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Inner-Φ-GCEP and Φ-GCEP-inner inverses

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Abstract. Inspired by the concepts of Φ-GCEP and Φ-*GCEP inverses as generalizations of the gMP, *gMP and MP inverse, the aim of this paper is to extend notions of inner-gMP, gMP-inner, inner-*gMP, *gMP-inner, 1MP and MP1 inverses using Φ-GCEP and Φ-*GCEP inverses. Thus, four new types of generalized inverses are introduced and called inner-Φ-GCEP, Φ-GCEP-inner, inner-Φ-*GCEP and Φ-*GCEP-inner inverses. As particular kinds of these new inverses, inner-GCEP, GCEP-inner, inner-*GCEP and *GCEP-inner inverses are established. Numerous properties, representations and characterizations of new inverses are presented as well. Applying our new inverses, we solve some linear equations and express their solutions.

1. Introduction

Through this paper, H, K and G are arbitrary Hilbert spaces, $\mathcal{B}(H,K)$ is the set of all bounded linear operators from H to K, and $\mathcal{B}(H) = \mathcal{B}(H,H)$. For $A \in \mathcal{B}(H,K)$, its null space, range and adjoint, respectively, are represented as $N(A)$, $R(A)$ and A^*. Denote by $P_{U,V}$ the projector onto U along V, where U and V are closed subspaces, and by P_U the orthogonal projector onto U.

Generalized inverses, firstly applied for solving various optimization and approximation problems, are very useful tools in singular differential and difference equations, Markov chain theory and so on [1]. For an operator $A \in \mathcal{B}(H,K)$, its Moore–Penrose inverse, denoted by $A^\dagger$, is unique if it exists and satisfies the following equations [1]: $AXA = A$, $XAX = X$, $(AX)^* = AX$, $(XA)^* = XA$. When only $AXA = A$ holds, A is regular and X is an inner inverse (or {1}-inverse) of A. The operator A is regular if and only if its Moore–Penrose inverse exists if and only if $R(A)$ is closed in K. The set of all inner inverses of A will be denoted by $A\{1\}$, and the set of all regular operators of $\mathcal{B}(H,K)$ by $\mathcal{B}(H,K)^-$. If T and S are closed subspaces of H and K, respectively, the outer inverse of A, denoted by $A^{(2)}_{T,S}$, is a unique (if it exists) solution to $XAX = X$, $R(X) = T$ and $N(X) = S$ [1]. The notation $A^{(1,2)}_{T,S}$ represents $A^{(2)}_{T,S}$ such that $AA^{(2)}_{T,S}A = A$. Denote by $\mathcal{B}(H,K)_{T,S}$ the set of all $A \in \mathcal{B}(H,K)$ such that $A^{(2)}_{T,S}$ exists. The symbol $\mathcal{B}(H,K)^-_{T,S}$ marks the subset of regular operators from $\mathcal{B}(H,K)_{T,S}$.

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The generalized Drazin inverse of $A \in \mathcal{B}(H)$, denoted by A^d [5], is a unique solution (if it exists) to $XAX = X$, $AX = XA$, $A - A^2X$ is quasinilpotent. We use $\mathcal{B}(H)^d$ to denote the set of all generalized Drazin invertible operators of $\mathcal{B}(H)$. The core–EP inverse of $A \in \mathcal{B}(H)^d$ [3, 6, 10], denoted by $A^{\textcircled{\dagger}}$, is a unique solution to the system of equations $XAX = X$, $R(X) = R(X^*) = R(A^d)$. Dually, there exists a unique *core–EP inverse $A_{\textcircled{\dagger}}$ of an operator $A \in \mathcal{B}(H)^d$, which satisfies the equations $XAX = X$, $R(X) = R(X^*) = R((A^d)^*)$.

The generalized Moore-Penrose inverse was presented for a generalized Drazin invertible operator in [16] in order to generalize the notion of the Moore-Penrose inverse for an operator with closed range. Precisely, the generalized Moore–Penorse (or gMP) inverse of $A \in \mathcal{B}(H)^d$ is unique solution to

$$XAX = X, \qquad AX = A(A^{\textcircled{\dagger}}A)^{\dagger}A^{\textcircled{\dagger}} \qquad \text{and} \qquad XA = (A^{\textcircled{\dagger}}A)^{\dagger}A^{\textcircled{\dagger}}A,$$

and it is given by

$$A^{\diamond} = (A^{\textcircled{\dagger}}A)^{\dagger}A^{\textcircled{\dagger}}.$$

The dual gMP (or *gMP) inverse of A [16] is expressed as

$$A_{\diamond} = A_{\textcircled{\dagger}}(AA_{\textcircled{\dagger}})^{\dagger}.$$

Notice that, for $A \in \mathcal{B}(H)^{\#}$, both gMP and *gMP inverses of A are equal to the Moore–Penrose inverse $A^{\dagger}$ [16]. Recent results about the gMP inverse were proposed in [2, 9, 16, 17].

To extend the concept of the core–EP inverse, a new generalized inverse was introduced in [11] for rectangular matrices and is called the generalized core–EP inverse. The definition of the generalized core–EP inverse is stated now for corresponding bounded linear operators. Let $A \in \mathcal{B}(H,K)_{T,S}$. The generalized core–EP (or GCEP) inverse of A is defined as a uniquely determined solution to

$$XAX = X, \qquad XA = A^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger}A \qquad \text{and} \qquad AX = AA^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger},$$

which is represented by

$$A^{\textcircled{2}}_{T,S} := A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}.$$

The generalized *core–EP (or *GCEP) inverse of A is defined by

$$A_{\textcircled{2}}^{T,S} := \left(A^{(2)}_{T,S}A\right)^{\dagger}A^{(2)}_{T,S}.$$

If $A^{(2)}_{T,S} = A^{\dagger}$, both GCEP and *GCEP inverses are equal to the Moore-Penrose inverse of A. For $H = K$ and $A^{(2)}_{T,S} = A^d$, by [8], recall that, the GCEP and *GCEP inverses reduces to the core–EP and *core–EP inverses of A, respectively. In the case that $H = K$ and $A^{(2)}_{T,S} = A_{\textcircled{\dagger}}$ (or $A^{(2)}_{T,S} = A^{\textcircled{\dagger}}$), the GCEP (or *GCEP) becomes the *gMP (or gMP) inverse of A.

The Φ-GCEP inverse of A was presented in [12] for rectangular matrices applying a matrix Φ in the definition of the GCEP inverse instead of $A^{(2)}_{T,S}$. For $A \in \mathcal{B}(H,K)$ and $\Phi \in \mathcal{B}(G,H)$ such that $A\Phi$ is regular, the Φ-GCEP inverse of A can be defined as

$$A^{\textcircled{2},\Phi} = \Phi(A\Phi)^{\dagger}.$$

For $A \in \mathcal{B}(H,K)$ and $\Phi \in \mathcal{B}(K,G)$ such that ΦA is regular, the Φ-*GCEP inverse of A can be represented by

$$A_{\textcircled{2},\Phi} = (\Phi A)^{\dagger}\Phi.$$

In the case that $A \in \mathcal{B}(H,K)_{T,S}$ and $\Phi = A^{(2)}_{T,S}$, the Φ-GCEP (or Φ-*GCEP) inverse is equal to the GCEP (or *GCEP) inverse of A.

Composing the inner and Moore–Penrose inverses in [4], the 1MP and MP1 inverses were defined for rectangular complex matrices. Let $A \in \mathcal{B}(H,K)$ be regular and $A^- \in A\{1\}$ be arbitrary but fixed. Then the 1MP inverse of A is given by

$$A^{-,\dagger} = A^-AA^{\dagger},$$

and the MP1 inverse of A is represented by

$$A^{\dagger,-} = A^{\dagger}AA^{-}.$$

Significant results about 1MP and MP1 inverses can be found in [4, 7, 13].

The concepts of the inner-gMP and gMP-inner inverses were presented in [15] as generalizations of the 1MP and MP1 inverses, respectively. Let $A \in \mathcal{B}(H)^{d,-}$ and $A^{-} \in A\{1\}$ be arbitrary but fixed. Then the inner-gMP inverse of A is a unique solution to the system

$$XAX = X, \qquad AX = AA^{\diamond} \qquad \text{and} \qquad XA = A^{-}AA^{\diamond}A, \tag{1}$$

which is represented by

$$A^{-,\diamond} = A^{-}AA^{\diamond}.$$

The gMP-inner inverse of A is expressed as

$$A^{\diamond,-} = A^{\diamond}AA^{-}.$$

Note that, if $A \in \mathcal{B}(H)^{\#}$, inner-gMP and gMP-inner inverses coincide with 1MP and MP1 inverses, respectively. Dually, the inner-*gMP and *gMP-inner inverses are defined by $A_{-,\diamond} = A^{-}AA_{\diamond}$ and $A_{\diamond,-} = A_{\diamond}AA^{-}$, respectively, in [15].

Motivated by the Φ-GCEP inverse, GCEP inverses and their duals as generalizations of the Moore–Penrose inverse, we introduce new generalized inverses for a bounded linear regular operator on Hilbert spaces, extending concepts of inner-gMP, gMP-inner, inner-*gMP, *gMP-inner, 1MP and MP1 inverses. By associating the Φ-GCEP with the inner inverse, we define both inner-Φ-GCEP and Φ-GCEP-inner inverses of that operator. Precisely, by replacing the Moore–Penorse inverse with a Φ-GCEP inverse in the definition of a 1MP (MP1) inverse, we obtain its inner-Φ-GCEP (Φ-GCEP-inner) inverse. As special cases of the inner-Φ-GCEP and Φ-GCEP-inverses, we establish the definitions of the inner-GCEP and GCEP-inner inverses. We present many characterizations and representations of new generalized inverses. Finally, substituting the Φ-GCEP inverse with the Φ-*GCEP inverse, we give definitions for the inner-Φ-*GCEP and Φ-*GCEP-inner inverses. Applying our new inverses, we solve several kinds of linear equations, and one of them is an extension of normal equation related to the least–squares solution.

The paper is organized in the following way. Inner-Φ-GCEP and Φ-GCEP-inner inverses, as well as inner-GCEP and GCEP-inner inverses, are defined and investigated in Section 2. In Section 3, the definitions and properties of inner-Φ-*GCEP, Φ-*GCEP-inner, inner-*GCEP and *GCEP-inner inverses are presented. Solvability of linear equations based on our new generalized inverses is part of Section 4.

2. Inner-Φ-GCEP and Φ-GCEP-inner inverses

Associating inner and Φ-GCEP inverses of a given operator, we present new classes of generalized inverses and generalized notions of the inner-*gMP and *gMP-inner inverses as well as of the 1MP and MP1 inverses.

Theorem 2.1. *Let $A \in \mathcal{B}(H,K)^{-}$ and $\Phi \in \mathcal{B}(G,H)$ such that $A\Phi$ is regular and $A^{-} \in A\{1\}$ is arbitrary but fixed. Then*

(a) *the system*

$$XAX = X, \qquad AX = AA^{②,\Phi} \qquad \text{and} \qquad XA = A^{-}AA^{②,\Phi}A \tag{2}$$

has a uniquely determined solution expressed by $X = A^{-}AA^{②,\Phi}$,

(b) *the system*

$$XAX = X, \qquad AX = AA^{②,\Phi}AA^{-} \qquad \text{and} \qquad XA = A^{②,\Phi}A$$

has a uniquely determined solution expressed by $X = A^{②,\Phi}AA^{-}$.

Proof. (a) Note that $A^{\textcircled{2},\Phi}AA^{\textcircled{2},\Phi} = \Phi(A\Phi)^{\dagger}A\Phi(A\Phi)^{\dagger} = \Phi(A\Phi)^{\dagger} = A^{\textcircled{2},\Phi}$. If $X = A^{-}AA^{\textcircled{2},\Phi}$, we have $XA = A^{-}AA^{\textcircled{2},\Phi}A$ and $AX = (AA^{-}A)A^{\textcircled{2},\Phi} = AA^{\textcircled{2},\Phi}$, which yields $X(AX) = (XA)A^{\textcircled{2},\Phi} = A^{-}A(A^{\textcircled{2},\Phi}AA^{\textcircled{2},\Phi}) = A^{-}AA^{\textcircled{2},\Phi} = X$. It follows that $X = A^{-}AA^{\textcircled{2},\Phi}$ is a solution of (2).

Assume that X is a solution of (2). From

$$X = (XA)X = A^{-}AA^{\textcircled{2},\Phi}(AX) = A^{-}A(A^{\textcircled{2},\Phi}AA^{\textcircled{2},\Phi}) = A^{-}AA^{\textcircled{2},\Phi},$$

we obtain that the system (2) has unique solution $X = A^{-}AA^{\textcircled{2},\Phi}$.

The part (b) can be checked similarly as part (a). □

Definition 2.2. *Let $A \in \mathcal{B}(H,K)^{-}$ and $\Phi \in \mathcal{B}(G,H)$ such that $A\Phi$ is regular and $A^{-} \in A\{1\}$ is arbitrary but fixed.*

(a) *The inner-Φ-GCEP inverse of A is defined by*

$$A^{-,\textcircled{2},\Phi} = A^{-}AA^{\textcircled{2},\Phi}.$$

(b) *The Φ-GCEP-inner inverse of A is defined by*

$$A^{\textcircled{2},\Phi,-} = A^{\textcircled{2},\Phi}AA^{-}.$$

By the definition of Φ-GCEP inverse, remark that

$$A^{-,\textcircled{2},\Phi} = A^{-}A\Phi(A\Phi)^{\dagger} \quad \text{and} \quad A^{\textcircled{2},\Phi,-} = \Phi(A\Phi)^{\dagger}AA^{-}.$$

As a consequence of Theorem 2.1, we can define the inner-GCEP and GCEP-inner inverses which are particular kinds of inner-Φ-GCEP and Φ-GCEP-inner inverses.

Corollary 2.3. *Let $A \in \mathcal{B}(H,K)^{-}_{T,S}$ and $A^{-} \in A\{1\}$ be arbitrary but fixed. Then*

(a) *the system*

$$XAX = X, \qquad AX = AA^{\textcircled{2}}_{T,S} \qquad \text{and} \qquad XA = A^{-}AA^{\textcircled{2}}_{T,S}A$$

has a uniquely determined solution expressed by $X = A^{-}AA^{\textcircled{2}}_{T,S}$,

(b) *the system*

$$XAX = X, \qquad AX = AA^{\textcircled{2}}_{T,S}AA^{-} \qquad \text{and} \qquad XA = A^{\textcircled{2}}_{T,S}A$$

has a uniquely determined solution expressed by $X = A^{\textcircled{2}}_{T,S}AA^{-}$.

Definition 2.4. *Let $A \in \mathcal{B}(H,K)^{-}_{T,S}$ and $A^{-} \in A\{1\}$ be arbitrary but fixed.*

(a) *The inner-GCEP inverse of A is defined by*

$$A^{-,\textcircled{2}}_{T,S} = A^{-}AA^{\textcircled{2}}_{T,S}.$$

(b) *The GCEP-inner inverse of A is defined by*

$$A^{\textcircled{2},-}_{T,S} = A^{\textcircled{2}}_{T,S}AA^{-}.$$

A list of special cases of inner-GCEP and GCEP-inner inverses follows:

- when $A^{(2)}_{T,S} = A^{\dagger}$, $A^{-,\textcircled{2}}_{T,S} = A^{-,\dagger}$ and $A^{\textcircled{2},-}_{T,S} = A^{\dagger,-}$;
- for $H = K$ and $A^{(2)}_{T,S} = A_{\textcircled{d}}$, $A^{\textcircled{2}}_{T,S} = A_{\diamond}$, $A^{-,\textcircled{2}}_{T,S} = A_{-,\diamond}$ and $A^{\textcircled{2},-}_{T,S} = A_{\diamond,-}$;
- if $H = K$ and $A^{(2)}_{T,S} = A^{d}$, $A^{\textcircled{2}}_{T,S} = A^{\textcircled{d}}$, $A^{-,\textcircled{2}}_{T,S} = A^{-}AA^{\textcircled{d}}$ and $A^{\textcircled{2},-}_{T,S} = A^{\textcircled{d}}AA^{-}$ [14];

Some representations for inner-GCEP and GCEP-inner inverses follow by [11, Corollary 2.1 and Corollary 2.3].

Corollary 2.5. *Let $A \in \mathcal{B}(H,K)^{-}_{T,S}$ and $A^{-} \in A\{1\}$ be arbitrary but fixed. Then*

$$A^{-,②}_{T,S} = A^{-}AA^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger} = A^{-}AA^{(2)}_{T,S}P_{R(AA^{(2)}_{T,S})} = A^{-}A(AA^{(2)}_{T,S}(A^{(2)}_{T,S})^{\dagger})^{\dagger}$$

and

$$A^{②,-}_{T,S} = A^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger}AA^{-} = A^{(2)}_{T,S}P_{R(AA^{(2)}_{T,S})}AA^{-} = (AA^{(2)}_{T,S}(A^{(2)}_{T,S})^{\dagger})^{\dagger}AA^{-}.$$

Example 2.6. *For complex matrices*

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \Phi = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

we have

$$A\Phi = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad (A\Phi)^{\dagger} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A^{②,\Phi} = \Phi(A\Phi)^{\dagger} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A^{-} = \begin{bmatrix} 1-2a & -2b & s \\ a & b & c \\ 0 & 1 & f \end{bmatrix},$$

where $a,b,c,f,s \in \mathbb{C}$ are arbitrary. Therefore,

$$A^{-,②,\Phi} = A^{-}AA^{②,\Phi} = \begin{bmatrix} 1-2a & 0 & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$A^{②,\Phi,-} = A^{②,\Phi}AA^{-} = \begin{bmatrix} \frac{1}{3} & 0 & \frac{s+2c}{3} \\ \frac{1}{3} & 0 & \frac{s+2c}{3} \\ 0 & 0 & 0 \end{bmatrix}.$$

Theorem 2.1 yields the following properties for inner-Φ-GCEP inverses.

Theorem 2.7. *Let $A \in \mathcal{B}(H,K)^{-}$ and $\Phi \in \mathcal{B}(G,H)$ be such that $A\Phi$ is regular and $A^{-} \in A\{1\}$ is arbitrary but fixed. For $X \in \mathcal{B}(K,H)$, the following statements are equivalent:*

(i) $X = A^{-,②,\Phi}$;

(ii) $XAX = X$, $AXA = AA^{②,\Phi}A$, $AX = AA^{②,\Phi}$ *and* $XA = A^{-}AA^{②,\Phi}A$;

(iii) $XA = A^{-}AA^{②,\Phi}A$ *and* $XAA^{②,\Phi} = X$;

(iv) $XA = A^{-}AA^{②,\Phi}A$ *and* $XA\Phi(A\Phi)^{\dagger} = X$;

(v) $XAA^{\dagger} = A^{-}AA^{②,\Phi}AA^{\dagger}$ *and* $XA\Phi(A\Phi)^{\dagger} = X$;

(vi) $XAA^{*} = A^{-}AA^{②,\Phi}AA^{*}$ *and* $XA\Phi(A\Phi)^{\dagger} = X$;

(vii) $XA\Phi = A^{-}A\Phi$ *and* $XA\Phi(A\Phi)^{\dagger} = X$;

(viii) $AX = AA^{②,\Phi}$ *and* $A^{-}AA^{②,\Phi}AX = X$;

(ix) $AX = AA^{②,\Phi}$ *and* $A^{-}AX = X$;

(x) $A^\dagger AX = A^\dagger AA^{②,\Phi}$ *and* $A^- AX = X$;

(xi) $A^* AX = A^* AA^{②,\Phi}$ *and* $A^- AX = X$;

(xii) $A^{②,\Phi} AX = A^{②,\Phi}$ *and* $A^- AA^{②,\Phi} AX = X$;

(xiii) $(A\Phi)^\dagger AX = (A\Phi)^\dagger$ *and* $A^- A\Phi(A\Phi)^\dagger AX = X$;

(xiv) $(A\Phi)^* AX = (A\Phi)^*$ *and* $A^- A\Phi(A\Phi)^\dagger AX = X$;

(xv) $XAA^{②,\Phi}AX = X$, $AA^{②,\Phi}AXAA^{②,\Phi}A = AA^{②,\Phi}A$, $AA^{②,\Phi}AX = AA^{②,\Phi}$ *and* $XAA^{②,\Phi}A = A^- AA^{②,\Phi}A$;

(xvi) $XAA^{②,\Phi}AX = X$, $AA^{②,\Phi}AX = AA^{②,\Phi}$ *and* $XAA^{②,\Phi}A = A^- AA^{②,\Phi}A$.

Proof. (i) ⇒ (ii): Using Theorem 2.1, this part trivially follows.

(ii) ⇒ (iii): $X = X(AX) = XAA^{②,\Phi}$ follows from $XAX = X$ and $AX = AA^{②,\Phi}$.

(iii) ⇒ (iv): By the definition of $A^{②,\Phi}$ and the assumption $XAA^{②,\Phi} = X$, we obtain $X = XAA^{②,\Phi} = XA\Phi(A\Phi)^\dagger$.

(iv) ⇒ (i): Since $XA = A^- AA^{②,\Phi}A$ and $XA\Phi(A\Phi)^\dagger = X$, we conclude that

$$X = (XA)\Phi(A\Phi)^\dagger = A^- AA^{②,\Phi}AA^{②,\Phi} = A^- AA^{②,\Phi} = A^{-,②,\Phi}.$$

We complete this proof similarly. □

Similar results hold for the inner-GCEP inverses.

Corollary 2.8. *Let* $A \in \mathcal{B}(H,K)^-_{T,S}$ *and* $A^- \in A\{1\}$ *be arbitrary but fixed. For* $X \in \mathcal{B}(K,H)$*, the following statements are equivalent:*

(i) $X = A^{-,②}_{T,S}$;

(ii) $XAX = X$, $AXA = AA^{②}_{T,S}A$, $AX = AA^{②}_{T,S}$ *and* $XA = A^- AA^{②}_{T,S}A$;

(iii) $XA = A^- AA^{②}_{T,S}A$ *and* $XAA^{②}_{T,S} = X$;

(iv) $XA = A^- AA^{②}_{T,S}A$ *and* $XAA^{(2)}_{T,S}(AA^{(2)}_{T,S})^\dagger = X$;

(v) $XAA^\dagger = A^- AA^{②}_{T,S}AA^\dagger$ *and* $XAA^{(2)}_{T,S}(AA^{(2)}_{T,S})^\dagger = X$;

(vi) $XAA^* = A^- AA^{②}_{T,S}AA^*$ *and* $XAA^{(2)}_{T,S}(AA^{(2)}_{T,S})^\dagger = X$;

(vii) $XAA^{(2)}_{T,S} = A^- AA^{(2)}_{T,S}$ *and* $XAA^{(2)}_{T,S}(AA^{(2)}_{T,S})^\dagger = X$;

(viii) $AX = AA^{②}_{T,S}$ *and* $A^- AA^{②}_{T,S}AX = X$;

(ix) $AX = AA^{②}_{T,S}$ *and* $A^- AX = X$;

(x) $A^\dagger AX = A^\dagger AA^{②}_{T,S}$ *and* $A^- AX = X$;

(xi) $A^* AX = A^* AA^{②}_{T,S}$ *and* $A^- AX = X$;

(xii) $A^{②}_{T,S}AX = A^{②}_{T,S}$ *and* $A^- AA^{②}_{T,S}AX = X$;

(xiii) $(AA^{(2)}_{T,S})^\dagger AX = (AA^{(2)}_{T,S})^\dagger$ *and* $A^- AA^{(2)}_{T,S}(AA^{(2)}_{T,S})^\dagger AX = X$;

(xiv) $(AA^{(2)}_{T,S})^* AX = (AA^{(2)}_{T,S})^*$ *and* $A^- AA^{(2)}_{T,S}(AA^{(2)}_{T,S})^\dagger AX = X$;

(xv) $XAA^{\textcircled{2}}_{T,S}AX = X$, $AA^{\textcircled{2}}_{T,S}AXAA^{\textcircled{2}}_{T,S}A = AA^{\textcircled{2}}_{T,S}A$, $AA^{\textcircled{2}}_{T,S}AX = AA^{\textcircled{2}}_{T,S}$ *and* $XAA^{\textcircled{2}}_{T,S}A = A^{-}AA^{\textcircled{2}}_{T,S}A$;

(xvi) $XAA^{\textcircled{2}}_{T,S}AX = X$, $AA^{\textcircled{2}}_{T,S}AX = AA^{\textcircled{2}}_{T,S}$ *and* $XAA^{\textcircled{2}}_{T,S}A = A^{-}AA^{\textcircled{2}}_{T,S}A$.

If $A \in \mathcal{B}(H,K)^{-}_{R(B),N(C)}$, for adequate operators B and C, and $A^{-} \in A\{1\}$ be arbitrary but fixed, notice, by Corollary 2.8(vii), that $X = A^{-,\textcircled{2}}_{R(B),N(C)}$ if and only if $XAB = A^{-}AB$ and $XAA^{(2)}_{R(B),N(C)}(AA^{(2)}_{R(B),N(C)})^{\dagger} = X$.

Analogously to Theorem 2.7, we characterize the Φ-GCEP-inner inverse.

Theorem 2.9. *Let $A \in \mathcal{B}(H,K)^{-}$ and $\Phi \in \mathcal{B}(G,H)$ be such that $A\Phi$ is regular and $A^{-} \in A\{1\}$ is arbitrary but fixed. For $X \in \mathcal{B}(K,H)$, the following statements are equivalent:*

(i) $X = A^{\textcircled{2},\Phi,-}$;

(ii) $XAX = X$, $AXA = AA^{\textcircled{2},\Phi}A$, $AX = AA^{\textcircled{2},\Phi}AA^{-}$ *and* $XA = A^{\textcircled{2},\Phi}A$;

(iii) $AX = AA^{\textcircled{2},\Phi}AA^{-}$ *and* $A^{\textcircled{2},\Phi}AX = X$;

(iv) $AX = AA^{\textcircled{2},\Phi}AA^{-}$ *and* $\Phi(A\Phi)^{\dagger}AX = X$;

(v) $A^{\dagger}AX = A^{\dagger}AA^{\textcircled{2},\Phi}AA^{-}$ *and* $\Phi(A\Phi)^{\dagger}AX = X$;

(vi) $A^{*}AX = A^{*}AA^{\textcircled{2},\Phi}AA^{-}$ *and* $\Phi(A\Phi)^{\dagger}AX = X$;

(vii) $(A\Phi)^{\dagger}AX = (A\Phi)^{\dagger}AA^{-}$ *and* $\Phi(A\Phi)^{\dagger}AX = X$;

(viii) $(A\Phi)^{*}AX = (A\Phi)^{*}AA^{-}$ *and* $\Phi(A\Phi)^{\dagger}AX = X$;

(ix) $XA = A^{\textcircled{2},\Phi}A$ *and* $X = XAA^{\textcircled{2},\Phi}AA^{-}$;

(x) $XA = A^{\textcircled{2},\Phi}A$ *and* $XAA^{-} = X$;

(xi) $XAA^{\dagger} = A^{\textcircled{2},\Phi}AA^{\dagger}$ *and* $XAA^{-} = X$;

(xii) $XAA^{*} = A^{\textcircled{2},\Phi}AA^{*}$ *and* $XAA^{-} = X$;

(xiii) $XAA^{\textcircled{2},\Phi} = A^{\textcircled{2},\Phi}$ *and* $X = XAA^{\textcircled{2},\Phi}AA^{-}$;

(xiv) $XA\Phi(A\Phi)^{\dagger} = \Phi(A\Phi)^{\dagger}$ *and* $X = XA\Phi(A\Phi)^{\dagger}AA^{-}$;

(xv) $XAA^{\textcircled{2},\Phi}AX = X$, $AA^{\textcircled{2},\Phi}AXAA^{\textcircled{2},\Phi}A = AA^{\textcircled{2},\Phi}A$, $AA^{\textcircled{2},\Phi}AX = AA^{\textcircled{2},\Phi}AA^{-}$ *and* $XAA^{\textcircled{2},\Phi}A = A^{\textcircled{2},\Phi}A$;

(xvi) $XAA^{\textcircled{2},\Phi}AX = X$, $AA^{\textcircled{2},\Phi}AX = AA^{\textcircled{2},\Phi}AA^{-}$ *and* $XAA^{\textcircled{2},\Phi}A = A^{\textcircled{2},\Phi}A$.

Corollary 2.10. *Let $A \in \mathcal{B}(H,K)^{-}_{T,S}$ and $A^{-} \in A\{1\}$ be arbitrary but fixed. For $X \in \mathcal{B}(K,H)$, the following statements are equivalent:*

(i) $X = A^{\textcircled{2},-}_{T,S}$;

(ii) $XAX = X$, $AXA = AA^{\textcircled{2}}_{T,S}A$, $AX = AA^{\textcircled{2}}_{T,S}AA^{-}$ *and* $XA = A^{\textcircled{2}}_{T,S}A$;

(iii) $AX = AA^{\textcircled{2}}_{T,S}AA^{-}$ *and* $A^{\textcircled{2}}_{T,S}AX = X$;

(iv) $AX = AA^{\textcircled{2}}_{T,S}AA^{-}$ *and* $A^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger}AX = X$;

(v) $A^{\dagger}AX = A^{\dagger}AA^{\textcircled{2}}_{T,S}AA^{-}$ *and* $A^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger}AX = X$;

(vi) $A^{*}AX = A^{*}AA^{\textcircled{2}}_{T,S}AA^{-}$ *and* $A^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger}AX = X$;

(vii) $(AA^{(2)}_{T,S})^{\dagger}AX = (AA^{(2)}_{T,S})^{\dagger}AA^{-}$ *and* $A^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger}AX = X$;

(viii) $(AA^{(2)}_{T,S})^*AX = (AA^{(2)}_{T,S})^*AA^-$ *and* $A^{(2)}_{T,S}(AA^{(2)}_{T,S})^\dagger AX = X$;

(ix) $XA = A^{②}_{T,S}A$ *and* $X = XAA^{②}_{T,S}AA^-$;

(x) $XA = A^{②}_{T,S}A$ *and* $XAA^- = X$;

(xi) $XAA^\dagger = A^{②}_{T,S}AA^\dagger$ *and* $XAA^- = X$;

(xii) $XAA^* = A^{②}_{T,S}AA^*$ *and* $XAA^- = X$;

(xiii) $XAA^{②}_{T,S} = A^{②}_{T,S}$ *and* $X = XAA^{②}_{T,S}AA^-$;

(xiv) $XAA^{(2)}_{T,S} = A^{(2)}_{T,S}$ *and* $X = XAA^{(2)}_{T,S}(AA^{(2)}_{T,S})^\dagger AA^-$;

(xv) $XAA^{②}_{T,S}AX = X$, $AA^{②}_{T,S}AXAA^{②}_{T,S}A = AA^{②}_{T,S}A$, $AA^{②}_{T,S}AX = AA^{②}_{T,S}AA^-$ *and* $XAA^{②}_{T,S}A = A^{②}_{T,S}A$;

(xvi) $XAA^{②}_{T,S}AX = X$, $AA^{②}_{T,S}AX = AA^{②}_{T,S}AA^-$ *and* $XAA^{②}_{T,S}A = A^{②}_{T,S}A$.

For $A \in \mathcal{B}(H,K)^-_{R(B),N(C)}$, where B and C are corresponding operators, and arbitrary but fixed $A^- \in A\{1\}$, Corollary 2.10(xiv) implies that $X = A^{R(B),N(C)}_{-,②}$ if and only if $XAB = B$ and $X = XAA^{(2)}_{R(B),N(C)}(AA^{(2)}_{R(B),N(C)})^\dagger AA^-$.

Theorem 2.1 implies that $A^{-,②,\Phi}$ and $A^{②,\Phi,-}$ are outer inverses of A, and Theorem 2.7 and Theorem 2.9 yield that $A^{-,②,\Phi}$ and $A^{②,\Phi,-}$ are both inner and outer inverses of $AA^{②,\Phi}A$. Therefore, we have the next characterizations of projectors defined by $A^{-,②,\Phi}$ and $A^{②,\Phi,-}$ and new representations for $A^{-,②,\Phi}$ and $A^{②,\Phi,-}$.

Lemma 2.11. *Let $A \in \mathcal{B}(H,K)^-$ and $\Phi \in \mathcal{B}(G,H)$ be such that $A\Phi$ is regular and $A^- \in A\{1\}$ is arbitrary but fixed. The following statements hold:*

(i) $AA^{-,②,\Phi}$ *is the orthogonal projector onto* $R(A\Phi)$;

(ii) $A^{-,②,\Phi}A$ *is a projector onto* $R(A^-A\Phi)$ *along* $N((A\Phi)^*A)$;

(iii) $A^{-,②,\Phi} = A^{(2,3)}_{R(A^-A\Phi),N((A\Phi)^*)} = (AA^{②,\Phi}A)^{(1,2,3)}_{R(A^-A\Phi),N((A\Phi)^*)}$;

(iv) $AA^{②,\Phi,-}$ *is a projector onto* $R(A\Phi)$ *along* $N((A\Phi)^*AA^-)$;

(v) $A^{②,\Phi,-}A$ *is a projector onto* $R(\Phi(A\Phi)^*)$ *along* $N((A\Phi)^*A)$;

(vi) $A^{②,\Phi,-} = A^{(2)}_{R(\Phi(A\Phi)^*),N((A\Phi)^*AA^-)} = (AA^{②,\Phi}A)^{(1,2)}_{R(\Phi(A\Phi)^*),N((A\Phi)^*AA^-)}$.

Proof. (i) Based on Theorem 2.1, $AA^{-,②,\Phi} = AA^{②,\Phi} = A\Phi(A\Phi)^\dagger$ is the orthogonal projector onto $R(A\Phi(A\Phi)^\dagger) = R(A\Phi)$.

(ii) Because $A^{-,②,\Phi} = A^{-,②,\Phi}AA^{-,②,\Phi}$ by Theorem 2.1, we deduce that $A^{-,②,\Phi}A$ is a projection. Given that $A^{-,②,\Phi}A = A^-AA^{②,\Phi}A = A^-A\Phi(A\Phi)^\dagger A$, it follows that $R(A^{-,②,\Phi}A) \subseteq R(A^-A\Phi)$. Furthermore,

$$R(A^-A\Phi) = R(A^-A\Phi(A\Phi)^\dagger A\Phi) \subseteq R(A^-A\Phi(A\Phi)^\dagger A) = R(A^{-,②,\Phi}A),$$

i.e. $R(A^{-,②,\Phi}A) = R(A^-A\Phi)$. Clearly,

$$N(A^{-,②,\Phi}A) = N(A^-A\Phi(A\Phi)^\dagger A) = N((A\Phi)^\dagger A) = N((A\Phi)^*A).$$

(iii) Evidently, $R(A^{-,②,\Phi}) = R(A^{-,②,\Phi}A) = R(A^-A\Phi)$ and $N(A^{-,②,\Phi}) = N(AA^{-,②,\Phi}) = R(A\Phi)^\perp = N((A\Phi)^*)$.

In a similar manner, we finish this proof. □

Lemma 2.11 gives the next properties of inner-GCEP and GCEP-inner inverses.

Corollary 2.12. *Let $A \in \mathcal{B}(H,K)^-$ and $A^- \in A\{1\}$ arbitrary but fixed. The following statements hold:*

(i) $AA^{-,②}_{T,S}$ *is the orthogonal projector onto* $R(AA^{(2)}_{T,S})$;

(ii) $A^{-,②}_{T,S}A$ *is a projector onto* $R(A^-AA^{(2)}_{T,S})$ *along* $N((AA^{(2)}_{T,S})^*A)$;

(iii) $A^{-,②}_{T,S} = A^{(2,3)}_{R(A^-AA^{(2)}_{T,S}),N((AA^{(2)}_{T,S})^*)} = (AA^{②}_{T,S}A)^{(1,2,3)}_{R(A^-AA^{(2)}_{T,S}),N((AA^{(2)}_{T,S})^*)}$;

(iv) $AA^{②,-}_{T,S}$ *is a projector onto* $R(AA^{(2)}_{T,S})$ *along* $N((AA^{(2)}_{T,S})^*AA^-)$;

(v) $A^{②,-}_{T,S}A$ *is a projector onto* T *along* $N((AA^{(2)}_{T,S})^*A)$;

(vi) $A^{②,-}_{T,S} = A^{(2)}_{T,N((AA^{(2)}_{T,S})^*AA^-)} = (AA^{②}_{T,S}A)^{(1,2)}_{T,N((AA^{(2)}_{T,S})^*AA^-)}$.

Also, we can consider $A^{-,②,\Phi}$ and $A^{②,\Phi,-}$ as solutions of some restricted equations.

Theorem 2.13. *Let* $A \in \mathcal{B}(H,K)^-$ *and* $\Phi \in \mathcal{B}(G,H)$ *be such that* $A\Phi$ *is regular and* $A^- \in A\{1\}$ *is arbitrary but fixed. Then, for* $X \in \mathcal{B}(K,H)$,

(i) $A^{-,②,\Phi}$ *is a unique solution to the restricted equation*

$$AX = P_{R(A\Phi)} \quad \text{and} \quad R(X) \subseteq R(A^-A); \tag{3}$$

(ii) $A^{②,\Phi,-}$ *is a unique solution to the restricted equation*

$$AX = P_{R(A\Phi),N((A\Phi)^*AA^-)} \quad \text{and} \quad R(X) \subseteq R(\Phi(A\Phi)^*). \tag{4}$$

Proof. (i) According to Lemma 2.11, (4) has a solution $A^{-,②,\Phi}$.

If X and Z are two solutions of (4), then $R(X - Z) \subseteq R(A^-A)$ and $A(X - Z) = 0$ yield $R(X - Z) \subseteq R(A^-A) \cap N(A^-A) = \{0\}$. Hence, $A^{-,②,\Phi} = X = Z$ is unique solution to (4).

In a similar way, we prove (ii). □

Corollary 2.14 is a consequence of Theorem 2.13.

Corollary 2.14. *Let* $A \in \mathcal{B}(H,K)^-$ *and* $A^- \in A\{1\}$ *arbitrary but fixed. Then, for* $X \in \mathcal{B}(K,H)$,

(i) $A^{-,②}_{T,S}$ *is a unique solution to the restricted equation* $AX = P_{R(AA^{(2)}_{T,S})}$ *and* $R(X) \subseteq R(A^-A)$;

(ii) $A^{②,-}_{T,S}$ *is a unique solution to the restricted equation* $AX = P_{R(AA^{(2)}_{T,S}),N((AA^{(2)}_{T,S})^*AA^-)}$ *and* $R(X) \subseteq T$.

Theorem 2.15. *Let* $A \in \mathcal{B}(H,K)^-$ *and* $\Phi \in \mathcal{B}(G,H)$ *be such that* $A\Phi$ *is regular and* $A^- \in A\{1\}$ *is arbitrary but fixed. Then, for* $X \in \mathcal{B}(K,H)$,

(i) $A^{-,②,\Phi}$ *is a unique solution to the restricted equation*

$$XA = P_{R(A^-A\Phi),N((A\Phi)^*A)} \quad \text{and} \quad R(X^*) \subseteq R(A\Phi); \tag{5}$$

(ii) $A^{②,\Phi,-}$ *is a unique solution to the restricted equation*

$$XA = P_{R(\Phi(A\Phi)^*),N((A\Phi)^*A))} \quad \text{and} \quad R(X^*) \subseteq R((AA^-)^*).$$

Proof. (i) Lemma 2.11 gives that $A^{-,②,\Phi}A = P_{R(A^-A\Phi),N((A\Phi)^*A)}$. Since $A^{-,②,\Phi} = A^-AA^{②,\Phi} = A^-A\Phi(A\Phi)^\dagger$, we get $(A^{-,②,\Phi})^* = A\Phi(A\Phi)^\dagger(A^-)^*$. Thus, $A^{-,②,\Phi}$ is a solution to (5).

If (5) has two solutions X and Z, then $A^*(X^* - Z^*) = 0$ and so

$$R(X^* - Z^*) \subseteq N(A^*) \cap R(A\Phi) \subseteq N((A\Phi)^*) \cap R(A\Phi) = R(A\Phi)^\perp \cap R(A\Phi) = \{0\}.$$

Hence, $X = Z$ and (5) has the unique solution $A^{-,②,\Phi}$.

Part (ii) can be verified analogously. □

Corollary 2.16. *Let $A \in \mathcal{B}(H,K)^-$ and $A^- \in A\{1\}$ arbitrary but fixed. Then, for $X \in \mathcal{B}(K,H)$,*

(i) $A^{-,ⓐ}_{T,S}$ *is unique solution to the restricted equation* $XA = P_{R(A^-AA^{(2)}_{T,S}),N((AA^{(2)}_{T,S})^*A)}$ *and* $R(X^*) \subseteq R(AA^{(2)}_{T,S})$;

(ii) $A^{ⓐ,-}_{T,S}$ *is unique solution to the restricted equation* $XA = P_{T,N((AA^{(2)}_{T,S})^*A))}$ *and* $R(X^*) \subseteq R((AA^-)^*)$.

Necessary and sufficient conditions for inner-Φ-GCEP and Φ-GCEP-inner inverses to be an inner inverse of A, are studied.

Theorem 2.17. *Let $A \in \mathcal{B}(H,K)^-$ and $\Phi \in \mathcal{B}(G,H)$ be such that $A\Phi$ is regular and $A^- \in A\{1\}$ is arbitrary but fixed. Then the following claims are mutually equivalent:*

(i) $A = AA^{-,ⓐ,\Phi}A$;

(ii) $A = AA^{ⓐ,\Phi}A$;

(iii) $A = AA^{ⓐ,\Phi,-}A$;

(iv) $N(A^{ⓐ,\Phi}A) = N(A)$ $(\Leftrightarrow N((A\Phi)^*A) = N(A))$;

(v) $R(A) = R(AA^{ⓐ,\Phi})$ $(\Leftrightarrow R(A) = R(A\Phi))$.

Proof. (i)⇔(ii)⇔(iii): This part follows from the equalities $AA^{-,ⓐ,\Phi}A = AA^{ⓐ,\Phi}A = AA^{ⓐ,\Phi,-}A$. (ii)⇔(iv): Keeping in mind that $A^{ⓐ,\Phi}A = \Phi(A\Phi)^\dagger A$ is a projecton, we have

$$\begin{aligned} A = AA^{ⓐ,\Phi}A &\Leftrightarrow A(I - A^{ⓐ,\Phi}A) = 0 \Leftrightarrow A(I - \Phi(A\Phi)^\dagger A) = 0 \\ &\Leftrightarrow R(I - A^{ⓐ,\Phi}A) \subseteq N(A) \Leftrightarrow N(A^{ⓐ,\Phi}A) = N(A) \\ &\Leftrightarrow N(\Phi(A\Phi)^\dagger A) \subseteq N(A) \Leftrightarrow N((A\Phi)^*A) = N(A). \end{aligned}$$

(ii)⇔(v): Similarly to (ii)⇔(iv), knowing that $AA^{ⓐ,\Phi}$ is a projection. □

For arbitrary operator F, it is interesting to investigate when $A^{-,ⓐ,\Phi}$ coincides with A^-AF.

Theorem 2.18. *Let $A \in \mathcal{B}(H,K)^-$ and $\Phi \in \mathcal{B}(G,H)$ be such that $A\Phi$ is regular and $A^- \in A\{1\}$ is arbitrary but fixed. Then, for $F \in \Phi \in \mathcal{B}(K,H)$, the following claims are mutually equivalent:*

(i) $A^{-,ⓐ,\Phi} = A^-AF$;

(ii) $AA^{ⓐ,\Phi} = AF$;

(iii) $N(AA^{ⓐ,\Phi}) = N(AF)$ *and* $AA^{ⓐ,\Phi}A = AFA$;

(iv) $F = A^{ⓐ,\Phi} + (I - A^-A)E$ *for arbitrary* $E \in \mathcal{B}(K,H)$.

Proof. (i) ⇒ (ii): Clearly, $AF = A(A^-AF) = AA^{-,ⓐ,\Phi} = AA^{ⓐ,\Phi}$.

(ii) ⇒ (iii) and (iv) ⇒ (ii): These implications are evident.

(iii) ⇒ (i): Note that $R(I - AA^{ⓐ,\Phi}) = N(AA^{ⓐ,\Phi}) = N(AF)$ gives $AF = AFAA^{ⓐ,\Phi}$. Using $AA^{ⓐ,\Phi}A = AEA$, we get $A^{-,ⓐ,\Phi} = A^-AA^{ⓐ,\Phi} = A^-(AA^{ⓐ,\Phi}A)A^{ⓐ,\Phi} = A^-(AFAA^{ⓐ,\Phi}) = A^-AF$.

(ii) ⇒ (iv): All solutions F of $AA^{ⓐ,\Phi} = AF$, by [1, p. 52], are the sum of its particular solution and the general solutions to $AF = 0$, i.e. $F = A^{ⓐ,\Phi} + (I - A^-A)E$, for arbitrary $F \in \mathcal{B}(K,H)$. □

Analogous theorem can be verified for the Φ-GCEP-inner inverse.

Theorem 2.19. *Let $A \in \mathcal{B}(H,K)^-$ and $\Phi \in \mathcal{B}(G,H)$ be such that $A\Phi$ is regular and $A^- \in A\{1\}$ is arbitrary but fixed. Then, for $F \in \Phi \in \mathcal{B}(K,H)$, the following claims are mutually equivalent:*

(i) $A^{ⓐ,\Phi,-} = FAA^-$;

(ii) $A^{②,\Phi}A = FA$;

(iii) $\mathcal{R}(A^{②,\Phi}A) = \mathcal{R}(FA)$ *and* $AA^{②,\Phi}A = AFA$;

(iv) $F = A^{②,\Phi} + E(I - AA^-)$, *for arbitrary* $E \in \mathcal{B}(H)$.

If we denote by $A\{-, ②, \Phi\}$ and $A\{②, \Phi, -\}$, respectively, the sets of all inner-Φ-GCEP and Φ-GCEP-inner inverses of A, these sets can be described as follows.

Theorem 2.20. *For* $A \in \mathcal{B}(H, K)^-$ *and* $\Phi \in \mathcal{B}(G, H)$ *be such that* $A\Phi$ *is regular and* $A^- \in A\{1\}$ *is arbitrary but fixed, we have*

$$A\{-, ②, \Phi\} = \{A^{-,②,\Phi} + (I - A^-A)ZA\Phi(A\Phi)^\dagger : Z \in \mathcal{B}(K, H)\}$$

and

$$A\{②, \Phi, -\} = \{A^{②,\Phi,-} + \Phi(A\Phi)^\dagger AZ(I - AA^-) : Z \in \mathcal{B}(K, H)\}.$$

Proof. It is clear from $A\{1\} = \{A^- + Z - A^-AZAA^- : Z \in \mathcal{B}(K, H)\}$ [1]. □

3. Inner-Φ-*GCEP and Φ-*GCEP-inner inverses

Using the Φ-*GCEP inverse instead of the Φ-GCEP inverse in the definitions of inner-Φ-GCEP and Φ-GCEP-inner inverses, we introduce inner-Φ-*GCEP and Φ-*GCEP-inner inverses in this section. In this manner, we generalize the notion of the inner-gMP and gMP-inner inverses. We omit the proofs of the following results because they are similar to the corresponding results of Section 2.

Theorem 3.1. *Let* $A \in \mathcal{B}(H, K)^-$ *and* $\Phi \in \mathcal{B}(K, G)$ *such that* ΦA *is regular and* $A^- \in A\{1\}$ *is arbitrary but fixed. Then*

(a) *the system*

$$XAX = X, \qquad AX = AA_{②,\Phi} \qquad \text{and} \qquad XA = A^-AA_{②,\Phi}A$$

has a uniquely determined solution expressed by $X = A^-AA_{②,\Phi}$,

(b) *the system*

$$XAX = X, \qquad AX = AA_{②,\Phi}AA^- \qquad \text{and} \qquad XA = A_{②,\Phi}A$$

has a uniquely determined solution expressed by $X = A_{②,\Phi}AA^-$.

Definition 3.2. *Let* $A \in \mathcal{B}(H, K)^-$ *and* $\Phi \in \mathcal{B}(K, G)$ *such that* ΦA *is regular and* $A^- \in A\{1\}$ *is arbitrary but fixed.*

(a) *The inner-Φ-*GCEP inverse of A is defined by*

$$A_{-,②,\Phi} = A^-AA_{②,\Phi}.$$

(b) *The Φ-*GCEP-inner inverse of A is defined by*

$$A_{②,\Phi,-} = A_{②,\Phi}AA^-.$$

Theorem 3.1 gives the definitions of inner-*GCEP and *GCEP-inner inverses.

Corollary 3.3. *Let* $A \in \mathcal{B}(H, K)^-_{T,S}$ *and* $A^- \in A\{1\}$ *be arbitrary but fixed. Then*

(a) *the system*

$$XAX = X, \qquad AX = AA_{②}^{T,S} \qquad \text{and} \qquad XA = A^-AA_{②}^{T,S}A$$

has a uniquely determined solution expressed by $X = A^-AA_{②}^{T,S}$,

(b) *the system*

$$XAX = X, \qquad AX = AA_{\circledtwo}^{T,S}AA^{-} \qquad \text{and} \qquad XA = A_{\circledtwo}^{T,S}A$$

has a uniquely determined solution expressed by $X = A_{\circledtwo}^{T,S}AA^{-}$.

Definition 3.4. *Let* $A \in \mathcal{B}(H,K)^{-}_{T,S}$ *and* $A^{-} \in A\{1\}$ *be arbitrary but fixed.*

(a) *The inner-*GCEP inverse of A is defined by*

$$A_{-,\circledtwo}^{T,S} = A^{-}AA_{\circledtwo}^{T,S}.$$

(b) *The *GCEP-inner inverse of A is defined by*

$$A_{\circledtwo,-}^{T,S} = A_{\circledtwo}^{T,S}AA^{-}.$$

Particular kinds of inner-*GCEP and *GCEP-inner inverses are given:

- if $A^{(2)}_{T,S} = A^{\dagger}$, $A_{-,\circledtwo}^{T,S} = A^{-,\dagger}$ and $A_{\circledtwo,-}^{T,S} = A^{\dagger,-}$;
- when $H = K$ and $A^{(2)}_{T,S} = A^{\textcircled{\#}}$, $A_{\circledtwo}^{T,S} = A^{\diamond}$, $A_{-,\circledtwo}^{T,S} = A^{-,\diamond}$ and $A_{\circledtwo,-}^{T,S} = A^{\diamond,-}$;
- for $H = K$ and $A^{(2)}_{T,S} = A^{d}$, $A_{\circledtwo}^{T,S} = A_{\textcircled{d}}$, $A_{-,\circledtwo}^{T,S} = A^{-}AA_{\textcircled{d}}$ and $A_{\circledtwo,-}^{T,S} = A_{\textcircled{d}}AA^{-}$.

The following properties for inner-Φ-*GCEP inverses are the consequence of Theorem 3.1.

Theorem 3.5. *Let* $A \in \mathcal{B}(H,K)^{-}$ *and* $\Phi \in \mathcal{B}(K,G)$ *such that* ΦA *is regular and* $A^{-} \in A\{1\}$ *is arbitrary but fixed. For* $X \in \mathcal{B}(K,H)$*, the following statements are equivalent:*

(i) $X = A_{-,\circledtwo,\Phi}$;

(ii) $XAX = X$, $AXA = AA_{\circledtwo,\Phi}A$, $AX = AA_{\circledtwo,\Phi}$ *and* $XA = A^{-}AA_{\circledtwo,\Phi}A$;

(iii) $XA = A^{-}AA_{\circledtwo,\Phi}A$ *and* $XAA_{\circledtwo,\Phi} = X$;

(iv) $XA = A^{-}AA_{\circledtwo,\Phi}A$ *and* $XA(\Phi A)^{\dagger}\Phi = X$;

(v) $XAA^{\dagger} = A^{-}AA_{\circledtwo,\Phi}AA^{\dagger}$ *and* $XA(\Phi A)^{\dagger}\Phi = X$;

(vi) $XAA^{*} = A^{-}AA_{\circledtwo,\Phi}AA^{*}$ *and* $XA(\Phi A)^{\dagger}\Phi = X$;

(vii) $XA(\Phi A)^{\dagger} = A^{-}A(\Phi A)^{\dagger}$ *and* $XA(\Phi A)^{\dagger}\Phi = X$;

(viii) $XA(\Phi A)^{*} = A^{-}A(\Phi A)^{*}$ *and* $XA(\Phi A)^{\dagger}\Phi = X$;

(ix) $AX = AA_{\circledtwo,\Phi}$ *and* $A^{-}AA_{\circledtwo,\Phi}AX = X$;

(x) $AX = AA_{\circledtwo,\Phi}$ *and* $A^{-}AX = X$;

(xi) $A^{\dagger}AX = A^{\dagger}AA_{\circledtwo,\Phi}$ *and* $A^{-}AX = X$;

(xii) $A^{*}AX = A^{*}AA_{\circledtwo,\Phi}$ *and* $A^{-}AX = X$;

(xiii) $A_{\circledtwo,\Phi}AX = A_{\circledtwo,\Phi}$ *and* $A^{-}AA_{\circledtwo,\Phi}AX = X$;

(xiv) $(\Phi A)^{\dagger}\Phi AX = (\Phi A)^{\dagger}\Phi$ *and* $A^{-}A(\Phi A)^{\dagger}\Phi AX = X$;

(xv) $XAA_{\circledtwo,\Phi}AX = X$, $AA_{\circledtwo,\Phi}AXAA_{\circledtwo,\Phi}A = AA_{\circledtwo,\Phi}A$, $AA_{\circledtwo,\Phi}AX = AA_{\circledtwo,\Phi}$ *and* $XAA_{\circledtwo,\Phi}A = A^{-}AA_{\circledtwo,\Phi}A$;

(xvi) $XAA_{\circledtwo,\Phi}AX = X$, $AA_{\circledtwo,\Phi}AX = AA_{\circledtwo,\Phi}$ *and* $XAA_{\circledtwo,\Phi}A = A^{-}AA_{\circledtwo,\Phi}A$.

Analogous properties stand for the inner-*GCEP inverses.

Corollary 3.6. *Let $A \in \mathcal{B}(H,K)^{-}_{T,S}$ and $A^{-} \in A\{1\}$ arbitrary but fixed. For $X \in \mathcal{B}(K,H)$, the following statements are equivalent:*

(i) $X = A^{T,S}_{-,②}$;

(ii) $XAX = X$, $AXA = AA^{T,S}_{②}A$, $AX = AA^{T,S}_{②}$ and $XA = A^{-}AA^{T,S}_{②}A$;

(iii) $XA = A^{-}AA^{T,S}_{②}A$ and $XAA^{T,S}_{②} = X$;

(iv) $XA = A^{-}AA^{T,S}_{②}A$ and $XA(A^{(2)}_{T,S}A)^{\dagger}A^{(2)}_{T,S} = X$;

(v) $XAA^{\dagger} = A^{-}AA^{T,S}_{②}AA^{\dagger}$ and $XA(A^{(2)}_{T,S}A)^{\dagger}A^{(2)}_{T,S} = X$;

(vi) $XAA^{*} = A^{-}AA^{T,S}_{②}AA^{*}$ and $XA(A^{(2)}_{T,S}A)^{\dagger}A^{(2)}_{T,S} = X$;

(vii) $XA(A^{(2)}_{T,S}A)^{\dagger} = A^{-}A(A^{(2)}_{T,S}A)^{\dagger}$ and $XA(A^{(2)}_{T,S}A)^{\dagger}A^{(2)}_{T,S} = X$;

(viii) $XA(A^{(2)}_{T,S}A)^{*} = A^{-}A(A^{(2)}_{T,S}A)^{*}$ and $XA(A^{(2)}_{T,S}A)^{\dagger}A^{(2)}_{T,S} = X$;

(ix) $AX = AA^{T,S}_{②}$ and $A^{-}AA^{T,S}_{②}AX = X$;

(x) $AX = AA^{T,S}_{②}$ and $A^{-}AX = X$;

(xi) $A^{\dagger}AX = A^{\dagger}AA^{T,S}_{②}$ and $A^{-}AX = X$;

(xii) $A^{*}AX = A^{*}AA^{T,S}_{②}$ and $A^{-}AX = X$;

(xiii) $A^{T,S}_{②}AX = A^{T,S}_{②}$ and $A^{-}AA^{T,S}_{②}AX = X$

(xiv) $A^{(2)}_{T,S}AX = A^{(2)}_{T,S}$ and $A^{-}A(A^{(2)}_{T,S}A)^{\dagger}A^{(2)}_{T,S}AX = X$;

(xv) $XAA^{T,S}_{②}AX = X$, $AA^{T,S}_{②}AXAA^{T,S}_{②}A = AA^{T,S}_{②}A$, $AA^{T,S}_{②}AX = AA^{T,S}_{②}$ and $XAA^{T,S}_{②}A = A^{-}AA^{T,S}_{②}A$;

(xvi) $XAA^{T,S}_{②}AX = X$, $AA^{T,S}_{②}AX = AA^{T,S}_{②}$ and $XAA^{T,S}_{②}A = A^{-}AA^{T,S}_{②}A$.

In a similar fashion to Theorem 3.5, the following properties for Φ-*GCEP-inner inverse are established.

Theorem 3.7. *Let $A \in \mathcal{B}(H,K)^{-}$ and $\Phi \in \mathcal{B}(K,G)$ such that ΦA is regular and $A^{-} \in A\{1\}$ is arbitrary but fixed. For $X \in \mathcal{B}(K,H)$, the following statements are equivalent:*

(i) $X = A_{②,\Phi,-}$;

(ii) $XAX = X$, $AXA = AA_{②,\Phi}A$, $AX = AA_{②,\Phi}AA^{-}$ and $XA = A_{②,\Phi}A$;

(iii) $AX = AA_{②,\Phi}AA^{-}$ and $A_{②,\Phi}AX = X$;

(iv) $AX = AA_{②,\Phi}AA^{-}$ and $(\Phi A)^{\dagger}\Phi AX = X$;

(v) $A^{\dagger}AX = A_{②,\Phi}AA^{-}(= A^{\dagger}AA_{②,\Phi}AA^{-})$ and $(\Phi A)^{\dagger}\Phi AX = X$;

(vi) $A^{*}AX = A^{*}AA_{②,\Phi}AA^{-}$ and $(\Phi A)^{\dagger}\Phi AX = X$;

(vii) $\Phi AX = \Phi AA^{-}$ and $(\Phi A)^{\dagger}\Phi AX = X$;

(viii) $XA = A_{②,\Phi}A$ and $X = XAA_{②,\Phi}AA^{-}$;

(ix) $XA = A_{②,\Phi}A$ and $XAA^- = X$;

(x) $XAA^\dagger = A_{②,\Phi}AA^\dagger$ and $XAA^- = X$;

(xi) $XAA^* = A_{②,\Phi}AA^*$ and $XAA^- = X$;

(xii) $XAA_{②,\Phi} = A_{②,\Phi}$ and $X = XAA_{②,\Phi}AA^-$;

(xiii) $XA(\Phi A)^\dagger = (\Phi A)^\dagger$ and $XA(\Phi A)^\dagger \Phi AA^- = X$;

(xiv) $XA(\Phi A)^* = (\Phi A)^*$ and $XA(\Phi A)^\dagger \Phi AA^- = X$;

(xv) $XAA_{②,\Phi}AX = X$, $AA_{②,\Phi}AXAA_{②,\Phi}A = AA_{②,\Phi}A$, $AA_{②,\Phi}AX = AA_{②,\Phi}AA^-$ and $XAA_{②,\Phi}A = A_{②,\Phi}A$;

(xvi) $XAA_{②,\Phi}AX = X$, $AA_{②,\Phi}AX = AA_{②,\Phi}AA^-$ and $XAA_{②,\Phi}A = A_{②,\Phi}A$.

Corollary 3.8. *Let $A \in \mathcal{B}(H,K)^-_{T,S}$ and $A^- \in A\{1\}$ arbitrary but fixed. For $X \in \mathcal{B}(K,H)$, the following statements are equivalent:*

(i) $X = A^{T,S}_{②,-}$;

(ii) $XAX = X$, $AXA = AA^{T,S}_{②}A$, $AX = AA^{T,S}_{②}AA^-$ and $XA = A^{T,S}_{②}A$;

(iii) $AX = AA^{T,S}_{②}AA^-$ and $A^{T,S}_{②}AX = X$;

(iv) $AX = AA^{T,S}_{②}AA^-$ and $(A^{(2)}_{T,S}A)^\dagger A^{(2)}_{T,S}AX = X$;

(v) $A^\dagger AX = A^{T,S}_{②}AA^- (= A^\dagger AA^{T,S}_{②}AA^-)$ and $(A^{(2)}_{T,S}A)^\dagger A^{(2)}_{T,S}AX = X$;

(vi) $A^*AX = A^*AA^{T,S}_{②}AA^-$ and $(A^{(2)}_{T,S}A)^\dagger A^{(2)}_{T,S}AX = X$;

(vii) $A^{(2)}_{T,S}AX = A^{(2)}_{T,S}AA^-$ and $(A^{(2)}_{T,S}A)^\dagger A^{(2)}_{T,S}AX = X$;

(viii) $XA = A^{T,S}_{②}A$ and $X = XAA^{T,S}_{②}AA^-$;

(ix) $XA = A^{T,S}_{②}A$ and $XAA^- = X$;

(x) $XAA^\dagger = A^{T,S}_{②}AA^\dagger$ and $XAA^- = X$;

(xi) $XAA^* = A^{T,S}_{②}AA^*$ and $XAA^- = X$;

(xii) $XAA^{T,S}_{②} = A^{T,S}_{②}$ and $X = XAA^{T,S}_{②}AA^-$;

(xiii) $XA(A^{(2)}_{T,S}A)^\dagger = (A^{(2)}_{T,S}A)^\dagger$ and $XA(A^{(2)}_{T,S}A)^\dagger A^{(2)}_{T,S}AA^- = X$;

(xiv) $XA(A^{(2)}_{T,S}A)^* = (A^{(2)}_{T,S}A)^*$ and $XA(A^{(2)}_{T,S}A)^\dagger A^{(2)}_{T,S}AA^- = X$;

(xv) $XAA^{T,S}_{②}AX = X$, $AA^{T,S}_{②}AXAA^{T,S}_{②}A = AA^{T,S}_{②}A$, $AA^{T,S}_{②}AX = AA^{T,S}_{②}AA^-$ and $XAA^{T,S}_{②}A = A^{T,S}_{②}A$;

(xvi) $XAA^{T,S}_{②}AX = X$, $AA^{T,S}_{②}AX = AA^{T,S}_{②}AA^-$ and $XAA^{T,S}_{②}A = A^{T,S}_{②}A$.

4. Applications

We can use inner-Φ-GCEP, Φ-GCEP-inner, inner-Φ-*GCEP and Φ-*GCEP-inner inverses to solve several linear equations.

Theorem 4.1. *Let $A \in \mathcal{B}(H,K)^-$ and $\Phi \in \mathcal{B}(G,H)$ such that $A\Phi$ is regular and $A^- \in A\{1\}$ is arbitrary but fixed. For $b \in H$, the general solution to the equation*

$$Ax = AA^{②,\Phi}b \tag{6}$$

is given by

$$x = A^{-,②,\Phi}b + (I - A^-A)u, \tag{7}$$

where $u \in H$ is arbitrary.

Proof. Because $AA^{-,②,\Phi} = AA^{②,\Phi}$, we easily check that x expressed as (7), is a solution to (6).

When x is a solution to (6), then $A^{-,②,\Phi}b = A^-(AA^{②,\Phi}b) = A^-Ax$ yields $x = A^{-,②,\Phi}b + (I - A^-A)x$, that is, x is represented by (7). □

If $\Phi = A^\dagger = A^{②,\Phi}$, (6) holds if and only if $A^*Ax = A^*b$, which is the normal equation of $Ax = b$. Let us recall that x is a least-squares solution to $Ax = b$ (often used approximate solution in statistical applications [1]) if and only if x is a solution to the above mentioned normal equation.

For $b \in R(A\Phi) = R(AA^{②,\Phi})$ in Theorem 4.1, we have $AA^{②,\Phi}b = b$ and solvability of (8).

Corollary 4.2. *Let $A \in \mathcal{B}(H,K)^-$ and $\Phi \in \mathcal{B}(G,H)$ such that $A\Phi$ is regular and $A^- \in A\{1\}$ is arbitrary but fixed. For $b \in H$, the general solution to the equation*

$$Ax = b, \quad b \in R(A\Phi) \tag{8}$$

is given by

$$x = A^-b + (I - A^-A)u,$$

where $u \in H$ is arbitrary.

The equation (6) has a unique solution in the next case.

Theorem 4.3. *Let $A \in \mathcal{B}(H,K)^-$ and $\Phi \in \mathcal{B}(G,H)$ such that $A\Phi$ is regular and $A^- \in A\{1\}$ is arbitrary but fixed. For $b \in H$, the equation (6) has in $R(A^-A)$ a unique solution $A^{-,②,\Phi}b$.*

Proof. We observe, by Theorem 4.1, that $A^{-,②,\Phi}b \in R(A^-A)$ is a solution to (6).

Suppose that (6) has in $R(A^-A)$ two solutions x and v. From $x - v \in N(A) \cap R(A^-A) = N(A^-A) \cap R(A^-A) = \{0\}$, we deduce that $x = v$. □

Similarly, we show the following results.

Theorem 4.4. *Let $A \in \mathcal{B}(H,K)^-$ and $\Phi \in \mathcal{B}(G,H)$ such that $A\Phi$ is regular and $A^- \in A\{1\}$ is arbitrary but fixed. For $b \in H$, the general solution to the equation*

$$A^{②,\Phi}Ax = A^{②,\Phi}b \tag{9}$$

is given by

$$x = A^{-,②,\Phi}b + (I - A^{-,②,\Phi}A)u,$$

where $u \in H$ is arbitrary. In addition, the equation (9) has in $R(A^-A\Phi)$ a unique solution $A^{-,②,\Phi}b$.

The equation (10) can be solved utilizing the Φ-GCEP-inner inverse.

Theorem 4.5. *Let $A \in \mathcal{B}(H,K)^-$ and $\Phi \in \mathcal{B}(G,H)$ such that $A\Phi$ is regular and $A^- \in A\{1\}$ is arbitrary but fixed. For $b \in H$, the general solution to the equation*

$$A^{②,\Phi}Ax = A^{②,\Phi,-}b \tag{10}$$

is given by

$$x = A^{②,\Phi,-}b + (I - A^{②,\Phi}A)u,$$

where $u \in H$ is arbitrary. In addition, the equation (10) *has in $R(\Phi(A\Phi)^*)$ a unique solution $A^{②,\Phi,-}b$.*

We give an example which confirms Theorem 4.1 and Theorem 4.5.

Example 4.6. *Let A be given as in Example 2.6, $b = [3\ 0\ 3]^*$ and $u = [u_1\ u_2\ u_3]^*$. Because*

$$\begin{aligned} x &= A^{-,②,\Phi}b + (I - A^-A)u \\ &= \begin{bmatrix} 3(1-2a) + 2au_1 - 2(1-2a)u_2 + 2bu_3 \\ 3a - au_1 + (1-2a)u_2 - bu_3 \\ 0 \end{bmatrix}, \end{aligned}$$

we get

$$Ax = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = AA^{②,\Phi}b.$$

According to Theorem 4.3, $A^{-,②,\Phi}b = [3(1-2a)\ 3a\ 0]^*$ is unique solution to (6) in

$$R(A^-A) = \{[(1-2a)y_1 + 2(1-2a)y_2 - 2by_3 \quad ay_1 + 2ay_2 + by_3 \quad y_3]^* \ : \ y_1, y_2, y_3 \in \mathbb{C}\}.$$

By calculations,

$$\begin{aligned} x &= A^{②,\Phi,-}b + (I - A^{②,\Phi}A)u \\ &= \begin{bmatrix} 1 + s + 2c + \frac{2}{3}u_1 - \frac{2}{3}u_2 \\ 1 + s + 2c - \frac{1}{3}u_1 + \frac{1}{3}u_2 \\ u_3 \end{bmatrix}, \end{aligned}$$

gives

$$A^{②,\Phi}Ax = \begin{bmatrix} 1+s+2c \\ 1+s+2c \\ 0 \end{bmatrix} = A^{②,\Phi,-}b.$$

By Theorem 4.5, (10) has unique solution $A^{②,\Phi,-}b = [1+s+2c\ \ 1+s+2c\ \ 0]^*$ in

$$R(\Phi(A\Phi)^*) = \left\{ \begin{bmatrix} \frac{1}{3}y_1 & \frac{1}{3}y_1 & 0 \end{bmatrix}^* \ : \ y_1 \in \mathbb{C} \right\}.$$

Using inner-Φ-*GCEP, we can verify solvability of next equations in an analogous manner.

Theorem 4.7. *Let $A \in \mathcal{B}(H,K)^-$ and $\Phi \in \mathcal{B}(K,G)$ such that ΦA is regular and $A^- \in A\{1\}$ is arbitrary but fixed. For $b \in H$, the general solution to the equation*

$$Ax = AA_{②,\Phi}b \tag{11}$$

is given by

$$x = A_{-,②,\Phi}b + (I - A^-A)u,$$

where $u \in H$ is arbitrary. In addition, the equation (11) *has in $R(A^-A)$ a unique solution $A_{-,②,\Phi}b$.*

In the case that $b \in R(A(\Phi A)^*)(= R(AA_{②,\Phi}))$ in Theorem 4.7, we show the following result.

Corollary 4.8. *Let $A \in \mathcal{B}(H,K)^-$ and $\Phi \in \mathcal{B}(K,G)$ such that ΦA is regular and $A^- \in A\{1\}$ is arbitrary but fixed. For $b \in H$, the general solution to the equation*

$$Ax = b, \quad b \in R(A(\Phi A)^*)$$

is given by

$$x = A^- b + (I - A^- A)u,$$

where $u \in H$ is arbitrary.

The equations solved by Φ-*GCEP-inner inverse are given now.

Theorem 4.9. *Let $A \in \mathcal{B}(H,K)^-$ and $\Phi \in \mathcal{B}(K,G)$ such that ΦA is regular and $A^- \in A\{1\}$ is arbitrary but fixed. For $b \in H$, the general solution to the equation*

$$A_{\textcircled{2},\Phi} A x = A_{\textcircled{2},\Phi,-} b \tag{12}$$

is given by

$$x = A_{\textcircled{2},\Phi,-} b + (I - A_{\textcircled{2},\Phi} A)u,$$

where $u \in H$ is arbitrary. In addition, the equation (12) *has in $R((\Phi A)^*)$ a unique solution $A_{\textcircled{2},\Phi,-} b$.*

Corollary 4.10. *Let $A \in \mathcal{B}(H,K)^-$ and $\Phi \in \mathcal{B}(K,G)$ such that ΦA is regular and $A^- \in A\{1\}$ is arbitrary but fixed. For $b \in H$, the general solution to the equation*

$$A_{\textcircled{2},\Phi} A x = A_{\textcircled{2},\Phi} b, \quad b \in R(A)$$

is given by

$$x = A_{\textcircled{2},\Phi} b + (I - A_{\textcircled{2},\Phi} A)u,$$

where $u \in H$ is arbitrary.

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