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Some coupled fixed point theorems for (ψ, ϕ)-contraction with applications to fractals

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Abstract. In this paper, we obtain coupled fixed point theorem for (ψ, ϕ)-contractions under some generalized conditions on the real valued functions ψ and ϕ defined on $(0, \infty)$. Also, we present a generalized version of coupled fixed point theorem for the same (ψ, ϕ)-contractions. A new approach to fractal generation by using the relation between fractals and fixed points is given in the light of these fixed point theorems. We establish a new type of iterated function system consisting of generalized (ψ, ϕ)-contractions. We also extend those results to coupled fractals. This article also provides examples to support and validate the main theorems.

1. Introduction and Preliminaries

Banach contraction principle, proved in 1922 by Banach, is a famous fixed point theorem. Numerous fields of mathematics, as well as other fields of science and technology, have found use for this theorem. Many mathematicians generalized Banach contraction principle in different ways. They led to exciting results in fixed point theory. Boyd and Wong [5] came up with an extension of Banach contraction principle in 1969.

Theorem 1.1. *[5] Given a metric space (X, d), let T be a self-mapping on X that fulfills the following condition:*

$$d\left(T(x), T(y)\right) \leq \phi\left(d(x, y)\right) \quad \text{for each } x, y \in X,$$

where $\phi : \mathbb{R}^+ \to [0, \infty)$ is upper semi-continuous from the right and satisfies the condition $0 \leq \phi(t) < t$ for $t > 0$. If (X, d) is complete, then T possesses a fixed point x_0 in X, which is unique, and the iterative sequence $\{T^n(x)\}$ converges to x_0 for any $x \in X$.

Later in 1975, Matkowski [9] proved another variant of Boyd- Wong fixed point theorem. In this variant, the continuity of the function ϕ is replaced with some more general condition.

Theorem 1.2. *[9] Let (X, d) is a metric space. Given a map $T : X \to X$ that satisfies*

$$d\left(T(x), T(y)\right) \leq \phi\left(d(x, y)\right) \quad \text{for each } x, y \in X,$$

where $\phi : (0, \infty) \to (0, \infty)$ is nondecrasing and satisfies the condition $\lim_{n\to\infty} \phi^n(t) = 0$ for $t > 0$. Then T has one and only one fixed point x_0 in X if (X, d) is complete. Further, the iterative sequence $\{T^n(x)\}$ converges to x_0 for any $x \in X$.

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One of the recent generalizations of Banach contraction principle was given by Proinov [19] for generalized (ψ, ϕ)- contractions. We will discuss some of the ideas introduced by Proinov.

Definition 1.3. *[19] Given a metric space (X, d), let $T : X \to X$ be a map. Then T is said to be (ψ, ϕ)-contraction if it satisfies the condition*

$$\psi\left(d\left(T(x), T(y)\right)\right) \leq \phi\left(d\left(x, y\right)\right), \quad \text{for all } x, y \in X \text{ with } d\left(T(x), T(y)\right) > 0, \tag{1}$$

provided the functions $\psi, \phi : (0, \infty) \to \mathbb{R}$ fulfil the condition $\phi(t) < \psi(t)$ for $t > 0$.

The main result given by Proinov is,

Theorem 1.4. *[19] On a metric space (X, d) which is complete, let $T : X \to X$ be a self-mapping that fulfills the condition(1), provided the functions $\psi, \phi : (0, \infty) \to \mathbb{R}$ satisfies the following requirements:*

(i) ψ *is nondecreasing;*

(ii) $\phi(t) < \psi(t)$ *for every* $t > 0$;

(iii) $\limsup_{t \to \epsilon+} \phi(t) < \psi(\epsilon+)$.

Then T has a fixed point $\xi \in X$ which is unique and the iterative sequence $\{T^n(x)\}$ converges to ξ for any $x \in X$.

In 2021, Popescu [18] proved another generalization by modifying and improving some results proved by Proinov.

Theorem 1.5. *[18] On a metric space (X, d) which is complete, let $T : X \to X$ be a self-mapping that fulfills the condition(1), provided the functions $\psi, \phi : (0, \infty) \to \mathbb{R}$ follows the below constraints:*

(i) $\phi(t) < \psi(t)$ *for every* $t > 0$;

(ii) $\inf_{t > \epsilon} \psi(t) > -\infty$ *for every* $\epsilon > 0$;

(iii) if there are two converging sequences $\{\psi(t_n)\}$ and $\{\phi(t_n)\}$ with the same limit and $\{\psi(t_n)\}$ is strictly decreasing, then $t_n \to 0$ as $n \to \infty$;

(iv) $\limsup_{t \to \epsilon+} \phi(t) < \liminf_{t \to \epsilon+} \psi(t)$ *for all* $\epsilon > 0$;

(v) the graph of S is closed or $\limsup_{t \to 0+} \phi(t) < \min\left\{\liminf_{t \to \epsilon} \psi(t),\ \psi(\epsilon)\right\}$ *for any* $\epsilon > 0$.

Then T possesses a unique fixed point $\xi \in X$ and the iterative sequence $\{T^n(x)\}$ converges to ξ for every $x \in X$.

Coupled and common fixed point results of generalized contractions are also of great importance. They are very helpful in establishing solutions to a system of integral and differential equations. Some of the recent developments in this direction can be found in [6, 13–15, 21, 22].

In this paper, we extend the above fixed point theorems by Proinov and Popescu to coupled fixed point problems which usually discuss about the fixed points of maps of the kind $T : X \times X \to X$ where X is a complete metric space. Also, we prove a generalized version of the coupled fixed point theorem by replacing the product of a complete metric space by a product of two different complete metric spaces, say $X \times Y$. As a major application of fixed point theory, we extend our work to the theory of fractals too. Here we present a new method to construct fractals using generalized (ψ, ϕ)-contractions, for which we use the idea of new iterated function system consisting of generalized (ψ, ϕ)-contractions. This is different from the classical way of generating fractals given by Hutchinson and Barnsley [4], which uses Banach contraction principle. Towards the end of the paper, inspired from the coupled fixed point results, we demonstrate the existence of a unique coupled self-similar set for generalized (ψ, ϕ)-contraction mappings.

2. Coupled fixed point theorem for (ψ, ϕ)-contraction

In this section, we give our main results which extend fixed point theorems by Proinov and Popescu to coupled fixed point problems.

Definition 2.1. *Let (X, d) be a metric space and $\psi, \phi : (0, \infty) \to \mathbb{R}$ be two maps. A map $T : X \times X \to X$ is said to be jointly (ψ, ϕ)-contraction if for $z = (x, y), w = (u, v) \in X \times X$, with* $\max\{d\left(T\left(x, y\right), T\left(u, v\right)\right), d\left(T\left(y, x\right), T\left(v, u\right)\right)\} > 0$, *then*

$$\psi\left(d\left(Tz, Tw\right)\right) \leq \phi\left(\max\{d(x, u), d(y, v)\}\right) \tag{2}$$

For a complete metric space (X, d) we define $X^* = X \times X$. Define $d^* : X^* \times X^* \to \mathbb{R}$ such that $d^*\left((x, y), (u, v)\right) = \max\{d(x, u), d(y, v)\}$. It can easily be seen that, completeness of the metric space (X^*, d^*) follows from the completeness of (X, d).

For the map $T : X^* \to X$, we can define the iterative sequence as follows:

$$\begin{cases} T^2(x, y) &= T\left(T(x, y), T(y, x)\right) \\ T^3(x, y) &= T\left(T^2(x, y), T^2(y, x)\right) \\ &\vdots \\ T^n(x, y) &= T\left(T^{n-1}(x, y), T^{n-1}(y, x)\right) \end{cases}$$

Theorem 2.2. *Let (X, d) be a complete metric space and $T : X^* \to X$ satisfies condition(2) provided $\psi, \phi : (0, \infty) \to \mathbb{R}$ follows the constraints:*

(i) *ψ is nondecreasing;*

(ii) *$\phi(t) < \psi(t)$ for every $t > 0$;*

(iii) *$\limsup_{t \to \epsilon+} \phi(t) < \psi(\epsilon+)$ for every $\epsilon > 0$.*

Then there exists an element $\xi^* = (x^*, y^*) \in X^*$, which is unique, such that

$$\begin{cases} x^* = T(x^*, y^*) \\ y^* = T(y^*, x^*) \end{cases}$$

and the iterative sequences $x_n = T^n(x, y)$ and $y_n = T^n(y, x)$ converge to x^* and y^* respectively, for any $(x, y) \in X^*$.

Proof. Since (X, d) is a complete metric space, from the earlier discussion, we have (X^*, d^*) is also complete. Now we define a map $T^* : X^* \to X^*$ such that $T^*(x, y) = (T(x, y), T(y, x))$ for $(x, y) \in X^*$. Let $z = (x, y)$, $w = (u, v) \in X^*$ with $d^*(T^*z, T^*w) > 0$. If $d^*(T^*z, T^*w) = \max\{d(T(x, y), T(u, v)), d\left(T(y, x), T(v, u)\right)\} = d\left(T(x, y), T(u, v)\right)$, then from condition(2) we get,

$$\begin{aligned} \psi\left(d^*\left(T^*z, T^*w\right)\right) &= \psi\left(d\left(T(x, y), T(u, v)\right)\right) \\ &\leq \phi\left(\max\{d(x, u), d(y, v)\}\right) \\ &= \phi\left(d^*(z, w)\right). \end{aligned}$$

On the other hand, assume $d^*(T^*z, T^*w) = \max\{d(T(x, y), T(u, v)), d(T(y, x), T(v, u))\} = d(T(y, x), T(v, u))$. Proceeding as above, we get $\psi\left(d^*\left(T^*z, T^*w\right)\right) \leq \phi\left(d^*(z, w)\right)$. Thus it follows that, for any z, $w \in X^*$ with $d^*(T^*z, T^*w) > 0$, we have $\psi\left(d^*\left(T^*z, T^*w\right)\right) \leq \phi\left(d^*(z, w)\right)$, which means the self-mapping T^* satisfies condition(1) in the complete metric space (X^*, d^*). Then, from condition *(i)*- *(iii)* in the hypothesis and Theorem 1.4, we can conclude that there exists a unique $\xi^* = (x^*, y^*) \in X^*$ such that $T^*\xi^* = \xi^*$. That is, $(T(x^*, y^*), T(y^*, x^*)) = (x^*, y^*)$, which implies that $x^* = T(x^*, y^*)$, $y^* = T(y^*, x^*)$. Also for any $z = (x, y) \in X^*$, the iterative sequence $\{T^{*n}(z)\}$ converges to ξ^*. That is, the iterative sequences $x_n = T^n(x, y)$ and $y_n = T^n(y, x)$ converge to x^* and y^* respectively for any $(x, y) \in X^*$. This completes the proof. □

In the same way, we can have an extension of Theorem 1.5 for coupled fixed points.

Theorem 2.3. *Given a complete metric space (X,d), $T : X^* \to X$ be a map satisfying the condition(2) where $\psi, \phi : (0,\infty) \to \mathbb{R}$ satisfying the conditions:*

(i) $\phi(t) < \psi(t)$ *for any* $t > 0$;

(ii) $\inf_{t>\epsilon} \psi(t) > -\infty$ *for any* $\epsilon > 0$;

(iii) if $\{\psi(t_n)\}$ *and* $\{\phi(t_n)\}$ *are sequences that converge to the same limit and* $\{\psi(t_n)\}$ *is strictly decreasing, then* $t_n \to 0$ *as* $n \to \infty$;

(iv) $\limsup_{t\to\epsilon+} \phi(t) < \liminf_{t\to\epsilon+} \psi(t)$ *for any* $\epsilon > 0$;

(v) graph of T is closed or $\limsup_{t\to 0+} \phi(t) < \min\left\{\liminf_{t\to\epsilon} \psi(t), \psi(\epsilon)\right\}$ *for any* $\epsilon > 0$.

Then there exists a unique point $\xi^* = (x^*, y^*) \in X^*$ *such that*

$$\begin{cases} x^* = T(x^*, y^*) \\ y^* = T(y^*, x^*) \end{cases}$$

and the iterative sequences $x_n = T^n(x,y)$ *and* $y_n = T^n(y,x)$ *converge to* x^* *and* y^* *respectively, for any* $(x,y) \in X^*$.

Proof. Define a map $T^* : X^* \to X^*$ such that $T^*(x,y) = (T(x,y), T(y,x))$ for $(x,y) \in X^*$. Then from the proof of Theorem 2.2, it is clear that T^* satisfies condition(1). Suppose that T has a closed graph. Then for any sequences $\{x_n\}, \{y_n\}$ in X such that $x_n \to x$, $y_n \to y$, $T(x_n, y_n) \to \alpha$ and $T(y_n, x_n) \to \beta$ as $n \to \infty$, it will be true that $T(x,y) = \alpha$ and $T(y,x) = \beta$. From this, we can conclude that for any sequence $\{(x_n, y_n)\} \subset X^*$ with $(x_n, y_n) \to (x,y)$ and $T^*(x_n, y_n) \to (\alpha, \beta)$ we get, $T^*(x,y) = (\alpha,\beta)$, which means T^* has a closed graph. Thus if the graph of T is closed then graph of T^* is also closed. Hence the result follows from the conditions *(i)- (v)* in the hypothesis and Theorem 1.5. □

Example 2.4. *Let* $M = \left\{\frac{1}{2^n} : n \in \mathbb{Z}^+ \cup \{0\}\right\}$ *and* $X = M \cup \{0\}$. *Define* $d : X \times X \to \mathbb{R}$ *by* $d(x,y) = |x-y|$. *It can be easily verified that* (X,d) *is a complete metric space. Also, consider the complete metric space* $X \times X$ *with the maximum metric given by*

$$d^*((x,y),(u,v)) = \max\left\{|x-u|, \left|y-v\right|\right\} \text{ for any } (x,y),(u,v) \in X \times X.$$

Define a map $T : X \times X \to X$ *such that*

$$T(x,y) = \begin{cases} \frac{1}{2^{\min\{m,n\}+1}} & \text{if } (x,y) \in M \times M \\ \frac{1}{2^{n+1}} & \text{if } (x,y) \in (M \times \{0\}) \cup (\{0\} \times M) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Also, define $\psi, \phi : (0,\infty) \to \mathbb{R}$ *as follows:*

$$\psi(t) = \begin{cases} \frac{t}{2} & \text{if } t \in (0, \frac{1}{2}) \\ \frac{3t}{2} & \text{if } t \in [\frac{1}{2}, 1) \\ 3t & \text{if } t \geq 1 \end{cases} \quad \text{and} \quad \phi(t) = \begin{cases} \frac{t}{4} & \text{if } t \in (0, \frac{1}{2}) \\ t & \text{if } t \in [\frac{1}{2}, 1) \\ 2t & \text{if } t \geq 1 \end{cases}$$

Here the functions ψ *and* ϕ *satisfy the conditions (i)- (v) of Theorem 2.3.*
Case 1*: For* $m,n,p,q \geq 0$ *with* $\min\{m,n\} \neq \min\{p,q\}$, *we have*

$$\psi\left(d\left(T\left(\frac{1}{2^m}, \frac{1}{2^n}\right), T\left(\frac{1}{2^p}, \frac{1}{2^q}\right)\right)\right) = \psi\left(\left|\frac{1}{2^{\min\{m,n\}+1}} - \frac{1}{2^{\min\{p,q\}+1}}\right|\right)$$
$$= \frac{1}{4}\left|\frac{1}{2^{\min\{m,n\}}} - \frac{1}{2^{\min\{p,q\}}}\right|$$

and

$$\phi\left(d^*\left(\left(\frac{1}{2^m},\frac{1}{2^n}\right),\left(\frac{1}{2^p},\frac{1}{2^q}\right)\right)\right)=\phi\left(\max\left\{\left|\frac{1}{2^m}-\frac{1}{2^p}\right|,\left|\frac{1}{2^n}-\frac{1}{2^q}\right|\right\}\right)$$
$$=\begin{cases}\frac{1}{4}\max\left\{\left|\frac{1}{2^m}-\frac{1}{2^p}\right|,\left|\frac{1}{2^n}-\frac{1}{2^q}\right|\right\} & \text{if } m,n,p,q>0\\ \max\left\{\left|\frac{1}{2^m}-\frac{1}{2^p}\right|,\left|\frac{1}{2^n}-\frac{1}{2^q}\right|\right\} & \text{otherwise.}\end{cases}$$

Case 2: *For $m,n,p\geq 0$ with $\min\{m,n\}\neq p$, we get*

$$\psi\left(d\left(T\left(\frac{1}{2^m},\frac{1}{2^n}\right),T\left(\frac{1}{2^p},0\right)\right)\right)=\psi\left(\left|\frac{1}{2^{\min\{m,n\}+1}}-\frac{1}{2^{p+1}}\right|\right)=\frac{1}{4}\left|\frac{1}{2^{\min\{m,n\}}}-\frac{1}{2^p}\right|$$

and

$$\phi\left(d^*\left(\left(\frac{1}{2^m},\frac{1}{2^n}\right),\left(\frac{1}{2^p},0\right)\right)\right)=\phi\left(\max\left\{\left|\frac{1}{2^m}-\frac{1}{2^p}\right|,\frac{1}{2^n}\right\}\right)$$
$$=\begin{cases}\frac{1}{4}\max\left\{\left|\frac{1}{2^m}-\frac{1}{2^p}\right|,\frac{1}{2^n}\right\} & \text{if } \max\left\{\left|\frac{1}{2^m}-\frac{1}{2^p}\right|,\frac{1}{2^n}\right\}<\frac{1}{2}\\ \max\left\{\left|\frac{1}{2^m}-\frac{1}{2^p}\right|,\frac{1}{2^n}\right\} & \text{if } \max\left\{\left|\frac{1}{2^m}-\frac{1}{2^p}\right|,\frac{1}{2^n}\right\}\geq\frac{1}{2}\\ 2 & \text{if } n=0.\end{cases}$$

Case 3: *If $m,p\geq 0$ and $m\neq p$, we get*

$$\psi\left(d\left(T\left(\frac{1}{2^m},0\right),T\left(\frac{1}{2^p},0\right)\right)\right)=\psi\left(\left|\frac{1}{2^{m+1}}-\frac{1}{2^{p+1}}\right|\right)=\frac{1}{4}\left|\frac{1}{2^m}-\frac{1}{2^p}\right|$$

and

$$\phi\left(d^*\left(\left(\frac{1}{2^m},0\right),\left(\frac{1}{2^p},0\right)\right)\right)=\phi\left(\max\left\{\left|\frac{1}{2^m}-\frac{1}{2^p}\right|,0\right\}\right)$$
$$=\begin{cases}\frac{1}{4}\left|\frac{1}{2^m}-\frac{1}{2^p}\right| & \text{if } \left|\frac{1}{2^m}-\frac{1}{2^p}\right|<\frac{1}{2}\\ \left|\frac{1}{2^m}-\frac{1}{2^p}\right| & \text{if } \left|\frac{1}{2^m}-\frac{1}{2^p}\right|\geq\frac{1}{2}.\end{cases}$$

On the other hand

$$\psi\left(d\left(T\left(\frac{1}{2^m},0\right),T\left(0,\frac{1}{2^p}\right)\right)\right)=\psi\left(\left|\frac{1}{2^{m+1}}-\frac{1}{2^{p+1}}\right|\right)=\frac{1}{4}\left|\frac{1}{2^m}-\frac{1}{2^p}\right|$$

and

$$\phi\left(d^*\left(\left(\frac{1}{2^m},0\right),\left(0,\frac{1}{2^p}\right)\right)\right)=\phi\left(\max\left\{\frac{1}{2^m},\frac{1}{2^p}\right\}\right)$$
$$=\begin{cases}\frac{1}{4}\max\left\{\frac{1}{2^m},\frac{1}{2^p}\right\} & \text{if } \max\left\{\frac{1}{2^m},\frac{1}{2^p}\right\}<\frac{1}{2}\\ \frac{1}{2} & \text{if } \max\left\{\frac{1}{2^m},\frac{1}{2^p}\right\}=\frac{1}{2}\\ 2 & \text{if } \max\left\{\frac{1}{2^m},\frac{1}{2^p}\right\}=1.\end{cases}$$

Case 4: *For $m,n>0$, we have*

$$\psi\left(d\left(T\left(\frac{1}{2^m},\frac{1}{2^n}\right),T(0,0)\right)\right)=\psi\left(\frac{1}{2^{\min\{m,n\}+1}}\right)=\frac{1}{4}\left(\frac{1}{2^{\min\{m,n\}}}\right)$$

and

$$\phi\left(d^*\left(\left(\frac{1}{2^m},\frac{1}{2^n}\right),(0,0)\right)\right)=\phi\left(\max\left\{\frac{1}{2^m},\frac{1}{2^n}\right\}\right)$$
$$=\begin{cases}\frac{1}{4}\max\left\{\frac{1}{2^m},\frac{1}{2^n}\right\} & \text{if } \max\left\{\frac{1}{2^m},\frac{1}{2^n}\right\}<\frac{1}{2}\\ \frac{1}{2} & \text{if } \max\left\{\frac{1}{2^m},\frac{1}{2^n}\right\}=\frac{1}{2}.\end{cases}$$

Case 5: *For $m > 0$, we get*

$$\psi\left(d\left(T\left(1,\frac{1}{2^m}\right),T(0,0)\right)\right)=\psi\left(\frac{1}{2^{\min\{0,m\}+1}}\right)=\psi\left(\frac{1}{2}\right)=\frac{3}{4}$$

and

$$\phi\left(d^*\left(\left(1,\frac{1}{2^m}\right),(0,0)\right)\right)=\phi\left(\max\left\{1,\frac{1}{2^m}\right\}\right)=\phi(1)=2$$

Also,

$$\psi\left(d\left(T\left(\frac{1}{2^m},0\right),T(0,0)\right)\right)=\psi\left(\frac{1}{2^{m+1}}\right)=\frac{1}{4}\left(\frac{1}{2^m}\right)$$

and

$$\phi\left(d^*\left(\left(\frac{1}{2^m},0\right),(0,0)\right)\right)=\phi\left(\frac{1}{2^m}\right)=\begin{cases}\frac{1}{4}\left(\frac{1}{2^m}\right) & \text{if } m>1\\ \frac{1}{2} & \text{if } m=1.\end{cases}$$

Case 6:

$$\psi\left(d\left(T(1,0),T(0,0)\right)\right)=\psi\left(d\left(T(\frac{1}{2^0},0),T(0,0)\right)\right)=\psi\left(\frac{1}{2}\right)=\frac{3}{4}$$

and

$$\phi\left(d^*\left((1,0),(0,0)\right)\right)=\phi(1)=2.$$

From all the above discussed cases, we can observe that, for any $(x,y),(u,v)\in X^*$, we have

$$\psi\left(d\left(T(x,y),T(u,v)\right)\right)\leq\phi\left(d^*\left((x,y),(u,v)\right)\right)$$

Hence T satisfies condition(2) and all the hypotheses of Theorem 2.3. It is evident that $(x,y)=(0,0)$ a coupled fixed point of T. Moreover, one can observe that it is the only coupled fixed point of T.

3. Extended coupled fixed points of (ψ,ϕ)-contractions

In this section we will consider more general problem in coupled fixed points. Here, instead of having a product of same complete metric space, we will deal with a product of two different complete metric spaces.

Given two complete metric spaces (X,d) and (Y,ρ), we define $Z=X\times Y$ and a function $\mu:Z\to\mathbb{R}$ such that

$$\mu(z,w)=\mu\left((x,y),(u,v)\right)=\max\{d(x,u),\rho(y,v)\}\text{ for any } z=(x,y),\ w=(u,v)\in Z. \tag{3}$$

We can easily observe that, since (X,d) and (Y,ρ) are complete metric spaces, the metric space (Z,μ) is also complete.

Definition 3.1. *Given two complete metric spaces (X,d) and (Y,ρ), let $T:X\times Y\to X$ and $S:X\times Y\to Y$ be two maps. T and S are said to be extended jointly (ψ,ϕ)-contractions if for $z=(x,y),\ w=(u,v)\in X\times Y$ with $\max\{d\left(T(x,y),T(u,v)\right),\rho\left(S(x,y),S(u,v)\right)\}>0$, we have*

$$\begin{aligned}\psi(d(Tz,Tw))&\leq\phi(\max\{d(x,u),\rho(y,v)\})\\ \psi(\rho(Sz,Sw))&\leq\phi(\max\{d(x,u),\rho(y,v)\})\end{aligned} \tag{4}$$

where $\psi,\phi:(0,\infty)\to\mathbb{R}$ satisfy $\phi(t)<\psi(t)$ for every $t>0$.

Lemma 3.2. *Given two complete metric spaces (X, d) and (Y, ρ), let $T : X \times Y \to X$ and $S : X \times Y \to Y$ be extended jointly (ψ, ϕ)-contractions, where $\psi, \phi : (0, \infty) \to \mathbb{R}$. Let $F_{TS} : X \times Y \to X \times Y$ be defined by $F_{TS}(x, y) = (T(x, y), S(x, y))$ for $(x, y) \in X \times Y$. Then F_{TS} is a (ψ, ϕ)-contraction.*

Proof. Let $Z = X \times Y$ and μ be the metric defined as in equation(3). Note that (Z, μ) is a complete metric space. Let $z = (x, y),\ w = (u, v) \in Z$. Then

$$\begin{aligned} \psi\left(\mu\left(F_{TS}(z), F_{TS}(w)\right)\right) &= \psi\left(\mu\left(\left(T(x,y), S(x,y)\right), \left(T(u,v), S(u,v)\right)\right)\right) \\ &= \psi\left(\max\{d\left(T(x,y), T(u,v)\right), \rho\left(S(x,y), S(u,v)\right)\}\right) \end{aligned} \tag{5}$$

Suppose that $\mu\left(F_{TS}(z), F_{TS}(w)\right) = d\left(T(x, y), T(u, v)\right)$. Then by condition(4), equation(5) becomes:

$$\begin{aligned} \psi\left(\mu\left(F_{TS}(z), F_{TS}(w)\right)\right) &= \psi\left(d\left(T(x,y), T(u,v)\right)\right) \\ &\leq \phi\left(\max\{d(x,u), \rho(y,v)\}\right) \\ &= \phi\left(\mu(z,w)\right). \end{aligned}$$

On the other hand, if $\mu\left(F_{TS}(z), F_{TS}(w)\right) = \rho\left(S(x, y), S(u, v)\right)$, then by a similar argument as above we get,

$$\mu\left(F_{TS}(z), F_{TS}(w)\right) \leq \phi\left(\mu(z,w)\right).$$

Hence the map F_{TS} is a (ψ, ϕ)-contraction. □

We define iterative sequences for the maps T and S as follows:

$$\begin{cases} T^2(x,y) &= T\left(T(x,y), S(x,y)\right) \\ S^2(x,y) &= S\left(T(x,y), S(x,y)\right) \\ T^3(x,y) &= T\left(T^2(x,y), S^2(x,y)\right) \\ S^3(x,y) &= S\left(T^2(x,y), S^2(x,y)\right) \\ &\vdots \\ T^n(x,y) &= T\left(T^{n-1}(x,y), S^{n-1}(x,y)\right) \\ S^n(x,y) &= S\left(T^{n-1}(x,y), S^{n-1}(x,y)\right) \end{cases}$$

where $(x, y) \in X \times Y$.
Now we can extend Theorem 1.4 to extended jointly (ψ, ϕ)-contractions to get a more generalized coupled fixed points.

Theorem 3.3. *Given two complete metric spaces (X, d) and (Y, ρ), let $T : X \times Y \to X$ and $S : X \times Y \to Y$ be extended jointly (ψ, ϕ)-contractions. If the functions ψ, ϕ satisfy the conditions:*

(i) *ψ is nondecreasing;*

(ii) *$\phi(t) < \psi(t)$ for every $t > 0$;*

(iii) *$\limsup_{t \to \epsilon+} \phi(t) < \psi(\epsilon+)$ for every $\epsilon > 0$,*

then there exists a unique element $\xi^* = (x^*, y^*) \in X \times Y$ such that, $x^* = T(x^*, y^*)$ and $y^* = S(x^*, y^*)$. Moreover, the sequences $x_n = T^n(x, y)$ and $y_n = S^n(x, y)$ converge to x^* and y^* respectively for any $(x, y) \in X \times Y$.

Proof. Define a map $F_{TS} : X \times Y \to X \times Y$ by $F_{TS}(x, y) = (T(x, y), S(x, y))$ for $(x, y) \in X \times Y$. Then by Lemma 3.2, we have F_{TS} is a (ψ, ϕ)-contraction. Hence the conditions *(i)-(iii)* in the hypotheses and Theorem 1.4 establish the result. □

Theorem 3.4. *Given two complete metric spaces (X, d) and (Y, ρ), let $T : X \times Y \to X$ and $S : X \times Y \to Y$ be extended jointly (ψ, ϕ)-contractions. Suppose that the functions ψ, ϕ satisfy the constraints:*

(i) $\phi(t) < \psi(t)$ *for every* $t > 0$;

(ii) $\inf_{t>\epsilon} \psi(t) > -\infty$ *for all* $\epsilon > 0$;

(iii) if there are two sequences $\{\psi(t_n)\}$ *and* $\{\phi(t_n)\}$ *having the same limit and* $\{\psi(t_n)\}$ *is strictly decreasing, then* $t_n \to 0$ *as* $n \to \infty$;

(iv) $\limsup_{t\to\epsilon+} \phi(t) < \liminf_{t\to\epsilon+} \psi(t)$ *for any* $\epsilon > 0$;

(v) either the graphs of T and S are closed or $\limsup_{t\to 0+} \phi(t) < \min\left\{\liminf_{t\to\epsilon} \psi(t), \psi(\epsilon)\right\}$ *for any* $\epsilon > 0$.

Then there exists a unique point $\xi^* = (x^*, y^*) \in X \times Y$ *such that,* $x^* = T(x^*, y^*)$ *and* $y^* = S(x^*, y^*)$. *Moreover, the sequences* $x_n = T^n(x, y)$ *and* $y_n = S^n(x, y)$ *converge to* x^* *and* y^* *respectively for any* $(x, y) \in X \times Y$.

Proof. Define a map $F_{TS} : X \times Y \to X \times Y$ by $F_{TS}(x, y) = (T(x, y), S(x, y))$ for $(x, y) \in X \times Y$. Then by Lemma 3.2, we have F_{TS} is a (ψ, ϕ)-contraction. Suppose that the graphs of T and S are closed. Then for any sequence $\{(x_n, y_n)\}$ in $X \times Y$ with $(x_n, y_n) \to (x, y)$, $T(x_n, y_n) \to \alpha$ and $S(x_n, y_n) \to \beta$, we have $T(x, y) = \alpha$ and $S(x, y) = \beta$. Since $T(x_n, y_n) \to \alpha$ and $S(x_n, y_n) \to \beta$, we get $F_{TS}(x_n, y_n) = (T(x_n, y_n), S(x_n, y_n)) \to (\alpha, \beta)$. Also, $T(x, y) = \alpha$ and $S(x, y) = \beta$ implies $F_{TS}(x, y) = (\alpha, \beta)$. Hence we are able to conclude that if $\{(x_n, y_n)\}$ is a sequence in $X \times Y$ with $(x_n, y_n) \to (x, y)$ and $F_{TS}(x_n, y_n) = (T(x_n, y_n), S(x_n, y_n)) \to (\alpha, \beta)$ then $F_{TS}(x, y) = (\alpha, \beta)$. This implies that F_{TS} has a closed graph if T and S have closed graphs. With this fact along with conditions *(i)*- *(v)* in the hypotheses and Theorem 1.5 we can complete the proof. □

Even more generally, suppose the maps T and S defined above satisfy the following contractive conditions:

$$\begin{aligned} &\psi(d(Tz, Tw)) \le \phi_1(\max\{d(x, u), \rho(y, v)\}) \\ &\psi(\rho(Sz, Sw)) \le \phi_2(\max\{d(x, u), \rho(y, v)\}) \end{aligned} \tag{6}$$

for $z = (x, y),\ w = (u, v) \in X \times Y$ with $\max\{d\,(T(x, y), T(u, v))\,, \rho\,(S(x, y), S(u, v))\} > 0$, where the functions $\psi, \phi_1, \phi_2 : (0, \infty) \to \mathbb{R}$ are such that $\phi_i(t) < \psi(t)$ for $i = 1, 2$ and for every $t > 0$. Let us define a function $\phi : (0, \infty) \to \mathbb{R}$ as $\phi(t) = \max\{\phi_1(t), \phi_2(t)\}$ for $t > 0$. Since $\phi_i(t) < \psi(t)$ for $i = 1, 2$ and for every $t > 0$, we get $\phi(t) < \psi(t)$ for every $t > 0$. Then we can have the following Lemma.

Lemma 3.5. *Given two complete metric spaces* (X, d) *and* (Y, ρ), *let* $Z = X \times Y$. *Let* $T : Z \to X$ *and* $S : Z \to Y$ *satisfy condition(6). Define a map* $F_{TS} : Z \to Z$ *by* $F_{TS}(x, y) = (T(x, y), S(x, y))$ *for* $(x, y) \in Z$. *Then* F_{TS} *is a* (ψ, ϕ)*-contraction.*

Proof. Note that (Z, μ) is a complete metric space. Let $z = (x, y),\ w = (u, v) \in Z$. Then,

$$\begin{aligned} \psi\left(\mu\left(F_{TS}(z), F_{TS}(w)\right)\right) &= \psi\left(\mu\left((T(x, y), S(x, y))\,, (T(u, v), S(u, v))\right)\right) \\ &= \psi\left(\max\{d\left(T(x, y), T(u, v)\right), \rho\left(S(x, y), S(u, v)\right)\}\right) \end{aligned} \tag{7}$$

Suppose $\mu\,(F_{TS}(z), F_{TS}(w)) = d\,(T(x, y), T(u, v))$, then by condition(6), equation(7) becomes:

$$\begin{aligned} \psi\left(\mu\left(F_{TS}(z), F_{TS}(w)\right)\right) &= \psi\left(d\left(T(x, y), T(u, v)\right)\right) \\ &\le \phi_1\left(\max\{d(x, u), \rho(y, v)\}\right) \\ &= \phi_1\left(\mu(z, w)\right) \\ &\le \phi\left(\mu(z, w)\right). \end{aligned}$$

On the other hand, if $\mu\,(F_{TS}(z), F_{TS}(w)) = \rho\,(S(x, y), S(u, v))$, then again by condition(6), equation(7) becomes:

$$\begin{aligned} \psi\left(\mu\left(F_{TS}(z), F_{TS}(w)\right)\right) &= \psi\left(\mu\left(S(x, y), S(u, v)\right)\right) \\ &\le \phi_2\left(\max\{d(x, u), \rho(y, v)\}\right) \\ &= \phi_2\left(\mu(z, w)\right) \\ &\le \phi\left(\mu(z, w)\right). \end{aligned}$$

Thus in both cases we have $\mu\,(F_{TS}(z), F_{TS}(w)) \le \phi\,(\mu(z, w))$. Hence the map F_{TS} is a (ψ, ϕ)-contraction. □

Our aim is to produce a fixed point theorem for maps T and S satisfying the contractive condition(6). Prior to that, we must demonstrate the following Lemma.

Lemma 3.6. *Let $\phi_1, \phi_2 : (0, \infty) \to \mathbb{R}$ be two functions such that for $t_0 > 0$, $\limsup_{t \to t_0} \phi_i(t)$ exists for $i = 1, 2$. If we define $\phi(t) = \max\{\phi_1(t), \phi_2(t)\}$, then $\limsup_{t \to t_0} \phi(t) \leq \max\left\{\limsup_{t \to t_0} \phi_1(t), \limsup_{t \to t_0} \phi_2(t)\right\}$.*

Proof. By the definition and properties of limit supremum of functions, since $t_0 > 0$ and $\limsup_{t \to t_0} \phi_i(t)$ exists, the set

$$A_i = \left\{l \in \mathbb{R} : \exists \{t_n\}, t_n > 0, t_n \to t_0, t_n \neq t_0 \text{ for all } n > 0, \text{ such that } \phi_i(t_n) \to l\right\}$$

is non empty, and also $\limsup_{t \to t_0} \phi_i(t) = \max A_i$ for $i = 1, 2$.
Since $\limsup_{t \to t_0} \phi_i(t)$ exists, we have $\limsup_{t \to t_0} \phi(t)$ also exists. Now we define

$$A = \left\{l \in \mathbb{R} : \exists \{t_n\}, t_n > 0, t_n \to t_0, t_n \neq t_0 \text{ for all } n > 0, \text{ such that } \phi(t_n) \to l\right\}.$$

Then we have A is nonempty and $\limsup_{t \to t_0} \phi(t) = \max A$.
Let $l \in A$. Then by definition, there exists a sequence $\{t_n\}$ in $(0, \infty)$ with $t_n \to t_0, t_n \neq t_0$ for all n such that $\phi(t_n) \to l$. Thus for $\epsilon > 0$ one can find an $N \in \mathbb{N}$ such that $\left|\phi(t_n) - l\right| < \epsilon$ whenever $n \geq N$. That is, $\left|\max\left\{\phi_1(t_n), \phi_2(t_n)\right\} - l\right| < \epsilon$ whenever $n \geq N$. For $k \in \mathbb{N}$, define $N_k(\phi_i) = \left\{n \in \mathbb{N} : \left|\phi_i(t_n) - l\right| < \frac{1}{k}\right\}$ for $i = 1, 2$. For each k, either $N_k(\phi_1)$ or $N_k(\phi_2)$ is nonempty. Also, $N_{k+1}(\phi_i) \subseteq N_k(\phi_i)$ for each k and $i = 1, 2$. Thus for at least one $i = 1, 2$, we get $N_k(\phi_i) \neq \emptyset$ for all k . Without loss of generality, assume that $N_k(\phi_1) \neq \emptyset$ for all k. Then by picking $n_k \in N_k(\phi_1)$, we get a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $\left|\phi_1(t_{n_k}) - l\right| < \frac{1}{m}$ for all $k \geq m$. This implies that $\phi_1(t_{n_k}) \to l$. Hence $l \in A_1$. By this, we can conclude that, if $l \in A$ then $l \in A_1 \cup A_2$, which yields that $\limsup_{t \to t_0} \phi(t) = \max A \leq \max\{\max A_1, \max A_2\} = \max\{\limsup_{t \to t_0} \phi_1(t), \limsup_{t \to t_0} \phi_2(t)\}$ □

This leads us to the following conclusion in light of these Lemmas.

Theorem 3.7. *Given two complete metric spaces (X, d) and (Y, ρ), let $T : X \times Y \to X$ and $S : X \times Y \to Y$ be two maps satisfying condition(6). If the functions ψ, ϕ_1 and ϕ_2 follow the constraints:*

(i) *ψ is nondecreasing;*

(ii) *$\phi_i(t) < \psi(t)$ for every $t > 0$ and $i = 1, 2$;*

(iii) *$\limsup_{t \to \epsilon+} \phi_i(t) < \psi(\epsilon+)$ for every $\epsilon > 0$.*

Then there exists a unique point $\xi^* = (x^*, y^*) \in X \times Y$ such that,

$$\begin{cases} x^* = T(x^*, y^*) \\ y^* = S(x^*, y^*) \end{cases}$$

and the sequences $x_n = T^n(x, y)$ and $y_n = S^n(x, y)$ converge to x^* and y^* respectively for any $(x, y) \in X \times Y$.

Proof. We define a map $F_{TS} : X \times Y \to X \times Y$ by $F_{TS}(x, y) = (T(x, y), S(x, y))$ for $(x, y) \in X \times Y$. Then by Lemma 3.2, we have F_{TS} is a (ψ, ϕ)-contraction where $\phi(t) = \max\{\phi_1(t), \phi_2(t)\}$ for $t > 0$. Since $\phi_i(t) < \psi(t)$ for all $t > 0$ and $i = 1, 2$, we get $\phi(t) < \psi(t)$ for every $t > 0$. Now, from condition *(ii)* in the hypothesis and Lemma 3.6, we get $\limsup_{t \to \epsilon+} \phi(t) < \psi(\epsilon+)$ for every $\epsilon > 0$. Hence the functions ψ and ϕ satisfy conditions *(i)-(iii)* in the hypothesis of Theorem 1.4. Then the result follows from the Theorem 1.4 and Lemma 3.5. □

4. Iterated Function Systems with (ψ, ϕ)-contractions

According to Barnsley [4], fractals can be mathematically identified as the fixed points of some set maps. For the rest of this article, $\mathcal{H}(X)$ represents the collection of all nonempty compact subsets of the complete metric space (X, d).

For any $A, B \in \mathcal{H}(X)$, we define distance between the sets A and B as

$$\begin{aligned} D(A, B) &= \max\{d(x, B) : x \in A\} \\ &= \max_{x\in A} \min_{y\in B} d(x, y). \end{aligned}$$

Now we define the Hausdorff distance in $\mathcal{H}(X)$ as $h_d(A, B) = \max\{D(A, B), D(B, A)\}$. The fact that h_d defines a metric on $\mathcal{H}(X)$ can be verified easily. Additionally, we possess the following Lemma:

Lemma 4.1. *[4] $(\mathcal{H}(X), h_d)$ is a complete metric space if (X, d) is a complete metric space. Moreover, for any Cauchy sequence $\{A_n\}$ in $\mathcal{H}(X)$ the limit is given by,*

$$A = \lim_{n\to\infty} A_n = \left\{x \in X : \exists \text{ a Cauchy sequence } \{x_n\} \text{ in } X \text{ such that } x_n \in A_n \text{ and } \lim_{n\to\infty} x_n = x\right\}.$$

The space $(\mathcal{H}(X), h_d)$ is usually called as the space of fractals.

Lemma 4.2. *[23, 24] If $\{A_i : i = 1, 2, \ldots n\}$, $\{B_i : i = 1, 2, \ldots n\}$ be two finite collections in $\mathcal{H}(X)$, then $h_d\left(\bigcup_{i=1}^{n} A_i, \bigcup_{i=1}^{n} B_i\right) \leq \max\{h_d(A_i, B_i) : i = 1, 2, \ldots n\}$.*

The technique, of Barnsley [4], of generating an IFS can not be employed on every generalized contractions. Some of the problems occurring while proving the results on IFS consisting of Kannan, Chaterjea and Reich type contractions has been discussed by Van Dung et al.[27]. But at the same time, many generalized contractions has been extended to the fractal space. More results on generation of IFSs consisting of variety of contraction maps can be found in [8, 10, 15, 16, 25, 26]. Motivated from all these literature, we will generate the IFS consisting of generalized (ψ, ϕ)-contractions in this section. Also, we prove the existence of attractors of these IFSs.

We first establish the extension of a continuous generalized (ψ, ϕ)-contraction to the fractal space.

Lemma 4.3. *Let (X, d) be a complete metric space and $w : X \to X$ be a continuous map satisfying the condition(1) with the nondecreasing control functions ψ, ϕ. Then $\hat{w} : \mathcal{H}(X) \to \mathcal{H}(X)$ defined by $\hat{w}(A) = w(A) = \bigcup_{x\in A}\{w(x)\}$ for $A \in \mathcal{H}(X)$ also satisfies condition(1) in $(\mathcal{H}(X), h_d)$.*

Proof. Let $A, B \in \mathcal{H}(X)$ such that $h_d(\hat{w}(A), \hat{w}(B)) > 0$. Suppose that $h_d(\hat{w}(A), \hat{w}(B)) = D(\hat{w}(A), \hat{w}(B)) = \sup_{x\in A}\inf_{y\in B} d(w(x), w(y)) > 0$. Since w and d are continuous and A is compact, there exist $a \in A$ for which $D(\hat{w}(A), \hat{w}(B)) = \inf_{y\in B} d(w(a), w(y)) > 0$, which implies that $d(w(a), w(y)) > 0$ for every $y \in B$. Thus for any $y \in B$,

$$\begin{aligned} \psi\left(D\left(\hat{w}(A), \hat{w}(B)\right)\right) &= \psi\left(\inf_{y\in B} d\left(w(a), w(y)\right)\right) \\ &\leq \psi\left(d\left(w(a), w(y)\right)\right) \\ &\leq \phi\left(d(a, y)\right). \end{aligned}$$

Let $b \in B$ be such that $d(a,b) = \inf_{y\in B} d(a,y)$. Since ϕ is nondecreasing,

$$\begin{aligned}\psi\left(h_d\left(\hat{w}(A),\hat{w}(B)\right)\right) &= \psi\left(D\left(\hat{w}(A),\hat{w}(B)\right)\right)\\ &\le \phi\left(d(a,b)\right)\\ &= \phi\left(\inf_{y\in B} d(a,y)\right)\\ &\le \phi\left(\sup_{x\in A}\inf_{y\in B} d(x,y)\right)\\ &\le \phi\left(h_d(A,B)\right).\end{aligned}$$

On the other hand, if we assume
$h_d(\hat{w}(A),\hat{w}(B)) = D(\hat{w}(B),\hat{w}(A)) = \sup_{y\in B}\inf_{x\in A} d\left(w(y),w(x)\right) > 0$, we can proceed as above and get
$\psi\left(h_d(\hat{w}(A),\hat{w}(B))\right) \le \phi\left(h_d(A,B)\right)$. This completes the proof. □

Remark 4.4. *The map $\hat{w}$ in Lemma 4.3 is called the fractal operator generated by w. Continuity of the map w is required to make sure that $\hat{w}$ maps $\mathcal{H}(X)$ to itself. A fixed point $A^* \in \mathcal{H}(X)$ of the map $\hat{w}$, if it exists, is called an attractor or a self-similar set of w.*

Theorem 4.5. *Let (X,d) be a complete metric space and $w : X \to X$ be a continuous map. If the map w satisfies condition(1), where $\psi,\phi : (0,\infty) \to \mathbb{R}$ follows the constraints:*

(i) *ψ,ϕ are nondecreasing;*

(ii) *$\phi(t) < \psi(t)$ for all $t > 0$;*

(iii) $\limsup_{t\to\epsilon+} \phi(t) < \psi(\epsilon+)$,

then there exists a unique attractor, say $A^* \in \mathcal{H}(X)$, for w. Moreover, for any $A \in \mathcal{H}(X)$ the sequence $\{A_n\}$ in $\mathcal{H}(X)$, given by $A_n = w^n(A)$, converges to A^*.

Proof. Let $\hat{w}$ be the fractal operator generated by w. By Lemma 4.3 it is clear that $\hat{w}$ satisfies condition(1). Then, conditions *(i)*- *(iii)* in the hypothesis and Theorem 1.4 guarantee a fixed point for $\hat{w}$, say $A^* \in \mathcal{H}(X)$, which is unique. Moreover, for any $A \in \mathcal{H}(X)$, the sequence $\{A_n\}$, where $A_n = \hat{w}^n(A)$, converges to A^*. This proves the theorem. □

Next lemma will prove a result on the property of graph of the function $\hat{w}$.

Lemma 4.6. *Given a continuous map $w : X \to X$ on a complete metric space (X,d). If $\hat{w}$ is the fractal operator generated by w, then $\hat{w}$ has a closed graph.*

Proof. Consider an arbitrary sequence (A_n) in $\mathcal{H}(X)$ such that $A_n \to A$ and $\hat{w}(A_n) \to B$. The proof is complete if we prove $B = \hat{w}(A)$. By Lemma 4.1, we have

$$A = \lim_{n\to\infty} A_n = \left\{x \in X : \exists \text{ a Cauchy sequence } \{x_n\} \text{ in } X \text{ such that } x_n \in A_n,\ \lim_{n\to\infty} x_n = x\right\} \text{ and}$$

$$B = \lim_{n\to\infty} \hat{w}(A_n) = \left\{x \in X : \exists \text{ a Cauchy sequence } \{x_n\} \text{ such that } x_n \in \hat{w}(A_n),\ \lim_{n\to\infty} x_n = x\right\}.$$

Let $x \in A$, then a Cauchy sequence $\{x_n\}$ in X, with $x_n \in A_n$ exists, such that $x_n \to x$. Since w is continuous, we get $w(x_n) \to w(x)$. In other words, there is a sequence $\{w(x_n)\}$ where $w(x_n) \in \hat{w}(A_n)$ such that $w(x_n) \to w(x)$. Then by definition of B, $w(x) \in B$, which implies that $w(A) \subseteq B$.

On the other hand, let $x \in B$, then one can find a Cauchy sequence $\{x_n\}$ in X where $x_n \in \hat{w}(A_n)$ such that $x_n \to x$. Then for each n, there exists $y_n \in A_n$ such that $x_n = w(y_n)$ and hence we can write $w(y_n) \to x$. Since

(A_n) is a Cauchy sequence in $\mathcal{H}(X)$, for $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $h_d(A_n, A_m) < \frac{\epsilon}{3}$ for every $n, m \geq N$. Then we have both $D(A_n, A_m) < \frac{\epsilon}{3}$ and $D(A_m, A_n) < \frac{\epsilon}{3}$ for all $n, m \geq N$. By definition, $D(A_n, A_m) < \frac{\epsilon}{3}$ implies $d(y_n, A_m) < \frac{\epsilon}{3}$ for every $n, m \geq N$. Choose $n, m \geq N$. Since d is continuous and A_m is compact, there exist $a \in A_m$ such that, $d(y_n, a) = D(y_n, A_m) < \frac{\epsilon}{3}$ for every $n \geq N$. By a similar argument, for $D(A_m, A_n) < \frac{\epsilon}{3}$ we get a $b \in A_n$ such that $d(y_m, b) < \frac{\epsilon}{3}$ for every $m \geq N$. From the condition $D(A_n, A_m) < \frac{\epsilon}{3}$, we get $d(a, b) < \frac{\epsilon}{3}$. Therefore, $d(y_n, y_m) \leq d(y_n, a) + d(a, b) + d(b, y_m) < \epsilon$ for every $n, m \geq N$, which implies that $\{y_n\}$ is a Cauchy sequence in X. Then we get a $y \in X$ such that $y_n \to y$ as $n \to \infty$. Thus we obtain $y \in A$. Also, by the continuity of w we get, $w(y_n) \to w(y)$. This implies that $x = w(y)$. Thus $x \in \hat{w}(A)$, which proves that $B \subseteq \hat{w}(A)$. This finishes the proof. □

We can establish the following result as an application of Theorem 1.5.

Theorem 4.7. *Let $w : X \to X$ be a continuous map on a complete metric space (X, d) that satisfies the condition(1), provided the nondecreasing maps $\psi, \phi : (0, \infty) \to \mathbb{R}$ fulfil the following constraints:*

(i) $\phi(t) < \psi(t)$ *for all* $t > 0$;

(ii) $\inf_{t>\epsilon} \psi(t) > -\infty$ *for every* $\epsilon > 0$;

(iii) if there exist two convergent sequences $\{\psi(t_n)\}$ and $\{\phi(t_n)\}$ with the same limit and $\{\psi(t_n)\}$ is nonincreasing, then $t_n \to 0$ as $n \to \infty$;

(iv) $\limsup_{t\to\epsilon+} \phi(t) < \liminf_{t\to\epsilon+} \psi(t)$ *for any* $\epsilon > 0$.

Then there exists a unique attractor, say $A^* \in \mathcal{H}(X)$, for w. Moreover, for any $A \in \mathcal{H}(X)$ the sequence $\{A_n\}$ in $\mathcal{H}(X)$, given by $A_n = w^n(A)$, converges to A^*.

Proof. Let $\hat{w}$ be the fractal operator generated by w. By Lemma 4.3, it is clear that $\hat{w}$ satisfies condition(1). Then, conditions *(i)*- *(iv)* in the hypothesis along with Lemma 4.6 and Theorem 1.5 guarantee a fixed point for $\hat{w}$, say $A^* \in \mathcal{H}(X)$, which is unique. Moreover, for any $A \in \mathcal{H}(X)$, the sequence $\{A_n\}$, where $A_n = \hat{w}^n(A)$, converges to A^*. This proves the theorem. □

Now, instead of a single function w on X, we consider an iterated function system (IFS) $\{X; w_1, w_2, \cdots, w_N\}$ where $w_i : X \to X$ for $i = 1, 2, \cdots, N$ are continuous and satisfies condition(1). The function $W : \mathcal{H}(X) \to \mathcal{H}(X)$, defined by $W(A) = \bigcup_{i=1}^{N} \hat{w}_i(A)$, is called the fractal operator generated by the IFS $\{X; w_1, w_2, \cdots, w_N\}$. A point $A \in \mathcal{H}(X)$ such that $W(A) = \bigcup_{i=1}^{N} \hat{w}_i(A) = A$, a fixed point of W, is called an attractor of the IFS.

Let us consider a more general situation.

Lemma 4.8. *Let $\{X; w_1, w_2, \cdots, w_N\}$ be an IFS on a complete metric space (X, d), where $w_i : X \to X$ are continuous maps satisfy the condition:*

$$\psi(d(w_i(x), w_i(y))) \leq \phi_i(d(x, y)), \tag{8}$$

where $\psi, \phi_i : (0, \infty) \to \mathbb{R}$ are nondecreasing for $i = 1, 2, \cdots, N$. If W is the fractal operator generated by the IFS, then it satisfies the condition(1), where $\phi(t) = \max_{1\leq i\leq N} \phi_i(t)$ for $t \in (0, \infty)$.

Proof. Let $A, B \in \mathcal{H}(X)$. Then, we have $\psi\left(h_d\left(\hat{w}_i(A), \hat{w}_i(B)\right)\right) \leq \phi_i\left(h_d(A, B)\right)$ for $i = 1, 2, \cdots, N$. By Lemma 4.2, we have

$$\begin{aligned} 0 \leq h_d\left(W(A), W(B)\right) &= h_d\left(\bigcup_{i=1}^{N} \hat{w}_i(A), \bigcup_{i=1}^{N} \hat{w}_i(B)\right) \\ &\leq \max_{1\leq i\leq N} h_d\left(\hat{w}_i(A), \hat{w}_i(B)\right) \\ &= h_d\left(\hat{w}_j(A), \hat{w}_j(B)\right), \end{aligned}$$

for some $j \in \{1, 2, \cdots N\}$. Since both ψ and ϕ_j are nondecreasing, we get

$$\begin{aligned}\psi\left(h_d\left(W(A), W(B)\right)\right) &\leq \psi\left(h_d\left(\hat{w}_j(A), \hat{w}_j(B)\right)\right) \\ &\leq \phi_j\left(h_d(A, B)\right) \\ &\leq \phi\left(h_d(A, B)\right).\end{aligned}$$

This completes the proof. □

We get the following theorem as a direct consequence of this lemma and Theorem 1.4.

Theorem 4.9. *Let $\{X; w_1, w_2, \cdots, w_N\}$ be an IFS on a complete metric space (X, d), where $w_i : X \to X$ are continuous maps satisfy the condition(8). If the maps ψ, ϕ_i, for $i = 1, 2, \cdots, N$, are nondecreasing and satisfy the constraints:*

(i) $\phi_i(t) < \psi(t)$ *for every* $t > 0$;

(ii) $\limsup_{t \to \epsilon+} \phi_i(t) < \psi(\epsilon+)$ *for every* $\epsilon > 0$,

then there exists a unique attractor, $A^* \in \mathcal{H}(X)$, for the IFS. Moreover, for any $A \in \mathcal{H}(X)$, the iterated sequence $A_n = W^n(A)$ converges to A^*.

Proof. Define a map $\phi : (0, \infty) \to \mathbb{R}$ such that $\phi(t) = \max_{1 \leq i \leq N} \phi_i(t)$ for $t \in (0, \infty)$. Then from condition *(i)* in the hypothesis, it is clear that,

$$\phi(t) = \max_{1 \leq i \leq N} \phi_i(t) \leq \psi(t), \quad \text{for every } t > 0.$$

Now, from condition *(ii)* in the hypothesis and a similar argument in Lemma 3.6 we get, $\limsup_{t \to \epsilon+} \phi(t) < \psi(\epsilon+)$ for every $\epsilon > 0$. Thus the functions ψ and ϕ satisfies conditions *(i)*- *(iii)* of Theorem 1.4. Then the proof follows immediately from Theorem 1.4 along with Lemma 4.8. □

Now we consider an example:

Example 4.10. *Consider $\mathbb{R}$ with Euclidean metric, which is a complete metric space. We define two maps $w_1, w_2 : \mathbb{R} \to \mathbb{R}$ as:*

$$w_1(x) = \begin{cases} \frac{2}{3}x & \text{if } x \geq 0 \\ \frac{-2}{3}x & \text{if } x < 0 \end{cases} \quad \text{and} \quad w_2(x) = \begin{cases} \frac{1}{3}x + \frac{2}{3} & \text{if } x \geq 0 \\ \frac{-1}{3}x + \frac{2}{3} & \text{if } x < 0 \end{cases}$$

Define three functions $\psi, \phi_1, \phi_2 : (0, \infty) \to \mathbb{R}$ as:

$$\psi(t) = \begin{cases} 2t & \text{if } 0 < t \leq 1 \\ 3t & \text{if } t > 1 \end{cases}, \quad \phi_1(t) = \begin{cases} \frac{3}{2}t & \text{if } 0 < t \leq 1 \\ 2t & \text{if } t > 1 \end{cases} \quad \text{and} \quad \phi_2(t) = \begin{cases} t & \text{if } 0 < t \leq 1 \\ \frac{5}{2}t & \text{if } t > 1 \end{cases}$$

It can be easily observed that the maps w_1 and w_2 are continuous and ψ, ϕ_1 and ϕ_2 are nondecreasing and satisfy the conditions (i) and (ii) of Theorem 4.9.
Consider the maps w_1, ψ and ϕ_1. Let $x, y \in \mathbb{R}$
***Case 1**: For $x, y \geq 0$ or $x, y < 0$, we get*

$$\psi\left(d\left(w_1(x), w_1(y)\right)\right) = \psi\left(\left|\frac{2}{3}x - \frac{2}{3}y\right|\right) = \begin{cases} \frac{4}{3}\left|x - y\right| & \text{if } \left|x - y\right| \leq 1 \\ 2\left|x - y\right| & \text{if } \left|x - y\right| > 1. \end{cases}$$

***Case 2**: If $x \geq 0$ and $y < 0$ we have*

$$\psi\left(d\left(w_1(x), w_1(y)\right)\right) = \psi\left(\left|\frac{2}{3}x + \frac{2}{3}y\right|\right) = \begin{cases} \frac{4}{3}\left|x + y\right| & \text{if } \left|x + y\right| \leq 1 \\ 2\left|x + y\right| & \text{if } \left|x + y\right| > 1. \end{cases}$$

Now, for any $x, y \in \mathbb{R}$ we have

$$\phi_1\left(d(x,y)\right) = \begin{cases} \frac{3}{2}\left|x-y\right| & \text{if } \left|x-y\right| \le 1 \\ 2\left|x-y\right| & \text{if } \left|x-y\right| > 1. \end{cases}$$

Thus from the above cases it is clear that, for any $x, y \in \mathbb{R}$, $\psi\left(d\left(w_1(x), w_2(y)\right)\right) \le \phi_1\left(d(x,y)\right)$. Similarly, if we consider the maps w_2, ψ and ϕ_2 we can have the following cases.
***Case 1**: For $x, y \ge 0$ or $x, y < 0$, we have*

$$\psi\left(d\left(w_2(x), w_2(y)\right)\right) = \psi\left(\left|\frac{1}{3}x - \frac{1}{3}y\right|\right) = \begin{cases} \frac{2}{3}\left|x-y\right| & \text{if } \left|x-y\right| \le 1 \\ \left|x-y\right| & \text{if } \left|x-y\right| > 1. \end{cases}$$

***Case 2**: If $x \ge 0$ and $y < 0$, we get*

$$\psi\left(d\left(w_2(x), w_2(y)\right)\right) = \psi\left(\left|\frac{1}{3}x + \frac{1}{3}y\right|\right) = \begin{cases} \frac{2}{3}\left|x+y\right| & \text{if } \left|x+y\right| \le 1 \\ \left|x+y\right| & \text{if } \left|x+y\right| > 1. \end{cases}$$

Now, for any $x, y \in \mathbb{R}$ we have

$$\phi_2\left(d(x,y)\right) = \begin{cases} \left|x-y\right| & \text{if } \left|x-y\right| \le 1 \\ \frac{5}{2}\left|x-y\right| & \text{if } \left|x-y\right| > 1. \end{cases}$$

Here also, we can observe that, for any $x, y \in \mathbb{R}$, $\psi\left(d\left(w_2(x), w_2(y)\right)\right) \le \phi_2\left(d\left(x, y\right)\right)$. Thus the IFS $\{\mathbb{R}; w_1, w_2\}$ along with the maps ψ, ϕ_1 and ϕ_2 satisfies the condition(8) and the hypothesis of Theorem 4.9. Hence by Theorem 4.9, the map $W : \mathcal{H}(\mathbb{R}) \to \mathcal{H}(\mathbb{R})$ defined by $W(A) = w_1(A) \cup w_2(A)$ satisfies condition(1) with the functions ψ and $\phi = \max\{\phi_1, \phi_2\}$. Also we can observe that $A = [0,1]$ is the unique attractor of this IFS. We have $w_1([0,1]) = [0, \frac{2}{3}]$ and $w_2([0,1]) = [\frac{2}{3}, 1]$. Hence, $W([0,1]) = w_1([0,1]) \cup w_2([0,1]) = [0,1]$.

5. Applications to Coupled Fractals

This section is reserved for the discussion about the existence and uniqueness of coupled self- similar sets for jointly (ψ, ϕ)-contractions.

Theorem 5.1. *Let $w : X \times X \to X$ be a continuous map on a complete metric space (X, d) that satisfies the condition(2) where $\psi, \phi : (0, \infty) \to \mathbb{R}$ follow the conditions:*

- *(i) ψ, ϕ are nondecreasing;*
- *(ii) $\phi(t) < \psi(t)$ for every $t > 0$;*
- *(iii) $\limsup_{t \to \epsilon+} \phi(t) < \psi(\epsilon+)$ for every $\epsilon > 0$.*

Then there exists a unique element $(A^*, B^*) \in \mathcal{H}(X) \times \mathcal{H}(X)$ such that

$$\begin{cases} A^* = w(A^*, B^*) \\ B^* = w(B^*, A^*) \end{cases}$$

Moreover, for any element $(A, B) \in \mathcal{H}(X) \times \mathcal{H}(X)$, the sequences

$$\begin{cases} A_n = w^n(A, B) \\ B_n = w^n(B, A) \end{cases}$$

converge to A^* and B^* respectively.

Proof. Let $X^* = X \times X$. We define the operator $w^* : X^* \to X^*$ such that $w^*(x, y) = (w(x, y), w(y, x))$. By Theorem 2.2, it is clear that w^* satisfies the condition(1) in the complete metric space (X^*, d^*). Let $W^* : \mathcal{H}(X^*) \to \mathcal{H}(X^*)$ be the fractal operator generated by w^*. Then Theorem 4.3 tells that the operator W^* satisfies condition(1) in the complete metric space $(\mathcal{H}(X^*), h_{d^*})$. Hence the result follows from Theorem 4.5. □

Our next result deals with existence of coupled self- similar set for an IFS.

Theorem 5.2. *Let $w_i : X^* \to X$ be continuous and jointly (ψ, ϕ_i)-contractions for $i = 1, 2, \cdots, N$ and ψ, ϕ_i satisfy the following conditions:*

(i) *ψ, ϕ_i are nondecreasing for $i = 1, 2, \cdots, N$;*

(ii) *$\phi_i(t) < \psi(t)$ for every $t > 0$;*

(iii) *$\limsup_{t\to\epsilon+} \phi_i(t) < \psi(\epsilon+)$ for every $\epsilon > 0$.*

Then there exists a unique pair $(A^*, B^*) \in \mathcal{H}(X) \times \mathcal{H}(X)$ such that

$$\begin{cases} A^* = \bigcup_{i=1}^{N} w_i(A^*, B^*) \\ B^* = \bigcup_{i=1}^{N} w_i(B^*, A^*). \end{cases}$$

Moreover, for any $(A, B) \in \mathcal{H}(X) \times \mathcal{H}(X)$, the sequences

$$\begin{cases} A_{n+1} = \bigcup_{i=1}^{N} w_i(A_n, B_n) \\ B_{n+1} = \bigcup_{i=1}^{N} w_i(B_n, A_n) \end{cases}$$

converge to A^* and B^* respectively.

Proof. Define $w_i^* : X^* \to X^*$ by $w_i^*(x, y) = (w_i(x, y), w_i(y, x))$ for $i = 1, 2, \cdots, N$. Then from the proof of Theorem 2.2 we can say each map w_i^* satisfies condition(1) with functions ψ, ϕ_i for $i = 1, 2, \cdots, N$. Now we define, the fractal operator generated by the IFS $\{X^*; w_1^*, w_2^*, \cdots, w_N^*\}$, $W^* : \mathcal{H}(X^*) \to \mathcal{H}(X^*)$ by $W^*(A, B) = \bigcup_{i=1}^{N} \hat{w}_i^*(A, B)$. From Lemma 4.8, it is clear that W^* satisfies condition(1) with $\phi = \max_{1\le i\le N} \phi_i$. Then the proof can be completed by using Theorem 4.9. □

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