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Impulsive conformable evolution equations in Banach spaces with fractional semigroup

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Abstract. In this paper, we explore the existence, uniqueness, and stability of mild solutions in impulsive differential equations featuring conformable fractional derivatives. Our principal findings leverage fractional semigroup theory, complemented by some fixed-point theorems. Additionally, we provide a practical example to demonstrate the relevance of our theoretical outcomes.

1. Introduction

Fractional calculus deals with integrals and derivatives of non-integer orders, extending classical calculus. Its application spans various fields including mechanics, physics, chemistry, engineering, and other scientific disciplines, making it a crucial area of study [1–4]. Various fractional derivative types like Riemann–Liouville, Caputo, Hadamard, Caputo–Hadamard and conformable have been developed, each contributing significantly to its advancement [5–11].

Impulsive FDEs are a special class of differential equations that involve fractional derivatives and impulses. In these equations, the fractional derivative captures the memory effects and non-local behavior of the system, while the impulses represent sudden changes or discontinuities in the system. For this reason and others, this class has increasingly been used in various applications. In particular, impulsive FDEs are widely used in various scientific fields for modeling dynamic phenomena with abrupt changes. Their applications span mechanical systems, electrical engineering, chemical reactions, and fluid dynamics, playing a crucial role in understanding dynamic systems facing sudden variations [12–16]. The Cauchy problem associated with these equations garners significant interest from numerous researchers [17–22].

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For instance, Shaochun Ji and Shu Wen [23] have demonstrated the existence of mild solutions for the Cauchy problem

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), \quad t \in J = [0, T], \ t_k \neq t, \ i = 1, ..., p \\ u(t_i^+) - u(t_i^-) = I_i(u(t_k)), \ i = 1, ..., p \\ u(0) = g(u) + u_0, \end{cases}$$

where the operator A generates an infinitesimal of C_0 semigroup $T(t)$ on a Banach space $\mathcal{X}$, $f : [0, T] \times \mathcal{X} \to \mathcal{X}$, $0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T$, $I_i : \mathcal{X} \to \mathcal{X}, i = 1, 2, \cdots, p$ are impulsive functions and $g : \mathcal{PC}([0, T], \mathcal{X}) \to \mathcal{X}$. Wang et al. [24] have demonstrated the existence and uniqueness of mild solutions for the ensuing fractional Cauchy problem within the context of the Caputo fractional derivative fractional initial value problem.

Rabhi et al [25] have studied the existence of a mild solution, investigating it using fractional semigroups for the following conformable fractional initial value problem

$$\begin{cases} \mathfrak{D}^{(\alpha)} u(t) - Au(t) = f(t, u(t)), \quad t > t_0 \\ u(t_0) = u_0. \end{cases}$$

In this paper, we investigate the solutions of an abstract differential equations using fractional semigroups involving conformable fractional derivatives for $0 < \varrho \leq 1$ and instantaneous impulses in a Banach space $(\mathcal{X}, \|.\|)$:

$$\begin{cases} \mathfrak{D}^{(\varrho)} \psi(t) = A\psi(t) + \phi(t, \psi(t)), \quad t \in J = [0, T], \ t_k \neq t, \ k = 1, ..., p \\ \psi(t_k^+) - \psi(t_k^-) = \varphi_k(\psi(t_k)), \ k = 1, ..., p \\ \psi(0) = \psi_0, \end{cases} \tag{1}$$

where the operator A generates an infinitesimal of fractional C_0-ϱ-semigroup $\mathfrak{I}_\varrho(t)$ and $\psi_0 \in \mathcal{X}$. $\psi(t_k^+) - \psi(t_k^-) = \varphi_k(\psi(t_k))$ means the impulsive condition, with $\psi(t_k^+), \psi(t_k^-)$ are the right and left limits of $\psi(.)$ at $t = t_k$. The functions $\phi : J \times \mathcal{X} \to \mathcal{X}$, $\varphi_k : \mathcal{X} \to \mathcal{X}$ satisfy certain assumptions which will be specified later.

In many branches of mathematics, physics, engineering, and other sciences, semigroup properties are essential for problem solving and analysis. The following are major important roles for the significance of semigroup properties in mathematical analysis [26–29]:

- Semigroups are frequently encountered in the context of dynamical and linear systems. For the stability analysis of these systems, knowledge of the semigroup properties—such as contractivity and the generation property—is crucial.
- Understanding the behavior of solutions to the ordinary, fractional, and partial differential equations over time requires an understanding of semigroup properties including continuity, positivity, and dissipativity.
- Functional analysis provides powerful techniques including spectral theory and operator semigroup theory, which can be applied with an understanding of semigroup properties.
- Engineers and scientists may evaluate and forecast the behavior of complex systems, create control strategies, and maximize performance by modeling physical systems using semigroups properties.

This work is significant as semi-group operators provide a versatile framework for studying dynamic systems, playing a crucial role in mathematical modeling and analysis [26, 30, 31].

2. Preliminaries

This section will explore conformable fractional derivatives and introduce the concept of a fractional semigroup for more details [7, 31–33]. This semigroup serves as a generalized version of the classical semigroup, with its generator based on the conformable derivative. Important properties of the fractional semigroup and its analysis will be presented. Additionally, it covers essential background details concerning the Kuratowski measure of noncompactness along with Mönch's fixed-point theorem.

Definition 2.1. *[7] Let $\psi : [0,\infty) \longrightarrow \mathbb{R}$ be a real valued function. Then the "conformable fractional derivative" of ψ of order $\varrho \in (0,1]$, at $t > 0$ is defined by*

$$\mathfrak{D}_t^{\varrho}\psi(t) = \lim_{s\to 0} \frac{\psi(t + st^{1-\varrho}) - \psi(t)}{s}. \tag{2}$$

When the limit exists, we say that ψ is $\varrho-$differentiable at t.
If ψ is $\varrho-$differentiable on the interval $(0,b]$, $b > 0$ and limit of $\mathfrak{D}_t^{\varrho}\psi(t)$ exists as t approaches 0^+, then we define $\mathfrak{D}_t^{\varrho}\psi(0)$ as the limit of $\mathfrak{D}_t^{\varrho}\psi(t)$ as t approaches 0^+.
The conformable fractional integral of ψ of order ϱ is defined by

$$\mathfrak{I}_{\varrho}^{a}(\psi)(t) = \mathfrak{I}_{1}^{a}(t^{1-\varrho}\psi(t)) = \int_a^t \frac{\psi(s)}{s^{1-\varrho}}\, ds, \tag{3}$$

where the integral is the usual Riemann improper integral. When $a = 0$, we write simply $\mathfrak{I}_{\varrho}^{a} = \mathfrak{I}_{\varrho}$.

Definition 2.2. *[31] Let $\varrho \in (0,1]$. For a Banach space $\mathcal{X}$, a family $\{\mathfrak{T}_{\varrho}(t)\}_{t\geq 0} \subseteq \mathcal{L}(\mathcal{X})$ is called a fractional $\varrho-$semigroup (or $\varrho-$semigroup) of operators if*

- *(i)* $\mathfrak{T}_{\varrho}(0) = I$,
- *(ii)* $\mathfrak{T}_{\varrho}(t+s)^{\frac{1}{\varrho}} = \mathfrak{T}_{\varrho}(t^{\frac{1}{\varrho}})\mathfrak{T}_{\varrho}(s^{\frac{1}{\varrho}})$ *for all* $t, s \in [0,\infty)$.

Clearly, if $\varrho = 1$, then $1-$semigroups are just the usual semigroups. A fractional $\varrho-$semigroup $\{\mathfrak{T}_{\varrho}(t)\}_{t\geq 0} \in \mathcal{L}(\mathcal{X})$ on a Banach space $\mathcal{X}$ is called uniformly continuous if

$$\lim_{t\to 0^+} \|\mathfrak{T}_{\varrho}(t) - I\| = 0.$$

An $\varrho-$semigroup $\mathfrak{T}_{\varrho}(t)$ is called a C_0-ϱ-semigroup, if for each $x \in \mathcal{X}$, $\mathfrak{T}_{\varrho}(t)x \to x$ as $t \to 0^+$.
The conformable $\varrho-$derivative of $\mathfrak{T}_{\varrho}(t)$ at $t = 0$ is called the $\varrho-$infinitesimal generator of the fractional $\varrho-$semigroup $\mathfrak{T}_{\varrho}(t)$, with domain equals:

$$\{\, x \in \mathcal{X}, \lim_{t\to 0^+} \mathfrak{D}_t^{\varrho}\mathfrak{T}_{\varrho}(t)x \text{ exists } \}.$$

We will write A for such a generator.

Theorem 2.3. *[34] Let $\mathfrak{T}_{\varrho}(t)$ be a $C_0 - \varrho-$semigroup where $\varrho \in (0,1]$. There exist constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|\mathfrak{T}_{\varrho}(t)\| \leq Me^{\omega t^{\varrho}} \quad \text{for} \quad 0 \leq t \leq \infty. \tag{4}$$

Theorem 2.4. *[34] Let $\mathfrak{T}_{\varrho}(t)$ be a $C_0 - \varrho-$semigroup where $\varrho \in (0,1]$ and let A be its $\varrho-$infinitesimal generator. Then*

- *a)* For $x \in \mathcal{X}$, $\displaystyle\lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_t^{t+\epsilon t^{1-\varrho}} \frac{1}{s^{1-\varrho}}\mathfrak{T}_{\varrho}(s)x\, ds = \mathfrak{T}_{\varrho}(t)x \quad$ *for every* $\quad t > 0$.
- *b)* For $x \in \mathcal{X}$, $\displaystyle\int_0^t \frac{1}{s^{1-\varrho}}\mathfrak{T}_{\varrho}(s)x\, ds \in D(A)$ *and* $\displaystyle A(\int_0^t \frac{1}{s^{1-\varrho}}\mathfrak{T}_{\varrho}(s)x\, ds) = \mathfrak{T}_{\varrho}(t)x - x$.
- *c)* For $x \in D(A)$, $\mathfrak{T}_{\varrho}(t)x \in D(A)$ *and* $\mathfrak{D}_t^{\sigma}\mathfrak{T}_{\varrho}(t)x = A\mathfrak{T}_{\varrho}(t)x = \mathfrak{T}_{\varrho}(t)Ax$.
- *d)* For $x \in D(A)$, $\displaystyle\mathfrak{T}_{\varrho}(t)x - \mathfrak{T}_{\varrho}(s)x = \int_s^t \frac{1}{u^{1-\sigma}}\mathfrak{T}_{\varrho}(u)Ax\, du = \int_s^t \frac{1}{u^{1-\varrho}}A\mathfrak{T}_{\varrho}(u)x\, du$.

Definition 2.5. *[35] Let $\mathcal{X}$ be a Banach space and Ω the family of bounded subsets of $\mathcal{X}$. The Kuratowski measure of noncompactness is the map $\vartheta : \Omega \to \mathbb{R}^+$ defined by*

$$\vartheta(F) = \inf\{\epsilon > 0 : F \subseteq \bigsqcup_{k=1}^{n} F_k \text{ and } diam(F_k) \leq \epsilon\}$$

If F, F_1, and F_2 are bounded subsets in $\mathcal{X}$, then

1. $\vartheta(\emptyset) = 0$.
2. $\vartheta(F) = 0 \Leftrightarrow F$ is relatively compact.
3. $\vartheta(F + x) = \vartheta(F)$ for any $x \in X$.
4. $\vartheta(F) = \vartheta(\overline{F}) = \vartheta(convF)$.
5. $\vartheta(\lambda F) = |\lambda|\vartheta(F), \lambda \in \mathbb{R}$.
6. $F_1 \subseteq F_2 \Rightarrow \vartheta(F_1) \leq \vartheta(F_2)$.
7. $\vartheta(F_1 + F_2) \leq \vartheta(F_1) + \vartheta(F_2)$.
8. $\vartheta(F_1 \cup F_2) \leq \max(\vartheta(F_1), \vartheta(F_2))$.

Here $\overline{F}$ and *convF* denote the closure and the convex hull of the bounded set F, respectively.

Lemma 2.6. *[36] If $\Delta \subseteq C(I, \mathcal{X})$ is an equicontinuous and bounded set, then the function $\vartheta(\Delta(t))$ is continuous for $t \in I$, and $\vartheta(\int_I u(s)ds) \leq \int_I \vartheta(u(s))ds$ for any $u \in \Delta$.*

Theorem 2.7. ***(Mönch's Fixed Point Theorem)****[37]*
Let F be a bounded, closed and convex subset of a Banach space $\mathcal{X}$ such that $0 \in F$, and $\Upsilon : F \to F$ be a continuous satisfying Mönch's condition, i.e., $\Delta = \overline{conv}(\Upsilon(\Delta) \cup \{0\}) \Rightarrow \vartheta(\Delta) = 0$. Then Υ has a fixed point.

3. Main results

Let $\mathcal{PC}$ is the space of functions $\psi(.)$ defined from $[0, T]$ into $\mathcal{X}$ such that $\psi(.)$ is continuous on each interval $]t_k, t_{k+1}]$ and $\psi(t_k^+), \psi(t_k^-)$ exist.
Evidently, $\mathcal{PC}(J, \mathcal{X})$ is a Banach space with norm $\|\psi\| = \sup\{\|\psi(t)\|,\ t \in J\}$. According to Theorem 2.3, there exists a constant $M > 0$ such that

$$\|\mathfrak{I}_\varrho(t)\| \leq M, \quad t \in J.$$

3.1. Representation of mild solution

We introduce the formula for the mild solution of Problem (1) using fractional semigroups.

Lemma 3.1. *Let the initial value problem:*

$$\begin{cases} \mathfrak{D}_t^\varrho \psi(t) = A\psi(t) + \phi(t, \psi(t)) & t \in J \\ \psi(0) = \psi_0. \end{cases} \tag{5}$$

Then

$$\psi(t) = \mathfrak{I}_\varrho(t)\psi_0 + \int_0^t \frac{1}{s^{1-\varrho}} \mathfrak{I}_\varrho(t^\varrho - s^\varrho)^{\frac{1}{\varrho}} \phi(s, \psi(s))\, ds.$$

is the mild solution of the initial value problem (5).

Proof. Let A be a generator of C_0-ϱ-semigroup $\mathfrak{I}_\varrho$. If ψ is the solution of (5), then the $\mathcal{X}$ valued-function $\aleph(s) = \mathfrak{I}_\varrho(t^\varrho - s^\varrho)^{\frac{1}{\varrho}}\psi(s)$ is ϱ−deffirentiable for $0 < s < t$ and

$$\begin{aligned}\mathfrak{D}_s^\varrho\aleph(s) &= -A\mathfrak{I}_\varrho(t^\varrho - s^\varrho)^{\frac{1}{\varrho}}\psi(s) + \mathfrak{I}_\varrho(t^\varrho - s^\varrho)^{\frac{1}{\varrho}}\mathfrak{D}_s^\varrho\psi(s)\\ &= -A\mathfrak{I}_\varrho(t^\varrho - s^\varrho)^{\frac{1}{\varrho}}\psi(s) + \mathfrak{I}_\varrho(t^\varrho - s^\varrho)^{\frac{1}{\varrho}}[A\psi(s) + \phi(s,\psi(s))]\\ &= \mathfrak{I}_\varrho(t^\varrho - s^\varrho)^{\frac{1}{\varrho}}\phi(s,\psi(s)).\end{aligned} \tag{6}$$

Now applying $\mathfrak{I}_\varrho^0$ to (6), we have

$$\mathfrak{I}_\varrho^0(\mathfrak{D}_s^\varrho\aleph(s))(t) = \mathfrak{I}_\varrho(t^\varrho - t^\varrho)^{\frac{1}{\varrho}}\psi(t) - \mathfrak{I}_\varrho\psi_0 = \int_0^t \frac{1}{s^{1-\varrho}}\mathfrak{I}_\varrho(t^\varrho - s^\varrho)^{\frac{1}{\varrho}}\phi(s,\psi(s))\,ds.$$

So

$$\psi(t) = \mathfrak{I}_\varrho(t)\psi_0 + \int_0^t \frac{1}{s^{1-\varrho}}\mathfrak{I}_\varrho(t^\varrho - s^\varrho)^{\frac{1}{\varrho}}\phi(s,\psi(s))\,ds. \tag{7}$$

□

We assume that the solution of equation (1) is such that at the point of discontinuity t_k, we have $\psi(t_k^-) = \psi(t_k)$. Hence, one has

$$\psi(t_1^-) = \mathfrak{I}_\varrho(t_1)\psi_0 + \int_0^{t_1} \frac{1}{s^{1-\varrho}}\mathfrak{I}_\varrho(t_1^\varrho - s^\varrho)^{\frac{1}{\varrho}}\phi(s,\psi(s))\,ds.$$

For $t \in (t_1, t_2]$, using the fractional semigroup in equation (1), we obtain

$$\begin{aligned}\psi(t) &= \mathfrak{I}_\varrho(t^\varrho - t_1^\varrho)^{\frac{1}{\varrho}}\psi(t_1^+) + \int_{t_1}^t \frac{1}{s^{1-\varrho}}\mathfrak{I}_\varrho(t^\varrho - s^\varrho)^{\frac{1}{\varrho}}\phi(s,\psi(s))\,ds\\ &= \mathfrak{I}_\varrho(t^\varrho - t_1^\varrho)^{\frac{1}{\varrho}}[\psi(t_1^-) + \varphi_1(\psi(t_1))] + \int_{t_1}^t \frac{1}{s^{1-\varrho}}\mathfrak{I}_\varrho(t^\varrho - s^\varrho)^{\frac{1}{\varrho}}\phi(s,\psi(s))\,ds.\end{aligned}$$

Replacing $\psi(t_1^-)$ by its expression in the above equation, we get

$$\begin{aligned}\psi(t) &= \mathfrak{I}_\varrho(t^\varrho - t_1^\varrho)^{\frac{1}{\varrho}}[\mathfrak{I}_\varrho(t_1)\psi_0 + \int_0^{t_1} \frac{1}{s^{1-\varrho}}\mathfrak{I}_\varrho(t_1^\varrho - s^\varrho)^{\frac{1}{\varrho}}\phi(s,\psi(s))\,ds + \varphi_1(\psi(t_1))]\\ &\quad + \int_{t_1}^t \frac{1}{s^{1-\varrho}}\mathfrak{I}_\varrho(t^\varrho - s^\varrho)^{\frac{1}{\varrho}}\phi(s,\psi(s))\,ds.\end{aligned}$$

By using a computation, the above equation becomes

$$\psi(t) = \mathfrak{I}_\varrho(t)\psi_0 + \mathfrak{I}_\varrho(t^\varrho - t_1^\varrho)^{\frac{1}{\varrho}}\varphi_1(\psi(t_1)) + \int_0^t \frac{1}{s^{1-\varrho}}\mathfrak{I}_\varrho(t^\varrho - s^\varrho)^{\frac{1}{\varrho}}\phi(s,\psi(s))\,ds.$$

In particular, for $t = t_2^-$, one has

$$\psi(t_2^-) = \mathfrak{I}_\varrho(t_2)\psi_0 + \mathfrak{I}_\varrho(t_2^\varrho - t_1^\varrho)^{\frac{1}{\varrho}}\varphi_1(\psi(t_1)) + \int_0^{t_2} \frac{1}{s^{1-\varrho}}\mathfrak{I}_\varrho(t_2^\varrho - s^\varrho)^{\frac{1}{\varrho}}\phi(s,\psi(s))\,ds.$$

As the same, for $t \in (t_2, t_3]$, we obtain

$$\begin{aligned}\psi(t) &= \mathfrak{I}_\varrho(t^\varrho - t_2^\varrho)^{\frac{1}{\varrho}}\psi(t_2^+) + \int_{t_2}^t \frac{1}{s^{1-\varrho}}\mathfrak{I}_\varrho(t^\varrho - s^\varrho)^{\frac{1}{\varrho}}\phi(s,\psi(s))\,ds\\ &= \mathfrak{I}_\varrho(t^\varrho - t_2^\varrho)^{\frac{1}{\varrho}}[\psi(t_2^-) + \varphi_2(\psi(t_2))] + \int_{t_2}^t \frac{1}{s^{1-\varrho}}\mathfrak{I}_\varrho(t^\varrho - s^\varrho)^{\frac{1}{\varrho}}\phi(s,\psi(s))\,ds.\end{aligned}$$

Hence, replacing $\psi(t_2^-)$ by its expression, we have

$$\begin{aligned}\psi(t) &= \Im_\varrho(t^\varrho - t_2^\varrho)^{\frac{1}{\varrho}}[\Im_\varrho(t_2)\psi_0 + \Im_\varrho(t_2^\varrho - t_1^\varrho)^{\frac{1}{\varrho}}\varphi_1(\psi(t_1)) \\ &+ \int_0^{t_2} \frac{1}{s^{1-\varrho}}\Im_\varrho(t_2^\varrho - s^\varrho)^{\frac{1}{\varrho}}\phi(s,\psi(s))\,ds + \varphi_2(\psi(t_2))] \\ &+ \int_{t_2}^{t} \frac{1}{s^{1-\varrho}}\Im_\varrho(t^\varrho - s^\varrho)^{\frac{1}{\varrho}}\phi(s,\psi(s))\,ds.\end{aligned}$$

Using a computation, we get

$$\begin{aligned}\psi(t) &= \Im_\varrho(t)\psi_0 + \Im_\varrho(t^\varrho - t_1^\varrho)^{\frac{1}{\varrho}}\varphi_1(\psi(t_1)) + \Im_\varrho(t^\varrho - t_2^\varrho)^{\frac{1}{\varrho}}\varphi_2(\psi(t_2)) \\ &+ \int_0^{t} \frac{1}{s^{1-\varrho}}\Im_\varrho(t^\varrho - s^\varrho)^{\frac{1}{\varrho}}\phi(s,\psi(s))\,ds.\end{aligned}$$

Repeating the same process, we obtain the following conformable fractional formula

$$\psi(t) = \Im_\varrho(t)\psi_0 + \sum_{0<t_k<t}\Im_\varrho(t^\varrho - t_k^\varrho)^{\frac{1}{\varrho}}\varphi_k(\psi(t_k)) + \int_0^t \frac{1}{s^{1-\varrho}}\Im_\varrho(t^\varrho - s^\varrho)^{\frac{1}{\varrho}}\phi(s,\psi(s))\,ds.$$

Definition 3.2. *A function $\psi \in \mathcal{PC}$ is called a mild solution of conformable fractional Cauchy problem (1) if*

$$\psi(t) = \Im_\varrho(t)\psi_0 + \sum_{0<t_k<t}\Im_\varrho(t^\varrho - t_k^\varrho)^{\frac{1}{\varrho}}\varphi_k(\psi(t_k)) + \int_0^t \frac{1}{s^{1-\varrho}}\Im_\varrho(t^\varrho - s^\varrho)^{\frac{1}{\varrho}}\phi(s,\psi(s))\,ds.$$

3.2. Existence of mild solutions

In this section, we apply a technique based on noncompactness measure assumption. In the following, we prove existence results, for the problem (1) by using a Mönch's fixed point theorem. Let us introduce the following hypotheses:

(H1) $\Im_\varrho(t)$ is compact for $t > 0$ in the Banach space $\mathcal{X}$.

(H2) The function $\phi(.,\psi(.)) : [0,T] \to \mathcal{X}$ is continuous, for all $\psi \in \mathcal{PC}$.

(H3) The function $\phi(t,.) : \mathcal{X} \to \mathcal{X}$ is continuous, and there exists $\mu \in C(J,\mathbb{R}^+)$ such that
i) $\|\phi(t,\psi(t))\| \leq \mu(t)\|\psi\|$ for each $t \in J$, $\psi \in \mathcal{PC}$.
ii) For each $t \in J$ and each bounded set $V \subset \mathcal{X}$, we have $\vartheta(\phi(t,V)) \leq \mu(t)\vartheta(V)$.

(H4) The mapping $\varphi_k : \mathcal{X} \to \mathcal{X}$ is continuous for $k = 1,...,p$, and for each $k = 1,...,p$ there exists a positive constant $\zeta_k > 0$ such that
i) $\|\varphi_k(\psi)(t)\| \leq \zeta_k\|\psi\|$ for each $t \in J$, $\psi \in \mathcal{PC}$.
ii) For each bounded set $V \subset \mathcal{X}$, we have $\vartheta(\varphi_k(V)) \leq \zeta_k\vartheta(V)$.

Let $B_\rho = \{\psi \in \mathcal{PC}(J,\mathcal{X}) : \|\psi\| \leq \rho\}$ for any $\rho > 0$. Then B_ρ is clearly a bounded closed and convex subset in $\mathcal{PC}(J,\mathcal{X})$. We define the operator Υ by

$$\Upsilon(\psi)(t) = \Im_\varrho(t)\psi_0 + \sum_{0<t_k<t}\Im_\varrho(t^\varrho - t_k^\varrho)^{\frac{1}{\varrho}}\varphi_k(\psi(t_k)) + \int_0^t \frac{1}{s^{1-\varrho}}\Im_\varrho(t^\varrho - s^\varrho)^{\frac{1}{\varrho}}\phi(s,\psi(s))\,ds.$$

Obviously, $\psi \in B_\rho$ is a mild solution of (1) if and only if the operator Υ has a fixed point on B_ρ, i.e., there exists $\psi \in B_\rho$ satisfies $\psi = \Upsilon\psi$.

Lemma 3.3. *If (H2)–(H4) hold, then Υ is continuous in B_ρ and maps B_ρ into B_ρ for any $\rho > 0$ satisfies*

$$\frac{M\|\psi_0\|}{1 - M(\frac{T^\varrho}{\varrho}\|\mu\| + \sum_{k=1}^{p}\zeta_k)} \leq \rho. \tag{8}$$

Proof. **Claim**: Υ maps B_ρ into B_ρ.
For $t \in J$ and for any $\psi \in B_\rho$, by using (H3)(i) and (H4)(i), we get

$$\begin{aligned}
\|\Upsilon(\psi(t))\| &\leq \|\mathfrak{I}_\varrho(t)\|\|\psi_0\| + \sum_{0<t_k<t}\|\mathfrak{I}_\varrho(t^\varrho - t_k^\varrho)^{\frac{1}{\varrho}}\|\|\varphi_k(\psi(t_k))\| + \int_0^t \frac{1}{s^{1-\varrho}}\|\mathfrak{I}_\varrho(t^\varrho - s^\varrho)^{\frac{1}{\varrho}}\|\|\phi(s,\psi(s))\|\,ds \\
&\leq M\|\psi_0\| + M\|\psi\|\sum_{k=0}^{k=p}\zeta_k + M\int_0^t \frac{1}{s^{1-\varrho}}\mu(t)\|\psi\|\,ds \\
&\leq M\|\psi_0\| + M\|\psi\|\sum_{k=0}^{k=p}\zeta_k + M\|\mu\|\|\psi\|\int_0^t \frac{1}{s^{1-\varrho}}\,ds \\
&\leq M\|\psi_0\| + M\|\psi\|\sum_{0<t_k<t}\zeta_k + M\frac{T^\varrho}{\varrho}\|\mu\|\|\psi\| \\
&\leq \rho[1 - M(\frac{T^\varrho}{\varrho}\|\mu\| + \sum_{k=1}^{p}\zeta_k)] + M(\sum_{0<t_k<t}\zeta_k + \frac{T^\varrho}{\varrho}\|\mu\|)\rho \\
&= \rho.
\end{aligned}$$

Claim: Υ is continuous in B_ρ.
Let ψ_n be a sequence such that $\psi_n \to \psi$ in $\mathcal{PC}(J,\mathcal{X})$. Then for each $t \in J$

$$\begin{aligned}
\|\Upsilon(\psi_n)(t) - \Upsilon(\psi)(t)\| \leq &\sum_{0<t_k<t}\|\mathfrak{I}_\varrho(t^\varrho - t_k^\varrho)^{\frac{1}{\varrho}}\|\|\varphi_k(\psi_n(t_k)) - \varphi_k(\psi(t_k))\| \\
&+ \int_0^t \frac{1}{s^{1-\varrho}}\|\mathfrak{I}_\varrho(t^\varrho - s^\varrho)^{\frac{1}{\varrho}}\|\|\phi(s,\psi_n(s)) - \phi(s,\psi(s))\|\,ds.
\end{aligned}$$

Using assumption (H1)(i), $s^{\varrho-1}\|\phi(s,\psi_n(s)) - \phi(s,\psi(s))\| \leq 2s^{\varrho-1}\mu(s)$ and $\phi(s,\psi_n(s)) \to \phi(s,\psi(s))$ as $n \to +\infty$. The Lebesgue dominated convergence theorem proves that
$\int_0^t \frac{1}{s^{1-\varrho}}\|\phi(s,\psi_n(s)) - \phi(s,\psi(s))\|\,ds \to 0$ as $n \to +\infty$. According to continuity of the function φ, we deduce that $\lim_{n\to+\infty}\|\varphi_k(\psi_n(t_k)) - \varphi_k(\psi(t_k))\| = 0$. Hence, Υ is continuous. □

Lemma 3.4. *If (H1)–(H4) hold, then $\Upsilon(B_\rho)$ is bounded and equicontinuous.*

Proof. By lemma 3.3, it is obvious that $\Upsilon(B_\rho) \subset \mathcal{PC}(J,\mathcal{X})$ is bounded.
For $\psi \in B_\rho$ and $\tau_{k_1}, \tau_{k_2} \in J$ such that $t_k \leq \tau_{k_1} \leq \tau_{k_2} \leq t_{k+1}$, for $k = 1, ..., p$ where $t_{p+1} = T$. We have

$$\begin{aligned}
\|\Upsilon(\psi)(\tau_{k_2}) - \Upsilon(\psi)(\tau_{k_1})\| \leq &\|\mathfrak{I}_\varrho(\tau_{k_2})\psi_0 - \mathfrak{I}_\varrho(\tau_{k_1})\psi_0\| \\
&+ \|\sum_{j=1}^{k}\mathfrak{I}_\varrho(\tau_{k_2}^\varrho - t_j^\varrho)^{\frac{1}{\varrho}}\varphi_k(\psi(t_j)) - \sum_{j=1}^{k}\mathfrak{I}_\varrho(\tau_{k_1}^\varrho - t_j^\varrho)^{\frac{1}{\varrho}}\varphi_k(\psi(t_j))\| \\
&+ \|\int_0^{\tau_{k_2}} \frac{1}{s^{1-\varrho}}\mathfrak{I}_\varrho(\tau_{k_2}^\varrho - s^\varrho)^{\frac{1}{\varrho}}\phi(s,\psi(s))\,ds \\
&- \int_0^{\tau_{k_1}} \frac{1}{s^{1-\varrho}}\mathfrak{I}_\varrho(\tau_{k_1}^\varrho - s^\varrho)^{\frac{1}{\varrho}}\phi(s,\psi(s))\,ds\|.
\end{aligned}$$

Using a computation, we get

$$\begin{aligned}\|\Upsilon(\psi)(\tau_{k_2}) - \Upsilon(\psi)(\tau_{k_1})\| \leq &\|(\mathfrak{I}_\varrho(\tau_{k_2}^\varrho - \tau_{k_1}^\varrho)^{\frac{1}{\varrho}} - I)\mathfrak{I}_\varrho(\tau_{k_1}^\varrho)^{\frac{1}{\varrho}}\psi_0\| \\ &+ \|\sum_{j=1}^{k}(\mathfrak{I}_\varrho(\tau_{k_2}^\varrho - \tau_{k_1}^\varrho)^{\frac{1}{\varrho}} - I)\mathfrak{I}_\varrho(\tau_{k_1}^\varrho - t_j^\varrho)^{\frac{1}{\varrho}}\varphi_k(\psi(t_j))\| \\ &+ \|\int_0^{\tau_{k_1}} \frac{1}{s^{1-\varrho}}(\mathfrak{I}_\varrho(\tau_{k_2}^\varrho - \tau_{k_1}^\varrho)^{\frac{1}{\varrho}} - I)\mathfrak{I}_\varrho(\tau_{k_1}^\varrho - s^\varrho)^{\frac{1}{\varrho}}\phi(s,\psi(s))\,ds \\ &+ \int_{\tau_{k_1}}^{\tau_{k_2}} \frac{1}{s^{1-\varrho}}\mathfrak{I}_\varrho(\tau_{k_2}^\varrho - s^\varrho)^{\frac{1}{\varrho}}\phi(s,\psi(s))\,ds\|.\end{aligned}$$

Using assumptions (H1)(i) and (H2)(i), we obtain

$$\|\Upsilon(\psi)(\tau_{k_2}) - \Upsilon(\psi)(\tau_{k_1})\| \leq M\|\mathfrak{I}_\varrho(\tau_{k_2}^\varrho - \tau_{k_1}^\varrho)^{\frac{1}{\varrho}} - I\|(\|\psi_0\| + \sum_{j=1}^{p}\zeta_J\rho + \|\mu\|\rho\frac{T^\varrho}{\varrho}) + M\|\mu\|\rho\frac{\tau_{k_2}^\varrho - \tau_{k_1}^\varrho}{\varrho}.$$

The compactness of $(\mathfrak{I}_\varrho(t))$ assures that $\|\mathfrak{I}_\varrho(\tau_{k_2}^\varrho - \tau_{k_1}^\varrho)^{\frac{1}{\varrho}} - I\| \to 0$ as $\tau_{k_1} \to \tau_{k_2}$. So when for $\tau_{k_1} \to \tau_{k_2}$ the right side of the above inequality tends to zero. In a similar manner throughout the interval $[0, t_1]$. Since the finite union of equicontinuous sets is equicontinuous, thus $\Upsilon(B_\rho)$ is equicontinuous on $[0, T]$. □

Theorem 3.5. *Assume that (H1)–(H4) are hold, then impulsive differential equations (1) has at least one mild solution within the ball B_ρ, where the radius ρ satisfies (8).*

Proof. We know that B_ρ is closed and convex. From Lemmas 3.3 and 3.4, we know that Ψ is a continuous map from B_ρ into B_ρ and the set $\Upsilon(B_\rho)$ is bounded and equicontinuous. We shall prove that Υ satisfies the Mönch's condition 2.7. Now, let Δ be a subset of B_ρ such that $\Delta \subset \overline{conv}(\Upsilon(\Delta) \cup \{0\})$. Then Δ is bounded and equicontinuous and therefore the function $t \to \chi(t) = \vartheta(\Delta(t))$ is continuous on J. From (H3),(H4), and the properties of the measure we have for each $t \in J$:

$$\begin{aligned}\chi(t) &\leq \vartheta(\overline{conv}(\Upsilon(\Delta)(t) \cup \{0\})) = \vartheta(\Upsilon(\Delta)(t) \cup \{0\}) \\ &\leq \vartheta(\Upsilon(\Delta)(t)) \\ &\leq M\sum_{0<t_k<t}\vartheta(\varphi_k(\Delta(t_k))) + M\int_0^t \frac{1}{s^{1-\varrho}}\vartheta(\phi(s,\Delta(s)))\,ds \\ &\leq M\sum_{0<t_k<t}\zeta_k\vartheta(\Delta)(t_k) + M\int_0^t \frac{1}{s^{1-\varrho}}\mu(t)\vartheta(\Delta(s))\,ds \\ &= M\sum_{0<t_k<t}\zeta_k\chi(t_k) + M\int_0^t \frac{1}{s^{1-\varrho}}\mu(t)\chi(s)\,ds \\ &\leq \|\chi\|M(\sum_{k=1}^{p}\zeta_k + \|\mu\|\frac{T^\varrho}{\varrho}).\end{aligned}$$

This means that

$$\|\chi\|(1 - M(\sum_{k=1}^{p}\zeta_k + \|\mu\|\frac{T^\varrho}{\varrho})) \leq 0.$$

By (8), it follows that $\|\chi\| = 0$, that is $\chi(t) = 0$ for each $t \in J$, and then $\Delta(t)$ is relatively compact in $\mathcal{PC}(J, \mathcal{X})$. In view of the Ascoli-Arzela theorem, Δ is relatively compact in B_ρ. Applying now Theorem 2.7, we conclude that Υ has a fixed point which is a solution of the problem (1). □

3.3. Uniqueness of mild solutions

To obtain the uniqueness of the mild solution, we need the following assumption:

(A1) There exists a constant $\eta > 0$ such that

$$\|\phi(t,\psi) - \phi(t,\overline{\psi})\| \leq \eta \|\psi - \overline{\psi}\|, \text{ for each } t \in J, \forall \psi, \overline{\psi} \in \mathcal{PC}.$$

(A2) There exist constants $\varsigma_k > 0$ such that

$$\|\varphi(\psi) - \varphi(\overline{\psi})\| \leq \varsigma_k \|\psi - \overline{\psi}\|, \text{ for each } k = 1, ..., p, \forall \psi, \overline{\psi} \in \mathcal{PC}.$$

Theorem 3.6. *If assumptions (A1) and (A2) are satisfied,, then the conformable fractional Cauchy problem (1) has an unique mild solution, provided that*

$$M(\eta \frac{T^\varrho}{\varrho} + \sum_{k=1}^{p} \varsigma_k) < 1.$$

Proof. Define the operator $\Upsilon : \mathcal{PC} \to \mathcal{PC}$ by

$$\Upsilon(\psi)(t) = \Im_\varrho(t)\psi_0 + \sum_{0<t_k<t} \Im_\varrho(t^\varrho - t_k^\varrho)^{\frac{1}{\varrho}} \varphi_k(\psi(t_k)) + \int_0^t \frac{1}{s^{1-\varrho}} \Im_\varrho(t^\varrho - s^\varrho)^{\frac{1}{\varrho}} \phi(s, \psi(s))\, ds.$$

We will show that Υ is a contraction, consider $\psi, \overline{\psi} \in \mathcal{PC}$. Thus, for $t \in J$, we have:

$$\begin{aligned}
\|\Upsilon(\psi)(t) - \Upsilon(\overline{\psi})(t)\| &\leq \sum_{0<t_k<t} \|\Im_\varrho(t^\varrho - t_k^\varrho)^{\frac{1}{\varrho}}\| \|\varphi_k(\psi(t_k)) - \varphi_k(\overline{\psi}(t_k))\| \\
&\quad + \int_0^t \frac{1}{s^{1-\varrho}} \|\Im_\varrho(t^\varrho - s^\varrho)^{\frac{1}{\varrho}}\| \|\phi(s, \psi(s)) - \phi(s, \overline{\psi}(s))\|\, ds \\
&\leq M[\sum_{0<t_k<t} \varsigma_k \|\psi(t_k) - \overline{\psi}(t_k)\| + \eta \int_0^t \frac{1}{s^{1-\varrho}} \|\psi(s) - \overline{\psi}(s)\|\, ds] \\
&\leq M[\eta \frac{T^\varrho}{\varrho} + \sum_{k=1}^{p} \varsigma_k] \|\psi - \overline{\psi}\|.
\end{aligned}$$

Since $M[\eta \frac{T^\varrho}{\varrho} + \sum_{k=1}^{p} \varsigma_k] < 1$, then Υ is a contraction operator on the Banach space $(\mathcal{PC}, \|.\|)$. Therefore, employing the Banach Fixed Point Theorem, we can affirm that the operator Υ possesses an unique fixed point in $\mathcal{PC}$, which represents the mild solution of the conformable fractional Cauchy problem (1). □

Theorem 3.7. *If we assume that the conditions of Theorem (3.6) are satisfied, with ψ_0 and $\overline{\psi}_0$ belonging to the set $\mathcal{X}$, and if ψ and $\overline{\psi}$ represent the solutions corresponding to ψ_0 and $\overline{\psi}_0$ respectively, then we can approximate the following estimate:*

$$\|\psi - \overline{\psi}\| \leq \frac{M}{1 - M[\eta \frac{T^\varrho}{\varrho} + \sum_{k=1}^{p} \varsigma_k]} \|\psi_0 - \overline{\psi}_0\|.$$

Proof. For $t \in J$, we have

$$\|\psi(t) - \overline{\psi}(t)\| \leq M\|\psi_0 - \overline{\psi}_0\| + M[\eta \frac{T^\varrho}{\varrho} + \sum_{k=1}^{p} \varsigma_k] \|\psi - \overline{\psi}\|.$$

Taking the supremum, we get

$$\|\psi - \overline{\psi}\| \leq M\|\overline{\psi}_0 - \overline{\psi}_0\| + M[\eta \frac{T^{\varrho}}{\varrho} + \sum_{k=1}^{p} \varsigma_k]\|\psi - \overline{\psi}\|.$$

Therefore, we can infer the desired approximation as follows:

$$\|\psi - \overline{\psi}\| \leq \frac{M}{1 - M[\eta \frac{T^{\varrho}}{\varrho} + \sum_{k=1}^{p} \varsigma_k]}\|\psi_0 - \overline{\psi}_0\|.$$

□

Example 3.8. *We consider the impulsive conformable partial differential equation of the form*

$$\begin{cases} \mathfrak{D}^{\frac{1}{2}}\omega(t,\lambda) = \dfrac{\partial^2\omega(t,\lambda)}{\partial\lambda^2} + \dfrac{e^{-t}|\omega(t,\lambda)|}{(5+e^t)(1+|\omega(t,\lambda)|)}, \quad (t,\lambda) \in [0,1]\times[0,1], \ t \neq \dfrac{1}{2} \\ \omega(t,0) = \omega(t,1) = 0 \\ \omega(0,\lambda) = \psi_0 \\ \lim\limits_{s\to 0^+} \omega(\dfrac{1}{2}+s,\lambda) = \lim\limits_{s\to 0^+} \omega(\dfrac{1}{2}-s,\lambda) + \dfrac{|\omega(\frac{1}{2},\lambda)|}{2+|\omega(\frac{1}{2},\lambda)|}. \end{cases} \tag{9}$$

Let $\mathcal{X} = L^2([0,1])$, and define the operator A as follows $A = \frac{\partial^2}{\partial\lambda^2}$ with its domain

$$D(A) = \{u \in X, u', u'' \text{ are absolutely continuous and } u'' \in X, u(0) = u(1) = 0\}.$$

From [26], A is the generator of the C_0-α-semigroup $\mathfrak{I}_{\varrho}(t)u$ and $\|\mathfrak{I}_{\varrho}(t)\| \leq 1$. Next, we consider the change $\psi(t)(\lambda) = \omega(t,\lambda)$ and the following notations:

$$\phi(t,\psi) = \frac{e^{-t}|\psi(t)|}{(5+e^t)(1+|\psi(t)|)},$$

$$\varphi_1(\psi(\frac{1}{2})) = \frac{|\psi(\frac{1}{2})|}{2+|\psi(\frac{1}{2})|}.$$

The equation (9) transforms into the following form:

$$\begin{cases} \mathfrak{D}^{\frac{1}{2}}\psi(t) = A\psi(t) + \phi(t,\psi(t)), \quad t \in J = [0,T], \ t \neq \frac{1}{2} \\ \psi(\frac{1}{2}^+) - \psi(\frac{1}{2}^-) = \varphi_1(\psi(\frac{1}{2})) \\ \psi(0) = \psi_0. \end{cases} \tag{10}$$

For each $t \in J, \forall \psi, \overline{\psi} \in \mathcal{PC}$, we have

$$\begin{aligned} \|\phi(t,\psi) - \phi(t,\overline{\psi})\| &= \frac{e^{-t}}{5+e^t}\|\frac{|\psi(t)|}{1+|\psi(t)|} - \frac{|\overline{\psi}(t)|}{1+|\overline{\psi}(t)|}\| \\ &\leq \frac{e^{-t}}{5+e^t}\|\psi - \overline{\psi}\| \\ &\leq \frac{1}{5}\|\psi - \overline{\psi}\|, \end{aligned}$$

and

$$\begin{aligned}\|\varphi_1(\psi)-\varphi_1(\overline{\psi})\| &= \|\frac{|\psi(\frac{1}{2})|}{2+|\psi(\frac{1}{2})|}-\frac{|\overline{\psi}(\frac{1}{2})|}{2+|\overline{\psi}(\frac{1}{2})|}\\ &\leq \frac{1}{2}\|\psi-\overline{\psi}\|.\end{aligned}$$

We have $M=1, T=1, \varrho=\frac{1}{2}, \eta=\frac{1}{5}$ *and* $\varsigma_1=\frac{1}{2}$. *It is clear that* $M(\eta\frac{T^\varrho}{\varrho}+\varsigma_1)=\frac{9}{10}<1$, *therefore, all the conditions of Theorem 3.6 are satisfied. Hence the problem (10) has a unique mild solution.*

Example 3.9. *Consider the following fractional pantograph-differential equation with impulsive by:*

$$\begin{cases}\mathfrak{D}^{\frac{1}{2}}\psi(t)=-\frac{1}{2}\psi(t)++\frac{\sin(t)|\psi(t)|}{(t+4)^2(1+|\psi(t)|)}+\frac{\cos(t)}{(e^t+3)^2}\psi(\frac{t}{4}), \quad t\in[0,1],\ t\neq\frac{1}{2}\\ \psi(\frac{1}{2}^+)-\psi(\frac{1}{2}^-)=\frac{|\psi(\frac{1}{2})|}{16+|\psi(\frac{1}{2})|}\\ \psi(0)=\psi_0.\end{cases} \tag{11}$$

Let $A=-\frac{1}{2}$. *A is the generator of the* C_0-$\frac{1}{2}$-*semigroup* $\mathfrak{I}_{\frac{1}{2}}(t)=e^{-\sqrt{t}}$ *and* $\|\mathfrak{I}_{\frac{1}{2}}(t)\|\leq e$.
Consider the following notations:

$$\phi(t,\psi(t),\psi(\frac{t}{4}))=\frac{\sin(t)|\psi(t)|}{(t+4)^2(1+|\psi(t)|)}+\frac{\cos(t)}{(e^t+3)^2}\psi(\frac{t}{4}),$$

$$\varphi_1(\psi(\frac{1}{2}))=\frac{|\psi(\frac{1}{2})|}{16+|\psi(\frac{1}{2})|}.$$

Then for any $\psi,\overline{\psi}\in\mathcal{PC}$ *and* $t\in[0,1]$, *we obtain*

$$\|\phi(t,\psi(t),\psi(\frac{t}{4}))-\phi(t,\overline{\psi}(t),\overline{\psi}(\frac{t}{4}))\|\leq\frac{1}{8}\|\psi-\overline{\psi}\|,$$

$$\|\varphi_1(\psi)-\varphi_1(\overline{\psi})\|\leq\frac{1}{16}\|\psi-\overline{\psi}\|.$$

It is clear that $M(\eta\frac{T^\varrho}{\varrho}+\varsigma_1)=\frac{5e}{16}<1$. *Hence problem (11) has a unique solution on* $[0,1]$.

Conclusion

In this study, we have found a mild solution to fractional impulsive evolutionary differential equations, primarily employing fractional semigroup analysis. The main outcome has been derived through the utilization of measures of non-compactness, in addition to Banach's and Mönch's theorems. The concepts introduced in this paper have the potential to be applied to various other models in fields such as physics, biology, chemistry, economics, and beyond.

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