xi}\left\{{}^{C}D_{\tilde{a}}^{\varsigma,\mu,\xi}u(t)\right\}(v) &= \frac{v^{n-\varsigma}e^{-\mu\xi(\tilde{a})}}{(1+\mu v)^{n-\varsigma+1}}\left[\left(\frac{v}{1+\mu v}\right)^{-n}\mathbb{S}_{\xi}\left\{e^{\mu\xi(t)}u(t)\right\}\left(\frac{v}{1+\mu v}\right)\right.\\
&\quad\left.-\sum_{k=0}^{n-1}\left(\frac{v}{1+\mu v}\right)^{-n+k}\left(e^{\mu\xi(\cdot)}u(\cdot)\right)_{\xi}^{[k]}(\tilde{a})\right]\\
&= \frac{v^{-\varsigma}}{(1+\mu v)^{1-\varsigma}}e^{-\mu\xi(\tilde{a})}\mathbb{S}\left\{e^{\mu(t+\xi(\tilde{a}))}u\left(\xi^{-1}(t+\xi(\tilde{a}))\right)\right\}\left(\frac{v}{1+\mu v}\right)\\
&\quad - e^{-\mu\xi(\tilde{a})}\sum_{k=0}^{n-1}\frac{v^{k-\varsigma}}{(1+\mu v)^{k-\varsigma+1}}u_{\mu,\xi}^{[k]}(\tilde{a})\\
&= \left(\frac{v}{1+\mu v}\right)^{-\varsigma}\mathbb{S}\left\{u\left(\xi^{-1}(t+\xi(\tilde{a}))\right)\right\}(v) - e^{-\mu\xi(\tilde{a})}\sum_{k=0}^{n-1}\frac{v^{k-\varsigma}}{(1+\mu v)^{k-\varsigma+1}}u_{\mu,\xi}^{[k]}(\tilde{a})\\
&= \left(\frac{1+\mu v}{v}\right)^{\varsigma}\mathbb{S}_{\xi}\{u(t)\}(v) - e^{-\mu\xi(\tilde{a})}\sum_{k=0}^{n-1}\frac{(1+\mu v)^{\varsigma-k-1}}{v^{\varsigma-k}}u_{\mu,\xi}^{[k]}(\tilde{a}).
\end{aligned}$$

□

4. Appliction to differential equations

The solution formula for differential equations with tempered ξ-Caputo fractional derivative is obtained in this section by using the generalized Sumudu transform. We specifically take into account the following initial value problem (IVP):

$$ {}^{C}D_{\tilde{a}}^{\varsigma,\mu,\xi}u(t) = \Phi(t,u(t)), \quad t \geq \tilde{a}, \ \ n-1<\varsigma<n\in\mathbb{N}, \mu\geq 0, \tag{9}$$

$$ u_{\mu,\xi}^{[k]}(\tilde{a}) = u_{\tilde{a}}^{k}, \quad k=0,1,\ldots,n-1 \tag{10}$$

where $u_{\tilde{a}}^{k}$ are constants, $\xi \in C^{n}[\tilde{a},+\infty)$ is such that for all $t \geq \tilde{a}$, $\xi'(t) > 0$, and $\Phi \in C([\tilde{a},+\infty)\times\mathbb{R},\mathbb{R})$. Denote that $u_{\mu,\xi}^{[0]}(t) = e^{\mu\xi(t)}u(t)$.

Definition 4.1. *A function $u \in C^{n}[\tilde{a},+\infty)$ is a solution to the IVP* (9)-(10), *if ${}^{C}D_{\tilde{a}}^{\varsigma,\mu,\xi}u(t)$ exists and be in $C(\tilde{a},+\infty)$, and $u(t)$ satisfies* (9)-(10).

It is necessary to confirm that each of Lemma 3.2's assumptions is true before utilizing the generalized Sumudu transform.

Lemma 4.2. *[16] Let $n-1<\varsigma<n$, $n\in\mathbb{N}$, $\mu\geq 0$, $\xi\in C^{n}[\tilde{a},+\infty)$ such that for all $t\geq\tilde{a}$, $\xi'(t)>0$. If the right-hand side $\Phi\in C([\tilde{a},+\infty)\times\mathbb{R},\mathbb{R})$ of* (9) *is of ξ-exponential order, then $u_{\mu,\xi}^{[k]}$, $k=0,1,\ldots,n-1$ are all of ξ-exponential order, where $u(t)$ is a solution to the IVP* (9)-(10).

To prove that a solution is suitably bounded, we will look at this follows linear equation:

$$ {}^{C}D_{\tilde{a}}^{\varsigma,\mu,\xi}u(t) = Au(t) + \phi(t), \quad t\geq\tilde{a}, n-1<\varsigma<n\in\mathbb{N}, \tag{11}$$

where $A\in\mathbb{R}$ and $\phi\in C[\tilde{a},+\infty)$.

Lemma 4.3. *[16] Let $\xi\in C^{n}[\tilde{a},+\infty)$ be such that for all $t\geq\tilde{a}$, $\xi'(t)>0$, and $\mu\geq 0$, $n-1<\varsigma<n$, $n\in\mathbb{N}$. If $\phi\in C[\tilde{a},+\infty)$ is of ξ-exponential order, then the solution $u(t)$ to the IVP* (11)-(10) *is of ξ-exponential order.*

Theorem 4.4. *Let $\xi \in C^n[\tilde{a}, +\infty)$ such that for all $t \geq \tilde{a}$, $\xi'(t) > 0$, and $n-1 < \varsigma < n$ for some $n \in \mathbb{N}, \mu \geq 0$. If $\phi \in C[\tilde{a}, +\infty)$ is of $\xi(t)$-exponential order, then a solution $u(t)$ to the IVP* (11)-(10) *takes the following form:*

$$u(t) = \mathrm{e}^{-\mu\xi(t)} \sum_{k=0}^{n-1} (\xi(t) - \xi(\tilde{a}))^k E_{\varsigma,k+1} \left((\xi(t) - \xi(\tilde{a}))^{\varsigma} A\right) u_{\tilde{a}}^k$$
$$+ \int_{\tilde{a}}^{t} (\xi(t) - \xi(\tau))^{\varsigma-1} \mathrm{e}^{-\mu(\xi(t)-\xi(\tau))} \xi'(\tau) E_{\varsigma,\varsigma} \left((\xi(t) - \xi(\tau))^{\varsigma} A\right) f(\tau) d\tau,$$

where $E_{\varsigma,\beta}(z) = \sum_{p=0}^{+\infty} \frac{z^j}{\Gamma(\varsigma p+\beta)}$ is the Mittag-Leffler function.

Proof. The generalized Sumudu transform may be applied to Equation (11) in accordance with Lemmas 4.2 and 4.3. Given $U(v) = \mathbb{S}_{\xi}\{u(t)\}(v)$ and $\Phi(v) = \mathbb{S}_{\xi}\{\phi(t)\}(v)$, we derive, by Lemma 3.2,

$$\left(\frac{1+\mu v}{v}\right)^{\varsigma} U(v) - \mathrm{e}^{-\mu\xi(\tilde{a})} \sum_{k=0}^{n-1} \frac{(1+\mu v)^{\varsigma-k-1}}{v^{\varsigma-k}} u_{\mu,\xi}^{[k]}(\tilde{a}) = AU(v) + \Phi(v),$$

whenever v is large enough. Thus,

$$U(v) = \left(\left(\frac{1+\mu v}{v}\right)^{\varsigma} - A\right)^{-1} \left[\mathrm{e}^{-\mu\xi(\tilde{a})} \sum_{k=0}^{n-1} \frac{(1+\mu v)^{\varsigma-k-1}}{v^{\varsigma-k}} u_{\mu,\xi}^{[k]}(\tilde{a}) + \Phi(v)\right]$$
$$= \left(1 - A\left(\frac{v}{1+\mu v}\right)^{\varsigma}\right)^{-1} \left[\mathrm{e}^{-\mu\xi(\tilde{a})} \sum_{k=0}^{n-1} \frac{v^k}{(1+\mu v)^{k+1}} u_{\mu,\xi}^{[k]}(\tilde{a}) + \Phi(v) \left(\frac{v}{1+\mu v}\right)^{\varsigma}\right].$$

Through expanding into series $\left(1 - A\left(\frac{v}{1+\mu v}\right)^{\varsigma}\right)^{-1}$, we obtain

$$U(v) = \mathrm{e}^{-\mu\xi(\tilde{a})} \sum_{k=0}^{n-1} \sum_{p=0}^{+\infty} \left(A\left(\frac{v}{1+\mu v}\right)^{\varsigma}\right)^p \frac{v^k}{(1+\mu v)^{k+1}} u_{\mu,\xi}^{[k]}(\tilde{a})$$
$$+ \sum_{p=0}^{+\infty} \left(A\left(\frac{v}{1+\mu v}\right)^{\varsigma}\right)^p \Phi(v) \left(\frac{v}{1+\mu v}\right)^{\varsigma}.$$

Therefore,

$$U(t) = \mathrm{e}^{-\mu\xi(\tilde{a})} \sum_{k=0}^{n-1} A_k u_{\tilde{a}}^k + A_{\Phi}, \quad t \geq \tilde{a},$$

where

$$A_k = \sum_{p=0}^{+\infty} A^p \mathbb{S}_{\xi}^{-1} \left\{\frac{v^{\varsigma p+k}}{(1+\mu v)^{\varsigma p+k+1}}\right\}(t), \quad k = 0, 1, \ldots, n-1,$$
$$A_{\phi} = \sum_{p=0}^{+\infty} A^p \mathbb{S}_{\xi}^{-1} \left\{\left(\frac{v}{(1+\mu v)}\right)^{(p+1)\varsigma} \Phi(v)\right\}(t).$$

The symbol $\mathbb{S}_{\xi}^{-1}$ denotes the inverse of $\mathbb{S}_{\xi}$ in this case. The same characteristic of the classical Sumudu transform also implies that the inverse in the set of continuous functions is unique due to Theorem 2.4 and attributes of ξ. It is evident from Lemma 2.5 assertion 2 that for every $k = 0, 1, \ldots, n-1$,

$$\begin{aligned} A_k &= \sum_{p=0}^{+\infty} A^p \frac{(\xi(t)-\xi(\tilde{a}))^{k+\varsigma p} \mathrm{e}^{-\mu(\xi(t)-\xi(\tilde{a}))}}{\Gamma(k+1+\varsigma p)} \\ &= (\xi(t)-\xi(\tilde{a}))^k \mathrm{e}^{-\mu(\xi(t)-\xi(\tilde{a}))} \sum_{p=0}^{+\infty} \frac{A^p (\xi(t)-\xi(\tilde{a}))^{\varsigma p}}{\Gamma(k+1+\varsigma p)} \\ &= (\xi(t)-\xi(\tilde{a}))^k \mathrm{e}^{-\mu(\xi(t)-\xi(\tilde{a}))} E_{\varsigma,k+1}\left((\xi(t)-\xi(\tilde{a}))^{\varsigma} A\right). \end{aligned}$$

Subsequently, we utilize both statements of Lemma 2.5, to deduce

$$\begin{aligned} A_\Phi &= \left(\sum_{p=0}^{+\infty} A^p \mathbb{S}_\xi^{-1} \left\{ v \mathbb{S}_\xi \left\{ \frac{[\xi(.)-\xi(\tilde{a})]^{(p+1)\varsigma-1} \, \mathrm{e}^{-\mu(\xi(.)-\xi(\tilde{a}))}}{\Gamma[(p+1)\varsigma]} \right\} \mathbb{S}_\xi \left\{ \phi(.) \right\} \right\} \right)(t) \\ &= \left(\sum_{p=0}^{+\infty} A^p \frac{[\xi(.)-\xi(\tilde{a})]^{(p+1)\varsigma-1} \, \mathrm{e}^{-\mu(\xi(.)-\xi(\tilde{a}))}}{\Gamma[(p+1)\varsigma]} *_\xi \phi \right)(t) \\ &= \left(f *_\xi [\xi(.)-\xi(\tilde{a})]^{\varsigma-1} \, \mathrm{e}^{-\mu(\xi(.)-\xi(\tilde{a}))} \sum_{p=0}^{+\infty} \frac{A^p \, [\xi(.)-\xi(\tilde{a})]^{\varsigma p}}{\Gamma[(p+1)\varsigma]} \right)(t) \\ &= \left(f *_\xi [\xi(.)-\xi(\tilde{a})]^{\varsigma-1} \, \mathrm{e}^{-\mu(\xi(.)-\xi(\tilde{a}))} E_{\varsigma,\varsigma}\left([\xi(.)-\xi(\tilde{a})]^{\varsigma} A\right) \right)(t) \\ &= \int_{\tilde{a}}^t (\xi(t)-\xi(\tau))^{\varsigma-1} \mathrm{e}^{-\mu(\xi(t)-\xi(\tau))} \xi'(\tau) E_{\varsigma,\varsigma}\left((\xi(t)-\xi(\tau))^{\varsigma} A\right) f(\tau) d\tau. \end{aligned}$$

which completes the proof. □

Remark 4.5. *For a given function $\phi \in C[\tilde{a}, +\infty)$, the statement of Theorem is still true. For further information, see [16, Theorem 4].*

Example 4.6. *We deduce from Theorem 4.4 and Remark 4.5 that for $0 < \varsigma < 1$, the Cauchy problem*

$$\begin{aligned} &{}^C D_a^{\varsigma,\mu,\xi} u(t) - q u(t) = \phi(t), \quad t \geq \tilde{a}, \;\; q \in \mathbb{R}, \mu \geq 0, \\ &u(\tilde{a}) = u_{\tilde{a}}, \end{aligned}$$

for $\xi \in C^1[\tilde{a}, +\infty)$ such that $0 < \xi'(t)$, and $\phi \in C[\tilde{a}, +\infty)$, has the solution

$$\begin{aligned} u(t) =& u_{\tilde{a}} \mathrm{e}^{-\mu(\xi(t)-\xi(\tilde{a}))} E_{\varsigma,1}\left(q(\xi(t)-\xi(\tilde{a}))^{\varsigma}\right) \\ &+ \int_{\tilde{a}}^t (\xi(t)-\xi(\tau))^{\varsigma-1} \mathrm{e}^{-\mu(\xi(t)-\xi(\tau))} \xi'(\tau) E_{\varsigma,\varsigma}\left(q(\xi(t)-\xi(\tau))^{\varsigma}\right) \phi(\tau) d\tau \end{aligned}$$

Note that in this case $u_{\mu,\xi}^{[0]}(\tilde{a}) = \mathrm{e}^{\mu\xi(\tilde{a})} u(\tilde{a}) = \mathrm{e}^{\mu\xi(\tilde{a})} u_{\tilde{a}}$.
For $\mu = 0$, we have:

$$u(t) = u_{\tilde{a}} E_\varsigma\left(q(\xi(t)-\xi(\tilde{a}))^{\varsigma}\right) + \int_{\tilde{a}}^t (\xi(t)-\xi(\tau))^{\varsigma-1} \xi'(\tau) E_{\varsigma,\varsigma}\left(q(\xi(t)-\xi(\tau))^{\varsigma}\right) \phi(\tau) d\tau,$$

which is the same formila from [6].
For $\mu = 0, \xi(t) = \frac{t^\rho}{\rho}, \tilde{a} = 0$ we have:

$$u(t) = u_0 E_\varsigma\left(q\left(\frac{t^\rho}{\rho}\right)^{\varsigma}\right) + \int_0^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\varsigma-1} E_{\varsigma,\varsigma}\left(q\left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\varsigma}\right) \phi(\tau) \frac{d\tau}{\tau^{1-\rho}},$$

which coincides with the solution in [3].

5. Conclution

The Sumudu transform is characterized by its ability to preserve units, making it suitable for problem-solving without resorting to the frequency domain. This property is particularly advantageous for applications in physical problems. In this work, we applied the generalized Sumudu transform to the tempered ξ-Hilfer fractional integral and the tempered ξ-Caputo fractional derivative. Then, these results was used to address a non homogeneous linear fractional differential equation in an initial value problem involving the tempered ξ-Caputo fractional derivative of a non-integer general order ς.

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