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Two-parameters semigroups for linear evolution equations

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Abstract. We establish sufficient conditions insuring the existence of mild solutions for a class of initial boundary value problem of time homogeneous evolution system of parabolic type defined on a Banach space. A concrete real-world example is provided to illustrate the result. A first application of the obtained results deals with a system of Volterra integro-differential equations. A second one concerns a class of elliptic second order differential equations.

1. Introduction and Preliminaries

Let X be a Banach space and $\mathcal{L}(X)$ the algebra of bounded linear operators on X. For a given initial value $u_0 \in X$, we consider the problem of finding a function $u : \mathbb{R}_+ \to X$, solution of the homogeneous Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = Au(t), \quad t \geq 0, \\ \\ u(0) = u_0, \end{cases} \tag{1}$$

where $A : \mathcal{D}(A) \subset X \to X$ is a linear operator. We assume that the problem (1) is well-posed. This means the existence of a unique solution to (1) for a large class of initial values u_0 and the continuous dependence upon them. Hence, by results of Engel and Nagel in [7], the unbounded operator A should generate a strongly continuous (one-parameter) semigroup (or C_0-semigroup) $\big(T(t)\big)_{t\geq 0}$. Consequently, the system of equations

$$\begin{cases} T(t+s) = T(t)T(s), \quad t,s \geq 0, \\ \\ T(0) = I. \end{cases} \tag{2}$$

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holds and the orbit map

$$\xi_{u_0} : t \mapsto \xi_{u_0}(t) = T(t)u_0 \tag{3}$$

is continuous from $\mathbb{R}_+$ into X, for every $u_0 \in \mathcal{D}(A)$.

We recall here the definition of a generator of a strongly continuous semigroup. The interested reader can consult, for example, the books [1, 9] and [10] and the references therein for more more details about the subject.

Definition 1.1. *The generator $A : \mathcal{D}(A) \subset X \to X$ of a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ on a Banach space X is the operator*

$$Ax := \xi_x(0) = \lim_{h\downarrow 0} \frac{1}{h}\Big(T(h)x - x\Big), \quad \forall x \in \mathcal{D}(A),$$

where

$$\mathcal{D}(A) := \Big\{x \in X : \xi_x \text{ is differentiable}\Big\}.$$

One can obtain solutions of (1) by using the following proposition.

Proposition 1.2. *[7] Let A be the generator of the strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$. Then, for every $u_0 \in \mathcal{D}(A)$, the orbit map*

$$u(t) = T(t)u_0, \qquad t \geq 0. \tag{4}$$

is the unique solution of the associated abstract Cauchy problem (1).

Definition 1.3. *Let A be the generator of a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ on X and take $u_0 \in X$. The function $u(.)$, defined by* (4) *is called the mild solution of* (1).

It is well known that if A is a bounded operator, then the function u defined by (4) is continuously differentiable and satisfies (1). It is called a classical solution of (1).

Obviously, every classical solution of (1) is also a mild solution. In particular, this implies the uniqueness of the classical solution of the problem. Moreover, we can deduce the following,

Remark 1.4. *Let $A \in \mathcal{L}(X)$. We suppose that A generates a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$. Then, for every $u_0 \in X$, the orbit map u defined by* (4) *is the unique classical solution of the associated Cauchy problem* (1).

Proposition 1.5. *[7] Let X be a Banach space, A be a generator of a strongly continuous semigroup on X and $B \in \mathcal{L}(X)$. Then, the operator $A + B$ is the generator of a strongly continuous semigroup on X.*

We are especially interested in a class of operators which are relatively bounded with respect to each other. We recall the following definitions.

Definition 1.6. *Let X, Y, Z be Banach spaces and consider the linear operators $A : \mathcal{D}(A) \subset X \to Y$ and $B : \mathcal{D}(B) \subset X \to Z$. A is called relatively bounded with respect to B (or B-bounded) if $\mathcal{D}(B) \subset \mathcal{D}(A)$ and there exist constants $a_A, b_A \geq 0$ such that*

$$\|Ax\| \leq a_A\|x\| + b_A\|Bx\|, \quad \forall x \in \mathcal{D}(B).$$

The infimum δ_A of all b_A so that this holds for some $a_A \geq 0$ is called relative bound of A with respect to B (or B-bound of A).

Definition 1.7. *Let $\delta \geq 0$ and consider the linear operators $A : \mathcal{D}(A) \subset X \to X$, $B : \mathcal{D}(B) \subset Y \to X$, $C : \mathcal{D}(C) \subset X \to Y$ and $D : \mathcal{D}(D) \subset Y \to Y$. The block operators matrix*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is called diagonally dominant of order δ if C is A-bounded with A-bound δ_C, B is D-bounded with D-bound δ_B, and $\delta = \max\{\delta_B, \delta_C\}$.

2. A System of Parabolic Equations

2.1. Mathematical model and main result

Let X and Y be two Banach spaces. For $\mu, \nu \in \mathbb{C}$, we are looking for mild solutions to the initial value problem for the time homogeneous evolution system of parabolic type

$$\begin{cases} \dfrac{\partial u}{\partial t} = Au + \mu Bv + f(t), \quad t \geq 0, \\ \dfrac{\partial v}{\partial t} = Dv + \nu Cu + g(t), \quad t \geq 0, \\ \Big(u(0), v(0)\Big) = (u_0, v_0), \end{cases} \tag{5}$$

where $A : \mathcal{D}(A) \subset X \to X$, $B : \mathcal{D}(B) \subset Y \to X$, $C : \mathcal{D}(C) \subset X \to Y$ and $D : \mathcal{D}(D) \subset Y \to Y$ are linear operators and (u_0, v_0) is a given initial value of $\Big(\mathcal{D}(A) \cap \mathcal{D}(C)\Big) \oplus \Big(\mathcal{D}(D) \cap \mathcal{D}(B)\Big)$.

To this aim, we consider the block operators matrices $\mathcal{A}_{\mu,\nu}$ defined on the space

$$\mathcal{D}(\mathcal{A}_{\mu,\nu}) = \Big(\mathcal{D}(A) \cap \mathcal{D}(C)\Big) \oplus \Big(\mathcal{D}(D) \cap \mathcal{D}(B)\Big)$$

and having the representation

$$\mathcal{A}_{\mu,\nu} = \begin{pmatrix} A & \mu B \\ \nu C & D \end{pmatrix}. \tag{6}$$

We enunciate our first main result as follows

Theorem 2.1. *If $\mathcal{A}_{\mu,\nu}$ is the generator of a strongly continuous semigroup on $X \times Y$, then for every $(u_0, v_0) \in \mathcal{D}(\mathcal{A}_{\mu,\nu})$, the problem* (5) *admits a unique mild solution.*

Proof. To prove our result, we need to find a new state space Z and a new operator $\mathcal{B}_{\mu,\nu} : \mathcal{D}(\mathcal{B}_{\mu,\nu}) \subset X \to X$ generating a semigroup $\Big(\mathcal{T}_{\mu,\nu}(t)\Big)_{t\geq 0}$ on Z, such that the solutions of (5) can be obtained using Proposition 1.5. Here, we denote by Z the Banach space

$$Z = X \times Y \times H^1(\mathbb{R}_+, X) \times H^1(\mathbb{R}_+, Y).$$

We define on Z the block operators matrices given by

$$\mathcal{B}_{\mu,\nu} = \begin{pmatrix} \mathcal{A}_{\mu,\nu} & \mathcal{K}_0 \\ 0 & \mathcal{D}_t \end{pmatrix},$$

where

$$\mathcal{K}_0 = \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_0 \end{pmatrix} \quad \textit{and} \quad \mathcal{D}_t = \begin{pmatrix} \partial/\partial t & 0 \\ 0 & \partial/\partial t \end{pmatrix}. \tag{7}$$

Both $\mathcal{K}_0$ and $\mathcal{D}_t$ are definded on $H^1(\mathbb{R}_+, X) \times H^1(\mathbb{R}_+, Y)$.

We denote by U_0 the matrix defined on Z by

$$U_0 = \begin{pmatrix} u_0 \\ v_0 \\ f(0) \\ g(0) \end{pmatrix}. \tag{8}$$

We also consider on Z the matrix

$$U(t) = \begin{pmatrix} u(t) \\ v(t) \\ f(t) \\ g(t) \end{pmatrix}, \qquad t \geq 0. \tag{9}$$

Then, the system (5) can be rewritten as,

$$\begin{cases} \dfrac{\partial U}{\partial t} = \mathcal{B}_{\mu,\nu} U(t), \quad t \geq 0, \\ \\ U(0) = U_0. \end{cases} \tag{10}$$

Now, since $\mathcal{D}_t$ is bounded in $H^1(\mathbb{R}_+, X) \times H^1(\mathbb{R}_+, Y)$ and $\mathcal{A}_{\mu,\nu}$ is the generator of a strongly continuous semigroup on $X \times Y$, then by Proposition 1.5, it is that the block operators matrices given by

$$\mathcal{M}_{\mu,\nu} = \begin{pmatrix} \mathcal{A}_{\mu,\nu} & 0 \\ 0 & \mathcal{D}_t \end{pmatrix} = \begin{pmatrix} \mathcal{A}_{\mu,\nu} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{D}_t \end{pmatrix}$$

is also the generator of a strongly continuous semigroup on Z. Again, $\mathcal{K}_0$ is bounded in $H^1(\mathbb{R}_+, X) \times H^1(\mathbb{R}_+, Y)$. So, using an other time Proposition 1.5, we deduce that the block operators matrices

$$\mathcal{B}_{\mu,\nu} = \mathcal{M}_{\mu,\nu} + \begin{pmatrix} 0 & \mathcal{K}_0 \\ 0 & 0 \end{pmatrix}$$

generates a strongly continuous semigroup $\left(\mathcal{T}_{\mu,\nu}(t)\right)_{t \geq 0}$ on Z.

Finally, thanks to Proposition 1.2, the orbit map

$$\mathcal{U}(t) = \mathcal{T}_{\mu,\nu}(t)\mathcal{U}_0, \qquad t \geq 0. \tag{11}$$

is the unique mild solution of the associated abstract Cauchy problem (10).

The two first equations of $\mathcal{U}(t)$ give us the unique mild solution (u, v) of the problem (5) on $X \times Y$. □

Remark 2.2. *If, in addition, $\mathcal{B}_{\mu,\nu}$ is bounded in Z, then similarly to Remark 1.4, it becomes that $\mathcal{U}(t)$ is the unique classical solution of problem* (10).

2.2. *A real-world illustrative example*

A concrete real-world example that can be modeled by the type of systems described by (5) and strongly continuous semigroup theory is the heat diffusion in a composite material, consisting of two different substances with varying thermal properties, for instance, a combination of a metal and an insulating material. In such scenario, the temperature distribution within the composite material can be accurately represented using (5). In such model,

1. $u(t, x)$ and $v(t, x)$ are the respective temperatures in the metal part and the insulating part of the material at time t and position x.
2. The operators A and D represent the thermal conductivities respectively in the metal and the insulation. They describe how heat diffuses within the metal and the insulation.
3. B and C represent the heat transfer operators between the metal and the insulating part.
4. The parameter μ is the rate of heat transfer between the metal and the insulation.
5. The parameter ν is the rate of heat transfer between the insulation and the metal.

6. $f(t)$ and $g(t)$ can model any external heat sources or sinks in the system, such as heat lamps, cooling systems or chemical reactions.
7. The initial conditions (u_0, v_0) could represent the initial temperature distribution in the composite material.

Finding mild solutions to the system (5) is essential for understanding how heat propagates and is exchanged within the composite material. The concept of a strongly continuous semigroup is used to ensure the well-posedness of the mathematical model, guaranteeing that solutions exist and are unique for a given set of initial conditions and external inputs.

2.3. Some consequences and related results

We consider the block operators matrices

$$\mathcal{M} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \quad \text{and} \quad \mathcal{N}_{\mu,\nu} = \begin{pmatrix} 0 & \mu B \\ \nu C & 0 \end{pmatrix}. \tag{12}$$

Then, we have the following result,

Corollary 2.3. *Suppose that $\mathcal{A}_{\mu,\nu}$ is diagonally dominant of order δ, for some $\delta \geq 0$ and that the block operators matrix $\mathcal{M}$ is a generator of a C_0-semigroup. Then, for every $(u_0, v_0) \in \mathcal{D}(\mathcal{A}_{\mu,\nu})$, the problem* (5) *admits a unique mild solution.*

Proof. Under these hypotheses on $\mathcal{A}_{\mu,\nu}$ and $\mathcal{M}$, we have by [7] that $\mathcal{N}_{\mu,\nu}$ is $\mathcal{M}$-bounded with $\mathcal{M}$-bound δ. Since $\mathcal{M}$ is the generator of a C_0-semigroup, then using Proposition 1.5, we deduce that the operator $\mathcal{A}_{\mu,\nu}$ is the generator of a strongly continuous semigroup on $X \times Y$. The result follows from Theorem 2.1. □

Corollary 2.4. *Suppose that $X = Y$, $\mathcal{A}_{\mu,\nu}$ is diagonally dominant of order δ, for some $\delta \geq 0$, and that the following situation holds:*

either (A is a generator of a C_0-semigroup and D is bounded)
or (D is a generator of a C_0-semigroup and A is bounded).

Then, for every $(u_0, v_0) \in \mathcal{D}(\mathcal{A}_{\mu,\nu})$, the problem (5) *admits a unique mild solution.*

Proof. In this situation, the block operators matrices $\mathcal{M}$ is a generator of a C_0-semigroup. Since, in addition, $\mathcal{A}_{\mu,\nu}$ is diagonally dominant of order δ, for some $\delta \geq 0$, then we obtain the result by Corollary 2.3. □

Theorem 2.5. *Let $0 \leq \min\{\mu, \nu\} \leq \max\{\mu, \nu\} < 1$ and suppose that*

1. *$\mathcal{M}$ is the generator of a strongly continuous semigroup on the Banach space $X \times Y$ and $\mathcal{N}_{\mu,\nu}$ is $\mathcal{M}$-bounded.*
2. *There exists $\alpha > 0$, such that for any $\lambda > 0$ and for all $(x, y) \in \mathcal{D}(\mathcal{A}_{\mu,\nu})$,*

$$\min\Big\{\|(\lambda - A)x\|^2 + \|(\lambda - D)y\|^2,\ \|(\lambda x - \mu By\|^2 + \|(\lambda y - \nu Cx\|^2\Big\} \geq \alpha\Big(\|x\|^2 + \|y\|^2\Big). \tag{13}$$

Then, for every $(u_0, v_0) \in \mathcal{D}(\mathcal{A}_{\mu,\nu})$, the problem (5) *admits a unique mild solution.*

Proof. Suppose that both operators $\mathcal{M}$ and $\mathcal{N}_{\mu,\nu}$ are dissipative. Then, by Theorem 2.7 in [7], we have that for every $0 \leq \mu, \nu < 1$, the operator $\mathcal{A}_{\mu,\nu}$ is a generator of a strongly continuous semigroup on $X \times Y$.

Now, thanks to Theorem 2.1, for every $(u_0, v_0) \in X \times Y$, the problem (5) admits a unique mild solution. The proof is achieved. □

In the next, for a given $r > 0$, we denote by $\mathcal{B}(0, r)$ the ball of $\mathbb{C}$ centered et the origin and with radius r, i.e.,

$$B(0, r) = \{x \in \mathbb{C},\ |x| < r\}.$$

Then one can prove the following,

Theorem 2.6. *Suppose that $\mathcal{M}$ is the generator of an analytic semigroup on the Banach space $X \times Y$ and $\mathcal{N}_{\mu,\nu}$ is $\mathcal{M}$-bounded. Then, there exists $r > 0$, such that for every $(u_0, v_0) \in \mathcal{D}(\mathcal{A}_{\mu,\nu})$, and for any $\mu, \nu \in B(0, r)$, the problem* (5) *admits a unique mild solution.*

Proof. If $\mathcal{M}$ generates an analytic semigroup, then thanks to Theorem 2.10 in [7], there exists $r > 0$ such that $\mathcal{A}_{\mu,\nu}$ is a generator of a strongly continuous semigroup on $X \times Y$, for any $|\mu| < r$ and $|\nu| < r$.

So, using Theorem 2.1, we deduce that for every $(u_0, v_0) \in X \times Y$, the problem (5) admits a unique mild solution. □

Theorem 2.7. *Suppose that for any $\mu, \nu \in \mathbb{C}$, the operatos matix $\mathcal{N}_{\mu,\nu}$ is bounded and that $\mathcal{M}$ is the generator of a strongly continuous semigroup on the Banach space $X \times Y$. Then, for every $(u_0, v_0) \in \mathcal{D}(\mathcal{A}_{\mu,\nu})$, the problem* (5) *admits a unique mild solution.*

Proof. If $\mathcal{N}_{\mu,\nu}$ is bounded on $X \times Y$, then using Theorem 1.3 in [7], we deduce that for every $\mu, \nu \in C$, the operator $\mathcal{A}_{\mu,\nu}$ generates a strongly continuous semigroup on $X \times Y$.

Again by Theorem 2.1, we have that, for every $(u_0, v_0) \in X \times Y$, the problem (5) admits a unique classical solution. The proof is achieved. □

Remark 2.8. *The third case can also be achieved if, in Theorem 2.10 of [7], the $\mathcal{M}$-bound a_0 of $\mathcal{N}_{\mu,\nu}$ satisfies $a_0 = 0$.*

3. A system of Volterra integro-differential equations

Volterra integral and integro-differential equations arise in many physical applications such as glass-forming process, nanohydrodynamics, heat transfer, diffusion process in general, neutron diffusion and biological species coexisting together with increasing and decreasing rates of generating, and wind ripple in the desert. The linear equations, in which we are interested, appears in the form

$$u'(t) = Au(t) + \alpha \int_0^t C(t-s)u(s)ds + f(t), \qquad t \geq 0, \tag{14}$$

where α is a constant parameter, A and $C(t)$ are given linear operators. The operator $K(t, s) = C(t - s)$ is called the kernel or the nucleus of the integral equation. For a classification and more details about more general classes of these equations, one can consult, for example [15] and the references therin.

A well known approach to transform (14) into an abstract Cauchy problem of the form (1) has been established by Miller [13]. This technique was then used by some researchers like Grimmer et al. [2, 3], Desch and Schappacher [4], Kadiri et al. [5], Engel et al. [6, 7], Nagel and Sinestrari [14]...

In this section, we are looking for mild solutions of systems of Volterra integro-differential equations of the form

$$\begin{cases} \dot{u}(t) = Dv(t) + \displaystyle\int_0^t \alpha C_1(t-s)u(s)ds + f(t), & t \geq 0, \\ \dot{v}(t) = Au(t) + \displaystyle\int_0^t \beta C_2(t-s)v(s)ds + g(t), & t \geq 0, \\ (u(0), v(0)) = (u_0, v_0), \end{cases} \tag{15}$$

where $\alpha, \beta \in \mathbb{C}$, A and D are linear operators defined as in section 1 on the Banach space X, (u_0, v_0) is a given initial value of $\mathcal{D}(A) \times \mathcal{D}(D) \subset X \times Y$, C_1 and C_2 are functions defined on $\mathbb{R}_+$ and such that $C_1(.)x \in H^1(\mathbb{R}_+, X)$ for all $x \in \mathcal{D}(A)$ and $C_2(.)x \in H^1(\mathbb{R}_+, Y)$ for all $x \in \mathcal{D}(B)$, $f : \mathbb{R}_+ \to X$ and $g : \mathbb{R}_+ \to Y$ are given functions and $u : \mathbb{R}_+ \to X$ and $v : \mathbb{R}_+ \to Y$ are unknown functions.

Denoting by $\mathcal{A}_{0,0}$ the operators matrices defined by (6), we can state the main result of this section as follows,

Theorem 3.1. *Suppose that* $\mathcal{A}_{0,0}$ *is a generator of strongly continuous semigroup on* $X \times Y$. *Then, for every* $(u_0, v_0) \in X \times Y$, *the problem* (15) *admits a unique mild solution.*

Proof. We consider again the space

$$Z = X \times Y \times H^1(\mathbb{R}_+, X) \times H^1(\mathbb{R}_+, Y).$$

The block operators matrices defined on Z is

$$\mathcal{L}_{\alpha,\beta} = \begin{pmatrix} \mathcal{A}_{0,0} & \mathcal{K}_0 \\ C_{\alpha,\beta} & \mathcal{D}_t \end{pmatrix},$$

where $\mathcal{K}_0$ and $\mathcal{D}_t$ are given by (7) and $C_{\alpha,\beta}$ is the operator matrix

$$C_{\alpha,\beta} = \begin{pmatrix} 0 & \beta C_2(.) \\ \alpha C_1(.) & 0 \end{pmatrix}.$$

We also denote by U_0 and, for $t \geq 0$, by $U(t)$ the matrices defined on Z as in (8). It becomes that the system (15) can be written,

$$\begin{cases} \ddot{U} = \mathcal{B}_{\alpha,\beta} U(t), \quad t \geq 0, \\ U(0) = U_0. \end{cases} \tag{16}$$

Now, as it was showed in the proof of Theorem 2.1, the block operators matrices $\mathcal{B}_{0,0}$ is also the generator of a strongly continuous semigroup on Z.

Again, $C_{\alpha,\beta}$ is bounded in $X \times Y$. So, using Proposition 1.5, we deduce that the block operators matrices

$$\mathcal{L}_{\alpha,\beta} = \mathcal{B}_{0,0} + \begin{pmatrix} 0 & 0 \\ C_{\alpha,\beta} & 0 \end{pmatrix}$$

generates a strongly continuous semigroup $\left(\mathcal{S}(t)\right)_{t\geq 0}$ on Z.

Finally, thanks to Proposition 1.2, the orbit map

$$\mathcal{U}(t) = \mathcal{S}(t)\mathcal{U}_0, \qquad t \geq 0. \tag{17}$$

is the unique mild solution of the associated abstract Cauchy problem (16).

The two first coordinates of $\mathcal{U}(t)$ give us the unique mild solution (u, v) of the problem (15) on $X \times Y$. □

Corollary 3.2. *Suppose that either A is a generator of strongly continuous semigroup on X and D is bounded on Y, or D is a generator of strongly continuous semigroup on Y and A is bounded on X. Then, for every* $(u_0, v_0) \in X \times Y$, *the problem* (15) *admits a unique mild solution.*

Proof. In this case, using Proposition 1.5, we deduce that the operator $\mathcal{A}_{0,0}$ is the generator of a strongly continuous semigroup on $X \times Y$. Now, for $\alpha, \beta \in \mathbb{C}$, we obtain the result by Theorem 3.1. □

4. Second order elliptic equations

We are also interested in giving sufficient conditions for the existence of solutions to the second order elliptic equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \nu A \dfrac{\partial u}{\partial t} + \mu B u + f(t), \quad t \geq 0, \\ \left(u(0), \dfrac{\partial u}{\partial t}(0)\right) = (u_0, v_0), \end{cases} \tag{18}$$

where A and B are linear operators defined as in section 1 with $X = Y$, (u_0, v_0) is a given initial value of $\big(\mathcal{D}(A) \times \mathcal{D}(A)\big) \subset (X \times X)$, $f : \mathbb{R}_+ \to X$ is a given function and $u : \mathbb{R}_+ \to X$ is an unknown function.

For $(\mu, \nu) \in \mathbb{C} \times \mathbb{C}^*$, we consider the block operators matrices $\widetilde{\mathcal{A}_{\mu,\nu}}$ with the representation

$$\widetilde{\mathcal{A}_{\mu,\nu}} = \begin{pmatrix} \nu A & \mu B \\ I & 0 \end{pmatrix}. \tag{19}$$

Our next result reads as follows

Theorem 4.1. *Suppose that $\widetilde{\mathcal{A}_{\mu,\nu}}$ is the generator of a strongly continuous semigroup on $X \times X$. Then, for every $(u_0, v_0) \in \big(\mathcal{D}(A) \times \mathcal{D}(A)\big)$, the problem* (18) *admits a unique mild solution.*

Proof. We will apply Theorem 2.1. To this aim, we transform (18) into a first order elliptic equation, by setting

$$\frac{\partial u}{\partial t} = v.$$

It follows that

$$\frac{\partial v}{\partial t} = \frac{\partial^2 u}{\partial t^2} = \nu A v + \mu B u + f(t).$$

Then, for $\mu, \nu \in \mathbb{C}^*$, the system (18) can be rewritten as follows,

$$\begin{cases} \dfrac{\partial v}{\partial t} = \nu A v + \mu B u + f(t), & t \geq 0, \\ \dfrac{\partial u}{\partial t} = v, & t \geq 0, \\ \big(v(0), u(0)\big) = (v_0, u_0). \end{cases} \tag{20}$$

On the space $\widetilde{Z} = X \times X \times H^1(\mathbb{R}_+, X)$, we define the following block operators matrices

$$\widetilde{\mathcal{B}_{\mu,\nu}} = \begin{pmatrix} \widetilde{\mathcal{A}_{\mu,\nu}} & \widetilde{\mathcal{K}_0} \\ 0 & \dfrac{\partial}{\partial t} \end{pmatrix},$$

where $\widetilde{\mathcal{A}_{\mu,\nu}}$ is defined by (19) and

$$\widetilde{\mathcal{K}_0} = \begin{pmatrix} \delta_0 \\ 0 \end{pmatrix}.$$

We denote by U_0 the matrix defined on $\widetilde{Z}$ by

$$U_0 = \begin{pmatrix} v_0 \\ u_0 \\ f(0) \end{pmatrix}.$$

We also consider on $\widetilde{Z}$ the matrix

$$U(t) = \begin{pmatrix} v(t) \\ u(t) \\ f(t) \end{pmatrix}, \qquad t \geq 0.$$

Then, the system (20) can be written in the following form,

$$\begin{cases} \dfrac{\partial U}{\partial t} = \widetilde{\mathcal{B}_{\mu,\nu}} U(t), \quad t \geq 0, \\ U(0) = U_0. \end{cases} \tag{21}$$

Now, since $\widetilde{\mathcal{A}_{\mu,\nu}}$ is the generator of a strongly continuous semigroup on $\widetilde{Z}$, then by Theorem 2.1, the problem (21) admits a unique mild solution, and so are equations (20) and (18). □

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