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Symmetric q-Appel polynomials via determinantal approches

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Abstract. This paper sets out to give a determinantal definition for symmetric q-Appel polynomials (symmetric under the interchange $q \leftrightarrow q^{-1}$) and justify some properties in the lights of the new definition.

1. Quantum and symmetric quantum calculus

Noteworthy, this study will be based on the forthcoming notions and notations of the q-theory (see [8] and [9]). Along this work, the parameter q is taken such that $q > 0$ and $q \neq 1$.

For all complex number a, the q-shifted factorials are given by

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{i=0}^{n-1}(1-aq^i) = (1-a)(1-aq)....(1-aq^{n-1}), \qquad n = 1,2,..... \tag{1}$$

The q-analogue of the complex number $x \in \mathbb{C}$ is defined by

$$[x]_q = \frac{1-q^x}{1-q}, \quad x \in \mathbb{C}, \tag{2}$$

and

$$\widetilde{[x]}_q = \frac{q^x - q^{-x}}{q-q^{-1}}, \quad x \in \mathbb{C}. \tag{3}$$

Also, we denote

$$[n]_q! = \prod_{k=1}^{n}[k]_q = \frac{(q;q)_n}{(1-q)^n} \quad \text{for } n \geq 1 \text{ and } [0]_q! = 1 \tag{4}$$

and

$$\widetilde{[n]}_q! = \prod_{k=1}^{n}\widetilde{[k]}_q \quad \text{for } n \geq 1, \text{ and } \widetilde{[0]}_q! = 1. \tag{5}$$

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The q-binomial coefficient is defined by

$$\left[\begin{array}{c} n \\ k \end{array}\right]_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad k = 0, 1, ..., n. \tag{6}$$

Similarly we can define the symmetric (symmetric under the interchange $q \longleftrightarrow q^{-1}$) q-binomial coefficient by

$$\widetilde{\left[\begin{array}{c} n \\ k \end{array}\right]}_q = \frac{\widetilde{[n]}_q!}{\widetilde{[k]}_q!\widetilde{[n-k]}_q!}, \quad k = 0, 1, ..., n. \tag{7}$$

The following relations are useful in the sequel

1. $\widetilde{[x]}_q = \widetilde{[x]}_{q^{-1}}$.
2. $\widetilde{[x]}_q = q^{-(x-1)}[x]_{q^2}$.
3. $\widetilde{\left[\begin{array}{c} n \\ k \end{array}\right]}_q = \widetilde{\left[\begin{array}{c} n \\ k \end{array}\right]}_{\frac{1}{q}}$.

The symmetric q-derivative $\widetilde{D}_q$ of a function f is defined by

$$(\widetilde{D}_q f)(x) = \frac{f(qx) - f(q^{-1}x)}{(q - q^{-1})x}, if x \neq 0, \tag{8}$$

$(\widetilde{D}_q f)(0) = f'(0)$ if $f'(0)$ exists.

$\widetilde{D}_q f$ and D_q are related as follows:

$$\widetilde{D}_q f(x) = D_{q^2} f(q^{-1}x) \tag{9}$$

where

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}. \tag{10}$$

The following properties hold ([9])

1. $\widetilde{D}_q x^n = \widetilde{[n]}_q x^{n-1}$,
2. $\widetilde{D}_q \widetilde{(x-a)}_q^n = \widetilde{[n]}_q \widetilde{(x-a)}_q^{n-1}$,
   where $\widetilde{(x-a)}_q^n = (x - q^{n-1}a)(x - q^{n-3}a)(x - q^{n-5}a)...(x - q^{-n+1}a)$ and $\widetilde{(x-a)}_q^0 = 1$.

In the special case $a = 0$, we have $\widetilde{(x-0)}_q^n = \widetilde{(x)}_q^n = x^n$.

A q-analogue of the Gauss binomial formula is given by

$$\widetilde{(x+a)}_q^n = \sum_{k=0}^{n} \widetilde{\left[\begin{array}{c} n \\ k \end{array}\right]}_q a^{n-k} x^k. \tag{11}$$

The symmetric q-integral or $\widetilde{q}$-integral is defined by ([9])

$$\int_0^a f(x) d_{\widetilde{q}} x = a(q^{-1} - q) \sum_{n=1,3,...} q^n f(q^n a), \tag{12}$$

$$\int_a^b f(x)d_{\widetilde{q}}x = \int_0^b f(x)d_{\widetilde{q}}x - \int_0^a f(x)d_{\widetilde{q}}x, \tag{13}$$

and

$$\int_0^\infty f(x)d_{\widetilde{q}}x = (q^{-1} - q) \sum_{n=\pm1,\pm3,\ldots} q^n f(q^n a). \tag{14}$$

From 8 and 12, we note that for any function f we have

1.

$$\int_0^a f(x)d_{\widetilde{q}}x = F(a) - F(0), \tag{15}$$

under the condition $\widetilde{D}_q F = f$, continuous at $x = 0$.

2.

$$\widetilde{D}_q \int_0^x f(t)d_{\widetilde{q}}t = f(x). \tag{16}$$

A symmetric q-analogue of the exponential function (symmetric under the interchange $q \leftrightarrow q^{-1}$) has been defined by ([13],[14])

$$\widetilde{e_q}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\widetilde{[n]_q!}}, \quad z \in \mathbb{C} \text{ and } q \in]0,1[\cup]1,+\infty[. \tag{17}$$

Note that we can consider $\widetilde{e_q}(z)$ as formal power series in the formal variable z satisfiying the relation

$$\lim_{q\to1} \widetilde{e_q}(z) = e^z.$$

In [6], the author secured the following result

$$\widetilde{e_q}(x+y) = \widetilde{e_q}(y)\widetilde{e_q}(x). \tag{18}$$

2. Symmetric q-Appel Polynomials

In literature, the history of Appell polynomials go back to Appel (1880) [2], and since then, Appell polynomials have been studied by many authors such that Throne [18], Sheffer [16], and Varma [19]. Inspiredby the previous works, , Al-Salam, in 1967, introduced the family of q-Appell polynomials $(A_{n,q}(x))_{n=0}^{\infty}$ and studied some of their properties [1].

According to his definition, the n-degree polynomials $A_{n,q}(x)$ are called q-Appell if they hold the following q-differential equation

$$D_q(A_{n,q}(x)) = [n]_q A_{n-1,q}(x); n = 0, 1, 2, \ldots \tag{19}$$

In1982, Srivastava provided more detais about the family of q-Appell polynomials [17], and since then ,they have been extensively studied from different perspectives [7, 15], various methods, like operator algebra, have been used to explore their properties [11]. In [12], Mahmudov derived the q-difference equations satisfied by sequence of q-Appell polynomials.

Inspired by the Costabile et al. s determinantal approach for defining Bernoulli polynomials as well as Appell polynomials [3, 4], Mahmudov et al. [10] introduced a new determinantal definition of q-Appell polynomials and proved new properties.

Motivated by [10], in this paper, we introduce and study a new q-analogue of Appell polynomials which is symmetric under the interchange $q \leftrightarrow q^{-1}$ called symmetric q-Appell polynomials then we give a new determinantal definition of symmetric q-Appell polynomials. Additionally, we prove some properties of the family of symmetric q-Appell polynomials using related algebraic approaches.

Definition 2.1. *The n-degree polynomials $\widetilde{A}_{n,q}(x)$ are called symmetric q-Appell if they hold the following q-differential equation*

$$\widetilde{D}_q(\widetilde{A}_{n,q}(x)) = \widetilde{[n]}_q \widetilde{A}_{n-1,q}(x); n = 1, 2, ... \tag{20}$$

Note that $\widetilde{A}_{0,q}(x)$ is a non zero constant let say $\widetilde{A}_{0,q}$.

Theorem 2.2. *The symmetric q-Appell polynomials satisfy the following relation*

$$\widetilde{A}_{n,q}(x) = \widetilde{A}_{n,q} + \widetilde{[n]}_q \widetilde{A}_{n-1,q}x + \widetilde{\begin{bmatrix} n \\ 2 \end{bmatrix}}_q \widetilde{A}_{n-2,q}x^2 + \widetilde{\begin{bmatrix} n \\ 3 \end{bmatrix}}_q \widetilde{A}_{n-3,q}x^3 + ... + \widetilde{A}_{0,q}x^n. \tag{21}$$

Proof. For $n = 1$, the relation 20 gives

$$\widetilde{D}_q(\widetilde{A}_{1,q}(x)) = \widetilde{[1]}_q \widetilde{A}_{0,q}(x) = \widetilde{A}_{0,q}.$$

Using 15, we obtain

$$\widetilde{A}_{1,q}(x) = \widetilde{A}_{0,q}x + \widetilde{A}_{1,q},$$

where $\widetilde{A}_{1,q}$ is an arbitrary constant.
By repeating the method above, we get $\widetilde{A}_{2,q}(x)$, as below by starting from the property 20 for q-Appell polynomials

$$\widetilde{D}_q(\widetilde{A}_{2,q}(x)) = \widetilde{[2]}_q \widetilde{A}_{1,q}(x) = \widetilde{[2]}_q \widetilde{A}_{0,q}x + \widetilde{[2]}_q \widetilde{A}_{1,q}.$$

Using symmetric q-integral 12, we get

$$\widetilde{A}_{2,q}(x) = \widetilde{A}_{0,q}x^2 + \widetilde{[2]}_q \widetilde{A}_{1,q}x + \widetilde{A}_{2,q},$$

where $\widetilde{A}_{2,q}$ is an arbitrary constant.
By induction on n and Application of similar method to the methods used for finding $\widetilde{A}_{1,q}(x)$, $\widetilde{A}_{2,q}(x)$ and continuing taking symmetric q-integral we have

$$\widetilde{A}_{n-1,q}(x) = \widetilde{A}_{n-1,q} + \widetilde{\begin{bmatrix} n-1 \\ 1 \end{bmatrix}}_q \widetilde{A}_{n-2,q}x + \widetilde{\begin{bmatrix} n-1 \\ 2 \end{bmatrix}}_q \widetilde{A}_{n-3,q}x^2 + ... + \widetilde{A}_{0,q}x^{n-1}.$$

Using the fact that for $n = 1, 2, 3, ...$, every $\widetilde{A}_{n,q}(x)$ satisfies the relation 20, we can write

$$\widetilde{D}_q(\widetilde{A}_{n,q}(x)) = \widetilde{[n]}_q \widetilde{A}_{n-1,q} + \widetilde{[n]}_q \widetilde{\begin{bmatrix} n-1 \\ 1 \end{bmatrix}}_q \widetilde{A}_{n-2,q}x + \widetilde{[n]}_q \widetilde{\begin{bmatrix} n-1 \\ 2 \end{bmatrix}}_q \widetilde{A}_{n-3,q}x^2 + ... + \widetilde{[n]}_q \widetilde{A}_{0,q}x^{n-1}.$$

Now, taking the symmetric q-integral of the symmetric q-differential equation above can lead to

$$\widetilde{A}_{n,q}(x) = \widetilde{A}_{n,q} + \widetilde{[n]}_q \widetilde{A}_{n-1,q}x + \frac{\widetilde{[n]}_q}{\widetilde{[2]}_q} \widetilde{\begin{bmatrix} n-1 \\ 1 \end{bmatrix}}_q \widetilde{A}_{n-2,q}x^2 + \frac{\widetilde{[n]}_q}{\widetilde{[3]}_q} \widetilde{\begin{bmatrix} n-1 \\ 2 \end{bmatrix}}_q \widetilde{A}_{n-3,q}x^3 + ... + \frac{\widetilde{[n]}_q}{\widetilde{[n]}_q} \widetilde{A}_{0,q}x^n,$$

where $\widetilde{A}_{n,q}$ is an arbitrary constant. Since

$$\frac{\widetilde{[n]}_q}{\widetilde{[i]}_q}\widetilde{\begin{bmatrix} n-1 \\ i-1 \end{bmatrix}}_q = \widetilde{\begin{bmatrix} n \\ i \end{bmatrix}}_q,$$

so for $n = 0, 1, 2, ...$, we have

$$\widetilde{A}_{n,q}(x) = \widetilde{A}_{n,q} + \widetilde{[n]}_q\widetilde{A}_{n-1,q}x + \widetilde{\begin{bmatrix} n \\ 2 \end{bmatrix}}_q\widetilde{A}_{n-2,q}x^2 + \widetilde{\begin{bmatrix} n \\ 3 \end{bmatrix}}_q\widetilde{A}_{n-3,q}x^3 + ... + \widetilde{A}_{0,q}x^n.$$

□

Note that there exists a one to one correspondence between the family of symmetric q-Appell polynomials $(\widetilde{A}_{n,q}(x))_{n=0}^{\infty}$ and the numerical sequence $(\widetilde{A}_{n,q})_{n=0}^{\infty}$, $\widetilde{A}_{n,q} \neq 0$. Moreover, every $\widetilde{A}_{n,q}(x)$ can be obtained recursively from $\widetilde{A}_{n-1,q}(x)$ for $n \geq 1$.

Also, symmetric q-Appell polynomials can be defined by means of generating function $\widetilde{A}_q(t)$, as follows

$$\widetilde{A}_q(x,t) = \widetilde{A}_q(t)\widetilde{e}_q(tx) = \sum_{n=0}^{\infty}\widetilde{A}_{n,q}(x)\frac{t^n}{\widetilde{[n]}_q!}, 0 < q < 1, \tag{22}$$

where

$$\widetilde{A}_q(t) = \sum_{n=0}^{\infty}\widetilde{A}_{n,q}\frac{t^n}{\widetilde{[n]}_q!}, \tag{23}$$

is an analytic function at $t = 0$, $\widetilde{A}_{n,q} = \widetilde{A}_{n,q}(0)$ and $\widetilde{e}_q(t) = \sum\limits_{n=0}^{\infty}\frac{t^n}{\widetilde{[n]}_q!}$.

Depending on the choice of the generating function $\widetilde{A}_q(t)$, we obtain different families of symmetric q-Appel polynomials. The following are some of them

1. By taking $\widetilde{A}_q(t) = \widetilde{[1]}_q = 1$, we obtain the family $\{1, x, x^2, ...\}$.
2. By taking $\widetilde{A}_q(t) = \frac{t}{\widetilde{e}_q(t)-1}$, we obtain the family of symmetric q-Bernoulli polynomials $\widetilde{B}_{n,q}(x)$.[5].
3. By taking $\widetilde{A}_q(t) = \frac{\widetilde{[2]}_q}{\lambda\widetilde{e}_q(t)+1}$, we obtain the family of symmetric (λ, q)-Euler polynomials $\widetilde{E}_{n,q}(x/\lambda)$. [5].
4. By taking $\widetilde{A}_q(t) = \left(\frac{\widetilde{[2]}_q}{\lambda\widetilde{e}_q(t)+1}\right)^r$, we obtain the family of higher-order symmetric (λ, q)-Euler polynomials $\widetilde{E}_{n,q}^{(r)}(x/\lambda)$. [5].

3. Symmetric q-Appel polynomials from determinantal point of view

Let consider the sequence $P_{n,q}(x)$ of n-degree q-polynomials defined by

$$\begin{cases} P_{0,q}(x) = \frac{1}{\beta_0} \\ P_{n,q}(x) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} 1 & x & x^2 & \cdots & x^{n-1} & x^n \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \widetilde{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_q \beta_1 & \cdots & \widetilde{\begin{bmatrix} n-1 \\ 1 \end{bmatrix}}_q \beta_{n-2} & \widetilde{\begin{bmatrix} n \\ 1 \end{bmatrix}}_q \beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \widetilde{\begin{bmatrix} n-1 \\ 2 \end{bmatrix}}_q \beta_{n-3} & \widetilde{\begin{bmatrix} n \\ 2 \end{bmatrix}}_q \beta_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_0 & \widetilde{\begin{bmatrix} n \\ n-1 \end{bmatrix}}_q \beta_1 \end{vmatrix}, \end{cases} \tag{24}$$

where $\beta_0, \beta_1, ..., \beta_n \in \mathbb{R}; \beta_0 \neq 0, n = 1, 2, 3...$
Then we have the following results.

Theorem 3.1. *$P_{n,q}(x)$ satisfies the following identity*

$$D_q(P_{n,q}(x)) = \widetilde{[n]}_q P_{n-1,q}(x), n = 1, 2, ...$$

To prove this theorem, we need to prove the following Lemma

Lemma 3.2. *Let consider the matrix $A_{n\times n}(x)$ with first order symmetric q-differentiable functions $a_{ij}(x)$ as elements. Then the symmetric q-derivative of $det(A_{n\times n}(x))$ is given by the following formula.*

$$\widetilde{D}_q(det(A_{n\times n}(x))) = \widetilde{D}_q(|a_{ij}(x)|) = \sum_{i=1}^{n} \begin{vmatrix} a_{11}(q^{-1}x) & a_{12}(q^{-1}x) & \cdots & a_{1n}(q^{-1}x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{i-1,1}(q^{-1}x) & a_{i-1,2}(q^{-1}x) & \cdots & a_{i-1,n}(q^{-1}x) \\ \widetilde{D}_q(a_{i,1}(x)) & \widetilde{D}_q(a_{i,2}(x)) & \cdots & \widetilde{D}_q(a_{i,n}(x)) \\ a_{i+1,1}(qx) & a_{i+1,2}(qx) & \cdots & a_{i+1,n}(qx) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}(qx) & a_{n,2}(qx) & \cdots & a_{n,n}(qx) \end{vmatrix} \tag{25}$$

Proof. Using the multi-linearity of the determinant, we obtain

$$\begin{aligned} & \det(R_1(qx), R_2(qx), ..., R_n(qx)) - \det(R_1(q^{-1}x), R_2(q^{-1}x), ..., R_n(q^{-1}x)) \\ = \; & \det(R_1(qx) - R_1(q^{-1}x), R_2(qx), ..., R_n(qx)) + \det(R_1(q^{-1}x), R_2(qx), ..., R_n(qx)) \\ - \; & \det(R_1(q^{-1}x), R_2(q^{-1}x), ..., R_n(q^{-1}x)) \\ = \; & \det(R_1(qx) - R_1(q^{-1}x), R_2(qx), ..., R_n(qx)) + \det(R_1(q^{-1}x), R_2(qx) - R_2(q^{-1}x), ..., R_n(qx)) \\ + \; & \det(R_1(q^{-1}x), R_2(q^{-1}x), ..., R_n(qx)) - \det(R_1(q^{-1}x), R_2(q^{-1}x), ..., R_n(q^{-1}x)) \\ = \; & \det(R_1(qx) - R_1(q^{-1}x), R_2(qx), ..., R_n(qx)) + \det(R_1(q^{-1}x), R_2(qx) - R_2(q^{-1}x), ..., R_n(qx)) \\ + \; & \det(R_1(q^{-1}x), R_2(q^{-1}x), R_3(qx) - R_3(q^{-1}x), ..., R_n(qx)) \\ + \; & \det(R_1(q^{-1}x), R_2(q^{-1}x), R_3(q^{-1}x), ..., R_n(qx)) - \det(R_1(q^{-1}x), R_2(q^{-1}x), ..., R_n(q^{-1}x)) \\ = \; & \det(R_1(qx) - R_1(q^{-1}x), R_2(qx), ..., R_n(qx)) + \det(R_1(q^{-1}x), R_2(qx) - R_2(q^{-1}x), ..., R_n(qx)) \\ + \; & \det(R_1(q^{-1}x), R_2(q^{-1}x), R_3(qx) - R_3(q^{-1}x), ..., R_n(qx)) \end{aligned}$$

$$+ \quad ... + \det(R_1(q^{-1}x), R_2(q^{-1}x), R_3(q^{-1}x), ..., R_n(qx) - R_n(q^{-1}x))$$
$$= \sum_{i=1}^{n} \det(R_1(q^{-1}x), R_2(q^{-1}x), ..., R_{i-1}(q^{-1}x), R_i(qx) - R_i(q^{-1}x), R_{i+1}(qx), ..., R_n(qx)),$$

where R_i is the i^{th} row of the determinant.
Dividing by $(q - q^{-1})x$, we obtain the desired result. □

Let prove theorem 3.1

Proof. Using Lemma 3.2, the symmetric q-derivative of determinant 24 with respect to x is given by

$$D_q(P_{n,q}(x)) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} 0 & 1 & \widetilde{[2]}_q x & \dots & \widetilde{[n]}_q x^{n-1} \\ \beta_0 & \beta_1 & \widetilde{\beta_2} & \dots & \widetilde{\beta_n} \\ 0 & \beta_0 & \widetilde{\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q} \beta_1 & \dots & \widetilde{\begin{bmatrix} n \\ 1 \end{bmatrix}_q} \beta_{n-1} \\ 0 & 0 & \beta_0 & \dots & \widetilde{\begin{bmatrix} n \\ 2 \end{bmatrix}_q} \beta_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \begin{bmatrix} \widetilde{n} \\ n-1 \end{bmatrix}_q \beta_1 \end{vmatrix} \tag{26}$$

Expanding the determinant 26 above along with the first column, we get

$$D_q(P_{n,q}(x)) = \frac{(-1)^{n-1}}{(\beta_0)^n} \begin{vmatrix} 1 & \widetilde{[2]}_q x & \dots & \widetilde{[n-1]}_q x^{n-2} & \widetilde{[n]}_q x^{n-1} \\ \beta_0 & \beta_1 & \dots & \widetilde{\begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q} \beta_{n-2} & \widetilde{\begin{bmatrix} n \\ 1 \end{bmatrix}_q} \beta_{n-1} \\ 0 & \beta_0 & \dots & \widetilde{\begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q} \beta_{n-3} & \widetilde{\begin{bmatrix} n \\ 2 \end{bmatrix}_q} \beta_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & \beta_0 & \begin{bmatrix} \widetilde{n} \\ n-1 \end{bmatrix}_q \beta_1. \end{vmatrix} \tag{27}$$

Using the fact that

$$\frac{\widetilde{[i-1]}_q}{\widetilde{[j]}_q} \begin{bmatrix} \widetilde{j} \\ i-1 \end{bmatrix}_q = \frac{\widetilde{[i-1]}_q \widetilde{[j]}_q!}{\widetilde{[j]}_q \widetilde{[i-1]}_q! \widetilde{[j-i+1]}_q!} = \frac{\widetilde{[j-1]}_q!}{\widetilde{[i-2]}_q! \widetilde{[j-i+1]}_q} = \begin{bmatrix} \widetilde{j-1} \\ i-2 \end{bmatrix}_q,$$

and multiplying the j^{th} column of the determinant 27 by $\frac{1}{\widetilde{[j]}_q}$, as well as the i^{th} row by $\widetilde{[i-1]}_q$ we obtain

$$D_q(P_{n,q}(x)) = \frac{(-1)^{n-1}}{(\beta_0)^n} \times \frac{\widetilde{[1]}_q!}{\widetilde{[0]}_q!} \times \frac{\widetilde{[2]}_q}{\widetilde{[1]}_q} \times ... \times \frac{\widetilde{[n]}_q}{\widetilde{[n-1]}_q} \times \begin{vmatrix} 1 & x & \dots & x^{n-2} & x^{n-1} \\ \beta_0 & \beta_1 & \dots & \widetilde{\beta_{n-2}} & \widetilde{\beta_{n-1}} \\ 0 & \beta_0 & \dots & \widetilde{\begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q} \beta_{n-3} & \widetilde{\begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q} \beta_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \beta_0 & \begin{bmatrix} \widetilde{n-1} \\ n-2 \end{bmatrix}_q \beta_1 \end{vmatrix}, \tag{28}$$

which completes the proof. □

Theorem 3.3. *The q-polynomials $P_{n,q}(x)$, defined in 26, can be expressed as*

$$P_{n,q}(x) = \sum_{j=0}^{n} \widetilde{\begin{bmatrix} n \\ j \end{bmatrix}}_q \alpha_{n-j} x^j, \tag{29}$$

where

$$\begin{cases} \alpha_0 = \frac{1}{\beta_0} \\ \alpha_j = \frac{(-1)^j}{(\beta_0)^{j+1}} \begin{vmatrix} \beta_1 & \widetilde{\beta_2} & \cdots & \widetilde{\beta_{j-1}} & \widetilde{\beta_j} \\ \beta_0 & \widetilde{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_q \beta_1 & \cdots & \widetilde{\begin{bmatrix} j-1 \\ 1 \end{bmatrix}}_q \beta_{j-2} & \widetilde{\begin{bmatrix} j \\ 1 \end{bmatrix}}_q \beta_{j-1} \\ 0 & \beta_0 & \cdots & \widetilde{\begin{bmatrix} j-1 \\ 2 \end{bmatrix}}_q \beta_{j-3} & \widetilde{\begin{bmatrix} j \\ 2 \end{bmatrix}}_q \beta_{j-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \beta_0 & \widetilde{\begin{bmatrix} j \\ j-1 \end{bmatrix}}_q \beta_1 \end{vmatrix}, \end{cases} \tag{30}$$

Proof. Expanding the determinant 24 along the first row, we obtain

$$P_{n,q}(x) = \frac{(-1)^{n+2}}{(\beta_0)^{n+1}} \begin{vmatrix} \beta_1 & \widetilde{\beta_2} & \cdots & \cdots & \widetilde{\beta_{n-1}} & \widetilde{\beta_n} \\ \beta_0 & \widetilde{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_q \beta_1 & \cdots & \cdots & \widetilde{\begin{bmatrix} n-1 \\ 1 \end{bmatrix}}_q \beta_{n-2} & \widetilde{\begin{bmatrix} n \\ 1 \end{bmatrix}}_q \beta_{n-1} \\ 0 & \beta_0 & \cdots & \cdots & \widetilde{\begin{bmatrix} n-1 \\ 2 \end{bmatrix}}_q \beta_{n-3} & \widetilde{\begin{bmatrix} n \\ 2 \end{bmatrix}}_q \beta_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & \beta_0 & \widetilde{\begin{bmatrix} n \\ n-1 \end{bmatrix}}_q \beta_1 \end{vmatrix}$$

$$+ \frac{(-1)^{n+3}}{(\beta_0)^{n+1}} x \begin{vmatrix} \beta_0 & \widetilde{\beta_2} & \cdots & \cdots & \widetilde{\beta_{n-1}} & \widetilde{\beta_n} \\ 0 & \widetilde{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_q \beta_1 & \cdots & \cdots & \widetilde{\begin{bmatrix} n-1 \\ 1 \end{bmatrix}}_q \beta_{n-2} & \widetilde{\begin{bmatrix} n \\ 1 \end{bmatrix}}_q \beta_{n-1} \\ 0 & \beta_0 & \cdots & \cdots & \widetilde{\begin{bmatrix} n-1 \\ 2 \end{bmatrix}}_q \beta_{n-3} & \widetilde{\begin{bmatrix} n \\ 2 \end{bmatrix}}_q \beta_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & \beta_0 & \widetilde{\begin{bmatrix} n \\ n-1 \end{bmatrix}}_q \beta_1 \end{vmatrix}$$

$$+\dots+\frac{(-1)^{2n+2}}{(\beta_0)^{n+1}}x^n\begin{vmatrix}\beta_0 & \beta_1 & \widetilde{\beta_2} & \dots & \dots & \widetilde{\beta_{n-1}}\\ 0 & \beta_0 & \begin{bmatrix}2\\1\end{bmatrix}_q\beta_1 & \dots & \dots & \begin{bmatrix}n-1\\1\end{bmatrix}_q\beta_{n-2}\\ 0 & 0 & \beta_0 & \dots & \dots & \begin{bmatrix}\widetilde{n-1}\\2\end{bmatrix}_q\beta_{n-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & \dots & \dots & \dots & 0 & \beta_0\end{vmatrix}.$$

By the definition of α_i in 30, the first determinant leads to obtain α_n, which is the coefficient of x^0. Also, the last determinant, which is the determinant of an upper triangular $n \times n$ matrix, will lead to obtain the coefficient of x^n as follows

$$\alpha_0 = \frac{(-1)^{2n+2}}{(\beta_0)^{n+1}}(\beta_0)^n = \frac{1}{\beta_0}.$$

To calculate the coefficient of x^j for $0 < j < n$, consider the following determinant

$$= \frac{(-1)^n}{(\beta_0)^{n+1}}(-1)^{j+2}\begin{vmatrix}\beta_0 & \beta_1 & \dots & \widetilde{\beta_{j-1}} & \widetilde{\beta_{j+1}} & \dots & \widetilde{\beta_n}\\ 0 & \beta_0 & \dots & \begin{bmatrix}j-1\\1\end{bmatrix}_q\beta_{j-2} & \begin{bmatrix}j+1\\1\end{bmatrix}_q\beta_j & \dots & \begin{bmatrix}n\\1\end{bmatrix}_q\beta_{n-1}\\ 0 & 0 & \dots & \begin{bmatrix}\widetilde{j-1}\\2\end{bmatrix}_q\beta_{j-3} & \begin{bmatrix}\widetilde{j+1}\\2\end{bmatrix}_q\beta_{j-1} & \dots & \begin{bmatrix}\widetilde{n}\\2\end{bmatrix}_q\beta_{n-2}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & \beta_0 & \begin{bmatrix}\widetilde{j+1}\\j-1\end{bmatrix}_q\beta_2 & \dots & \begin{bmatrix}n\\2\end{bmatrix}_q\beta_{n-j-1}\\ 0 & \dots & \dots & 0 & \begin{bmatrix}\widetilde{j+1}\\j\end{bmatrix}_q\beta_1 & \dots & \begin{bmatrix}\widetilde{n}\\j\end{bmatrix}_q\beta_{n-j}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & \dots & \dots & 0 & 0 & \dots & \begin{bmatrix}\widetilde{n}\\n-1\end{bmatrix}_q\beta_1\end{vmatrix}$$

$$= \frac{(-1)^{n+j}}{(\beta_0)^{n+1}}(\beta_0)^j\begin{vmatrix}\begin{bmatrix}\widetilde{j+1}\\j\end{bmatrix}_q\beta_1 & \dots & \begin{bmatrix}\widetilde{n-1}\\j\end{bmatrix}_q\beta_{n-j-1} & \begin{bmatrix}\widetilde{n}\\j\end{bmatrix}_q\beta_{n-j}\\ \beta_0 & \dots & \dots & \dots\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \dots & \beta_0 & \begin{bmatrix}\widetilde{n}\\n-1\end{bmatrix}_q\beta_1\end{vmatrix}$$

$$= \frac{(-1)^{n+j}}{(\beta_0)^{n-j+1}} \widetilde{\begin{bmatrix} j+1 \\ j \end{bmatrix}}_q \begin{vmatrix} \beta_1 & \widetilde{\begin{bmatrix} j+2 \\ j \end{bmatrix}}_q \beta_2 & \dots & \widetilde{\begin{bmatrix} n \\ j \end{bmatrix}}_q \beta_{n-j} \\ \frac{1}{\widetilde{\begin{bmatrix} j+1 \\ j \end{bmatrix}}_q} \beta_0 & \widetilde{\begin{bmatrix} j+2 \\ j+1 \end{bmatrix}}_q \beta_1 & \dots & \widetilde{\begin{bmatrix} n \\ j+1 \end{bmatrix}}_q \beta_{n-j-1} \\ 0 & \beta_0 & \dots & \dots \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \beta_0 & \widetilde{\begin{bmatrix} n \\ n-1 \end{bmatrix}}_q \beta_1 \end{vmatrix}$$

$$= \frac{(-1)^{n+j}}{(\beta_0)^{n-j+1}} \widetilde{\begin{bmatrix} j+1 \\ j \end{bmatrix}}_q \widetilde{\begin{bmatrix} j+2 \\ j \end{bmatrix}}_q \begin{vmatrix} \beta_1 & \beta_2 & \dots & \widetilde{\begin{bmatrix} n \\ j \end{bmatrix}}_q \beta_{n-j} \\ \frac{1}{\widetilde{\begin{bmatrix} j+1 \\ j \end{bmatrix}}_q} \beta_0 & \frac{\widetilde{\begin{bmatrix} j+2 \\ j+1 \end{bmatrix}}_q \beta_1}{\widetilde{\begin{bmatrix} j+2 \\ j \end{bmatrix}}_q} & \dots & \widetilde{\begin{bmatrix} n \\ j+1 \end{bmatrix}}_q \beta_{n-j-1} \\ 0 & \frac{1}{\widetilde{\begin{bmatrix} j+2 \\ j \end{bmatrix}}_q} \beta_0 & \dots & \dots \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \beta_0 & \widetilde{\begin{bmatrix} n \\ n-1 \end{bmatrix}}_q \beta_1 \end{vmatrix}$$

$$= \frac{(-1)^{n+j}}{(\beta_0)^{n-j+1}} \widetilde{\begin{bmatrix} j+1 \\ j \end{bmatrix}}_q \dots \widetilde{\begin{bmatrix} n \\ j \end{bmatrix}}_q \begin{vmatrix} \beta_1 & \beta_2 & \dots & \beta_{n-j} \\ \frac{1}{\widetilde{\begin{bmatrix} j+1 \\ j \end{bmatrix}}_q} \beta_0 & \frac{\widetilde{\begin{bmatrix} j+2 \\ j+1 \end{bmatrix}}_q \beta_1}{\widetilde{\begin{bmatrix} j+2 \\ j \end{bmatrix}}_q} & \dots & \frac{\widetilde{\begin{bmatrix} n \\ j+1 \end{bmatrix}}_q \beta_{n-j-1}}{\widetilde{\begin{bmatrix} n \\ j \end{bmatrix}}_q} \\ 0 & \frac{1}{\widetilde{\begin{bmatrix} j+2 \\ j \end{bmatrix}}_q} \beta_0 & \dots & \dots \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \frac{1}{\widetilde{\begin{bmatrix} n-1 \\ j \end{bmatrix}}_q} \beta_0 & \frac{\widetilde{\begin{bmatrix} n \\ n-1 \end{bmatrix}}_q}{\widetilde{\begin{bmatrix} n \\ j \end{bmatrix}}_q} \beta_1 \end{vmatrix}.$$

To obtain coefficient 1 for the term β_0 placed in the second row, we multiply this raw by $\widetilde{\begin{bmatrix} j+1 \\ j \end{bmatrix}}_q$. Using the fact that

$$\frac{\widetilde{\begin{bmatrix} j+2 \\ j+1 \end{bmatrix}}_q}{\widetilde{\begin{bmatrix} j+2 \\ j \end{bmatrix}}_q} \widetilde{\begin{bmatrix} j+1 \\ j \end{bmatrix}}_q = \widetilde{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_q$$

and

$$\frac{\widetilde{\begin{bmatrix} n \\ j+1 \end{bmatrix}}_q}{\widetilde{\begin{bmatrix} n \\ j \end{bmatrix}}_q}\widetilde{\begin{bmatrix} j+1 \\ j \end{bmatrix}}_q = \widetilde{\begin{bmatrix} n-j \\ 1 \end{bmatrix}}_q,$$

we get

$$= \frac{(-1)^{n+j}}{(\beta_0)^{n-j+1}} \frac{\widetilde{\begin{bmatrix} j+1 \\ j \end{bmatrix}}_q \cdots \widetilde{\begin{bmatrix} n \\ j \end{bmatrix}}_q}{\widetilde{\begin{bmatrix} j+1 \\ j \end{bmatrix}}_q} \times$$

$$\begin{vmatrix} \beta_1 & \beta_2 & \cdots & \beta_{n-j-1} & \beta_{n-j} \\ \beta_0 & \widetilde{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_q \beta_1 & \cdots & \widetilde{\begin{bmatrix} n-j-1 \\ 1 \end{bmatrix}}_q \beta_{n-j-2} & \widetilde{\begin{bmatrix} n-j \\ 1 \end{bmatrix}}_q \beta_{n-j-1} \\ 0 & \frac{1}{\widetilde{\begin{bmatrix} j+2 \\ j \end{bmatrix}}_q}\beta_0 & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \frac{1}{\widetilde{\begin{bmatrix} n-1 \\ j \end{bmatrix}}_q}\beta_0 & \frac{\widetilde{\begin{bmatrix} n \\ n-1 \end{bmatrix}}_q}{\widetilde{\begin{bmatrix} n \\ j \end{bmatrix}}_q}\beta_1 \end{vmatrix}.$$

We continue this method for each row. At the end we obtain

$$= \frac{(-1)^{n-j}}{(\beta_0)^{n-j+1}} \frac{\widetilde{\begin{bmatrix} j+1 \\ j \end{bmatrix}}_q \cdots \widetilde{\begin{bmatrix} n \\ j \end{bmatrix}}_q}{\widetilde{\begin{bmatrix} j+1 \\ j \end{bmatrix}}_q \cdots \widetilde{\begin{bmatrix} n-1 \\ j \end{bmatrix}}_q} \times$$

$$\times \begin{vmatrix} \beta_1 & \beta_2 & \cdots & \beta_{n-j-1} & \beta_{n-j} \\ \beta_0 & \widetilde{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_q \beta_1 & \cdots & \widetilde{\begin{bmatrix} n-j-1 \\ 1 \end{bmatrix}}_q \beta_{n-j-2} & \widetilde{\begin{bmatrix} n-j \\ 1 \end{bmatrix}}_q \beta_{n-j-1} \\ 0 & \beta_0 & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \beta_0 & \widetilde{\begin{bmatrix} n \\ n-1 \end{bmatrix}}_q \beta_1 \end{vmatrix} = \widetilde{\begin{bmatrix} n \\ j \end{bmatrix}}_q \alpha_{n-j}$$

.

whence the result. □

Corollary 3.4. *The q-polynomials $P_{n,q}(x)$ satisfy*

$$P_{n,q}(x) = \sum_{j=0}^{n} \widetilde{\begin{bmatrix} n \\ j \end{bmatrix}}_q P_{n-j,q}(0)x^j; \quad n = 0, 1, 2, ... \tag{31}$$

Proof. According to the definition 24, for $j = 0, 1, ..., n$, $P_{j,q}(0) = \alpha_j$, since

$$P_{j,q}(0) = \frac{(-1)^j}{(\beta_0)^{j+1}} \begin{vmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ \beta_0 & \beta_1 & \beta_2 & \dots & \beta_{j-1} & \beta_j \\ 0 & \beta_0 & \widetilde{\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q} \beta_1 & \dots & \widetilde{\begin{bmatrix} j-1 \\ 1 \end{bmatrix}_q} \beta_{j-2} & \widetilde{\begin{bmatrix} j \\ 1 \end{bmatrix}_q} \beta_{j-1} \\ 0 & 0 & \beta_0 & \dots & \widetilde{\begin{bmatrix} j-1 \\ 2 \end{bmatrix}_q} \beta_{j-3} & \widetilde{\begin{bmatrix} j \\ 2 \end{bmatrix}_q} \beta_{j-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \beta_0 & \widetilde{\begin{bmatrix} j \\ j-1 \end{bmatrix}_q} \beta_1 \end{vmatrix}$$

$$= \frac{(-1)^j}{(\beta_0)^{j+1}} \begin{vmatrix} \beta_1 & \beta_2 & \dots & \beta_{j-1} & \beta_j \\ \beta_0 & \widetilde{\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q} \beta_1 & \dots & \widetilde{\begin{bmatrix} j-1 \\ 1 \end{bmatrix}_q} \beta_{j-2} & \widetilde{\begin{bmatrix} j \\ 1 \end{bmatrix}_q} \beta_{j-1} \\ 0 & \beta_0 & \dots & \widetilde{\begin{bmatrix} j-1 \\ 2 \end{bmatrix}_q} \beta_{j-3} & \widetilde{\begin{bmatrix} j \\ 2 \end{bmatrix}_q} \beta_{j-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \beta_0 & \widetilde{\begin{bmatrix} j \\ j-1 \end{bmatrix}_q} \beta_1 \end{vmatrix} = \alpha_j.$$

Replacing $P_{n-j,q}(0)$, instead of α_{n-j} in relation 29, gives the desired result. □

Corollary 3.5. *The following relations hold for α_j's in relation 29*

$$\begin{cases} \alpha_0 = \frac{1}{\beta_0} \\ \alpha_j = -\frac{1}{\beta_0} \sum_{i=0}^{j-1} \widetilde{\begin{bmatrix} j \\ i \end{bmatrix}_q} \beta_{j-i} \alpha_i \qquad j = 1; 2; ...; n \end{cases} \tag{32}$$

Proof. The proof is done by expanding α_j, defined in relation 30, along with the first row and also applying a similar technique to the proof of theorem 3.3. □

Theorem 3.6. *Suppose that $\widetilde{A}_{n,q}(x)$ be the sequence of symmetric q-Appell polynomials with generating function $\widetilde{A}_q(t)$, defined in the relations 22 and 23. If $B_{0,q}, B_{1,q}, ..., B_{n,q}$, with $B_{0,q} \neq 0$ are the coefficients of q-Taylor series expansion of the function $\frac{1}{\widetilde{A}_q(t)}$, then for $n = 0, 1, ...$ we have*

$$\begin{cases} \widetilde{A}_{0,q}(x) = \frac{1}{B_{0,q}} \\ \widetilde{A}_{n,q} = \frac{(-1)^n}{(B_{0,q})^{n+1}} \begin{vmatrix} 1 & x & x^2 & \dots & \dots & x^{n-1} & x^n \\ B_{0,q} & B_{1,q} & B_{2,q} & \dots & \dots & B_{n-1,q} & B_{n,q} \\ 0 & B_{0,q} & \widetilde{\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q} B_{1,q} & \dots & \dots & \widetilde{\begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q} B_{n-2,q} & \widetilde{\begin{bmatrix} n \\ 1 \end{bmatrix}_q} B_{n-1,q} \\ 0 & 0 & B_{0,q} & \dots & \dots & \widetilde{\begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q} B_{n-3,q} & \widetilde{\begin{bmatrix} n \\ 2 \end{bmatrix}_q} B_{n-2,q} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \dots & B_{0,q} & \widetilde{\begin{bmatrix} n \\ n-1 \end{bmatrix}_q} B_{1,q} \end{vmatrix}, \end{cases} \tag{33}$$

Proof. Using 22 and 23, we obtain

$$\widetilde{A}_q(t)=\sum_{n=0}^{\infty}\widetilde{A}_{n,q}\frac{t^n}{\widetilde{[n]}_q!}=\widetilde{A}_{0,q}+\widetilde{A}_{1,q}t+A_{2,q}\frac{t^2}{\widetilde{[2]}_q!}+...+\widetilde{A}_{n,q}\frac{t^n}{\widetilde{[n]}_q!}+..., \tag{34}$$

and

$$\widetilde{A}_q(t)\widetilde{e}_q(tx)=\sum_{n=0}^{\infty}\widetilde{A}_{n,q}(x)\frac{t^n}{\widetilde{[n]}_q!}=\widetilde{A}_{0,q}(x)+\widetilde{A}_{1,q}(x)t+\widetilde{A}_{2,q}(x)\frac{t^2}{\widetilde{[2]}_q!}+...+\widetilde{A}_{n,q}(x)\frac{t^n}{\widetilde{[n]}_q!}+... \tag{35}$$

Let $B_q(t)=\frac{1}{\widetilde{A}_q(t)}$. Thus, taking in account the hypothesis of the theorem and the definition of q-Taylor series expansion of $B_q(t)$ at $a=0$ we get

$$B_q(t)=B_{0,q}+B_{1,q}\frac{t}{\widetilde{[1]}_q!}+B_{2,q}\frac{t^2}{\widetilde{[2]}_q!}+...+B_{n,q}\frac{t^n}{\widetilde{[n]}_q!}+..., \tag{36}$$

Cauchy product rule for the series production $\widetilde{A}_q(t)B_q(t)$ gives

$$\begin{aligned}1 &= \widetilde{A}_q(t)B_q(t)\\ &= \sum_{n=0}^{\infty}\widetilde{A}_{n,q}\frac{t^n}{\widetilde{[n]}_q!}\sum_{n=0}^{\infty}B_{n,q}\frac{t^n}{\widetilde{[n]}_q!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\widetilde{\begin{bmatrix} n\\ k\end{bmatrix}}_q\widetilde{A}_{k,q}B_{n-k,q}\frac{t^n}{\widetilde{[n]}_q!}.\end{aligned}$$

then,

$$\sum_{k=0}^{\infty}\widetilde{\begin{bmatrix} n\\ k\end{bmatrix}}_q\widetilde{A}_{k,q}B_{n-k,q}=\begin{cases}1 & for\ \ n=0,\\ 0 & for\ \ n>0.\end{cases}$$

which is equivalent to

$$\begin{cases} B_{0,q}=\frac{1}{\widetilde{A}_{0,q}}\\ B_{n,q}=-\frac{1}{\widetilde{A}_{0,q}}\left(\sum_{k=1}^{\infty}\widetilde{\begin{bmatrix} n\\ k\end{bmatrix}}_q\widetilde{A}_{k,q}B_{n-k,q}\right),\quad n=1,2,3,...\end{cases} \tag{37}$$

By multiplying both sides of identity 35 by $B_q(t)=\frac{1}{\widetilde{A}_q(t)}$, and replacing $\widetilde{e}_q(tx)$ by its q-Taylor series expansion, i. e. $\sum\limits_{n=0}^{\infty}x^n\frac{t^n}{\widetilde{[n]}_q!}$. We obtain

$$\begin{aligned}\sum_{n=0}^{\infty}x^n\frac{t^n}{\widetilde{[n]}_q!} &= \widetilde{e}_q(tx)\\ &= B_q(t)\sum_{n=0}^{\infty}\widetilde{A}_{n,q}(x)\frac{t^n}{\widetilde{[n]}_q!}\\ &= \sum_{n=0}^{\infty}B_{n,q}\frac{t^n}{\widetilde{[n]}_q!}\sum_{n=0}^{\infty}\widetilde{A}_{n,q}(x)\frac{t^n}{\widetilde{[n]}_q!}.\end{aligned}$$

Cauchy product rule in the last part of relation above leads to

$$\sum_{n=0}^{\infty}x^n\frac{t^n}{\widetilde{[n]}_q!}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\widetilde{\begin{bmatrix} n\\ k\end{bmatrix}}_q B_{n-k,q}\widetilde{A}_{k,q}(x)\frac{t^n}{\widetilde{[n]}_q!}. \tag{38}$$

Comparing the coefficients of $\frac{t^n}{\widetilde{[n]_q!}}$ in both sides of equation 38, we have

$$\sum_{k=0}^{\infty} \widetilde{\begin{bmatrix} n \\ k \end{bmatrix}}_q B_{n-k,q}\widetilde{A}_{k,q}(x) = x^n, \quad n = 0, 1, 2, ... \tag{39}$$

Writing identity 39 for $n = 0, 1, 2, ...$ leads to obtain the following infinite system in the parameter $\widetilde{A}_{k,q}(x)$

$$\begin{cases} B_{0,q}\widetilde{A}_{0,q}(x) = 1 \\ B_{1,q}\widetilde{A}_{0,q}(x) + B_{0,q}\widetilde{A}_{1,q}(x) = x \\ B_{2,q}\widetilde{A}_{0,q}(x) + \widetilde{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_q B_{1,q}\widetilde{A}_{1,q}(x) + B_{0,q}\widetilde{A}_{2,q}(x) = x^2 \\ \vdots \\ B_{n,q}\widetilde{A}_{0,q}(x) + \widetilde{\begin{bmatrix} n \\ 1 \end{bmatrix}}_q B_{n-1,q}\widetilde{A}_{1,q}(x) + ... + B_{0,q}\widetilde{A}_{n,q}(x) = x^n, \\ \vdots \end{cases} \tag{40}$$

The coefficient matrix of the infinite system 40 is lower triangular. By applying Cramer rule to only the first $n+1$ equations of this system. We obtain

$$\widetilde{A}_{n,q}(x) = \frac{\begin{vmatrix} B_{0,q} & 0 & 0 & \dots & 0 & 1 \\ B_{1,q} & B_{0,q} & 0 & \dots & 0 & x \\ B_{2,q} & \widetilde{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_q B_{1,q} & B_{0,q} & \dots & 0 & x^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{n-1,q} & \widetilde{\begin{bmatrix} n-1 \\ 1 \end{bmatrix}}_q B_{n-2,q} & \dots & \dots & B_{0,q} & x^{n-1} \\ B_{n,q} & \widetilde{\begin{bmatrix} n \\ 1 \end{bmatrix}}_q B_{n-1,q} & \dots & \dots & \widetilde{\begin{bmatrix} n-1 \\ 1 \end{bmatrix}}_q B_{1,q} & x^n \end{vmatrix}}{\begin{vmatrix} B_{0,q} & 0 & 0 & \dots & 0 & 0 \\ B_{1,q} & B_{0,q} & 0 & \dots & 0 & 0 \\ B_{2,q} & \widetilde{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_q B_{1,q} & B_{0,q} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{n-1,q} & \widetilde{\begin{bmatrix} n-1 \\ 1 \end{bmatrix}}_q B_{n-2,q} & \dots & \dots & B_{0,q} & 0 \\ B_{n,q} & \widetilde{\begin{bmatrix} n \\ 1 \end{bmatrix}}_q B_{n-1,q} & \dots & \dots & \widetilde{\begin{bmatrix} n-1 \\ 1 \end{bmatrix}}_q B_{1,q} & B_{0,q} \end{vmatrix}}$$

$$= \frac{1}{(B_{0,q})^{n+1}} \begin{vmatrix} B_{0,q} & 0 & 0 & \cdots & 0 & 1 \\ B_{1,q} & B_{0,q} & 0 & \cdots & 0 & x \\ B_{2,q} & \widetilde{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_q B_{1,q} & B_{0,q} & \cdots & 0 & x^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{n-1,q} & \widetilde{\begin{bmatrix} n-1 \\ 1 \end{bmatrix}}_q B_{n-2,q} & \cdots & \cdots & B_{0,q} & x^{n-1} \\ B_{n,q} & \widetilde{\begin{bmatrix} n \\ 1 \end{bmatrix}}_q B_{n-1,q} & \cdots & \cdots & \widetilde{\begin{bmatrix} n-1 \\ 1 \end{bmatrix}}_q B_{1,q} & x^n \end{vmatrix}.$$

By taking the transpose of the last determinant and then interchange i^{th} row of the obtained determinant with $(i+1)^{th}$ row, $i = 1, 2, ..., n$. We obtain the desired result that is exactly relation 33. □

Theorem 3.7. *The following facts are equivalent for the symmetric q-Appell polynomials:*
a) Symmetric q-Appell polynomials can be expressed by considering the relations 20 and 21.
b) Symmetric q-Appell polynomials can be expressed by considering the relations 22 and 23.
c) Symmetric q-Appell polynomials can be expressed by considering the determinantal relation 33.

Proof. ($a \Rightarrow b$) Suppose that relations 20 and 21 hold. Construct an infinite series $\sum_{n=0}^{\infty} \widetilde{A}_{n,q} \frac{t^n}{\widetilde{[n]_q!}}$ from all constants $\widetilde{A}_{n,q}$ used for defining $\widetilde{A}_{n,q}(x)$ in relation 21.
Now find the following Cauchy product

$$\begin{aligned} \sum_{n=0}^{\infty} \widetilde{A}_{n,q} \frac{t^n}{\widetilde{[n]_q!}} \widetilde{e}_q(tx) &= \sum_{n=0}^{\infty} \widetilde{A}_{n,q} \frac{t^n}{\widetilde{[n]_q!}} \sum_{n=0}^{\infty} x^n \frac{t^n}{\widetilde{[n]_q!}} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \widetilde{A}_{n-k,q} x^k \frac{t^n}{\widetilde{[n]_q!}}. \end{aligned}$$

From relation 21 we have

$$\sum_{k=0}^{\infty} \widetilde{A}_{n-k,q} x^k = \widetilde{A}_{n,q}(x),$$

then we find that

$$\sum_{n=0}^{\infty} \widetilde{A}_{n,q} \frac{t^n}{\widetilde{[n]_q!}} \widetilde{e}_q(tx) = \widetilde{A}_q(x,t),$$

whence the result.
($b \Rightarrow c$) The proof follows directly from Theorem 3.6.
($c \Rightarrow a$) The proof follows from Theorems 3.1 and 3.6. □

As the consequence of discussion above and particularly Theorem 3.7, we introduce the determinantal definition of symmetric q-Appell polynomials as follows

Definition 3.8. *Symmetric q-Appell polynomials* $(\widetilde{A}_{n,q}(x))_{n=0}^{\infty}$ *can be defined as*

$$\begin{cases} \widetilde{A}_{0,q}(x) = \frac{1}{B_{0,q}} \\ \widetilde{A}_{n,q}(x) = \frac{(-1)^n}{(B_{0,q})^{n+1}} \begin{vmatrix} 1 & x & x^2 & \cdots & \cdots & x^{n-1} & x^n \\ B_{0,q} & B_{1,q} & B_{2,q} & \cdots & \cdots & B_{n-1,q} & B_{n,q} \\ 0 & B_{0,q} & \widetilde{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_q B_{1,q} & \cdots & \cdots & \widetilde{\begin{bmatrix} n-1 \\ 1 \end{bmatrix}}_q B_{n-2,q} & \widetilde{\begin{bmatrix} n \\ 1 \end{bmatrix}}_q B_{n-1,q} \\ 0 & 0 & B_{0,q} & \cdots & \cdots & \widetilde{\begin{bmatrix} n-1 \\ 2 \end{bmatrix}}_q B_{n-3,q} & \widetilde{\begin{bmatrix} n \\ 2 \end{bmatrix}}_q B_{n-2,q} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & B_{0,q} & \widetilde{\begin{bmatrix} n \\ n-1 \end{bmatrix}}_q B_{1,q} \end{vmatrix} \end{cases} \tag{41}$$

where $B_{0,q}, B_{1,q}, B_{2,q}, ..., B_{n,q} \in \mathbb{R}$, $B_{0,q} \neq 0$ *and* $n = 1, 2, 3, ...$

4. Basic Properties of Symmetric q-Appell polynomials from determinantal point of view

In this section by using Definition 3.8, we review the basic properties of symmetric q-Appell polynomials.

Theorem 4.1. *Let* $(\widetilde{A}_{n,q}(x))_{n=0}^{\infty}$ *be a sequence of symmetric q-Appel polynomials, then*

$$A_{n,q}(x) = \frac{1}{B_{0,q}}(x^n - \sum_{k=0}^{n-1} \widetilde{\begin{bmatrix} n \\ k \end{bmatrix}}_q B_{n-k,q} A_{k,q}(x)), \quad n = 1, 2, 3, ... \tag{42}$$

Proof. By expanding the determinant in the Definition 3.8 along with the $(n+1)^{th}$ row, we obtain

$$A_{n,q}(x) = \frac{(-1)^n}{(B_{0,q})^{n+1}} \widetilde{\begin{bmatrix} n \\ n-1 \end{bmatrix}}_q B_{1,q} \times \begin{vmatrix} 1 & x & x^2 & \cdots & \cdots & x^{n-1} \\ B_{0,q} & B_{1,q} & B_{2,q} & \cdots & \cdots & B_{n-1,q} \\ 0 & B_{0,q} & \widetilde{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_q B_{1,q} & \cdots & \cdots & \widetilde{\begin{bmatrix} n-1 \\ 1 \end{bmatrix}}_q B_{n-2,q} \\ 0 & 0 & B_{0,q} & \cdots & \cdots & \widetilde{\begin{bmatrix} n-1 \\ 2 \end{bmatrix}}_q B_{n-3,q} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B_{0,q} & \widetilde{\begin{bmatrix} n-1 \\ n-2 \end{bmatrix}}_q B_{1,q} \end{vmatrix}$$

$$+ \frac{(-1)^n}{(B_{0,q})^{n+1}} B_{0,q} \times \begin{vmatrix} 1 & x & x^2 & \cdots & \cdots & x^{n-2} & x^n \\ B_{0,q} & B_{1,q} & B_{2,q} & \cdots & \cdots & B_{n-2,q} & B_{n,q} \\ 0 & B_{0,q} & \widetilde{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_q B_{1,q} & \cdots & \cdots & \widetilde{\begin{bmatrix} n-2 \\ 1 \end{bmatrix}}_q B_{n-3,q} & \widetilde{\begin{bmatrix} n \\ 1 \end{bmatrix}}_q B_{n-1,q} \\ 0 & 0 & B_{0,q} & \cdots & \cdots & \widetilde{\begin{bmatrix} n-2 \\ 2 \end{bmatrix}}_q B_{n-4,q} & \widetilde{\begin{bmatrix} n \\ 2 \end{bmatrix}}_q B_{n-2,q} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & B_{0,q} & \widetilde{\begin{bmatrix} n-1 \\ n-2 \end{bmatrix}}_q B_{2,q} \end{vmatrix}$$

$$= -\frac{1}{B_{0,q}}\widetilde{\begin{bmatrix} n \\ n-1 \end{bmatrix}}_q B_{1,q}A_{n-1,q}(x) + \frac{(-1)^{n+1}}{(B_{0,q})^n}\times$$

$$\begin{vmatrix} 1 & x & x^2 & \dots & \dots & x^{n-2} & x^n \\ B_{0,q} & B_{1,q} & B_{2,q} & \dots & \dots & B_{n-2,q} & B_{n,q} \\ 0 & B_{0,q} & \widetilde{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_q B_{1,q} & \dots & \dots & \widetilde{\begin{bmatrix} n-2 \\ 1 \end{bmatrix}}_q B_{n-3,q} & \widetilde{\begin{bmatrix} n \\ 1 \end{bmatrix}}_q B_{n-1,q} \\ 0 & 0 & B_{0,q} & \dots & \dots & \widetilde{\begin{bmatrix} n-2 \\ 2 \end{bmatrix}}_q B_{n-4,q} & \widetilde{\begin{bmatrix} n \\ 2 \end{bmatrix}}_q B_{n-2,q} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \dots & B_{0,q} & \widetilde{\begin{bmatrix} n-1 \\ n-2 \end{bmatrix}}_q B_{2,q} \end{vmatrix}$$

Repeating the same method for the last determinant

$$= -\frac{1}{B_{0,q}}\widetilde{\begin{bmatrix} n \\ n-1 \end{bmatrix}}_q B_{1,q}A_{n-1,q}(x) + \frac{(-1)^{n+1}}{(B_{0,q})^n}\widetilde{\begin{bmatrix} n-1 \\ n-2 \end{bmatrix}}_q B_{2,q}\times$$

$$\begin{vmatrix} 1 & x & x^2 & \dots & \dots & x^{n-3} & x^{n-2} \\ B_{0,q} & B_{1,q} & B_{2,q} & \dots & \dots & B_{n-3,q} & B_{n-2,q} \\ 0 & B_{0,q} & \widetilde{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_q B_{1,q} & \dots & \dots & \widetilde{\begin{bmatrix} n-3 \\ 1 \end{bmatrix}}_q B_{n-4,q} & \widetilde{\begin{bmatrix} n-2 \\ 1 \end{bmatrix}}_q B_{n-3,q} \\ 0 & 0 & B_{0,q} & \dots & \dots & \widetilde{\begin{bmatrix} n-3 \\ 2 \end{bmatrix}}_q B_{n-5,q} & \widetilde{\begin{bmatrix} n-2 \\ 2 \end{bmatrix}}_q B_{n-4,q} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \dots & B_{0,q} & \widetilde{\begin{bmatrix} n-2 \\ n-3 \end{bmatrix}}_q B_{1,q} \end{vmatrix}$$

$$= -\frac{1}{B_{0,q}}\widetilde{\begin{bmatrix} n \\ n-1 \end{bmatrix}}_q B_{1,q}A_{n-1,q}(x) + \frac{(-1)^{n+1}}{(B_{0,q})^n}\left(\widetilde{\begin{bmatrix} n-1 \\ n-2 \end{bmatrix}}_q B_{2,q}\frac{(B_{0,q})^{n-1}}{(-1)^{n-2}}A_{n-2,q}(x)\right)$$

$$+\frac{(-1)^{n-2}}{(B_{0,q})^{n-1}}\times\begin{vmatrix} 1 & x & x^2 & \dots & \dots & x^{n-3} & x^n \\ B_{0,q} & B_{1,q} & B_{2,q} & \dots & \dots & B_{n-3,q} & B_{n,q} \\ 0 & B_{0,q} & \widetilde{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_q B_{1,q} & \dots & \dots & \widetilde{\begin{bmatrix} n-3 \\ 1 \end{bmatrix}}_q B_{n-4,q} & \widetilde{\begin{bmatrix} n \\ 1 \end{bmatrix}}_q B_{n-1,q} \\ 0 & 0 & B_{0,q} & \dots & \dots & \widetilde{\begin{bmatrix} n-3 \\ 2 \end{bmatrix}}_q B_{n-5,q} & \widetilde{\begin{bmatrix} n \\ 2 \end{bmatrix}}_q B_{n-2,q} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \dots & B_{0,q} & \widetilde{\begin{bmatrix} n-1 \\ n-2 \end{bmatrix}}_q B_{2,q} \end{vmatrix}$$

$$= -\frac{1}{B_{0,q}}\widetilde{\begin{bmatrix} n \\ n-1 \end{bmatrix}}_q B_{1,q}A_{n-1,q}(x) - \frac{1}{B_{0,q}}\widetilde{\begin{bmatrix} n-1 \\ n-2 \end{bmatrix}}_q B_{2,q}A_{n-2,q}(x)$$

$$+\frac{(-1)^{n-2}}{(B_{0,q})^{n-1}}\times\begin{vmatrix} 1 & x & x^2 & \dots & \dots & x^{n-3} & x^n \\ B_{0,q} & B_{1,q} & \widetilde{B_{2,q}} & \dots & \dots & \widetilde{B_{n-3,q}} & \widetilde{B_{n,q}} \\ 0 & B_{0,q} & \widetilde{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_q B_{1,q} & \dots & \dots & \widetilde{\begin{bmatrix} n-3 \\ 1 \end{bmatrix}}_q B_{n-4,q} & \widetilde{\begin{bmatrix} n \\ 1 \end{bmatrix}}_q B_{n-1,q} \\ 0 & 0 & B_{0,q} & \dots & \dots & \widetilde{\begin{bmatrix} n-3 \\ 2 \end{bmatrix}}_q B_{n-5,q} & \widetilde{\begin{bmatrix} n \\ 2 \end{bmatrix}}_q B_{n-2,q} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \dots & B_{0,q} & \widetilde{\begin{bmatrix} n-1 \\ n-2 \end{bmatrix}}_q B_{2,q} \end{vmatrix}$$

Similar method gives

$$= -\frac{1}{B_{0,q}}\widetilde{\begin{bmatrix} n \\ n-1 \end{bmatrix}}_q B_{1,q}A_{n-1,q}(x) - \frac{1}{B_{0,q}}\widetilde{\begin{bmatrix} n-1 \\ n-2 \end{bmatrix}}_q B_{2,q}A_{n-2,q}(x) - \dots - \frac{1}{(B_{0,q})^2}\begin{vmatrix} 1 & x^n \\ B_{0,q} & B_{n,q} \end{vmatrix}$$

$$= -\frac{1}{B_{0,q}}\widetilde{\begin{bmatrix} n \\ n-1 \end{bmatrix}}_q B_{1,q}A_{n-1,q}(x) - \frac{1}{B_{0,q}}\widetilde{\begin{bmatrix} n-1 \\ n-2 \end{bmatrix}}_q B_{2,q}A_{n-2,q}(x) - \dots - \frac{1}{(B_{0,q})^2}(B_{n,q} - B_{0,q}x^n)$$

$$= -\frac{1}{B_{0,q}}\widetilde{\begin{bmatrix} n \\ n-1 \end{bmatrix}}_q B_{1,q}A_{n-1,q}(x) - \frac{1}{B_{0,q}}\widetilde{\begin{bmatrix} n-1 \\ n-2 \end{bmatrix}}_q B_{2,q}A_{n-2,q}(x) - \dots - \frac{1}{B_{0,q}}B_{n,q}A_{0,q}(x) + \frac{1}{B_{0,q}}x^n$$

$$= \frac{1}{B_{0,q}}\left(x^n - \sum_{k=0}^{n-1}\widetilde{\begin{bmatrix} n \\ k \end{bmatrix}}_q B_{n-k,q}A_{k,q}(x)\right). \quad \square$$

Corollary 4.2. *Powers of x can be expressed based on symmetric q-Appell polynomials as*

$$x^n = \sum_{k=0}^{n}\widetilde{\begin{bmatrix} n \\ k \end{bmatrix}}_q B_{n-k,q}A_{k,q}(x), \quad n = 1, 2, 3, \dots$$

Proof. The proof is the direct result of relation 42 in Theorem 4.1. □

Notation 4.3. *Let $P_n(x)$ and $Q_n(x)$ be two polynomials of degree n with $P_n(x)$ defined as in relation 24. Then for $n = 1, 2, 3, \dots$, we have*

$$(PQ)(x) = \frac{(-1)^n}{(\beta_0)^{n+1}}\begin{vmatrix} Q_0(x) & Q_1(x) & Q_2(x) & \dots & Q_{n-1}(x) & Q_n(x) \\ \beta_0 & \beta_1 & \widetilde{\beta_2} & \dots & \widetilde{\beta_{j-1}} & \widetilde{\beta_j} \\ 0 & \beta_0 & \widetilde{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_q \beta_1 & \dots & \widetilde{\begin{bmatrix} n-1 \\ n \end{bmatrix}}_q \beta_{n-2} & \widetilde{\begin{bmatrix} n \\ 1 \end{bmatrix}}_q \beta_{n-1} \\ 0 & 0 & \beta_0 & \dots & \widetilde{\begin{bmatrix} n-1 \\ 2 \end{bmatrix}}_q \beta_{n-3} & \widetilde{\begin{bmatrix} n \\ 2 \end{bmatrix}}_q \beta_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \beta_0 & \widetilde{\begin{bmatrix} n \\ n-1 \end{bmatrix}}_q \beta_1 \end{vmatrix}. \tag{43}$$

Theorem 4.4. *Let $(\widetilde{A}_{n,q}(x))_{n=0}^{\infty}$ and $(\widehat{A}_{n,q}(x))_{n=0}^{\infty}$ be two families of symmetric q-Appell polynomials. Then*
a) For every α and β ∈ ℝ, $(\alpha\widetilde{A}_{n,q}(x) + \beta\widehat{A}_{n,q}(x))_{n=0}^{\infty}$ is also a family of symmetric q-Appell polynomials.
b) $((\widetilde{A}\widehat{A})_{n,q}(x))_{n=0}^{\infty}$ is also a family of symmetric q-Appell polynomials.

Proof. a) The proof is the direct consequence of linear properties of determinant.
b) According to the determinantal definition of symmetric q-Appell polynomials given in Theorem 3.6

relation 33 and also notation 43, we have

$$(\widetilde{A}\widehat{A})_{n,q}(x) = \widetilde{A}_{n,q}(\widehat{A}_{n,q}(x))$$

$$= \frac{(-1)^n}{(B_{0,q})^{n+1}} \begin{vmatrix} \widehat{A}_{0,q}(x) & \widehat{A}_{1,q}(x) & \widehat{A}_{2,q}(x) & \dots & \dots & \widehat{A}_{n-1,q}(x) & \widehat{A}_{n,q}(x) \\ B_{0,q} & B_{1,q} & B_{2,q} & \dots & \dots & B_{n-1,q} & B_{n,q} \\ 0 & B_{0,q} & \widetilde{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_q B_{1,q} & \dots & \dots & \widetilde{\begin{bmatrix} n-1 \\ 1 \end{bmatrix}}_q B_{n-2,q} & \widetilde{\begin{bmatrix} n \\ 1 \end{bmatrix}}_q B_{n-1,q} \\ 0 & 0 & B_{0,q} & \dots & \dots & \widetilde{\begin{bmatrix} n-1 \\ 2 \end{bmatrix}}_q B_{n-3,q} & \widetilde{\begin{bmatrix} n \\ 2 \end{bmatrix}}_q B_{n-2,q} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \dots & B_{0,q} & \widetilde{\begin{bmatrix} n \\ n-1 \end{bmatrix}}_q B_{1,q} \end{vmatrix}.$$

Using formula 25 given in Lemma 3.2 we have

$$\widetilde{D}_q((\widetilde{A}\widehat{A})_{n,q}(x)) = \frac{(-1)^n}{(B_{0,q})^{n+1}} \times$$

$$\begin{vmatrix} \widetilde{D}_q(\widehat{A}_{0,q}(x)) & \widetilde{D}_q(\widehat{A}_{1,q}(x)) & \widetilde{D}_q(\widehat{A}_{2,q}(x)) & \dots & \dots & \widetilde{D}_q(\widehat{A}_{n-1,q}(x)) & \widetilde{D}_q(\widehat{A}_{n,q}(x)) \\ B_{0,q} & B_{1,q} & B_{2,q} & \dots & \dots & B_{n-1,q} & B_{n,q} \\ 0 & B_{0,q} & \widetilde{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_q B_{1,q} & \dots & \dots & \widetilde{\begin{bmatrix} n-1 \\ 1 \end{bmatrix}}_q B_{n-2,q} & \widetilde{\begin{bmatrix} n \\ 1 \end{bmatrix}}_q B_{n-1,q} \\ 0 & 0 & B_{0,q} & \dots & \dots & \widetilde{\begin{bmatrix} n-1 \\ 2 \end{bmatrix}}_q B_{n-3,q} & \widetilde{\begin{bmatrix} n \\ 2 \end{bmatrix}}_q B_{n-2,q} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \dots & B_{0,q} & \widetilde{\begin{bmatrix} n \\ n-1 \end{bmatrix}}_q B_{1,q} \end{vmatrix}.$$

Since $(\widehat{A}_{n,q}(x))_{n=0}^{\infty}$ is a family of symmetric q-Appell polynomials, according to relation 20 we have

$$\widetilde{D}_q(\widehat{A}_{n,q}(x)) = \widetilde{[n]}_q \widehat{A}_{n-1,q}(x); n = 0, 1, 2, ...$$

Therefore we can continue as $\widetilde{D}_q((A\widehat{A})_{n,q}(x)) = \frac{(-1)^n}{(B_{0,q})^{n+1}} \times$

$$\begin{vmatrix} 0 & \widehat{A}_{0,q}(x) & \widetilde{[2]}_q \widehat{A}_{1,q}(x) & \dots & \dots & \widetilde{[n-1]}_q \widehat{A}_{n-2,q}(x) & \widetilde{[n]}_q \widehat{A}_{n-1,q}(x) \\ B_{0,q} & B_{1,q} & B_{2,q} & \dots & \dots & B_{n-1,q} & B_{n,q} \\ 0 & B_{0,q} & \widetilde{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_q B_{1,q} & \dots & \dots & \widetilde{\begin{bmatrix} n-1 \\ 1 \end{bmatrix}}_q B_{n-2,q} & \widetilde{\begin{bmatrix} n \\ 1 \end{bmatrix}}_q B_{n-1,q} \\ 0 & 0 & B_{0,q} & \dots & \dots & \widetilde{\begin{bmatrix} n-1 \\ 2 \end{bmatrix}}_q B_{n-3,q} & \widetilde{\begin{bmatrix} n \\ 2 \end{bmatrix}}_q B_{n-2,q} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \dots & B_{0,q} & \widetilde{\begin{bmatrix} n \\ n-1 \end{bmatrix}}_q B_{1,q} \end{vmatrix}.$$

By expanding the last determinant along with the first column as follows

$$= \frac{(-1)^n}{(B_{0,q})^{n+1}}(-B_{0,q}) \begin{vmatrix} \widehat{A}_{0,q}(x) & \widetilde{[2]}_q\widehat{A}_{1,q}(x) & \dots & \dots & \widetilde{[n-1]}_q\widehat{A}_{n-2,q}(x) & \widetilde{[n]}_q\widehat{A}_{n-1,q}(x) \\ B_{1,q} & B_{2,q} & \dots & \dots & B_{n-1,q} & B_{n,q} \\ B_{0,q} & \widetilde{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_q B_{1,q} & \dots & \dots & \widetilde{\begin{bmatrix} n-1 \\ 1 \end{bmatrix}}_q B_{n-2,q} & \widetilde{\begin{bmatrix} n \\ 1 \end{bmatrix}}_q B_{n-1,q} \\ 0 & B_{0,q} & \dots & \dots & \widetilde{\begin{bmatrix} n-1 \\ 2 \end{bmatrix}}_q B_{n-3,q} & \widetilde{\begin{bmatrix} n \\ 2 \end{bmatrix}}_q B_{n-2,q} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & B_{0,q} & \widetilde{\begin{bmatrix} n \\ n-1 \end{bmatrix}}_q B_{1,q} \end{vmatrix}.$$

$= \widetilde{[n]}_q(\widetilde{A}\widehat{A})_{n-1,q}(x)$.

which means that $((\widetilde{A}\widehat{A})_{n,q}(x))_{n=0}^{\infty}$ belongs to the family of symmetric q-Appell polynomials too. □

Theorem 4.5. *For Symmetric q-Appell polynomials $A_{n,q}(x)$ we have*

$$A_{n,q}(x+y) = \sum_{i=0}^{n} \widetilde{\begin{bmatrix} n \\ i \end{bmatrix}}_q A_{i,q}(x) y^{n-i}; n = 0, 1, ... \tag{44}$$

Proof. Using the definition in 24 and the identity

$$\widetilde{(x+y)}_q^n = \sum_{k=0}^{n} \widetilde{\begin{bmatrix} n \\ k \end{bmatrix}}_q x^{n-k} y^k,$$

we obtain

$$A_{n,q}(x+y) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} 1 & (x+y)^1(& (x+y)^2 & \dots & (x+y)^{n-1} & (x+y)^n \\ \beta_0 & \beta_1 & \beta_2 & \dots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \widetilde{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_q \beta_1 & \dots & \widetilde{\begin{bmatrix} n-1 \\ 1 \end{bmatrix}}_q \beta_{n-2} & \widetilde{\begin{bmatrix} n \\ 1 \end{bmatrix}}_q \beta_{n-1} \\ 0 & 0 & \beta_0 & \dots & \widetilde{\begin{bmatrix} n-1 \\ 2 \end{bmatrix}}_q \beta_{n-3} & \widetilde{\begin{bmatrix} n \\ 2 \end{bmatrix}}_q \beta_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \beta_0 & \widetilde{\begin{bmatrix} n \\ n-1 \end{bmatrix}}_q \beta_1 \end{vmatrix}$$

$= \sum_{i=0}^{n} y^i \frac{(-1)^n}{(\beta_0)^{n+1}} \times$

$$\begin{vmatrix}
0 & 0 & \dots & 0 & \widetilde{\begin{bmatrix} i \\ i \end{bmatrix}}_q & \dots & \widetilde{\begin{bmatrix} n-1 \\ i \end{bmatrix}}_q x^{n-1-i} & \widetilde{\begin{bmatrix} n \\ i \end{bmatrix}}_q x^{n-i} \\
\beta_0 & \beta_1 & \dots & \beta_{i-1} & \beta_i & \dots & \beta_{n-1} & \beta_n \\
0 & \beta_0 & \dots & \beta_{i-2}\widetilde{\begin{bmatrix} i-1 \\ 1 \end{bmatrix}}_q & \widetilde{\begin{bmatrix} i \\ 1 \end{bmatrix}}_q \beta_{i-1} & \dots & \widetilde{\begin{bmatrix} n-1 \\ 1 \end{bmatrix}}_q \beta_{n-2} & \widetilde{\begin{bmatrix} n \\ 1 \end{bmatrix}}_q \beta_{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\dots & \dots & \dots & \beta_0 & \widetilde{\begin{bmatrix} i \\ i-1 \end{bmatrix}}_q \beta_1 & \dots & \dots & \widetilde{\begin{bmatrix} n \\ i-1 \end{bmatrix}}_q \beta_{n-i+1} \\
\dots & \dots & \dots & \dots & \beta_0 & \dots & \dots & \widetilde{\begin{bmatrix} n \\ i \end{bmatrix}}_q \beta_{n-i} \\
0 & \dots & \dots & \dots & \dots & \dots & \beta_0 & \widetilde{\begin{bmatrix} n \\ n-1 \end{bmatrix}}_q \beta_1
\end{vmatrix}$$

$$= \sum_{i=0}^{n} y^i \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix}
\widetilde{\begin{bmatrix} i \\ i \end{bmatrix}}_q & \widetilde{\begin{bmatrix} i+1 \\ i \end{bmatrix}}_q x^1 & \widetilde{\begin{bmatrix} i+2 \\ i \end{bmatrix}}_q x^2 & \dots & \widetilde{\begin{bmatrix} n-1 \\ i \end{bmatrix}}_q x^{n-1-i} & \widetilde{\begin{bmatrix} n \\ i \end{bmatrix}}_q x^{n-i} \\
\beta_0 & \widetilde{\begin{bmatrix} i+1 \\ i \end{bmatrix}}_q \beta_1 & \widetilde{\begin{bmatrix} i+2 \\ i \end{bmatrix}}_q \beta_2 & \dots & \widetilde{\begin{bmatrix} n-1 \\ i \end{bmatrix}}_q \beta_{n-i-1} & \widetilde{\begin{bmatrix} n \\ i \end{bmatrix}}_q \beta_{n-i} \\
0 & \beta_0 & \widetilde{\begin{bmatrix} i+2 \\ i+1 \end{bmatrix}}_q \beta_1 & \dots & \widetilde{\begin{bmatrix} n-1 \\ i+1 \end{bmatrix}}_q \beta_{n-i-2} & \widetilde{\begin{bmatrix} n \\ i+1 \end{bmatrix}}_q \beta_{n-i-1} \\
\vdots & \vdots & \dots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \beta_0 & \ddots & \vdots & \vdots \\
0 & \dots & \dots & 0 & \beta_0 & \widetilde{\begin{bmatrix} n \\ n-1 \end{bmatrix}}_q \beta_1
\end{vmatrix}.$$

Dividing the j^{th} column by $\widetilde{\begin{bmatrix} i+j-1 \\ i \end{bmatrix}}_q$ for $j = 2, ..., n-i+1$ and multiplying the h^{th} row by $\widetilde{\begin{bmatrix} i+h-2 \\ i \end{bmatrix}}_q$ for $h = 3, ..., n-i+1$,
we obtain

$$A_n(x+y) = \sum_{i=0}^{n} \frac{\widetilde{\begin{bmatrix} i+1 \\ i \end{bmatrix}}_q \cdots \widetilde{\begin{bmatrix} n \\ i \end{bmatrix}}_q}{\widetilde{\begin{bmatrix} i+1 \\ i \end{bmatrix}}_q \cdots \widetilde{\begin{bmatrix} n-1 \\ i \end{bmatrix}}_q} \frac{(-1)^{n-i}}{(\beta_0)^{n-i+1}} \times$$

$$\begin{vmatrix}
1 & x^1 & x^2 & \dots & x^{n-1-i} & x^{n-i} \\
\beta_0 & \beta_1 & \beta_2 & \dots & \beta_{n-i-1} & \beta_{n-i} \\
0 & \beta_0 & \widetilde{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_q \beta_1 & \dots & \widetilde{\begin{bmatrix} n-i-1 \\ 1 \end{bmatrix}}_q \beta_{n-i-2} & \widetilde{\begin{bmatrix} n-i \\ 1 \end{bmatrix}}_q \beta_{n-i-1} \\
\vdots & \vdots & \dots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \beta_0 & \ddots & \vdots & \vdots \\
0 & \dots & \dots & 0 & \beta_0 & \widetilde{\begin{bmatrix} n-i \\ n-i-1 \end{bmatrix}}_q \beta_1
\end{vmatrix}$$

$$= \sum_{i=0}^{n} \widetilde{\begin{bmatrix} n \\ i \end{bmatrix}}_q A_{n-i}(x) y^i = \sum_{i=0}^{n} \widetilde{\begin{bmatrix} n \\ i \end{bmatrix}}_q A_i(x) y^{n-i}.$$

□

Corollary 4.6. *(Forward Difference). For Symmetric q-Appell polynomials $A_n(x)$ we have*

$$\Delta A_n(x) \equiv A_n(x+1) - A_n(x) = \sum_{i=0}^{n-1} \widetilde{\begin{bmatrix} n \\ i \end{bmatrix}}_q A_i(x), \quad n = 0, 1, ... \tag{45}$$

Proof. The desired result follows from 44 with $y = 1$. □

Corollary 4.7. *(Multiplication Theorem). For Symmetric q-Appell polynomials $A_n(x)$ we have*

$$A_n(mx) = \sum_{i=0}^{n-1} \widetilde{\begin{bmatrix} n \\ i \end{bmatrix}}_q A_i(x)(m-1)^{n-i} x^{n-i}, \quad n = 0, 1, ... \text{ and } m = 1, 2, ... \tag{46}$$

Proof. The desired result follows from 44 with $y = x(m-1)$. □

Theorem 4.8. *(Symmetry). For Symmetric q-Appell polynomials $A_n(x)$ the following relation holds*

$$(A_n(h-x) = (-1)^n A_n(x)) \Leftrightarrow (A_n(h) = (-1)^n A_n(0)) \quad n = 0, 1, ... \text{ and } h \in \mathbb{R}. \tag{47}$$

Proof. (⇒) Follows from the hypothesis with $x = 0$.
(⇐) Using 44 we find

$$\begin{aligned} A_n(h-x) &= \sum_{i=0}^{n-1} \widetilde{\begin{bmatrix} n \\ i \end{bmatrix}}_q A_i(h)(-x)^{n-i} \\ &= (-1)^n \sum_{i=0}^{n-1} \widetilde{\begin{bmatrix} n \\ i \end{bmatrix}}_q A_i(h)(-1)^i x^{n-i} \\ &= (-1)^n \sum_{i=0}^{n-1} \widetilde{\begin{bmatrix} n \\ i \end{bmatrix}}_q A_{n-i}(h)(-1)^{n-i} x^i. \end{aligned}$$

Therefore, using the assumptions and 31, we have

$$\begin{aligned} A_n(h-x) &= (-1)^n \sum_{i=0}^{n-1} \widetilde{\begin{bmatrix} n \\ i \end{bmatrix}}_q A_{n-i}(0) x^i \\ &= (-1)^n A_n(x). \end{aligned}$$

□

Lemma 4.9. *For the numbers α_{2n+1} and β_{2n+1} we have*

$$(\alpha_{2n+1} = 0) \Leftrightarrow (\beta_{2n+1} = 0) \quad n = 0, 1, ... \tag{48}$$

Proof. As in 32, we know that

$$\begin{cases} \beta_0 = \frac{1}{\alpha_0} \\ \beta_n = -\frac{1}{\beta_0} \sum_{k=1}^{n} \widetilde{\begin{bmatrix} n \\ k \end{bmatrix}}_q \alpha_k \beta_{n-k} \quad n = 1; 2; ... \end{cases}$$

Hence

$$\begin{cases} \beta_1 = -\frac{1}{\alpha_0} \alpha_1 \beta_0 \\ \beta_{2n+1} = -\frac{1}{\alpha_0} \widetilde{\begin{bmatrix} 2n+1 \\ 1 \end{bmatrix}}_q \alpha_1 \beta_{2n} - \frac{1}{\beta_0} \sum_{k=1}^{n} \left[\widetilde{\begin{bmatrix} 2n+1 \\ 2k \end{bmatrix}}_q \alpha_{2k} \beta_{2(n-k)+1} + \widetilde{\begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}}_q \alpha_{2k+1} \beta_{2(n-k)} \right] \\ n = 1; 2; ... \end{cases}$$

and
$\alpha_{2n+1} = 0, \quad n = 0;1;...$

$$\Rightarrow \begin{cases} \beta_1 = 0 \\ \beta_{2n+1} = -\frac{1}{\beta_0}\sum_{k=1}^{n} \left[\begin{array}{c} \widetilde{2n+1} \\ 2k \end{array}\right]_q \alpha_{2k}\beta_{2(n-k)+1} \quad n = 1;2;... \end{cases}$$

$$\Rightarrow \beta_{2n+1} = 0 \quad n = 0;1;...$$

In the same way, again from 32, we have

$$\begin{cases} \alpha_0 = \frac{1}{\beta_0} \\ \alpha_n = -\frac{1}{\beta_0}\sum_{k=0}^{n-1} \left[\begin{array}{c} \widetilde{n} \\ k \end{array}\right]_q \alpha_k\beta_{n-k} \quad n = 1;2;... \end{cases}$$

As a consequence

$$\begin{cases} \alpha_1 = -\frac{1}{\beta_0}\alpha_0\beta_1 \\ \alpha_{2n+1} = -\frac{1}{\beta_0}\sum_{k=0}^{n-1}\left[\left[\begin{array}{c} \widetilde{2n+1} \\ 2k \end{array}\right]_q \beta_{2k}\alpha_{2(n-k)+1} + \left[\begin{array}{c} \widetilde{2n+1} \\ 2k+1 \end{array}\right]_q \alpha_{2k+1}\beta_{2(n-k)}\right] - \frac{1}{\beta_0}\left[\begin{array}{c} \widetilde{2n+1} \\ 2n \end{array}\right]_q \beta_1\alpha_{2n} \\ n = 1;2;... \end{cases}$$

and
$\beta_{2n+1} = 0, \quad n = 0;1;...$

$$\Rightarrow \begin{cases} \alpha_1 = 0 \\ \alpha_{2n+1} = -\frac{1}{\beta_0}\sum_{k=0}^{n-1} \left[\begin{array}{c} \widetilde{2n+1} \\ 2k+1 \end{array}\right]_q \alpha_{2k+1}\beta_{2(n-k)} \quad n = 1;2;... \end{cases}$$

$$\Rightarrow \alpha_{2n+1} = 0 \quad n = 0;1;...$$

□

Theorem 4.10. *For Symmetric q-Appell polynomials $A_n(x)$ the following relation holds*

$$(A_n(-x) = (-1)^n A_n(x)) \Leftrightarrow (\beta_{2n+1} = 0). \tag{49}$$

Proof. By Theorem 4.8 with $h = 0$ and Lemma 4.9, we find

$$(A_n(-x) = (-1)^n A_n(x)) \Leftrightarrow (A_n(0) = (-1)^n A_n(0)) \Leftrightarrow (A_{2n+1}(0) = 0) \Leftrightarrow (\alpha_{2n+1} = 0) \Leftrightarrow (\beta_{2n+1} = 0)$$

$n = 0;1;...$ □

Theorem 4.11. *For each $n \geq 1$ it is true that*

$$\int_0^x A_n(t)d_{\widetilde{q}}t = \frac{1}{n+1}[A_{n+1}(x) - A_{n+1}(0)] \tag{50}$$

and

$$\int_0^1 A_n(t)d_{\widetilde{q}}t = \frac{1}{n+1}\sum_{i=0}^{n} \left[\begin{array}{c} \widetilde{n+1} \\ i \end{array}\right]_q A_i(0). \tag{51}$$

Proof. Equality 50 follows from 20. Moreover, for $x = 1$ we find

$$\int_0^1 A_n(t)d_{\widetilde{q}}t = \frac{1}{n+1}[A_{n+1}(1) - A_{n+1}(0)] \tag{52}$$

and, using 44 with $x = 0$ and $y = 1$, we obtain

$$A_{n+1}(1) = \sum_{i=0}^{n+1} \left[\begin{array}{c} \widetilde{n+1} \\ i \end{array} \right]_q A_i(0), \tag{53}$$

so, by 53, relation 52 becomes

$$\begin{aligned} \int_0^1 A_n(t)d_{\widetilde{q}}t &= \frac{1}{n+1}\left[\sum_{i=0}^{n+1} \left[\begin{array}{c} \widetilde{n+1} \\ i \end{array} \right]_q A_i(0) - A_{n+1}(0)\right] \\ &= \frac{1}{n+1}\sum_{i=0}^{n} \left[\begin{array}{c} \widetilde{n+1} \\ i \end{array} \right]_q A_i(0). \end{aligned}$$

□

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