Filomat 38:26 (2024), 9289–9304
https://doi.org/10.2298/FIL2426289T

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Convergence of single projection method with inertial and self-adaptive techniques for variational inequalities

Duong Viet Thong[a,*], Dang Huy Ngan[a], Nguyen Thi An[a]

[a]*Faculty of Mathematical Economics, National Economics University, Hanoi, Vietnam*

Abstract. In this paper, we investigate pseudomonotone variational inequality problems in a real Hilbert space and propose a projection-type algorithm with an inertial technique for solving them. The proposed algorithm does not require prior knowledge of the Lipschitz constant of the mapping which governs the variational inequality. The weak convergence theorem for our algorithm is proved under pseudomonotonicity and Lipschitz continuity assumptions. We also establish the strong convergence theorem for this algorithm even the sequence converges in norm to the unique solution of the problem with an R-linear convergence rate under strong pseudomonotonicity and Lipschitz continuity hypotheses. Our obtained results in this work extend and improve the related results in the literature.

1. Introduction

This paper deals with a numerical approache to finding a solution to the *variational inequality problem* **(VI)** in a real Hilbert space H. Recall that problem **(VI)** is formulated as follows:

$$\text{Find } x^* \in C \text{ such that } \langle Fx^*, y - x^* \rangle \geq 0 \quad \text{for all} \quad y \in C,$$

where C is a nonempty, closed and convex subset of H and $F : H \to H$ is a given operator. The solution set of problem **(VI)** is denoted by $\mathrm{Sol}(C, F)$.

It is well known that problem **(VI)** is a central problem in nonlinear analysis and in optimization theory. It unifies many important concepts in, for instance, applied mathematics, economics, mathematical programming, mechanics, and transportation engineering (see, for example, [4, 5, 16, 22, 24]). Many authors have recently proposed to solve problem **(VI)** by applying various iterative methods [8–10, 18, 23, 28–30, 35, 44, 45, 47–49].

In recent years, several projection methods for solving the monotone variational inequality problem have been introduced. In among, the most famous projection method is the extragradient method which was first introduced by Korpelevich [26] and Antipin [3] for solving the saddle point problems. Then, this method

2020 *Mathematics Subject Classification*. Primary 65Y05; Secondary 65K15, 68W10, 47H05, 47J25.
Keywords. Inertial method, Tseng's extragradient method, variational inequality, weak convergence, convergence rate.
Received: 22 April 2022; Revised: 16 July 2024; Accepted: 23 September 2024
Communicated by Adrian Petrusel
ORCID iD: 0000-0003-1753-7237 (Duong Viet Thong), 0009-0001-9817-0643 (Dang Huy Ngan), 009-0004-8759-9825 (Nguyen Thi An).
* Corresponding author: Duong Viet Thong
Email addresses: thongduongviet@neu.edu.vn (Duong Viet Thong), ngandh@neu.edu.vn (Dang Huy Ngan), annt@neu.edu.vn (Nguyen Thi An)

was extended and modified to solve problem **(VI)** when the operator $F : H \to H$ is *monotone* and L-Lipschitz continuous on C. Recently, this method has been interested in many authors and many results have been investigated and related to it in Hilbert space are proposed under the monotonicity and Lipschitz continuity assumptions of the variational inequality operator (see, for example, [1, 7, 12–15, 25, 27, 34, 39, 41, 42, 46]).

We observe that the **(EGM)** needs to require the computation of two projections onto feasible set and of two values of the variational inequality operator per iteration. In general, this is very expensive and can affect the performance of the method when the operator F and the feasible set C have complicated structures. To our knowledge, one of the methods which reduces this obstacle is *Tseng's extragradient method* **(TEGM)** [40], which only need to compute one projection in each iterative step. Recently, the Tseng method for solving problem **(VI)** has received much attention from many authors (see, for example, [6, 43] and references therein).

In this work, we propose a new variant of Tseng's extragradient method for solving problem **(VI)** with a *pseudomonotone* (in the sense of [21]) associated operator. We use an inertial parameter which is different from the one in [33, 37, 38, 40] and self-adaptive step sizes which allow the proposed algorithm to work without prior knowledge of the Lipschitz constant of the variational inequality operator. Moreover, our results in this investigation also extend the results in [36, 37, 40, 43, 47] from the class of *monotone mappings* to the class of *pseudomonotone mappings*.

The structure of the paper is as follows. In Sect. 2, we recall some definitions and preliminary results for the use in what follows. Section 3 is devoted to the main results. Here we first propose Algorithm 3.1 and establish a sufficient condition for its weak convergence under pseudomonotonicity and Lipschitz continuity assumptions (Theorem 3.6). Next, This algorithm is also proved to be strongly convergent with an R-linear rate, but under more restrictive assumptions of k-strong pseudomonotonivity and Lipschitz continuity (Theorem 4.1). Final remarks and conclusions are given in Sect. 6.

2. Preliminaries

Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. The weak convergence of a sequence $\{x_n\}_{n=1}^{\infty}$ to x as $n \to \infty$ is denoted by $x_n \rightharpoonup x$ while the strong convergence of $\{x_n\}_{n=1}^{\infty}$ to x as $n \to \infty$ is denoted by $x_n \to x$. For each $x, y \in H$, we have

$$||x + y||^2 \le ||x||^2 + 2\langle y, x + y\rangle.$$

For each $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, which satisfies

$$||x - P_C x|| \le ||x - y|| \quad \forall y \in C.$$

The mapping P_C is called the *metric projection* of H onto C. It is known that P_C is nonexpansive (that is, 1-Lipschitz).

Lemma 2.1. ([19]) *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Then for any $x \in H$ and $z \in C$, we have*

$$z = P_C x \Longleftrightarrow \langle x - z, z - y\rangle \ge 0 \quad \forall y \in C.$$

Lemma 2.2. ([19]) *Let C be a closed and convex subset of a real Hilbert space H and let $x \in H$. Then the following two inequalities hold:*

(1) $||P_C x - P_C y||^2 \le \langle P_C x - P_C y, x - y\rangle$ *for all* $y \in H$;
(2) $||P_C x - y||^2 \le ||x - y||^2 - ||x - P_C x||^2$ *for all* $y \in C$.

Lemma 2.3. ([2]) *Let* $\{\varphi_n\}$, $\{\delta_n\}$ *and* $\{\alpha_n\}$ *be sequences in* $[0, +\infty)$ *such that*

$$\varphi_{n+1} \le \varphi_n + \alpha_n(\varphi_n - \varphi_{n-1}) + \delta_n \quad \forall n \ge 1, \ \sum_{n=1}^{+\infty} \delta_n < +\infty,$$

and such that there exists a real number α so that $0 \leq \alpha_n \leq \alpha < 1$ for all $n \in \mathbb{N}$. Then the following assertions hold:

(1) $\sum_{n=1}^{+\infty} [\varphi_n - \varphi_{n-1}]_+ < +\infty$, *where* $[t]_+ := \max\{t, 0\}$;

(2) *there exists* $\varphi^* \in [0, +\infty)$ *such that* $\lim_{n\to+\infty} \varphi_n = \varphi^*$.

Lemma 2.4. ([32]) *Let C be a nonempty subset of H and let $\{x_n\}$ be a sequence in H such that the following two conditions hold:*

(a) *for each $x \in C$, $\lim_{n\to\infty} \|x_n - x\|$ exists;*

(b) *every sequential weak cluster point of $\{x_n\}$ belongs to C.*

Then $\{x_n\}$ converges weakly to a point in C.

Lemma 2.5. ([11, Lemma 2.1]) *Consider problem (VI), where C is a nonempty, closed and convex subset of a real Hilbert space H, and the cost operator $F : C \to H$ is pseudomonotone and continuous. Then we have the following equivalence:*

$$x^* \in \mathrm{Sol}(C, F) \Longleftrightarrow \langle Fx, x - x^* \rangle \geq 0 \quad \forall x \in C.$$

Definition 2.6. [31] *A sequence $\{x_n\}$ in H is said to converge R-linearly to x^* with rate $\rho \in [0, 1)$ if there is a constant $c > 0$ such that*

$$\|x_n - x^*\| \leq c\rho^n \quad \forall n \in \mathbb{N}.$$

3. Weak convergence

To establish and prove our weak convergence theorem, we need the following conditions:

Condition 3.1. *The solution set* $\mathrm{Sol}(C, F)$ *is nonempty.*

Condition 3.2. *The mapping $F : H \to H$ is pseudomonotone on H, that is,*

$$\langle Fx, y - x \rangle \geq 0 \Longrightarrow \langle Fy, y - x \rangle \geq 0 \quad \forall x, y \in H.$$

Condition 3.3. *The mapping $F : H \to H$ is Lipschitz continuous with constant $L > 0$, that is, there exists a number $L > 0$ such that*

$$\|Fx - Fy\| \leq L\|x - y\| \quad \forall x, y \in H.$$

We now present our algorithm.

Algorithm 3.4. *Let $\mu, \delta \in \left(0, 1\right)$, $\lambda \in [0, 1)$ and $\tau_0 > 0$ be given, and let $z_0, z_1 \in H$ be arbitrary. Compute*

$$\begin{aligned} u_n &= z_n + \lambda(z_n - z_{n-1}), \\ y_{n+1} &= P_C(u_n - \tau_n F u_n), \\ z_{n+1} &= y_{n+1} - \tau_n(F y_{n+1} - F u_n). \end{aligned}$$

If

$$\tau_n \|F u_n - F y_{n+1}\| \leq \mu \|u_n - y_{n+1}\| \tag{1}$$

then $\tau_{n+1} := \tau_n$. Else, set $\tau_{n+1} := \delta\tau_n$.

We present the following lemma.

Lemma 3.5. ([20]) *Let $\{\tau_n\}$ be a sequence generated by (1). Then $\{\tau_n\}$ is nonincreasing and bounded away from zero.*

Theorem 3.6. *Assume that Conditions 3.1–3.3 hold and that the mapping $F : H \to H$ satisfies the following condition:*

$$\text{if } \{z_n\} \subset C,\ z_n \rightharpoonup z \text{ and } \liminf_{n\to\infty} \|Fz_n\| = 0 \text{ then } Fz = 0. \tag{2}$$

Assume, in addition, that the parameters λ and μ satisfy the conditions $0 \le \lambda < \frac{\sqrt{5}-1}{2}$ and $\mu < 1 - \lambda - \lambda^2$. Then the sequence $\{z_n\}$ generated by Algorithm 3.4 converges weakly to an element $x^* \in \mathrm{Sol}(C, F)$.

Proof. The proof is divided into four steps as follows:
Step 1.

$$\|z_{n+1} - x^*\|^2 \le \|u_n - x^*\|^2 - \left(1 - \mu^2\right)\|y_{n+1} - u_n\|^2 \quad \forall x^* \in \mathrm{Sol}(C, F). \tag{3}$$

Indeed, we have

$$\begin{aligned}
\|z_{n+1} - x^*\|^2 &= \|y_{n+1} - \tau_n(Fy_{n+1} - Fu_n) - x^*\|^2 \\
&= \|y_{n+1} - x^*\|^2 + \tau_n^2\|Fy_{n+1} - Fu_n\|^2 - 2 - \tau_n\langle y_{n+1} - x^*, Fy_{n+1} - Fu_n\rangle \\
&= \|u_n - x^*\|^2 + \|u_n - y_{n+1}\|^2 + 2\langle y_{n+1} - u_n, u_n - x^*\rangle + \tau_n^2\|Fy_{n+1} - Fu_n\|^2 \\
&\quad - 2 - \tau_n\langle y_{n+1} - x^*, Fy_{n+1} - Fu_n\rangle \\
&= \|u_n - x^*\|^2 + \|u_n - y_{n+1}\|^2 - 2\langle y_{n+1} - u_n, y_{n+1} - u_n\rangle + 2\langle y_{n+1} - u_n, y_{n+1} - x^*\rangle \\
&\quad + \tau_n^2\|Fy_{n+1} - Fu_n\|^2 - 2\tau_n\langle y_{n+1} - x^*, Fy_{n+1} - Fu_n\rangle \\
&= \|u_n - x^*\|^2 - \|u_n - y_{n+1}\|^2 + 2\langle y_{n+1} - u_n, y_{n+1} - x^*\rangle + \tau_n^2\|Fy_{n+1} - Fu_n\|^2 \\
&\quad - 2\tau_n\langle y_{n+1} - x^*, Fy_{n+1} - Fu_n\rangle.
\end{aligned} \tag{4}$$

Since $y_{n+1} = P_C(u_n - \tau_n Fu_n)$, we have

$$\langle y_{n+1} - u_n + \tau_n Fu_n, y_{n+1} - x^*\rangle \le 0$$

or, equivalently,

$$\langle y_{n+1} - u_n, y_{n+1} - x^*\rangle \le -\tau_n\langle Fu_n, y_{n+1} - x^*\rangle. \tag{5}$$

From (28) and (29), it follows that

$$\begin{aligned}
\|z_{n+1} - x^*\|^2 \le& \|u_n - x^*\|^2 - \|u_n - y_{n+1}\|^2 - 2\tau_n\langle Fu_n, y_{n+1} - x^*\rangle + \tau_n^2\|Fy_{n+1} - Fu_n\|^2 \\
&- 2\tau_n\langle y_{n+1} - x^*, Fy_{n+1} - Fu_n\rangle \\
=& \|u_n - x^*\|^2 - \|u_n - y_{n+1}\|^2 + \tau_n^2\|Fy_{n+1} - Fu_n\|^2 - 2\tau_n\langle y_{n+1} - x^*, Fy_{n+1}\rangle.
\end{aligned} \tag{6}$$

Since $x^* \in \mathrm{Sol}(C, F)$, we have $\langle Fx^*, y_{n+1} - x^*\rangle \ge 0$. It now follows from the pseudomonotonicity of the operator F that

$$\langle Fy_{n+1}, y_{n+1} - x^*\rangle \ge 0,$$

which, when combined with (30), implies that

$$\|z_{n+1} - x^*\|^2 = \|u_n - x^*\|^2 - \|u_n - y_{n+1}\|^2 + \tau_n^2\|Fy_{n+1} - Fu_n\|^2. \tag{7}$$

By Lemma 3.5, then there exists $N \in \mathbb{N}$ such that

$$\tau_n\|Fu_n - Fy_{n+1}\| \le \mu\|u_n - y_{n+1}\| \text{ and } \tau_{n+1} = \tau_n = \tau\ \forall n \ge N. \tag{8}$$

Combining (8) with (7), we now obtain

$$\|z_{n+1} - x^*\|^2 \le \|u_n - x^*\|^2 - \left(1 - \mu^2\right)\|y_{n+1} - u_n\|^2,$$

as claimed.

Step 2. Next, we show that the limit

$$\lim_{n\to\infty} \|z_n - x^*\| \text{ exists.}$$

Indeed, by the definition of z_{n+1}, we have

$$\|z_{n+1} - y_{n+1}\| = \|y_{n+1} - \tau_n(Fy_{n+1} - Fu_n) - y_{n+1}\| \le \tau_n\|Fy_{n+1} - Fu_n\| \le \mu\|y_{n+1} - u_n\|.$$

Therefore we have

$$\|z_{n+1} - u_n\| \le \|z_{n+1} - y_{n+1}\| + \|y_{n+1} - u_n\| \le \left(1 + \mu\right)\|y_{n+1} - u_n\|.$$

This implies that

$$\|y_{n+1} - u_n\| \ge \frac{1}{\left(1 + \mu\right)}\|z_{n+1} - u_n\|. \tag{9}$$

Let $x^* \in \mathrm{Sol}(C, F)$. Then, by **Step 1**, we have

$$\|z_{n+1} - x^*\|^2 \le \|u_n - x^*\|^2 - \left(1 - \mu^2\right)\|y_{n+1} - u_n\|^2. \tag{10}$$

It follows from (35) and (10) that

$$\begin{aligned}\|z_{n+1} - x^*\|^2 &\le \|u_n - x^*\|^2 - \frac{\left(1 - \mu^2\right)}{\left(1 + \mu\right)^2}\|z_{n+1} - u_n\|^2 \\ &= \|u_n - x^*\|^2 - \frac{\left(1 - \mu\right)}{\left(1 + \mu\right)}\|z_{n+1} - u_n\|^2. \end{aligned} \tag{11}$$

By the definition of u_n, we have

$$\begin{aligned}\|u_n - x^*\|^2 &= \|z_n + \lambda(z_n - z_{n-1}) - x^*\|^2 \\ &= \|(1 + \lambda)(z_n - x^*) - \lambda(z_{n-1} - x^*)\|^2 \\ &= (1 + \lambda)\|z_n - x^*\|^2 - \lambda\|z_{n-1} - x^*\|^2 + \lambda(1 + \lambda)\|z_n - z_{n-1}\|^2. \end{aligned} \tag{12}$$

It now follows from (36) and (12) that

$$\begin{aligned}\|z_{n+1} - x^*\|^2 &\le (1 + \lambda)\|z_n - x^*\|^2 - \lambda\|z_{n-1} - x^*\|^2 + \lambda(1 + \lambda)\|z_n - z_{n-1}\|^2 \\ &\quad - \frac{\left(1 - \mu\right)}{\left(1 + \mu\right)}\|z_{n+1} - u_n\|^2 \qquad (13) \\ &\le (1 + \lambda)\|z_n - x^*\|^2 - \lambda\|z_{n-1} - x^*\|^2 + \lambda(1 + \lambda)\|z_n - z_{n-1}\|^2. \qquad (14)\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\|z_{n+1} - u_n\|^2 &= \|z_{n+1} - z_n - \lambda(z_n - z_{n-1})\|^2 \\ &= \|z_{n+1} - z_n\|^2 + \lambda^2\|z_n - z_{n-1}\|^2 - 2\lambda\langle z_{n+1} - z_n, z_n - z_{n-1}\rangle \\ &\ge \|z_{n+1} - z_n\|^2 + \lambda^2\|z_n - z_{n-1}\|^2 - 2\lambda\|z_{n+1} - z_n\|\|z_n - z_{n-1}\| \\ &\ge (1 - \lambda)\|z_{n+1} - z_n\|^2 + \left(\lambda^2 - \lambda\right)\|z_n - z_{n-1}\|^2. \end{aligned} \tag{15}$$

Combining (13) and (15), we obtain for all $n \geq N$,

$$\begin{aligned}
\|z_{n+1} - x^*\|^2 &\leq (1+\lambda)\|z_n - x^*\|^2 - \lambda\|z_{n-1} - x^*\|^2 + \lambda(1+\lambda)\|z_n - z_{n-1}\|^2 \\
&\quad - \frac{\left(1-\mu\right)}{\left(1+\mu\right)}(1-\lambda)\|z_{n+1} - z_n\|^2 - \frac{\left(1-\mu\right)}{\left(1+\mu\right)}\left(\lambda^2 - \lambda\right)\|z_n - z_{n-1}\|^2 \\
&= (1+\lambda)\|z_n - x^*\|^2 - \lambda\|z_{n-1} - x^*\|^2 - \frac{\left(1-\mu\right)}{(1+\mu)}(1-\lambda)\|z_{n+1} - z_n\|^2 \\
&\quad + \left[\lambda(1+\lambda) - \frac{\left(1-\mu\right)}{\left(1+\mu\right)}\left(\lambda^2 - \lambda\right)\right]\|z_n - z_{n-1}\|^2 \\
&= (1+\lambda)\|z_n - x^*\|^2 - \lambda\|z_{n-1} - x^*\|^2 - \gamma\|z_{n+1} - z_n\|^2 + \mu\|z_n - z_{n-1}\|^2, \qquad (16)
\end{aligned}$$

where

$$\gamma := \frac{\left(1-\mu\right)}{\left(1+\mu\right)}(1-\lambda), \quad \mu := \left[\lambda(1+\lambda) - \frac{\left(1-\mu\right)}{\left(1+\mu\right)}\left(\lambda^2 - \lambda\right)\right].$$

Since $\mu, \lambda \in [0,1)$, it is not difficut to see that $\mu > 0$. Now set

$$\Gamma_n := \|z_n - x^*\|^2 - \lambda\|z_{n-1} - x^*\|^2 + \mu\|z_n - z_{n-1}\|^2.$$

It follows from (16) that

$$\Gamma_{n+1} - \Gamma_n \leq -(\gamma - \mu)\|z_{n+1} - z_n\|^2 \ \ \forall n \geq N. \qquad (17)$$

We also see that

$$\begin{aligned}
\gamma - \mu &= \left[\frac{\left(1-\mu\right)}{\left(1+\mu\right)}(1-\lambda) - \left(\lambda(1+\lambda) - \frac{\left(1-\mu\right)}{\left(1+\mu\right)}\left(\lambda^2 - \lambda\right)\right)\right] \\
&= \frac{1-\mu}{1+\mu}(1-\lambda) - \left(\lambda(1+\lambda) - \frac{1-\mu}{1+\mu}\left(\lambda^2 - \lambda\right)\right) \\
&= \frac{1-\mu}{1+\mu}(1-\lambda)^2 - \lambda(1+\lambda). \qquad (18)
\end{aligned}$$

Using the hypothesis $0 \leq \lambda < \frac{\sqrt{5}-1}{2}$ and $\mu < -\lambda^2 - \lambda + 1$, we see that $\frac{1-\mu}{1+\mu}(1-\lambda)^2 - \lambda(1+\lambda) < (1-\mu) - \lambda(1+\lambda) = -\lambda^2 - \lambda + 1 - \mu > 0$. This implies that

$$\gamma - \mu > 0.$$

Let $\delta := \gamma - \mu$. Then, combining (17) and (18), we get

$$\Gamma_{n+1} - \Gamma_n \leq -\delta\|z_{n+1} - z_n\|^2 \quad \forall n \geq N. \qquad (19)$$

Hence we have

$$\Gamma_{n+1} - \Gamma_n \leq 0 \quad \forall n \geq N.$$

Thus the sequence $\{\Gamma_n\}$ is decreasing for $n \geq N$. On the other hand, we have

$$\begin{aligned}
\Gamma_n &= \|z_n - x^*\|^2 - \lambda\|z_{n-1} - x^*\|^2 + \mu_n\|z_n - z_{n-1}\|^2 \\
&\geq \|z_n - x^*\|^2 - \lambda\|z_{n-1} - x^*\|^2.
\end{aligned}$$

This implies that

$$\begin{aligned}\|z_n - x^*\|^2 &\le \lambda\|z_{n-1} - x^*\|^2 + \Gamma_n\\ &\le \lambda\|z_{n-1} - x^*\|^2 + \Gamma_N\\ &\le \cdots\\ &\le \lambda^{n-N}\|z_N - x^*\|^2 + \Gamma_N(\lambda^{n-N-1} + \cdots + 1)\\ &\le \lambda^{n-N}\|z_N - x^*\|^2 + \frac{\Gamma_N}{1-\lambda}. \end{aligned} \tag{20}$$

We also have

$$\begin{aligned}\Gamma_{n+1} &= \|z_{n+1} - x^*\|^2 - \lambda\|z_n - x^*\|^2 + \mu\|z_{n+1} - z_n\|^2\\ &\ge -\lambda\|z_n - x^*\|^2.\end{aligned} \tag{21}$$

Using (20) and (21), we get

$$-\Gamma_{n+1} \le \lambda\|z_n - x^*\|^2 \le \lambda^{n-N+1}\|z_N - x^*\|^2 + \frac{\lambda\Gamma_N}{1-\lambda}.$$

It follows from (19) that

$$\begin{aligned}\gamma\sum_{n=N}^{k}\|z_{n+1} - z_n\|^2 &\le \Gamma_N - \Gamma_{k+1}\\ &\le \lambda^{k-N+1}\|z_N - x^*\|^2 + \frac{\Gamma_N}{1-\lambda}\\ &\le \|z_N - x^*\|^2 + \frac{\Gamma_N}{1-\lambda} \quad \forall k > N.\end{aligned}$$

This implies that

$$\sum_{n=1}^{\infty}\|z_{n+1} - z_n\|^2 < +\infty.$$

Using (14) and Lemma 2.3, we now see that

$$\lim_{n\to\infty}\|z_n - x^*\|^2 = l,$$

as claimed.

Step 3. Any sequential weakl cluster point of the sequence $\{z_n\}$ belongs to $\mathrm{Sol}(C, F)$.

Indeed, since $\lim_{n\to\infty}\|z_n - x^*\|$ exists, the sequence $\{z_n\}$ is bounded. We now choose a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \rightharpoonup z$.

We claim that $z \in \mathrm{Sol}(C, F)$. Indeed, since $\sum_{n=1}^{\infty}\|z_{n+1}-z_n\|^2 < +\infty$, it immediately follows that $\|z_{n+1}-z_n\| \to 0$. On the other hand, we have

$$\|z_{n+1} - u_n\|^2 = \|z_{n+1} - z_n\|^2 + \lambda^2\|z_n - z_{n-1}\|^2 - 2\lambda\langle z_{n+1} - z_n, z_n - z_{n-1}\rangle$$

and so we also have $\|z_{n+1} - u_n\| \to 0$. Using (12), we obtain

$$\lim_{n\to\infty}\|u_n - x^*\|^2 = l.$$

On the other hand, by (27), we get

$$(1-\mu^2)\|y_{n+1} - u_n\|^2 \le \|u_n - x^*\|^2 - \|z_{n+1} - x^*\|^2.$$

This implies that

$$\lim_{n\to\infty} \|y_{n+1} - u_n\| = 0.$$

We also have $u_n = z_n + \lambda(z_n - z_{n-1})$, which implies that

$$\|u_n - z_n\|^2 = \lambda^2 \|z_n - z_{n-1}\|^2 \to 0 \text{ as } n \to \infty.$$

Thus we obtain

$$\lim_{n\to\infty} \|u_n - z_n\| = 0.$$

From $z_{n_k} \rightharpoonup z$, $\lim_{n\to\infty} \|u_n - z_n\| = 0$ and $\lim_{n\to\infty} \|u_n - y_{n+1}\| = 0$, it follows that $u_{n_k} \rightharpoonup z$ and $y_{n_k+1} \rightharpoonup z$. We also have

$$\langle u_{n_k} - \tau_{n_k} F u_{n_k} - y_{n_k+1}, x - y_{n_k+1} \rangle \leq 0 \quad \forall x \in C,$$

or, equivalently,

$$\frac{1}{\tau_{n_k}} \langle u_{n_k} - y_{n_k+1}, x - y_{n_k+1} \rangle \leq \langle F u_{n_k}, x - y_{n_k+1} \rangle, \quad \forall x \in C.$$

Consequently, we have

$$\frac{1}{\tau_{n_k}} \langle u_{n_k} - y_{n_k+1}, x - y_{n_k+1} \rangle + \langle F u_{n_k}, y_{n_k+1} - u_{n_k} \rangle \leq \langle F u_{n_k}, x - u_{n_k} \rangle \quad \forall x \in C. \tag{22}$$

Since the sequence $\{u_{n_k}\}$ is weakly convergent, it is bounded. Since the operator F is Lipschitz continuous, it follows that the sequence $\{F u_{n_k}\}$ is bounded too. Since $\|u_{n_k} - y_{n_k+1}\| \to 0$, it follows that the sequence $\{y_{n_k+1}\}$ is also bounded. We also have $\tau_{n_k} \geq \min\{\tau_1, \frac{\mu}{L}\}$. Passing to the limit in (22) as $k \to \infty$, we get

$$\liminf_{k\to\infty} \langle F u_{n_k}, x - u_{n_k} \rangle \geq 0 \quad \forall x \in C. \tag{23}$$

Moreover, we have

$$\langle F y_{n_k+1}, x - y_{n_k+1} \rangle = \langle F y_{n_k+1} - F u_{n_k}, x - u_{n_k} \rangle + \langle F u_{n_k}, x - u_{n_k} \rangle + \langle F y_{n_k+1}, u_{n_k} - y_{n_k+1} \rangle. \tag{24}$$

Since $\lim_{k\to\infty} \|u_{n_k} - y_{n_k+1}\| = 0$ and F is L-Lipschitz continuous on H, it follows that

$$\lim_{k\to\infty} \|F u_{n_k} - F y_{n_k+1}\| = 0,$$

which, when combined with (23) and (24) implies that

$$\liminf_{k\to\infty} \langle F y_{n_k+1}, x - y_{n_k+1} \rangle \geq 0.$$

Next, we show that $z \in \mathrm{Sol(C, F)}$. We consider the following possible cases:

Case I: Suppose $\liminf_{k\to\infty} \|F y_{n_k+1}\| = 0$. Since $y_{n_k+1} \rightharpoonup z$ and by condition (2), we deduce that $Fz = 0$. Hence $z \in \mathrm{Sol(C, F)}$.

Case II: Let $\liminf_{k\to\infty} \|F y_{n_k+1}\| > 0$. We choose a decreasing sequence $\{\epsilon_k\}$ of positive numbers which tends to 0. For each $k \geq 0$, we denote by N_k the smallest positive integer such that

$$\langle F y_{n_j}, x - y_{n_j} \rangle + \epsilon_k \geq 0 \quad \forall j \geq N_k. \tag{25}$$

Since $\{\epsilon_k\}$ is decreasing, it is not difficult to see that the sequence $\{N_k\}$ is increasing. Furthermore, since $\{y_{N_k+1}\} \subset C$, we may suppose that $Fy_{N_k+1} \neq 0$ for each $k \geq 0$ (otherwise, y_{N_k+1} is a solution) and so, setting

$$v_{N_k} = \frac{Fy_{N_k+1}}{\|Fy_{N_k+1}\|^2},$$

we have $\langle Fy_{N_k+1}, v_{N_k}\rangle = 1$ for each $k \geq 0$. Now, we can deduce from (25) that, for each $k \geq 0$,

$$\langle Fy_{N_k+1}, x + \epsilon_k v_{N_k} - y_{N_k+1}\rangle \geq 0.$$

Since F is pseudomonotone on H, we get

$$\langle F(x + \epsilon_k v_{N_k}), x + \epsilon_k v_{N_k} - y_{N_k+1}\rangle \geq 0.$$

This implies that

$$\langle Fx, x - y_{N_k+1}\rangle \geq \langle Fx - F(x + \epsilon_k v_{N_k}), x + \epsilon_k v_{N_k} - y_{N_k+1}\rangle - \epsilon_k\langle Fx, v_{N_k}\rangle. \tag{26}$$

Next, we claim that $\lim_{k\to\infty} \epsilon_k v_{N_k} = 0$. Indeed, since $u_{n_k} \rightharpoonup z$ and $\lim_{k\to\infty} \|u_{n_k} - y_{n_k+1}\| = 0$, we obtain $y_{N_k+1} \rightharpoonup z$ as $k \to \infty$. On the other hand, since $\{y_{N_k+1}\} \subset \{y_{n_k+1}\}$ and $\epsilon_k \to 0$ as $k \to \infty$, we obtain

$$0 \leq \limsup_{k\to\infty} \|\epsilon_k v_{N_k}\| = \limsup_{k\to\infty} \left(\frac{\epsilon_k}{\|Fy_{n_k+1}\|}\right) \leq \frac{\limsup_{k\to\infty} \epsilon_k}{\liminf_{k\to\infty} \|Fy_{n_k+1}\|} = 0,$$

which implies that $\lim_{k\to\infty} \epsilon_k v_{N_k} = 0$, as claimed.

Now, letting $k \to \infty$, we see that the right-hand side of (26) tends to zero because F is uniformly continuous, the sequences $\{u_{N_k}\}$ and $\{v_{N_k}\}$ are bounded, and $\lim_{k\to\infty} \epsilon_k v_{N_k} = 0$. Thus we get

$$\liminf_{k\to\infty} \langle Fx, x - y_{N_k+1}\rangle \geq 0$$

and hence, for all $x \in C$, we have

$$\langle Fx, x - z\rangle = \lim_{k\to\infty} \langle Fx, x - y_{N_k+1}\rangle = \liminf_{k\to\infty} \langle Fx, x - y_{N_k+1}\rangle \geq 0.$$

Using Lemma 2.5, we can now conclude that $z \in \mathrm{Sol(C, F)}$, as claimed.

Step 4. we claim that the sequence $\{z_n\}$ converges weakly to some point in $\mathrm{Sol}(C, F)$. Indeed, we have already shown that, for every $x^* \in \mathrm{Sol(C, F)}$, the limit $\lim_{n\to\infty} \|z_n - x^*\|$ exists and that each sequential weak cluster point of the sequence $\{z_n\}$ belongs to $\mathrm{Sol(C, F)}$. Invoking Lemma 2.4, we see that the sequence $\{z_n\}$ converges weakly to an element in $\mathrm{Sol(C, F)}$, as asserted. This completes the proof. □ □

Remark 3.7. *In comparison with Theorem 3.1 in [37], our Theorem 3.6 provides the following improvements:*

- *The monotonicity of is replaced by its pseudo-monotonicity on H.*
- *The fixed step size is replaced by self-adaptive step size rule that is allowed our algorithm does not require the prior knowledge of the Lipschitz constant of the variational inequality mapping.*

4. Convergence rate of Algorithm 3.4

In this section, under some mild assumptions we present the linear convergence rate of Algorithm 3.4 when the cost function is strongly-pseudomonotone and Lipschitz continuous.

Theorem 4.1. *Assume that $F : H \to H$ is L-Lipschitz continuous on H and κ-strongly pseudo-monotone on C. Let $\theta, \gamma \in (0, 1)$ and λ be such that*

$$0 \leq \lambda \leq \min\left\{\frac{\xi}{\xi + 2}, \frac{\sqrt{(1 + \gamma\xi)^2 + 4\gamma\xi} - (1 + \gamma\xi)}{2}, (1 - \gamma)\left(1 - \frac{(1 - \mu)\theta}{2}\right)\right\}$$

where $\xi := \dfrac{1 - \mu}{1 + \mu}(1 - \theta)$. Then the sequence $\{x_n\}$ is generated by Algorithm 3.4 converges in norm to the unique solution x^* of the problem (VI) *with an R-linear rate.*

Proof. First, we claim the following inequality.

$$\|z_{n+1} - x^*\|^2 \leq \|u_n - x^*\|^2 - (1-\mu^2)\|y_{n+1} - u_n\|^2 - 2\tau_n\kappa\|y_{n+1} - x^*\|^2. \tag{27}$$

Indeed, we have

$$\begin{aligned}\|z_{n+1} - x^*\|^2 &= \|y_{n+1} - \tau_n(Fy_{n+1} - Fu_n) - x^*\|^2\\ &= \|y_{n+1} - x^*\|^2 + \tau_n^2\|Fy_{n+1} - Fu_n\|^2 - 2 - \tau_n\langle y_{n+1} - x^*, Fy_{n+1} - Fu_n\rangle\\ &= \|u_n - x^*\|^2 + \|u_n - y_{n+1}\|^2 + 2\langle y_{n+1} - u_n, u_n - x^*\rangle + \tau_n^2\|Fy_{n+1} - Fu_n\|^2\\ &\quad - 2 - \tau_n\langle y_{n+1} - x^*, Fy_{n+1} - Fu_n\rangle\\ &= \|u_n - x^*\|^2 + \|u_n - y_{n+1}\|^2 - 2\langle y_{n+1} - u_n, y_{n+1} - u_n\rangle + 2\langle y_{n+1} - u_n, y_{n+1} - x^*\rangle\\ &\quad + \tau_n^2\|Fy_{n+1} - Fu_n\|^2 - 2\tau_n\langle y_{n+1} - x^*, Fy_{n+1} - Fu_n\rangle\\ &= \|u_n - x^*\|^2 - \|u_n - y_{n+1}\|^2 + 2\langle y_{n+1} - u_n, y_{n+1} - x^*\rangle + \tau_n^2\|Fy_{n+1} - Fu_n\|^2\\ &\quad - 2\tau_n\langle y_{n+1} - x^*, Fy_{n+1} - Fu_n\rangle. \end{aligned} \tag{28}$$

Since $y_{n+1} = P_C(u_n - \tau_n Fu_n)$, it holds

$$\langle y_{n+1} - u_n + \tau_n Fu_n, y_{n+1} - x^*\rangle \leq 0$$

or equivalently,

$$\langle y_{n+1} - u_n, y_{n+1} - x^*\rangle \leq -\tau_n\langle Fu_n, y_{n+1} - x^*\rangle. \tag{29}$$

From (28) and (29), it follows that

$$\begin{aligned}\|z_{n+1} - x^*\|^2 \leq &\|u_n - x^*\|^2 - \|u_n - y_{n+1}\|^2 - 2\tau_n\langle Fu_n, y_{n+1} - x^*\rangle + \tau_n^2\|Fy_{n+1} - Fu_n\|^2\\ &- 2\tau_n\langle y_{n+1} - x^*, Fy_{n+1} - Fu_n\rangle\\ = &\|u_n - x^*\|^2 - \|u_n - y_{n+1}\|^2 + \tau_n^2\|Fy_{n+1} - Fu_n\|^2 - 2\tau_n\langle y_{n+1} - x^*, Fy_{n+1}\rangle. \end{aligned} \tag{30}$$

Since x^* is the solution of VI(C,F), we have $\langle Fx^*, x - x^*\rangle \geq 0$ for all $x \in C$. By the strong pseudomontonicity of F on C we have $\langle Fx, x - x^*\rangle \geq \kappa\|x - x^*\|^2$ for all $x \in C$.
Taking $x := y_{n+1} \in C$ we get

$$\langle Fy_{n+1}, x^* - y_{n+1}\rangle \leq -\kappa\|y_{n+1} - x^*\|^2. \tag{31}$$

From (30) and (31) we obtain

$$\begin{aligned}\|z_{n+1} - x^*\|^2 \leq &\|u_n - x^*\|^2 - \|u_n - y_{n+1}\|^2 + \tau_n^2\|Fy_{n+1} - Fu_n\|^2 - 2\tau_n\langle y_{n+1} - x^*, Fy_{n+1}\rangle\\ \leq &\|u_n - x^*\|^2 - \|u_n - y_{n+1}\|^2 + \tau_n^2\|Fy_{n+1} - Fu_n\|^2 - 2\tau_n\kappa\|y_{n+1} - x^*\|^2. \end{aligned} \tag{32}$$

Moreover, using (8) we have

$$\|Fu_n - Fy_{n+1}\| \leq \frac{\mu}{\tau_n}\|u_n - y_{n+1}\| \ \ \forall n. \tag{33}$$

Combining (32) and (33), we obtain

$$\|z_{n+1} - x^*\|^2 \leq \|u_n - x^*\|^2 - (1-\mu^2)\|y_{n+1} - u_n\|^2 - 2\tau_n\kappa\|y_{n+1} - x^*\|^2.$$

Next, we show that there exists $N \in \mathbb{N}$ and $\rho, \xi \in (0,1)$ such that

$$\|z_{n+1} - x^*\|^2 \leq \rho\|u_n - x^*\|^2 - \xi\|z_{n+1} - u_n\|^2 \ \ \forall n \geq N.$$

Indeed, thanks to the inequality 27, we have

$$\|z_{n+1} - x^*\|^2 \le \|u_n - x^*\|^2 - (1 - \mu^2)\|y_{n+1} - u_n\|^2 - 2\tau_n\kappa\|y_{n+1} - x^*\|^2.$$

Hence for any $\theta \in (0, 1)$ we can deduce

$$\|z_{n+1} - x^*\|^2 \le \|u_n - x^*\|^2 - (1 - \mu^2)(1 - \theta)\|y_{n+1} - u_n\|^2 - (1 - \mu^2)\theta\|y_{n+1} - u_n\|^2 - 2\tau_n\kappa\|y_{n+1} - x^*\|^2. \tag{34}$$

By the definition of z_{n+1} we have

$$\begin{aligned}\|z_{n+1} - y_{n+1}\| &= \|y_{n+1} - \tau_n(Fy_{n+1} - Fu_n) - y_{n+1}\| \\ &\le \tau_n\|Fy_{n+1} - Fu_n\| \\ &\le \mu\|y_{n+1} - u_n\|.\end{aligned}$$

Therefore

$$\|z_{n+1} - u_n\| \le \|z_{n+1} - y_{n+1}\| + \|y_{n+1} - u_n\| \le (1 + \mu)\|y_{n+1} - u_n\|.$$

This implies

$$\|y_{n+1} - u_n\| \ge \frac{1}{(1 + \mu)}\|z_{n+1} - u_n\|. \tag{35}$$

Substituting (35) into (34) we have for all n that

$$\begin{aligned}\|z_{n+1} - x^*\|^2 \le& \|u_n - x^*\|^2 - \frac{(1 - \mu^2)}{\left(1 + \mu\right)^2}(1 - \theta)\|z_{n+1} - u_n\|^2 - (1 - \mu^2)\theta\|y_{n+1} - u_n\|^2 - 2\tau_n\kappa\|y_{n+1} - x^*\|^2 \\ =& \|u_n - x^*\|^2 - \frac{\left(1 - \mu\right)}{\left(1 + \mu\right)}(1 - \theta)\|z_{n+1} - u_n\|^2 - \left(1 - \mu^2\right)\theta\|y_{n+1} - u_n\|^2 - 2\tau_n\kappa\|y_{n+1} - x^*\|^2 \end{aligned} \tag{36}$$

Using (8) then there exists $N \in \mathbb{N}$ such that $\tau_{n+1} = \tau_n = \tau$ for all $n \ge N$. Therefore, we deduce

$$\|z_{n+1} - x^*\|^2 \le \|u_n - x^*\|^2 - \frac{\left(1 - \mu\right)}{\left(1 + \mu\right)}(1 - \theta)\|z_{n+1} - u_n\|^2 - \left(1 - \mu^2\right)\theta\|y_{n+1} - u_n\|^2 - 2\tau\kappa\|y_{n+1} - x^*\|^2 \quad \forall n \ge N. \tag{37}$$

Let $\lambda := \min\left\{\frac{(1 - \mu^2)\theta}{2}, \tau\kappa\right\}$, we have

$$\left(1 - \mu^2\right)\theta \ge 2\lambda, \quad \text{and } \tau\kappa \ge \lambda.$$

Thus, using (37) we get for all $n \ge N$ that

$$\begin{aligned}\|z_{n+1} - x^*\|^2 \le& \|u_n - x^*\|^2 - \frac{1 - \mu}{1 + \mu}(1 - \theta)\|z_{n+1} - u_n\|^2 - 2\lambda(\|y_{n+1} - u_n\|^2 + \|y_{n+1} - x^*\|^2) \\ \le& \|u_n - x^*\|^2 - \frac{1 - \mu}{1 + \mu}(1 - \theta)\|z_{n+1} - u_n\|^2 - \lambda\|u_n - x^*\|^2 \\ \le& (1 - \lambda)\|u_n - x^*\|^2 - \frac{1 - \mu}{1 + \mu}(1 - \theta)\|z_{n+1} - u_n\|^2 \\ =& \rho\|u_n - x^*\|^2 - \xi\|z_{n+1} - u_n\|^2,\end{aligned} \tag{38}$$

where $\rho := 1 - \lambda \in (0, 1)$ and $\xi := \dfrac{1-\mu}{1+\mu}(1-\theta) \in (0, 1)$. Now, we prove that the iterative sequence generated by Algorithm 3.4 converges R-linearly to the unique solution of the problem Sol(C,F). We have

$$\begin{aligned}||u_n - x^*||^2 &= ||(1+\lambda)(z_n - x^*) - \lambda(z_{n-1} - x^*)||^2\\ &= (1+\lambda)||z_n - x^*||^2 - \lambda||z_{n-1} - x^*||^2 + \lambda(1+\lambda)||z_n - z_{n-1}||^2\end{aligned}$$

and

$$\begin{aligned}||z_{n+1} - u_n||^2 &= ||z_{n+1} - z_n - \lambda(z_n - z_{n-1})||^2\\ &= ||z_{n+1} - z_n||^2 + \lambda^2||z_n - z_{n-1}||^2 - 2\lambda\langle z_{n+1} - z_n, z_n - z_{n-1}\rangle\\ &\geq ||z_{n+1} - z_n||^2 + \lambda^2||z_n - z_{n-1}||^2 - 2\lambda||z_{n+1} - z_n||||z_n - z_{n-1}||\\ &\geq ||z_{n+1} - z_n||^2 + \lambda^2||z_n - z_{n-1}||^2 - \lambda||z_{n+1} - z_n||^2 - \lambda||z_n - z_{n-1}||^2\\ &\geq (1-\lambda)||z_{n+1} - z_n||^2 - \lambda(1-\lambda)||z_n - z_{n-1}||^2.\end{aligned}$$

Combining these inequalities with (38) we obtain

$$\begin{aligned}||z_{n+1} - x^*||^2 \leq{}& \rho(1+\lambda)||z_n - x^*||^2 - \rho\lambda||z_{n-1} - x^*||^2 + \rho\lambda(1+\lambda)||z_n - z_{n-1}||^2\\ &- \xi(1-\lambda)||z_{n+1} - z_n||^2 + \xi\lambda(1-\lambda)||z_n - z_{n-1}||^2 \quad \forall n \geq N,\end{aligned}$$

or equivalently

$$\begin{aligned}||z_{n+1} - x^*||^2 - \rho\lambda||z_n - x^*||^2 &+ \xi(1-\lambda)||z_{n+1} - z_n||^2\\ &\leq \rho\left[||z_n - x^*||^2 - \lambda||z_{n-1} - x^*||^2 + \xi(1-\lambda)||z_n - z_{n-1}||^2\right]\\ &\quad - (\rho\xi(1-\lambda) - \rho\lambda(1+\lambda) - \xi\lambda(1-\lambda))\,||z_n - z_{n-1}||^2 \quad \forall n \geq N.\end{aligned}$$

Setting

$$\Gamma_n := ||z_n - x^*||^2 - \lambda||z_{n-1} - x^*||^2 + \xi(1-\lambda)||z_n - z_{n-1}||^2,$$

since $\rho \in (0, 1)$ we can write

$$\begin{aligned}\Gamma_{n+1} &\leq ||z_{n+1} - x^*||^2 - \rho\lambda||z_n - x^*||^2 + \xi(1-\lambda)||z_{n+1} - z_n||^2\\ &\leq \rho\Gamma_n - (\rho\xi(1-\lambda) - \rho\lambda(1+\lambda) - \xi\lambda(1-\lambda))\,||z_n - z_{n-1}||^2 \quad \forall n \geq N. \end{aligned} \tag{39}$$

Now, we show that $\Gamma_n \geq 0$ for all n. Indeed, using condition (**??**) we have $\frac{1}{2}\xi(1-\lambda) - \lambda \geq 0$ and

$$\begin{aligned}\Gamma_n &= ||z_n - x^*||^2 - \lambda||z_{n-1} - x^*||^2 + \xi(1-\lambda)||z_n - z_{n-1}||^2\\ &= (1 - \xi(1-\lambda))||z_n - x^*||^2 - \lambda||z_{n-1} - x^*||^2 + \xi(1-\lambda)(||z_n - z_{n-1}||^2 + ||z_n - x^*||^2)\\ &\geq (1 - \xi(1-\lambda))||z_n - x^*||^2 - \lambda||z_{n-1} - x^*||^2 + \frac{1}{2}\xi(1-\lambda)||z_{n-1} - x^*||^2\\ &\geq (1 - \xi(1-\lambda))||z_n - x^*||^2 + [\frac{1}{2}\xi(1-\lambda) - \lambda]||z_{n-1} - x^*||^2\\ &\geq (1 - \xi(1-\lambda))||z_n - x^*||^2 \geq 0.\end{aligned}$$

Now, we prove that

$$\Gamma_{n+1} \leq \rho\Gamma_n.$$

Note that from (**??**) we have

$$\begin{aligned}\lambda &\leq (1-\gamma)\left(1 - \frac{(1-\mu)\theta}{2}\right)\\ &\leq (1-\gamma)(1-\alpha) = (1-\gamma)\rho,\end{aligned}$$

which implies

$$\xi\lambda(1-\lambda) \le (1-\gamma)\rho\xi(1-\lambda) = \rho\xi(1-\lambda) - \gamma\rho\xi(1-\lambda). \tag{40}$$

Since

$$\lambda \le \frac{\sqrt{(1+\gamma\xi)^2 + 4\gamma\xi} - (1+\gamma\xi)}{2}$$

it holds

$$\lambda^2 + (1+\gamma\xi)\lambda - \gamma\xi \le 0,$$

or equivalently

$$\lambda(1+\lambda) \le \gamma\xi(1-\lambda).$$

Hence

$$\rho\lambda(1+\lambda) \le \rho\gamma\xi(1-\lambda). \tag{41}$$

From (40) and (41) we deduce

$$\rho\xi(1-\lambda) - \rho\lambda(1+\lambda) - \xi\lambda(1-\lambda) \ge 0. \tag{42}$$

Combining (39) and (42) we deduce

$$\Gamma_{n+1} \le \rho\Gamma_n.$$

Thus

$$\Gamma_{n+1} \le \rho\Gamma_n \le ... \le \rho^{n-N+1}\Gamma_N,$$

that is

$$\|z_n - x^*\|^2 \le \frac{\Gamma_N}{\rho^N(1-\xi(1-\lambda))}\rho^n,$$

which implies that $\{z_n\}$ converges R-linearly to x^*, the unique solution of Sol(C,F). □

Remark 4.2. *Theorem 4.1 don't need to use condition* (2).

5. Numerical Illustrations

In this section, we present some numerical experiments in solving pseudomontone variational inequality problems. We compare our proposed algorithm with some well-known algorithms including Algorithms: TEGM [40], Algorithm 3.5 [37], the self-adaptive subgradient extragradient algorithm of Gibali [17] to prove the practicability of our proposed algorithm. All the numerical experiments are performed on an HP laptop with Intel(R) Core(TM)i5-6200U CPU 2.3GHz with 4 GB RAM. The programs are written in Matlab2015a.

Example 5.1. *Assume that $F : \mathbb{R}^m \to \mathbb{R}^m$ is defined by $F(x) = Mx + q$ with $M = NN^T + S + D$, N is an $m \times m$ matrix, S is an $m \times m$ skew-symmetric matrix, D is an $m \times m$ diagonal matrix , whose diagonal entries are positive (so M is positive definite), q is a vector in $\mathbb{R}^m$. The feasible set C is given by*

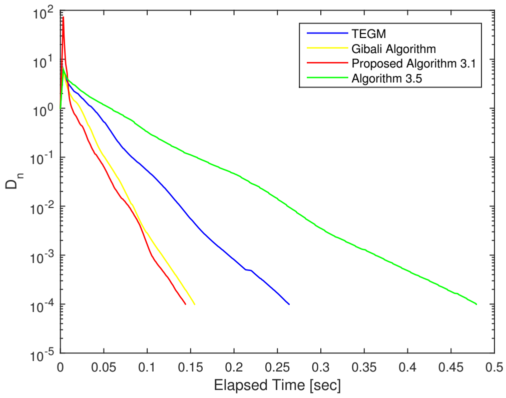
$$C := \{x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m : -1 \le x_i \le 1, i = 1, \cdots m\}.$$

It is clear that F is pseudomonotone and Lipschitz continuous with the Lipschitz constant $L = \|M\|$.

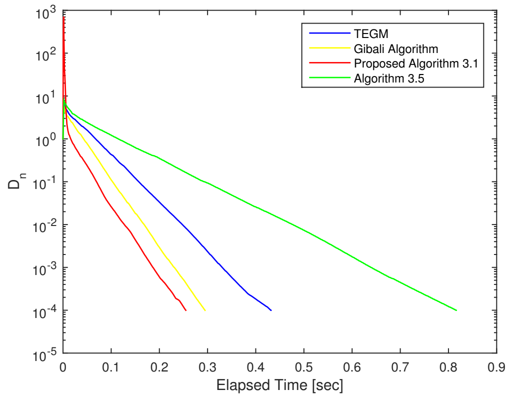
For the experiment N, S, D are randomly generated matrices such that S is skew-symmetric, D is a positive definite diagonal matrix. The process is started with the initial $x_0 = (1, ..., 1)^T \in \mathbb{R}^m$ and $x_1 = 0.5x_0$. To terminate algorithms, we use the condition $D_n = \|x_n\| \le \epsilon$ with $\epsilon = 10^{-4}$ or the number of iterations ≥ 2000 for all algorithms. We choose parameters as follows:

Proposed Algorithm 3.1.: $\lambda = 0.05, \tau_0 = 0.1, \delta = 0.5, \nu = 0.8$;
Tseng's extragradient method (TEGM) [40]: $\lambda = \frac{0.4}{\|M\|}$;
Algorithm 3.5 [37]: $\alpha_k = 0.05, \gamma = 1, l = 0.5, \nu = 0.8$;
Algorithm 3.1 of Gibali [17]: $\alpha_0 = 0.1, \beta = 0.5, \epsilon = 0.2$.

The numerical results are described in Figs. 1.



(a) Comparison of the CPU time of all Algorithms in Example 5.1 with m=50.



(b) Comparison of the CPU time of all Algorithms in Example 5.1 with m=80.

Figure 1: Comparison results of all Algorithms in Example 5.1.

Figure 1 compares the errors and execution times for the proposed algorithm against TEGM in [40], Algorithm 3.5 in [37], Algorithm of Gibali in [17]. The results demonstrate the superior performance of our algorithm compared to these existing methods.

6. Conclusions

In this work we have proposed a variant of the inertial extragradient algorithm which is called the inertial Tseng method for solving the variational inequality problem in real Hilbert spaces. First, we have presented weak convergence of the sequence generated by the proposed algorithm under the assumptions of pseudomonotonicity and Lipschitz continuity of the variational inequality operator. Second, the strong convergence theorem of this algorithm is also proved even with an R-linear rate of convergence, under strong pseudomonotonicity and Lipschitz continuity hypotheses. Our algorithm improves recent related results in the literature.

Acknowledgments

We would like to thank the Editor and the anonymous reviewers for their comments on our manuscript. These comments very much helped us revise and improve the original version of our paper. This research is funded by the National Economics University, Hanoi, Vietnam.

Conflict of interest

The authors declare that they have no conflict of interest.

References

[1] R. Abaidoo, E.K. Agyapong, Financial development and institutional quality among emerging economies, J. Econ. Dev. **24** (2022), 198-216.

[2] F. Alvarez, H. Attouch, An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping, Set-Valued Anal. **9** (2001), 3-11.

[3] A.S. Antipin, On a method for convex programs using a symmetrical modification of the Lagrange function, Ekonomika i Matematicheskie Metody. **12** (1976), 1164-1173.

[4] J.P. Aubin, I., Ekeland, Applied Nonlinear Analysis, Wiley, New York, 1984.

[5] C. Baiocchi, A. Capelo, Variational and Quasivariational Inequalities; Applications to Free Boundary Problems, Wiley, New York, 1984.

[6] R.I. Boţ, E.R. Csetnek, P.T. Vuong, The forward-backward-forward method from discrete and continuous perspective for pseudo-monotone variational inequalities in Hilbert spaces, Europ. J. Oper. Res. **287** (2020), 49-60.

[7] L.C. Ceng, N. Hadjisavvas, N.C. Wong, Strong convergence theorem by a hybrid extragradient-like approximation method for variational inequalities and fixed point problems, J. Glob. Optim. **46** (2010), 635-646.

[8] Y. Censor, A. Gibali, S. Reich, The subgradient extragradient method for solving variational inequalities in Hilbert spaces, J. Optim. Theory Appl. **148** (2011), 318-335.

[9] Y. Censor, A. Gibali, S. Reich, Strong convergence of subgradient extragradient methods for the variational inequality problem in Hilbert space, Optim. Meth. Softw. **26** (2011), 827-845.

[10] Y. Censor, A. Gibali, S. Reich, Extensions of Korpelevich's extragradient method for the variational inequality problem in Euclidean space, Optimization. **61** (2012), 1119-1132.

[11] R.W. Cottle, J.C. Yao, Pseudo-monotone complementarity problems in Hilbert space, J. Optim. Theory Appl. **75** (1992), 281-295

[12] S.V. Denisov, V.V. Semenov, L.M. Chabak, Convergence of the modified extragradient method for variational inequalities with non-Lipschitz operators, Cybern. Syst. Anal. **51** (2015), 757-765.

[13] Q.L. Dong, Y.Y. Lu, J.F. Yang, The extragradient algorithm with inertial effects for solving the variational inequality, Optimization. **65** (2016), 2217-2226.

[14] Q.L. Dong, H.B. Yuan, Y.J. Cho, Th.M. Rassias, Modified inertial Mann algorithm and inertial CQ-algorithm for nonexpansive mappings, Optim. Lett. **12** (2018), 87-102.

[15] Q.L. Dong, Y.J. Cho, L.L. Zhong, Th.M. Rassias, Inertial projection and contraction algorithms for variational inequalities, J. Glob. Optim. **70** (2018), 687-704.

[16] F. Facchinei, J.S. Pang, Finite-Dimensional Variational Inequalities and Complementarity Problems. Volume I. Springer Series in Operations Research, Springer, New York, 2003.

[17] A. Gibali, A new non-Lipschitzian projection method for solving variational inequalities in Euclidean spaces, J. Nonlinear Anal. Optim. **6** (2015), 41-51.

[18] A. Gibali, S. Reich, R. Zalas, Outer approximation methods for solving variational inequalities in Hilbert space, Optimization. **66** (2017), 417-437.

[19] K. Goebel, S. Reich, Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, Marcel Dekker, New York, 1984.

[20] T.N. Hai, Two modified extragradient algorithms for solving variational inequalities, J. Global Optim. **78** (2020), 91- 106.

[21] S. Karamardian, S. Schaible, Seven kinds of monotone maps, J. Optim. Theory Appl. **66** (1990), 37-46.

[22] D. Kinderlehrer, G. Stampacchia, An Introduction to Variational Inequalities and Their Applications, Academic Press, New York, 1980.

[23] R. Kraikaew, S. Saejung, Strong convergence of the Halpern subgradient extragradient method for solving variational inequalities in Hilbert spaces, J. Optim. Theory Appl. **163** (2014), 399-412.

[24] I.V. Konnov, Combined Relaxation Methods for Variational Inequalities, Springer-Verlag, Berlin, 2001.

[25] I.V. Konnov, Equilibrium formulations of relative optimization problems, Math. Meth. Oper. Res. **90** (2019), 137-152.

[26] G.M. Korpelevich, The extragradient method for finding saddle points and other problems, Ekonomikai Matematicheskie Metody. **12** (1976), 747-756.

[27] Y. Luo, J. Fan, S. Li, A self-adaptive inertial algorithm for bilevel pseudo-monotone variational inequality problems with non-Lipschitz mappings, Fixed Point Theory **25** (2024), 213-228.

[28] P.E. Maingé, A hybrid extragradient-viscosity method for monotone operators and fixed point problems, SIAM J. Control Optim. **47** (2008), 1499-1515.

[29] Y.V. Malitsky, V.V. Semenov, A hybrid method without extrapolation step for solving variational inequality problems, J. Glob. Optim. **61** (2015), 193-202.

[30] Y.V. Malitsky, Projected reflected gradient methods for monotone variational inequalities, SIAM J. Optim. **25** (2015), 502-520.

[31] J. M. Ortega, W. C. Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, New York, 1970.

[32] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. **73** (1967), 591-597.

[33] S. Reich, D.V. Thong, P. Cholamjiak, L.V. Long, Inertial projection-type methods for solving pseudomonotone variational inequality problems in Hilbert space, Numer Algor. **88** (2021), 813-835.

[34] Y. Shehu, O.S. Iyiola, D.V. Thong, N.T.C. Van, An inertial subgradient extragradient algorithm extended to pseudomonotone equilibrium problems, Math. Meth. Oper. Res. **93** (2021), 213-242.

[35] M.V. Solodov, B.F. Svaiter, A new projection method for variational inequality problems, SIAM J. Control Optim. **37** (1999), 765-776.

[36] D.V. Thong, D.V. Hieu, Weak and strong convergence theorems for variational inequality problems, Numer. Algor. **78** (2018), 1045-1060.

[37] D.V. Thong, D.V. Hieu, Modified Tseng's extragradient algorithms for variational inequality problems. J. Fixed Point Theory Appl. 2018, 20:152.

[38] D.V. Thong, P.T. Vuong, Modified Tseng's extragradient methods for solving pseudo-monotone variational inequalities, Optimziation. **68** (2019), 2207-2226.

[39] D.V. Thong, P.T. Vuong, R-linear convergence analysis of inertial extragradient algorithms for strongly pseudo-monotone variational inequalities, J. Comput. Appl. Math. **406** (2022), 114003.

[40] P. Tseng, A modified forward-backward splitting method for maximal monotone mappings, SIAM J. Control Optim. **38** (2000), 431-446.

[41] P.T. Vuong, On the weak convergence of the extragradient method for solving pseudo-monotone variational inequalities, J. Optim. Theory Appl. **176** (2018), 399-409.

[42] P.T. Vuong, Y. Shehu, Convergence of an extragradient-type method for variational inequality with applications to optimal control

problems. Numer. Algor. **81** (2019), 269-291.
[43] F.H. Wang, H.K. Xu, Weak and strong convergence theorems for variational inequality and fixed point problems with Tseng's extragradient method, Taiwan. J. Math. **16** (2012), 1125-1136.
[44] Y.M. Wang, Y.B. Xiao, X. Wang, Y.J. Cho, Equivalence of well-posedness between systems of hemivariational inequalities and inclusion problems, J. Nonlinear Sci. Appl. **9** (2016), 1178-1192.
[45] Y.B. Xiao, N.J. Huang, Y.J. Cho, A class of generalized evolution variational inequalities in Banach space, Appl. Math. Lett. **25** (2012), 914–920.
[46] Y. Yao, G. Marino, L. Muglia, A modified Korpelevich's method convergent to the minimum-norm solution of a variational inequality, Optimization. **63** (2014), 559-569.
[47] J. Yang, H. Liu, Strong convergence result for solving monotone variational inequalities in Hilbert space, Numer. Algor. **80** (2019), 741-752.
[48] J. Yang, H. Liu, Z. Liu, Modified subgradient extragradient algorithms for solving monotone variational inequalities, Optimization. **67** (2018), 2247-2258.
[49] J. Yang, H. Liu, A modified projected gradient method for monotone variational inequalities, J. Optim. Theory Appl. **179** (2018), 197-211.