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Applications on a subclass of parabolic starlike functions connected with Mittag-Leffler function

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Abstract. We familiarize in this paper a new family of starlike functions in parabolic domain related to the Mittag-Leffler function (MLF). By using this family of functions with a negative coefficient, we discuss coefficient estimates, extreme points, distortion bounds, closure theorem, radii of starlikeness and convexity. Moreover, the neighborhood, partial sums, and integral means of functions for this new family are studied.

1. Introduction and Preliminaries

Let $\mathbf{E}_\jmath$ be the function defined by

$$\mathbf{E}_\jmath(\zeta) := \sum_{n=0}^{\infty} \frac{\zeta^n}{\Gamma(\jmath n + 1)}, \ \zeta \in \mathbb{C}, \ \jmath \in \mathbb{C} \text{ with } \operatorname{Re} \jmath > 0,$$

that was presented by Mittag-Leffler [24] and are generally known as the *Mittag-Leffler function*(MLF). Wiman [40] defined its two-parameter version $\mathbf{E}_{\jmath,\ell}$ which generalizes widely used Mittag-Leffler function $\mathbf{E}_\jmath$ as

$$\mathbf{E}_{\jmath,\ell}(\zeta) := \sum_{n=0}^{\infty} \frac{\zeta^n}{\Gamma(\jmath n + \ell)}, \ \zeta \in \mathbb{C}, \ \jmath, \ell \in \mathbb{C}, \text{ with } \operatorname{Re} \jmath > 0, \ \operatorname{Re} \ell > 0.$$

When $\ell = 1$, it is abbreviated as $\mathbf{E}_\jmath(\zeta) = \mathbf{E}_{\jmath,1}(\zeta)$. Witness that the function $\mathbf{E}_{\jmath,\ell}$ comprises many well-known

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functions as extensions of the exponential, hyperbolic, and trigonometric functions for example,

$$\mathbf{E}_{1,1}(\zeta)=e^{\zeta},\quad \mathbf{E}_{1,2}(\zeta)=\frac{e^{\zeta}-1}{\zeta},\quad \mathbf{E}_{2,1}\left(\zeta^2\right)=\cosh\zeta,$$
$$\mathbf{E}_{2,1}\left(-\zeta^2\right)=\cos\zeta,\quad \mathbf{E}_{2,2}\left(\zeta^2\right)=\frac{\sinh\zeta}{\zeta},\quad \mathbf{E}_{2,2}\left(-\zeta^2\right)=\frac{\sin\zeta}{\zeta},$$
$$\mathbf{E}_{4}(\zeta)=\frac{1}{2}\left(\cos\zeta^{1/4}+\cosh\zeta^{1/4}\right),\quad \mathbf{E}_{3}(\zeta)=\frac{1}{2}\left[e^{\zeta^{1/3}}+2e^{-\frac{1}{2}\zeta^{1/3}}\cos\left(\frac{\sqrt{3}}{2}\zeta^{1/3}\right)\right].$$

It is of curiosity to note that by fixing $\jmath=1/2$ and $\ell=1$ we get

$$\mathbf{E}_{\frac{1}{2},1}(\zeta)=e^{\zeta^2}\cdot\operatorname{erfc}(-\zeta),$$

that is

$$\mathbf{E}_{\frac{1}{2},1}(\zeta)=e^{\zeta^2}\left(1+\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\zeta^{2n+1}\right).$$

The MLF have widespread applications in chemistry, physics, biology, engineering, and other applied sciences. The applications of these functions can be seen in n-fractional differential equations, stochastic systems, chaotic systems, statistical distributions and dynamical systems.

The MLF rises naturally in the solution of integral and fractional order differential equations, specifically in the investigations of fractional generalizing of kinetic equation, random walks, Lévy flights, super-diffusive transport and in the study of complex systems. For a potentially useful further investigation of generalized MLF, the reader is referred to [1, 2, 5, 6, 8, 11–14, 18–21, 26].

We note that the above generalized Mittag-Leffler function $\mathbf{E}_{\jmath,\ell}$ is not a member of family $\mathcal{A}$, where $\mathcal{A}$ represents the class of functions analytic in the open unit disk $\mathbb{U}:=\{\zeta\in\mathbb{C}:|\zeta|<1\}$ whose members are of the form

$$f(\zeta)=\zeta+\sum_{n=2}^{\infty}a_n\zeta^n,\ \zeta\in\mathbb{U}, \tag{1}$$

and normalized by the conditions $f'(0)-1=0=f(0)$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ whose members are univalent in $\mathbb{U}$. . Thus, it is natural to consider the following normalization of MLF due to Bansal and Prajapat [5]:

$$E_{\jmath,\ell}(\zeta):=\zeta\Gamma(\ell)\,\mathbf{E}_{\jmath,\ell}(\zeta)=\zeta+\sum_{n=2}^{\infty}\frac{\Gamma(\ell)}{\Gamma(\jmath(n-1)+\ell)}\zeta^n, \tag{2}$$

that holds for the parameters $\jmath,\ell\in\mathbb{C}$ with $\operatorname{Re}\ell>0$, $\operatorname{Re}\jmath>0$ and $\zeta\in\mathbb{C}$. Moreover, Srivastava and Tomovski [33] introduced the function $E^{\tau,\kappa}_{\jmath,\ell}(\zeta)(\zeta\in\mathbb{C})$ in the form

$$E^{\tau,\kappa}_{\jmath,\ell}(\zeta)=\sum_{n=0}^{\infty}\frac{(\tau)_{n\kappa}\zeta^n}{\Gamma(\jmath n+\ell)\,n!},$$

$(\jmath,\ell,\tau\in\mathbb{C};\Re(\jmath)>\max\{0,\Re(\kappa)-1\};\Re(\kappa)>0)$. Lately, Attiya[2] defined

$$M^{\tau,\kappa}_{\jmath,\ell}(\zeta)=\frac{\Gamma(\jmath+\ell)}{(\tau)_\kappa}\left(E^{\tau,\kappa}_{\jmath,\ell}(\zeta)-\frac{1}{\Gamma(\ell)}\right)$$

with $(\ell,\tau\in\mathbb{C};\Re(\jmath)>\max\{0,\Re(\kappa)-1\};\Re(\kappa)>0;\Re(\jmath)=0\ \textit{when}\ \Re(\kappa)=1\ \textit{with}\ \ell\neq 0)$ and gave a new linear operator

$$\mathcal{I}^{\tau,\kappa}_{\jmath,\ell}:\mathcal{A}\to\mathcal{A}$$

given by

$$\mathcal{I}^{\tau,\kappa}_{\jmath,\ell} f(\zeta) = M^{\tau,\kappa}_{\jmath,\ell}(\zeta) * f(\zeta)$$

where (∗) denotes the Hadamard product (or convolution) for functions $f, g \in \mathcal{A}$ where f be assumed as in (1) and $g(\zeta) = \zeta + \sum_{n=2}^{\infty} b_n \zeta^n$, then the *Hadamard product (or convolution)* of f and g is given by

$$(f * g)(\zeta) := \zeta + \sum_{n=2}^{\infty} a_n b_n \zeta^n, \ \zeta \in \mathbb{U}.$$

Thus,

$$\mathcal{I}^{\tau,\kappa}_{\jmath,\ell} f(\zeta) = \zeta + \sum_{n=2}^{\infty} \frac{\Gamma(\tau + n\kappa)\Gamma(\jmath + \ell)}{\Gamma(\tau + \kappa)\Gamma(n\jmath + \ell)\, n!} a_n \zeta^n, \ \zeta \in \mathbb{U}. \tag{3}$$

Shortly ,we let

$$\mathcal{I}^{\tau,\kappa}_{\jmath,\ell} f(\zeta) := \zeta + \sum_{n=2}^{\infty} \Lambda_n a_n \zeta^n \tag{4}$$

where

$$\Lambda_n = \frac{\Gamma(\tau + n\kappa)\Gamma(\jmath + \ell)}{\Gamma(\tau + \kappa)\Gamma(n\jmath + \ell)\, n!} \tag{5}$$

unless otherwise stated. Throughout our study we assume $\jmath, \ell$, are real-valued parameters and $\zeta \in \mathbb{U}$.

1.1. Subclasses of $\mathcal{S}$:

Robertson [27] defined and studied the two well- known subclasses namely *starlike functions of order* ξ $(0 \le \xi < 1)$, and *convex functions of order* ξ $(0 \le \xi < 1)$ as below :

$$\mathcal{S}^*(\xi) = \{f \in \mathcal{A} : \mathrm{Re}\left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right) > \xi, \ \zeta \in \mathbb{U}, \}$$

$$\mathcal{K}(\xi) = \{f \in \mathcal{A} : \mathrm{Re}\left(\frac{(\zeta f'(\zeta))'}{f'(\zeta)}\right) > \xi, \ \zeta \in \mathbb{U}\}$$

respectively. We also write $\mathcal{S}^*(0) =: \mathcal{S}^*$, where $\mathcal{S}^*$ represents the class of functions $f \in \mathcal{A}$ such that $f(\mathbb{D})$ is starlike domain with respect to the origin. Further, $\mathcal{K} := \mathcal{K}(0)$ signifies the well-known standard class of convex functions. By Alexander's duality relation (see [10]), it is a known fact that

$$f \in \mathcal{K} \Leftrightarrow \zeta f'(\zeta) \in \mathcal{S}^*.$$

In1975, Silverman [30] promote a new direction of study by defining a subclass$\mathcal{T}$ of $\mathcal{A}$, comprising of functions of the form

$$f(\zeta) = \zeta - \sum_{n=2}^{\infty} a_n \zeta^n, \quad a_n \ge 0, \zeta \in \mathbb{U} \tag{6}$$

and discussed extensively for the classes $\mathcal{T}^*(\xi) = \mathcal{S}^*(\xi) \cap \mathcal{T}$ and $\mathcal{C}(\xi) = \mathcal{K}(\xi) \cap \mathcal{T}$ the class of starlike and convex functions of order ξ with negative coefficients. In the year 1993, Goodman [17] hosted the theory of uniform convexity and uniform starlikeness for functions in $\mathcal{A}$.

Definition 1.1. *A function $f(\zeta)$ is uniformly convex (uniformly starlike) in $\mathbb{U}$ if $f(\zeta)$ is in $\mathcal{K}\,\mathcal{S}^*$ and has the property that for every circular arc υ contained in $\mathbb{U}$, with center ξ also in $\mathbb{U}$, the arc $f(\zeta)$ is convex (starlike) with respect to $f(\xi)$.*

For $-1 \leq \xi < 1$ and $\zeta \in \mathbb{U}$ a function $f \in \mathcal{S}$ is said to be in
(i) the class $\mathcal{S}_p(\xi)$ $\xi-$ parabolic starlike functions if it satisfies the condition

$$f \in \mathcal{S}_p \Leftrightarrow \left|\frac{\zeta f'(\zeta)}{f(\zeta)} - 1\right| \leq \text{Re}\left(\frac{\zeta f'(\zeta)}{f(\zeta)} - \xi\right)$$

(ii) the class $\mathcal{S}_p(\xi, k)$ $k-$starlike functions if it satisfies the condition

$$\text{Re}\left(\frac{\zeta f'(\zeta)}{f(\zeta)} - \gamma\right) > k\left|\frac{\zeta f'(\zeta)}{f(\zeta)} - 1\right|,\ k \geq 0$$

and
(iii) the class $\mathcal{UCV}(k, \xi)$, uniformly $k-$convex functions if it satisfies the condition

$$\text{Re}\left(\frac{(\zeta f'(\zeta))'}{f'(\zeta)} - \gamma\right) > k\left|\frac{\zeta f''(\zeta)}{f'(\zeta)}\right|,\ k \geq 0.$$

Ronning [28] familiarized the class $\mathcal{S}_p = \{\mathfrak{F} \in \mathcal{S}^* : \mathfrak{F}(\zeta) = \zeta\mathfrak{F}'(\zeta), f \in \mathcal{UCV}\}$. Geometrically $\mathcal{S}_P$ is the class of functions $\mathfrak{F}$ for which $\zeta\mathfrak{F}'(\zeta)/\mathfrak{F}(\zeta)$ has values in the interior of the parabola in the right half-plane symmetric almost the real axis with vertex at $(1/2, 0)$. Inspired by the earlier works of Goodman[16, 17] and Sokol et al.,[34] and the techniques followed in [3, 7, 30, 38]), in this article we present a new subclass of $k-$ starlike functions of order ξ based on generalized Mittag-Leffler function.

Definition 1.2. *For $0 \leq \vartheta \leq 1$, $0 \leq \xi < 1$ and $k \geq 0$, we let $\mathcal{MG}_k^*(\xi, \vartheta)$ be the subclass of $\mathcal{T}$ consisting of functions of the form (6) and satisfying the analytic criterion*

$$\text{Re}\ \{G_\vartheta(\zeta) - \xi\} > k\,|G_\vartheta(\zeta) - 1| \tag{7}$$

where

$$G_\vartheta(\zeta) = \frac{\zeta(\mathcal{I}_{\jmath,\ell}^{\tau,\kappa} f(\zeta))'}{(1-\vartheta)\zeta + \vartheta\zeta(\mathcal{I}_{\jmath,\ell}^{\tau,\kappa} f(\zeta))'} \tag{8}$$

$\zeta \in \mathbb{U}$, and $\mathcal{I}_{\jmath,\ell}^{\tau,\kappa} f(\zeta)$ is given by (4).

By fixing ϑ suitably, we present few following (new) subclasses of starlike and convex functions based on MLF:

Definition 1.3. *If $\vartheta = 0$, then*

$$\mathcal{USD}(\xi, k) := \left\{f \in \mathcal{T} : \text{Re}\left(\zeta(\mathcal{I}_{\jmath,\ell}^{\tau,\kappa} f(\zeta))' - \xi\right) > k\left|\zeta(\mathcal{I}_{\jmath,\ell}^{\tau,\kappa} f(\zeta))' - 1\right|,\ \ \zeta \in \mathbb{U}\right\} \tag{9}$$

Definition 1.4. *If $\vartheta = 1$, then*

$$\mathcal{USP}(\xi, k) := \left\{f \in \mathcal{T} : \text{Re}\left(\frac{\zeta(\mathcal{I}_{\jmath,\ell}^{\tau,\kappa} f(\zeta))'}{\mathcal{I}_{\jmath,\ell}^{\tau,\kappa} f(\zeta)} - \xi\right) > k\left|\frac{\zeta(\mathcal{I}_{\jmath,\ell}^{\tau,\kappa} f(\zeta))'}{\mathcal{I}_{\jmath,\ell}^{\tau,\kappa} f(\zeta)} - 1\right|, \zeta \in \mathbb{U}\right\}. \tag{10}$$

Definition 1.5. *If $\vartheta = 0, k = 0$ then*

$$\mathcal{SG}_p(\xi) := \left\{f \in \mathcal{T} : \text{Re}\left(\frac{\zeta(\mathcal{I}_{\jmath,\ell}^{\tau,\kappa} f(\zeta))'}{\mathcal{I}_{\jmath,\ell}^{\tau,\kappa} f(\zeta)}\right) > \xi,\ \ \zeta \in \mathbb{U}\right\}$$

Definition 1.6. *If $\vartheta = 1, k = 0$ then*

$$\mathcal{R}(\xi) := \left\{ f \in \mathcal{T} : \operatorname{Re}\left(\zeta(\mathcal{I}^{\tau,\kappa}_{\jmath,\ell} f(\zeta))'\right) > \xi, \quad \zeta \in \mathbb{U} \right\}.$$

In this paper we discuss certain characterization properties like results on coefficient bounds, closure property and extreme points for $f \in \mathcal{MG}_k^*(\xi, \vartheta)$. Besides for $f \in \mathcal{MG}_k^*(\xi, \vartheta)$ we discuss , radii properties under integral transforms, neighborhood results and integral means inequalities. results on subordination theorem.

2. Characterization Properties

For brevity we let

$$0 \le \vartheta \le 1, \quad 0 \le \xi < 1, \quad k \ge 0,$$

unless otherwise stated.

Theorem 2.1. *Let f be assumed as in (1) and if $f \in \mathcal{MG}_k^*(\xi, \vartheta)$ then*

$$\sum_{n=2}^{\infty} [n(1+k) - \vartheta(\xi + k)]\Lambda_n \, |a_n| \le 1 - \xi, \tag{11}$$

where Λ_n is given by (5).

Proof. Since $f \in \mathcal{MG}_k^*(\xi, \vartheta)$ it is enough show that

$$k \left| \frac{\zeta(\mathcal{I}^{\tau,\kappa}_{\jmath,\ell} f(\zeta))'}{(1-\vartheta)z + \vartheta \mathcal{I}^{\tau,\kappa}_{\jmath,\ell} f(\zeta)} - 1 \right| - \operatorname{Re}\left(\frac{\zeta(\mathcal{I}^{\tau,\kappa}_{\jmath,\ell} f(\zeta))'}{(1-\vartheta)\zeta + \vartheta \mathcal{I}^{\tau,\kappa}_{\jmath,\ell} f(\zeta)} - 1 \right) \le 1 - \xi.$$

We have

$$\begin{aligned}
& k \left| \frac{\zeta(\mathcal{I}^{\tau,\kappa}_{\jmath,\ell} f(\zeta))'}{(1-\vartheta)\zeta + \vartheta \mathcal{I}^{\tau,\kappa}_{\jmath,\ell} f(\zeta)} - 1 \right| - \operatorname{Re}\left(\frac{\zeta(\mathcal{I}^{\tau,\kappa}_{\jmath,\ell} f(\zeta))'}{(1-\vartheta)\zeta + \vartheta \mathcal{I}^{\tau,\kappa}_{\jmath,\ell} f(\zeta)} - 1 \right) \\
\le \; & (1+k) \left| \frac{\zeta(\mathcal{I}^{\tau,\kappa}_{\jmath,\ell} f(\zeta))'}{(1-\vartheta)\zeta + \vartheta \mathcal{I}^{\tau,\kappa}_{\jmath,\ell} f(\zeta)} - 1 \right| \\
= \; & \frac{(1+k) \sum_{n=2}^{\infty} (n-\vartheta)\Lambda_n |a_n| |\zeta|^{n-1}}{1 - \sum_{n=2}^{\infty} \vartheta \Lambda_n |a_n| |\zeta|^{n-1}} \\
= \; & \frac{(1+k) \sum_{n=2}^{\infty} (n-\vartheta)\Lambda_n |a_n|}{1 - \sum_{n=2}^{\infty} \vartheta \Lambda_n |a_n|}.
\end{aligned}$$

The previous expression is constrained above by $1 - \xi$ if

$$\sum_{n=2}^{\infty} [n(1+k) - \vartheta(\xi + k)]\Lambda_n |a_n| \le 1 - \xi$$

and the proof is complete. □

In next theorem, we give necessary and sufficient conditions for $f \in \mathcal{MG}_k^*(\xi, \vartheta)$.

Theorem 2.2. *Let $f \in \mathcal{T}$ be of the form (6) and $f \in \mathcal{MG}_k^*(\xi, \vartheta)$ if and only if*

$$\sum_{n=2}^{\infty}[n(1+k) - \vartheta(\xi + k)]\Lambda_n \, |a_n| \leq 1 - \xi, \tag{12}$$

where Λ_n are given by (5).

Proof. In interpretation of Theorem 2.1, we require only to show only the necessity. If $f \in \mathcal{MG}_k^*(\xi, \vartheta)$ and ζ is real then

$$\mathrm{Re}\left(\frac{1 - \sum\limits_{n=2}^{\infty} n\Lambda_n \, a_n \zeta^{n-1}}{1 - \sum\limits_{n=2}^{\infty} \vartheta\Lambda_n \, a_n \zeta^{n-1}} - \xi \right) > k \left| \frac{\sum\limits_{n=2}^{\infty} (n - \vartheta)\Lambda_n \, a_n \zeta^{n-1}}{1 - \sum\limits_{n=2}^{\infty} \vartheta\Lambda_n \, a_n \zeta^{n-1}} \right|.$$

Allowing $\zeta \to 1$ along the real axis, we get the desired inequality 12. □

In our current discussions for brevity we let

$$\aleph(\vartheta, \xi, k, n) = [n(1+k) - \vartheta(\xi + k)]\Lambda(n), \tag{13}$$

$$\aleph(\vartheta, \xi, k, 2) = [2(1+k) - \vartheta(\xi + k)]\Lambda(2), \tag{14}$$

$$\Lambda(2) = \frac{\Gamma(\tau + 2\kappa)\Gamma(\jmath + \ell)}{2\Gamma(\tau + \kappa)\Gamma(2\jmath + \ell)} \tag{15}$$

unless otherwise stated.

Corollary 2.3. *If $f \in \mathcal{MG}_k^*(\xi, \vartheta)$, then*

$$|a_n| \leq \frac{1 - \xi}{\aleph(\vartheta, \xi, k, n)}, \;\; 0 \leq \vartheta \leq 1, \;\; 0 \leq \xi < 1, k \geq 0.$$

Equality holds for the function $f(\zeta) = \zeta - \frac{1-\xi}{\aleph(\vartheta,\xi,k,n)}\zeta^n$.

Employing the techniques given in([9, 30] for $f \in \mathcal{MG}_k^*(\xi, \vartheta)$ one can straightforwardly prove the following results so we state the results without proof.

Theorem 2.4. (Distortion Bounds) *Let f be as assumed in (6)and $f \in \mathcal{MG}_k^*(\xi, \vartheta)$, then*

$$r - \frac{1 - \xi}{\aleph(\vartheta, \xi, k, 2)} r^2 \leq |f(\zeta)| \leq r + \frac{1 - \xi}{\aleph(\vartheta, \xi, k, 2)} r^2, \;\; |\zeta| = r \tag{16}$$

and

$$1 - \frac{2(1 - \xi)}{\aleph(\vartheta, \xi, k, 2)} r \leq |f'(\zeta)| \leq 1 + \frac{2(1 - \xi)}{\aleph(\vartheta, \xi, k, 2)} r, \;\; |\zeta| = r. \tag{17}$$

Equalities are sharp for $f(\zeta) = \zeta - \frac{1-\xi}{\aleph(\vartheta,\xi,k,2)}\zeta^2$, where $\aleph(\vartheta, \xi, k, 2)$ is as in (14)

Theorem 2.5. (Extreme Points):*Let*

$$f_1(\zeta) = \zeta \text{ and } \; f_n(\zeta) = \zeta - \frac{1 - \xi}{\aleph(\vartheta, \xi, k, n)}\zeta^n, \quad \text{for } \; n = 2, 3, 4, \ldots. \tag{18}$$

where $\aleph(k, \vartheta, \xi, n)$ is as given in (13) *are the extreme points of $\mathcal{MG}_k^*(\xi, \vartheta)$. Then $f \in \mathcal{MG}_k^*(\xi, \vartheta)$ if and only if it can be stated as*

$$f(\zeta) = \sum_{n=1}^{\infty} \omega_n f_n(\zeta), \qquad \omega_n \geq 0, \quad \sum_{n=1}^{\infty} \omega_n = 1. \tag{19}$$

Theorem 2.6. (Closure theorem) *Let $f_i(\zeta)$ $(i = 1, 2, \ldots m)$ be defined by*

$$f_i(\zeta) = \zeta - \sum_{n=2}^{\infty} a_{n,i}\, \zeta^n \text{ for } a_{n,\,i} \geq 0,\ \zeta \in \mathbb{U}. \tag{20}$$

and $f_i \in \mathcal{MG}_k^*(\vartheta, \xi_i)$ $(i = 1, 2, \ldots m)$ respectively. Then given $h(\zeta) = \zeta - \frac{1}{m}\sum\limits_{n=2}^{\infty}\left(\sum\limits_{i=1}^{m} a_{n,i}\right)\zeta^n$ is in $\mathcal{MG}_k^*(\xi, \vartheta)$, where $\xi = \min\limits_{1\leq i\leq m}\{\xi_i\}$ and $-1 \leq \xi_i < 1$.

Proof. Since $f_i(\zeta) \in \mathcal{MG}_k^*(\vartheta, \xi_i)$ $(i = 1, 2, 3, \ldots m)$ and by using Theorem 2.2, we get

$$\begin{aligned}
&\sum_{n=2}^{\infty} \aleph(k, \vartheta, \xi, n)\left(\frac{1}{m}\sum_{i=1}^{m} a_{n,i}\right) \\
&\quad = \frac{1}{m}\sum_{i=1}^{m}\left(\sum_{n=2}^{\infty} \aleph(k, \vartheta, \xi, n) a_{n,j}\right) \\
&\quad \leq \frac{1}{m}\sum_{i=1}^{m}(1 - \xi_i) \leq 1 - \xi
\end{aligned}$$

where $\aleph(k, \vartheta, \xi, n)$ is defined in (13) and again by Theorem 2.2, we have $h(\zeta) \in \mathcal{MG}_k^*(\xi, \vartheta)$, which completes the proof. □

3. Integral Transform of the class $\mathcal{MG}_k^*(\xi, \vartheta)$

Now for $f \in \mathcal{A}$ we show that the class $\mathcal{MG}_k^*(\xi, \vartheta)$ is closed under integral transform

$$\Xi_\eta(f)(\zeta) = \int_0^1 \eta(t)\frac{f(t\zeta)}{t}dt,$$

where v is a real valued, non-negative weight function normalized as $\int_0^1 \eta(t)dt = 1$. Fixing $v(t) = (c+1)t^c$, $c > -1$, then Ξ_η is become as the Bernardi operator[4].If we assume

$$\eta(t) = \frac{(c+1)^\delta}{\eta(\delta)} t^c \left(\log\frac{1}{t}\right)^{\delta-1}, \quad c > -1, \quad \delta \geq 0,$$

then Ξ_η is called the Komatu operator(see [22]).

In the following theorem we prove that the class $\mathcal{MG}_k^*(\xi, \vartheta)$ is closed under the transform $\Xi_\eta(f)(\zeta)$.

Theorem 3.1. *Let $f(\zeta) \in \mathcal{MG}_k^*(\xi, \vartheta)$. Then $\Xi_\eta(f)(\zeta) \in \mathcal{MG}_k^*(\xi, \vartheta)$.*

Proof. By definition, we have

$$\begin{aligned}
\Xi_\eta(f)(\zeta) &= \frac{(c+1)^\delta}{\eta(\delta)} \int_0^1 (-1)^{\delta-1} t^c (\log t)^{\delta-1}\left(\zeta - \sum_{n=2}^{\infty} a_n \zeta^n t^{n-1}\right)dt \\
&= \frac{(-1)^{\delta-1}(1+c)^\delta}{\eta(\delta)} \lim_{r\to 0^+}\left[\int_r^1 t^c (\log t)^{\delta-1}\left(\zeta - \sum_{n=2}^{\infty} a_n \zeta^n t^{n-1}\right)dt\right].
\end{aligned}$$

By simple calculation, we get

$$\Xi_\eta(f)(\zeta) = \zeta - \sum_{n=2}^{\infty} \left(\frac{c+1}{c+n}\right)^{\delta} a_n \zeta^n. \tag{21}$$

We need to prove that

$$\sum_{n=2}^{\infty} \frac{\aleph(k,\vartheta,\xi,n)}{1-\xi} \left(\frac{1+c}{c+n}\right)^{\delta} a_n \leq 1. \tag{22}$$

On the other hand by (12), $f \in \mathcal{MG}_k^*(\xi,\vartheta)$ if and only if

$$\sum_{n=2}^{\infty} \frac{\aleph(k,\vartheta,\xi,n)}{1-\xi} a_n \leq 1,$$

where $\aleph(k,\vartheta,\xi,n)$ is given in (13). Thus $\frac{1+c}{c+n} < 1$, so (22) holds and thus we complete the proof. □

The above theorem yields the subsequent theorem.

Theorem 3.2. *(i) If $f \in \mathcal{S}^*(\xi)$ then $\Xi_\eta(f)(\zeta) \in \mathcal{S}^*(\xi)$.*
(ii) If $f \in \mathcal{K}(\xi)$ is convex of order ξ then $\Xi_\eta(f) \in \mathcal{K}(\xi)$.

Theorem 3.3. *Let $f \in \mathcal{MG}_k^*(\xi,\vartheta)$, then $\Xi_\eta(f)(\zeta)$ is starlike of order $0 \leq \xi < 1$ in $|\zeta| < R_1$ where*

$$R_1 = \inf_n \left[\left(\frac{c+n}{c+1}\right)^{\delta} \frac{(1-\xi)\aleph(\vartheta,\xi,k,n)}{(n-\xi)(1-\xi)}\right]^{\frac{1}{n-1}} \quad (n \geq 2),$$

where $\aleph(k,\vartheta,\xi,n)$ as given in (13).

Proof. Since $\Xi_\eta(f)(\zeta)$ is starlike of order $0 \leq \xi < 1$ it suffices to show

$$\left|\frac{\zeta(\Xi_\eta(f)(\zeta))'}{\Xi_\eta(f)(\zeta)} - 1\right| < 1-\xi. \tag{23}$$

From (21) we have,

$$\begin{aligned}\left|\frac{\zeta(\Xi_\eta(f)(\zeta))'}{\Xi_\eta(f)(\zeta)} - 1\right| &= \left|\frac{\sum\limits_{n=2}^{\infty}(1-n)\left(\frac{c+1}{c+n}\right)^{\delta} a_n \zeta^{n-1}}{1-\sum\limits_{n=2}^{\infty}\left(\frac{c+1}{c+n}\right)^{\delta} a_n \zeta^{n-1}}\right| \\ &\leq \frac{\sum\limits_{n=2}^{\infty}(1-n)\left(\frac{c+1}{c+n}\right)^{\delta} a_n |\zeta|^{n-1}}{1-\sum\limits_{n=2}^{\infty}\left(\frac{c+1}{c+n}\right)^{\delta} a_n |\zeta|^{n-1}}.\end{aligned}$$

The above expression is bounded above by $1-\xi$ thus ,

$$|\zeta|^{n-1} < \left(\frac{c+n}{c+1}\right)^{\delta} \frac{(1-\xi)\aleph(\vartheta,\xi,k,n)}{(n-\xi)(1-\xi)}.$$

Hence, the proof is completed. □

By the fact that $f \in \mathcal{K} \Leftrightarrow \zeta f'(\zeta) \in \mathcal{S}^*$, we state the following:

Theorem 3.4. *Let $f \in \mathcal{MG}_k^*(\xi,\vartheta)$, then $\Xi_\eta(f)(\zeta) \in \mathcal{K}(\xi)$ in $|\zeta| < R_2$ where*

$$R_2 = \inf_n \left[\left(\frac{c+n}{c+1}\right)^{\delta} \frac{(1-\xi)\aleph(\vartheta,\xi,k,n)}{n(n-\xi)(1-\xi)}\right]^{\frac{1}{n-1}} \quad (n \geq 2),$$

where $\aleph(k,\vartheta,\xi,n)$ is given by (13).

4. Neighbourhood Results

The concept of neighborhoods of analytic functions was first introduced by Goodman [15] later, Ruscheweyh [29]. We now recall the definition of $\delta-$ neighbourhood [15, 29] and determine neighbourhood results for certain families of analytic functions.(also see[35–37]).We now extend the familiar concept of neighborhoods to the analytic functions of the family $f \in \mathcal{MG}_k^*(\xi, \vartheta)$ in this section.
First we recall the definition of the $\delta-$ neighbourhood of $f \in \mathcal{T}$ is given by

$$N_\delta(f) := \left\{ h \in \mathcal{T} : h(\zeta) = \zeta - \sum_{n=2}^{\infty} d_n \zeta^n \text{ and } \sum_{n=2}^{\infty} n|a_n - d_n| \leq \delta \right\}. \tag{24}$$

Mostly for the identity function $e(\zeta) = \zeta$, we have

$$N_\delta(e) := \left\{ h \in \mathcal{T} : g(\zeta) = \zeta - \sum_{n=2}^{\infty} d_n \zeta^n \text{ and } \sum_{n=2}^{\infty} n|d_n| \leq \delta \right\}. \tag{25}$$

Theorem 4.1. *If*

$$\delta := \frac{2(1-\xi)}{\aleph(\vartheta, \xi, k, 2)} \tag{26}$$

then $\mathcal{MG}_k^*(\xi, \vartheta) \subset N_\delta(e)$, where $\aleph(\vartheta, \xi, k, 2)$ is assumed as (14).

Proof. For $f \in \mathcal{MG}_k^*(\xi, \vartheta)$, Theorem 2.2 immediately yields

$$\aleph(\vartheta, \xi, k, 2) \sum_{n=2}^{\infty} a_n \leq 1 - \xi,$$

so that

$$\sum_{n=2}^{\infty} a_n \leq \frac{1-\xi}{\aleph(\vartheta, \xi, k, 2)}. \tag{27}$$

Additionally, from (12) and (27) that

$$\begin{aligned}
(k+1)\Lambda(2) \sum_{n=2}^{\infty} n a_n &\leq 1 - \xi + \vartheta(\xi + k)\Lambda(2) \sum_{n=2}^{\infty} a_n \\
&= 1 - \xi + \frac{\vartheta(\xi + k)(1-\xi)\Lambda(2}{\aleph(\vartheta, \xi, k, 2)} \\
&= (1-\xi)\left[1 + \frac{\vartheta(\xi + k)\Lambda(2}{[2(1+k) - \vartheta(\xi + k)]\Lambda(2)}\right] \\
(k+1)\Lambda(2) \sum_{n=2}^{\infty} n a_n &= 2(1-\xi)\left[\frac{1+k}{[2(1+k) - \vartheta(\xi + k)]\Lambda(2)}\right] \\
\sum_{n=2}^{\infty} n a_n &= \frac{2(1-\xi)}{[2(1+k) - \vartheta(\xi + k)]\Lambda(2)},
\end{aligned}$$

that is

$$\sum_{n=2}^{\infty} n a_n \leq \frac{2(1-\xi)}{\aleph(\vartheta, \xi, k, 2)} := \delta \tag{28}$$

which, in sight (25) proves Theorem. 4.1. □

Definition 4.2. *Let $f \in \mathcal{T}$ and we let $f \in \mathcal{MG}^*(\rho, \vartheta, \xi, k)$ if there exists a function $h \in \mathcal{MG}^*(\rho, \vartheta, \xi, k)$ such that*

$$\left|\frac{f(\zeta)}{h(\zeta)} - 1\right| < 1 - \rho, \quad (\zeta \in \mathbb{U}, \ \ 0 \le \rho < 1). \tag{29}$$

Theorem 4.3. *If $h \in \mathcal{MG}^*(\rho, \vartheta, \xi, k)$ and*

$$\rho = 1 - \frac{\delta \aleph(\vartheta, \xi, k, 2)}{2[(\aleph(\vartheta, \xi, k, 2) - (1 - \xi)]} \tag{30}$$

then

$$N_\delta(h) \subset \mathcal{MG}^*(\rho, \vartheta, \xi, k) \tag{31}$$

where $\aleph(\vartheta, \xi, k, 2)$ is defined in (14).

Proof. Assume that $f \in N_\delta(h)$, then from (24) we have

$$\sum_{n=2}^{\infty} n|a_n - d_n| \le \delta$$

which infers that

$$\sum_{n=2}^{\infty} |a_n - d_n| \le \frac{\delta}{2}.$$

Subsequently $h \in \mathcal{MG}_k^*(\xi, \vartheta)$, we have

$$\sum_{n=2}^{\infty} d_n = \frac{1 - \xi}{\aleph(\vartheta, \xi, k, 2)}$$

so that

$$\begin{aligned} \left|\frac{f(\zeta)}{h(\zeta)} - 1\right| &< \frac{\sum\limits_{n=2}^{\infty} |a_n - d_n|}{1 - \sum\limits_{n=2}^{\infty} d_n} \\ &\le \frac{\delta}{2} \times \frac{\aleph(\vartheta, \xi, k, 2)}{\aleph(\vartheta, \xi, k, 2) - (1 - \xi)} \\ &\le \frac{\delta \aleph(\vartheta, \xi, k, 2)}{2[(\aleph(\vartheta, \xi, k, 2) - (1 - \xi)]} \\ &= 1 - \rho, \end{aligned}$$

if that ρ is assumed precisely by (31), consequently by Definition 4.2, $f \in \mathcal{MG}^*(\rho, \vartheta, \xi, k)$ which concludes the proof. □

5. Integral Means

In [30], Silverman originate that the extremal over the family $\mathcal{T}$ is$f_2(\zeta) = \zeta - \frac{\zeta^2}{2}$. In[31] he conjectured the integral means inequality given by ,

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^\eta d\theta \le \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^\eta d\theta,$$

for all $f \in \mathcal{T}$, $\eta > 0$ and $0 < r < 1$ and settled in [32] by using the extremal function $f_2(\zeta)$. In [32], he also showed his conjecture for the subclasses $\mathcal{T}^*(\xi)$ and $C(\xi)$.

We recall the subsequent definition and the lemma to show our result on Integral means inequality.

Definition 5.1. (Subordination Principle)*[23]: Let f and g be functions analytic in $\mathbb{D}$. Then, we say that the function f is subordinated to g, if there exists a Schwarz function ϖ, analytic in $\mathbb{U}$ with $\varpi(0) = 0$ and $|\varpi(z)| < 1$, $\zeta \in \mathbb{U}$, such that*

$$f(\zeta) = g(\varpi(\zeta)),\ \zeta \in \mathbb{U},$$

and we symbolize this subordination by $f(\zeta) \prec g(\zeta)$. In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Lemma 5.2. *[23] If the functions $\mathfrak{f}, \mathfrak{g} \in \mathcal{A}$ with $\mathfrak{g} \prec \mathfrak{f}$, then for $\varrho > 0$, and $0 < r < 1$,*

$$\int_0^{2\pi} \left|\mathfrak{g}(re^{i\theta})\right|^{\varrho} d\theta \leq \int_0^{2\pi} \left|\mathfrak{f}(re^{i\theta})\right|^{\varrho} d\theta. \tag{32}$$

Using Lemma 5.2,Theorem 2.2 and Theorem 2.5, we prove the integral means inequality for $f \in \mathcal{MG}_k^*(\xi, \vartheta)$.

Theorem 5.3. *Suppose $f \in \mathcal{MG}_k^*(\xi, \vartheta)$, $\varrho > 0$, $0 \leq \vartheta \leq 1$, $0 \leq \xi < 1$, $k \geq 0$ and $f_2(\zeta)$ is defined by*

$$f_2(\zeta) = \zeta - \frac{1-\xi}{\aleph(\vartheta, \xi, k, 2)}\zeta^2,$$

where $\aleph(\vartheta, \xi, k, 2)$ is as in (14). Then for $\zeta = re^{i\theta}$, $0 < r < 1$, we have

$$\int_0^{2\pi} \left|f(\zeta)\right|^{\varrho} d\theta \leq \int_0^{2\pi} \left|f_2(\zeta)\right|^{\varrho} d\theta. \tag{33}$$

Proof. For $f \in \mathcal{T}$, the inequality (33) is equal to showing that

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} |a_n|\zeta^{n-1}\right|^{\varrho} d\theta \leq \int_0^{2\pi} \left|1 - \frac{1-\xi}{\aleph(\vartheta, \xi, k, 2)}\zeta\right|^{\varrho} d\theta.$$

By Lemma 5.2, it suffices to prove that

$$1 - \sum_{n=2}^{\infty} |a_n|\zeta^{n-1} \prec 1 - \frac{1-\xi}{\aleph(\vartheta, \xi, k, 2)}\zeta.$$

Setting

$$1 - \sum_{n=2}^{\infty} |a_n|\zeta^{n-1} = 1 - \frac{1-\xi}{\aleph(\vartheta, \xi, k, 2)}\varpi(\zeta), \tag{34}$$

and using (12), we obtain

$$\begin{aligned} |\varpi(\zeta)| &= \left|\sum_{n=2}^{\infty} \frac{\aleph(\vartheta, \xi, k, n)}{1-\xi}|a_n|\zeta^{n-1}\right| \\ &\leq |\zeta| \sum_{n=2}^{\infty} \frac{\aleph(\vartheta, \xi, k, n)}{1-\xi}|a_n| \\ &\leq |\zeta|, \end{aligned}$$

which completes the proof . □

6. Subordination Results

Now due to Wilf [39], we state subordinating factor sequence which are more essential for our discussion.

Definition 6.1. (Subordinating Factor Sequence)[39]: *A sequence* $\{b_n\}_{n=1}^{\infty}$ *of complex numbers is said to be a subordinating sequence if,* $f \in \mathcal{A}$ *given by* (1) *is holomorphic, univalent and convex in* $\mathbb{U}$*, then*

$$\sum_{n=1}^{\infty} b_n a_n \zeta^n \prec f(\zeta), \quad \zeta \in \mathbb{U}. \tag{35}$$

Lemma 6.2. *The sequence* $\{b_n\}_{n=1}^{\infty}$ *is a subordinating factor sequence if and only if*

$$Re \left\{ 1 + 2\sum_{n=1}^{\infty} b_n \zeta^n \right\} > 0, \quad \zeta \in \mathbb{U}. \tag{36}$$

Theorem 6.3. *Let* $f \in \mathcal{MG}_k^*(\xi, \vartheta)$ *and* $g(\zeta) \in \mathcal{K}$ *then*

$$\frac{\aleph(\vartheta, \xi, k, 2)}{2[1 - \xi + \aleph(\vartheta, \xi, k, 2)]}(f * g)(\zeta) \prec g(\zeta) \tag{37}$$

where $0 \leq \xi < 1$; $k \geq 0$ *and* $0 \leq \vartheta \leq 1$, *and*

$$Re\ \{f(\zeta)\} > -\frac{[1 - \xi + \aleph(\vartheta, \xi, k, 2)]}{\aleph(\vartheta, \xi, k, 2)}, \quad \zeta \in \mathbb{U}. \tag{38}$$

The constant factor $\frac{\aleph(\vartheta,\xi,k,2)}{2[1-\xi+\aleph(\vartheta,\xi,k,2)]}$ *in (37) cannot be substituted by a greater number.*

Proof. Since $f \in \mathcal{MG}_k^*(\xi, \vartheta)$ and assume that $g(\zeta) = \zeta + \sum_{n=2}^{\infty} b_n \zeta^n \in \mathcal{K}$. Then

$$\frac{\aleph(\vartheta, \xi, k, 2)}{2[1 - \xi + \aleph(\vartheta, \xi, k, 2)]}(f * g)(\zeta) = \frac{\aleph(\vartheta, \xi, k, 2)}{2[1 - \xi + \aleph(\vartheta, \xi, k, 2)]}\left(\zeta + \sum_{n=2}^{\infty} b_n a_n \zeta^n\right). \tag{39}$$

Therefore, by Definition 6.1, the subordination result holds if

$$\left\{\frac{\aleph(\vartheta, \xi, k, 2)}{2[1 - \xi + \aleph(\vartheta, \xi, k, 2)]}\right\}_{n=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In sight of Lemma 6.2, this is equal to the subsequent inequality

$$\text{Re}\left\{1 + \sum_{n=1}^{\infty} \frac{\aleph(\vartheta, \xi, k, 2)}{[1 - \xi + \aleph(\vartheta, \xi, k, 2)]} a_n \zeta^n\right\} > 0, \quad \zeta \in \mathbb{U}. \tag{40}$$

For $n \geq 2$ we note that $\frac{\aleph(k,\vartheta,\xi,n)}{1-\xi}$ is increasing function and in particular

$$\frac{\aleph(\vartheta, \xi, k, 2)}{1 - \xi} \leq \frac{\aleph(k, \vartheta, \xi, n)}{1 - \xi}, \quad n \geq 2,$$

therefore, for $|\zeta| = r < 1$, we have

$$\begin{aligned}
&\mathrm{Re}\left\{1 + \frac{\aleph(\vartheta,\xi,k,2)}{[1-\xi+\aleph(\vartheta,\xi,k,2)]}\sum_{n=1}^{\infty} a_n\zeta^n\right\}\\
&= \mathrm{Re}\left\{1 + \frac{\aleph(\vartheta,\xi,k,2)}{[1-\xi+\aleph(\vartheta,\xi,k,2)]}\zeta + \frac{\sum_{n=2}^{\infty}\aleph(\vartheta,\xi,k,2)a_n\zeta^n}{[1-\xi+\aleph(\vartheta,\xi,k,2)]}\right\}\\
&\geq 1 - \frac{\aleph(\vartheta,\xi,k,2)}{[1-\xi+\aleph(\vartheta,\xi,k,2)]}r - \frac{\sum_{n=2}^{\infty}|\aleph(k,\vartheta,\xi,n)a_n|\, r^n}{[1-\xi+\aleph(\vartheta,\xi,k,2)]}\\
&\geq 1 - \frac{\aleph(\vartheta,\xi,k,2)}{[1-\xi+\aleph(\vartheta,\xi,k,2)]}r - \frac{1-\xi}{[1-\xi+\aleph(\vartheta,\xi,k,2)]}r\\
&> 0, \quad |\zeta| = r < 1,
\end{aligned}$$

by the assertion (12) of Theorem 2.2 . This clearly proves (40) and hence (37) .

By fixing

$$g(\zeta) = \frac{\zeta}{1-\zeta} = \zeta + \sum_{n=2}^{\infty}\zeta^n \in \mathcal{K},$$

thus

$$f * g = F(\zeta) := \zeta - \frac{1-\xi}{\aleph(\vartheta,\xi,k,2)}\zeta^2$$

inequality (38) follows from (37) . Subsequently we consider the function

$$F(\zeta) := \zeta - \frac{1-\xi}{\aleph(\vartheta,\xi,k,2)}\zeta^2 \in \mathcal{MG}_k^*(\xi,\vartheta).$$

For this function (37) becomes

$$\frac{\aleph(\vartheta,\xi,k,2)}{2[1-\xi+\aleph(\vartheta,\xi,k,2)]}F(\zeta) \prec \frac{\zeta}{1-\zeta} = g(\zeta).$$

It is easily verified that

$$\min\left\{\mathrm{Re}\left(\frac{\aleph(\vartheta,\xi,k,2)}{2[1-\xi+\aleph(\vartheta,\xi,k,2)]}F(\zeta)\right)\right\} = -\frac{1}{2}, \quad \zeta \in \mathbb{U}.$$

This proves that the constant $\frac{\aleph(\vartheta,\xi,k,2)}{2[1-\xi+\aleph(\vartheta,\xi,k,2)]}$ cannot be substituted by a greater number. □

Concluding Remarks: Suitably fixing the parameters ϑ, ξ and k the results discussed in Theorems2.1 - 6.3 would find additional applications for $f \in \mathcal{T}$ for thefunction classesillustrated in Examples 1.3 to 1.6 which have not been studied so far. Further by fixing $\tau = 1; \kappa = 1; \jmath = 1/2$ and $\ell = 1$ we get error functions given by

$$I^{1,1}_{\frac{1}{2},1}(\zeta) = e^{\zeta^2}\left(1 + \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\zeta^{2n+1}\right).$$

which gives new study on the family of Starlike functions we left this as an exercise to interested readers.Miller and Bertram Ross[25] proposed the special function, which is called the Miller-Ross function defined as

$$\mathbf{E}_\zeta(\mathsf{s},\wp) = \zeta^{\mathsf{s}}e^{\wp\zeta}\Upsilon^*(\mathsf{s},\wp\zeta),$$

whereΥ^* is the incomplete gamma function (p.314, [25]). Using the properties of the incomplete gamma functions the Miller-Ross function can easily be written as

$$\mathbf{E}_\zeta(\mathsf{s},\wp) := \zeta^{\mathsf{s}} \sum_{n=0}^{\infty} \frac{(\zeta\wp)^n}{\Gamma(\mathsf{s}+n+1)},\ \mathsf{s},\wp \in \mathbb{C},\ \text{with } \operatorname{Re}\mathsf{s} > 0,\ \operatorname{Re}\wp > 0.$$

Which can be stated as

$$\mathbf{E}_\zeta(\mathsf{s},\wp) \equiv \zeta^{\mathsf{s}} \mathbf{E}_{1,1+\mathsf{s}}(\wp\zeta)$$

where in the right hand member $\mathbf{E}_{1,1+\mathsf{s}}(\wp\zeta)$ is the Mittag-Leffler of two parameters when $\jmath = 1$, and $\ell = \mathsf{s}+1$, as a conclusion in future one may consider Miller-Ross function and discuss the above proved characteristic functions for the subclasses of $\mathcal{S}$ or $\mathcal{T}$ defined in the unit disc.

References

[1] M.I. Abbas, M. A. Ragusa, *Nonlinear fractional differential inclusions with nonsingular Mittag-Leffler kernel*, AIMS Mathematics, 7 (11), 20328â€"20340, (2022);

[2] A. A. Attiya, *Some applications of Mittag-Leffler function in the unit disk*, Filomat, **30**(7) (2016), 2075–2081.

[3] M. K. Aouf and G. Murugusundaramoorthy,*On a subclass of uniformly convex functions defined by the Dziok-Srivastava operator,* Australian J. Math. Analysis and Appl., **5**(1) Art.3 (2008).

[4] S. D. Bernardi, *Convex and starlike univalent functions,* Trans. Amer. Math. Soc., **135** (1969), 429–446.

[5] D. Bansal and J. K. Prajapat, *Certain geometric properties of the Mittag-Leffler functions*, Complex Var. Elliptic Equ., **61**(3) (2016), 338–350.

[6] S. Boulaaras, A. Choucha, D. Ouchenane, M. Abdalla, A. Vazquez., *Solvability of the Moore-Gibson-Thompson equation with viscoelastic memory type II and integral condition*. Discrete and Continuous Dynamical Systems - S, 2023, 16(6): 1216-1241. doi: 10.3934/dcdss.2022151

[7] B.C.Carlson and S.B.Shaffer,*Starlike and prestarlike hypergrometric functions,* SIAM J. Math. Anal., **15** (1984), no. 4, 737–745.

[8] M. Caglar, K.R. Karthikeyan, G. Murugusundaramoorthy, *Inequalities on a class of analytic functions defined by generalized Mittag-Leffler function*, Filomat, 37 (19), 6277â€"6288, (2023)

[9] J.Dziok and H.M.Srivastava,*Certain subclasses of analytic functions associated with the generalized hypergeometric function,* Intergral Transform Spec. Funct., **14** (2003), 7 - 18.

[10] P. L. Duren, Univalent functions, Grundlehren der Mathematischen Wissenschaften Series, 259, Springer Verlag, New York, 1983.

[11] N. Doudi, S. Boulaaras, N. Mezouar, R.d Jan., *Global existence, general decay and blow-up for a nonlinear wave equation with logarithmic source term and fractional boundary dissipation*. Discrete and Continuous Dynamical Systems - S, 2023, **16**(6): 1323-1345. doi: 10.3934/dcdss.2022106

[12] S. Etemad, M.M. Matar, M.A. Ragusa, S. Rezapour, *Tripled fixed points and existence study to a tripled impulsive fractional differential system via measures of noncompactness*, Mathematics, 10 (1), 25, (2022);

[13] B. A. Frasin, *An application of an operator associated with generalized Mittag-Leffler function*, Konuralp J. Math., **7**(1) (2019), 199–202.

[14] B. A. Frasin, Tariq Al-Hawary and F. Yousef, *Some properties of a linear operator involving generalized Mittag-Leffler function*, Stud. Univ. Babeş-Bolyai Math., **65**(1) (2020), 67–75.

[15] A.W.Goodman, *Univalent functions and nonanalytic curves,* Proc. Amer. Math. Soc., (8)(1957), 598–601.

[16] A.W. Goodman,*On uniformly convex functions,* Ann. polon. Math., **56** (1991), 87 - 92.

[17] A.W. Goodman,*On uniformly starlike functions,* J. Math. Anal. & Appl., **155**(1991), 364 - 370.

[18] H. J. Haubold, A. M. Mathai and R. K. Saxena, *Mittag-Leffler functions and their applications*, J. Appl. Math., 2011.

[19] R. Jan, S. Boulaaras, S. Alyobi, K. Rajagopal, M. Jawad. *Fractional dynamics of the transmission phenomena of dengue infection with vaccination*. Discrete and Continuous Dynamical Systems - S, 2023, 16(8): 2096-2117. doi: 10.3934/dcdss.2022154

[20] R. Jan, S. Qureshi, S. Boulaaras, V. Pham, E. Hincal, R. Guefaifia. *Optimization of the fractional-order parameter with the error analysis for human immunodeficiency virus under Caputo operator*. Discrete and Continuous Dynamical Systems - S, 2023, 16(8): 2118-2140. doi: 10.3934/dcdss.2023010

[21] V. Kiryakova, Generalized fractional calculus and applications. Pitman Research Notes in Mathematics Series, 301. Longman Scientific & Technical, Harlow; co-published in the United States with John Wiley & Sons, Inc., New York, 1994.

[22] Y.C.Kim and F.Rønning, *Integral transform of certain subclass of analytic functions,* J. Math. Anal. Appl., **258** (2001), 466 - 489.

[23] J.E.Littlewood, *On inequalities in theory of functions,* Proc. London Math. Soc., **23** (1925), 481 - 519.

[24] G. M. Mittag-Leffler, *Sur la nouvelle fonction* $\mathbf{E}(x)$, C. R. Acad. Sci. Paris, **137** (1903), 554–558.

[25] K.S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*. John Wiley and Sons, New York Press(1993)

[26] S. A. Meziane, S. Boulaaras, T. Hadj-ammar, R. Guefaifia, R.Jan., *On the problem of dynamic bi-frictional contact of thermo-electro-viscoelastic materials*. Discrete and Continuous Dynamical Systems - S. doi: 10.3934/dcdss.2024037

[27] M. S. Robertson, *On the theory of univalent functions*, Ann. of Math. (2), **37**(2) (1936), 374–408.

[28] F.Rønning,*Uniformly convex functions and a corresponding class of starlike functions,* Proc. Amer. Math. Soc., **118** (1993), 189 - 196.

[29] S.Rucheweyh,*Neighborhoods of univalent functions,* Proc. Amer. Math. Soc., **81** (1981), 521-527.

[30] H. Silverman,*Univalent functions with negative coefficients,* Proc. Amer. Math. Soc. , **51** (1975), 109–116.

[31] H. Silverman,*A survey with open problems on univalent functions whose coefficients are negative,* Rocky Mt. J. Math., **21** (1991), 1099-1125.
[32] H. Silverman,*Integral means for univalent functions with negative coefficients,* Houston J. Math., **23** (1997), 169–174.
[33] H.M. Srivastava and Z. Tomovski, *Fractional calculus with an itegral operator containing a generalized Mittag-Leffler function in the kernal*, Appl. Math. Comp., **211**(2009), 198-210.
[34] J.Sokol, G.Murugusundaramootyhy and K.Thilagavathi,*Some inclusion properties of new subclass of starlike and convex functions associated with Hohlov Operator*,Kyungpook Math. J. **56**(2016), 597-610 http://dx.doi.org/10.5666/KMJ.2016.56.2.597
[35] T.Rosy,G. Murugusundaramoorthy, *Fractional calculus and their applications to certain subclass of uniformly convex functions*. Far East J. Math. Sci., **15**(2004), 231-242.
[36] T.Rosy, K.G.Subramanian and G.Murugusundaramoorthy,*Neighbourhoods and partial sums of starlike functions based on Ruscheweyh derivatives*,JIPAM,**4**,Issue4,Article64,(2003).http://jipam.vu.edu.au/
[37] H.Silverman,*Neighborhoods of classes of analytic function*, Far.East.J.Math. Sci., **3**(2) (1995),165-169.
[38] K.G.Subramanian, G.Murugusundaramoorthy, P.Balasubrahmanyam and H.Silverman, *Subclasses of uniformly convex and uniformly starlike functions.* Math. Japonica., **42** (3) (1995), 517-522.
[39] H. S. Wilf,*Subordinating factor sequence for convex maps of the unit circle,* Proc. Amer. Math. Soc., 12 (1961), 689-693.
[40] A. Wiman, *Über die Nullstellun der Funcktionen* $\mathbf{E}(x)$, Acta Math., **29** (1905), 217–134.
[41] R.Yamakawa *Current topics in analytic function theory* (H.M. Srivastava and S. Owa, Editors). (World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong) 1992; 393-402.