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On combining independent tests in case of conditional logistic distribution

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Abstract. Combining n independent tests of simple hypotheses, vs one-tailed alternative as n approaches infinity, in case of conditional normal distribution with probability density function $X|\theta \sim \mathscr{L}(\gamma\theta, 1)$, $\theta \in [a, \infty), a \geq 0$ for the case where $\theta_1, \theta_2, ...$ are distributed according to the distribution function (DF) G_θ was studied. Four nonparametric combination procedures (Fisher, logistic, sum of P-values and inverse normal) were compared via the exact Bahadur slope. We observed that the sum of the p-values consistently outperformed all other combing methods used in our study via EBS.

1. Introduction

In evolutionary biology, as in most branches of empirical science that have embraced a statistical approach, it is relatively common to have several independent tests of the same null hypothesis. Often we would like to combine the results of these tests to ask whether there is evidence from the collection of studies that we might reject the null hypothesis. The collection of methods known as meta-analysis gives many ways to do these combinations, including some techniques that combine p-values from multiple independent tests. There are many methods which are used for combining independent tests and they are compared by using different criteria viz., Exact Bahadur slope, Approximate Bahadur Slope, Pitman Efficiency, Local Power, Admissibility and others. Bahadur's stochastic comparison is one of the most common approaches in asymptotic relative efficiency for two test procedures in which the probabilities of the two types of errors (I and II) change with increasing sample size, and also with respect to the manner in which the alternatives under consideration are required to behave. In comparison of test procedures, suppose H_0 is to be tested. Typically, we represent H_0 as a specified family $\mathscr{F}_0$ of distributions for the data. For a test procedure T_n, the function $\gamma_n(T, F) = P_F(T_n \text{ rejects } H_0)$, for distribution functions F, represents the power function of T_n.

- For $H_0 : F \in \mathscr{F}_0$, $\gamma_n(T, F)$ represents the probability of a Type I error. The quantity $\alpha_n(T, \mathscr{F}_0) = \sup_{F \in \mathscr{F}_0} \gamma_n(T, F)$ is called the size of the test.
- For $H_A : F \notin \mathscr{F}_0$, the quantity $\beta_n(T, F) = 1 - \gamma_n(T, F)$ represents the probability of a Type II error.

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Usually, attention is confined to consistent tests, i.e. for a fixed $F \notin \mathscr{F}_0$, $\beta_n(T, F) \to 0$ as $n \to \infty$, and unbiased tests i.e. for $F \notin \mathscr{F}_0$, $\gamma_n(T, F) \geq \alpha_n(T, \mathscr{F}_0)$. A general way to compare two such test procedures is through their power functions. For two competing statistical procedures A and B, suppose that a desired performance criterion is specified. Let n_1 and n_2 be the respective sample sizes at which the two procedures "perform equivalently" with respect to the adopted criterion. Then the ratio n_1/n_2 is usually regarded as the relative efficiency, of procedure B relative to procedure A. Suppose that the specified performance criterion is tightened in a way that causes the required sample sizes n_1 and n_2 to tend to ∞, then ratio n_1/n_2 approaches limit L, where L represents the asymptotic relative efficiency of procedure B relative to procedure A. Note that the value L depends upon the particular performance criterion adopted. This limit represents the asymptotic relative efficiency (ARE) of procedure T_B relative to procedure T_A and is denoted by $e(T_B, T_A)$. We shall consider several performance criteria. Each entails specifications regarding: $\alpha = lim_n \alpha_n(T, \mathscr{F}_0)$, an alternative distribution $F^{(n)}$ allowed to depend on n, and $\beta = lim_n \beta_n(T, F^{(n)})$. In Bahadur's approach, the following behaviors are satisfied: the Type I error is $\alpha_n \to 0$, the Type II error is $\beta_n \to \beta > 0$, and the alternatives $F^{(n)} = F$ is fixed.

Several authors have considered the problem of combining n independent tests of hypothesis. [1] showed that any reasonable p-value combiner must be optimal against some alternative hypothesis. [3] studied six free-distribution methods (sum of p-values, inverse normal, logistic, Fisher, minimum of p-values and maximum of p-values) of combining infinite number of independent tests when the p-values are independent identically distributed (iid) random variables (rv's) distributed with uniform distribution under the null hypothesis versus triangular distribution with essential support $(0, 1)$ under the alternative hypothesis. They proved that the sum of p-values method is the best method. [4] combined infinite number of independent tests for testing simple hypotheses against one-sided alternative for normal and logistic distributions, they used four methods of combining tests namely, Fisher, logistic, sum of p-values and inverse normal. They showed that under the both distributions the inverse normal method is the best method. [5] studied six methods of combining independent tests. He showed under conditional shifted Exponential distribution that the inverse normal method is the best among six combination methods. [9] considered combining independent tests in case of conditional normal distribution with probability density function $X|\theta \sim N(\gamma\theta, 1)$, $\theta \in [a, \infty], a \geq 0$ when $\theta_1, \theta_2, ...$ have a distribution function (DF) F_θ. They showed that the inverse normal procedure is the best among four combination procedures via exact Bahadur slope. [7] considered combining n independent tests of simple hypothesis, vs one-tailed alternative as n approaches infinity, in case of Laplace distribution $\mathbb{L}(\gamma, 1)$. He showed that the sum of p-values procedure is better than all other procedures under the null hypothesis, and the inverse normal procedure is better than the other procedures under the alternative hypothesis. [6] considered combining n independent tests of simple hypothesis, vs one-tailed alternative as n approaches infinity, in case of log-logistic distribution. They showed that the sum of p-values procedure is better than all other procedures under the null hypothesis and under the alternative hypothesis. [8] considered the problem of combining n independent tests as $n \to \infty$ for testing a simple hypothesis in case of log-normal distribution. He showed that as $\xi \to 0$, the maximum of p-values is better than all other methods, followed in decreasing order by the inverse normal, logistic, the sum of p-values, Fisher and Tippett's procedure. Also, as $\xi \to \infty$ the worst method the sum of p-values and the other methods remain the same, since they have the same limit.

The conditional logistic distribution is largely applied to statistical modelling and analysis. For instance, the distribution is used to model the period until an event occurs, such as the start of a specific disease. Furthermore, the distribution can be used to analyse panel data while accounting for within-cluster correlation over time. It can also be utilized to accurately model the relationship between observations while estimating regression parameter coefficients in grouped data analysis.

To measure the strength of the observed sample as the evidence against the null hypothesis the significance level of the observed value of the test statistic is computed. This concept provides another way to compare two test procedures, the better procedure being the one which, when the alternative is true, on the average yields stronger evidence against the null hypothesis. [2] Introduced a notion of "stochastic comparison" and corresponding measure of asymptotic relative efficiency.

Consider iid observations $X_1, \ldots, X_n$ in a sample space, having a distribution with parameter $\theta \in \Theta$. Now consider testing the hypothesis $H_0 : \theta \in \Theta_0$ by a real-valued test statistic T_n, where H_0 is rejected for sufficiently large values of T_n. Let G_{θ_n} denote the DF of T_n under the $\theta-$distribution of $X_1, \ldots, X_n$. The level attained is the indicator of the significance of the observed data against the null hypothesis which is given by $L_n = L_n(X_1, \ldots, X_n) = sup_{\theta\in\Theta_0}[1 - G_{\theta_n}(T_n)]$, where $sup_{\theta\in\Theta_0}[1 - G_{\theta_n}(t)]$ is the maximum probability, under any one of the null hypothesis models, that the experiment will lead to a test statistic exceeding the value t. It is a decreasing function of t. Evaluated at the observed T_n, it represents the largest probability, under the possible null distributions, that a more extreme value than T_n would be observed in a repetition of the experiment. Thus the level attained is a random variable representing the degree to which the test statistic T_n, tends to reject H_0. The lower the value of the level attained, the greater the evidence against H_0. Also, [2] suggests stochastic comparison of two test sequences $T_A = T_{A_n}$ and $T_B = T_{B_n}$ in terms of their performances with respect to the level attained, as follows: Under the nonnull $\theta-$distribution, the test T_{A_n} is more successful than the test T_{B_n} with the sample $X_1, \ldots, X_n$ if $L_{A_n}(X_1, \ldots, X_n) < L_{B_n}(X_1, \ldots, X_n)$. Equivalently, defining $K_n = -2\log L_n$, T_{A_n} is more successful than T_{B_n} at the observed sample if $K_{A_n} > K_{B_n}$. In this case, for $\theta \in \Theta_0$, L_n converges in $\theta-$distribution to some nondegenerate random variable, and under an alternative $\theta \notin \Theta_0$, $L_n \to 0$ at an exponential rate of θ.

2. The specific problem

Consider n hypotheses of the form:

$$H_0^{(i)} : \eta_i = \eta_0^i,\ vs\ , H_1^{(i)} : \eta_i \in \Omega_i - \{\eta_0^i\} \tag{1}$$

such that $H_0^{(i)}$ is rejected for large values, $i = 1, 2, ..., n$ of some continuous random variable $T^{(i)}$. The n hypotheses are combined into one as

$$H_0^{(i)} : (\eta_1, ..., \eta_n) = (\eta_0^1, ..., \eta_0^n),\ vs\ , H_1^{(i)} : (\eta_1, ..., \eta_n) \in \left\{ \prod_{i=1}^{n} \Omega_i - \{(\eta_0^1, ..., \eta_0^n)\} \right\} \tag{2}$$

For $i = 1, 2, \ldots, n$ the p-value of the i-th test is given by

$$P_i(t) = P_{H_0^{(i)}}\left(T^{(i)} > t\right) = 1 - F_{H_0^{(i)}}(t) \tag{3}$$

where $F_{H_0^{(i)}}(t)$ is the DF of $T^{(i)}$ under $H_0^{(i)}$. Note that $P_i \sim U(0, 1)$ under $H_0^{(i)}$.
If considering the special case where $\eta_i = \theta$ and $\eta_0^i = \theta_0$ for $i = 1, \ldots, n$, and also assume that $T^{(1)}, \ldots, T^{(n)}$ are independent, then (1) reduces to

$$H_0 : \theta = \theta_0,\ vs\ , H_1 : \theta \in \Omega - \{\theta_0\} \tag{4}$$

It follows that the p-values $P_1, \ldots, P_n$ are also independent identically distributed random variables that have a $U(0, 1)$ distribution under H_0, and under H_1 have a distribution whose support is a subset of the interval $(0, 1)$ and is not a $U(0, 1)$ distribution. Therefore, if f is the probability density function (pdf) of P, then (4) is equivalent to

$$H_0 : P \sim U(0, 1),\ vs\ , H_1 : P \nsim U(0, 1) \tag{5}$$

where P has a pdf f with support a subset of the interval $(0, 1)$.
This study considers the case: $\eta_i = \gamma\theta_i$, $i = 1, \ldots, n$ where $\theta_1, \ldots, \theta_n$ are independent identically distributed with DF G_θ with support $[a, \infty)$, $a \geq 0$ and the following hypothesis is tested:

$$H_0 : \gamma = 0,\ vs\ , H_1 : \gamma > 0 \tag{6}$$

where the i-th problem is based on $T_1, \ldots, T_n$, which are independent random variables from a conditional logistic distribution with pdf $\mathscr{L}(\gamma\theta, 1)$ and $\theta_1, \ldots, \theta_n$ are independent identically distributed with DF G_θ with support $[a, \infty)$, $a \geq 0$.

We shall assume that the i-th problem in case of the conditional logistic distribution is based on $T_1^{(i)}, \dots, T_{(n_i)}^{(i)}$ which are independent r.v.'s. By sufficiency we may assume $n_i = 1$ and $T^{(i)} = X_i$ for $i = 1, \dots, n$. Then we consider the sequence $\left\{T^{(n)}\right\}$ of independent test statistics that is we will take a random sample $X_1, \dots, X_n$ of size n and let $n \to \infty$ and compare the four non-parametric methods via EBS. Although X_i is not sufficient for θ_i under $H_0^{(i)}$ for the other distributions, but we will assume $n_i = 1$; the reason for this is that, under the conditions specified in theorems 3.1 and 3.2 in the following section, we shall get additional tests in this situation in addition to those previously know and $T^{(i)} = X_i$ for $i = 1, \dots, n$.
The following test will be used in this paper:

$$\varphi_{Fisher} = \begin{cases} 1, & -2\sum_{i=1}^{n} \ln(P_i) > c \\ 0, & ow \end{cases}$$

$$\varphi_{logistic} = \begin{cases} 1, & -\sum_{i=1}^{n} \ln\left(\frac{P_i}{1-P_i}\right) > c \\ 0, & ow \end{cases}$$

$$\varphi_{Normal} = \begin{cases} 1, & -\sum_{i=1}^{n} \Phi^{-1}(P_i) > c \\ 0, & ow \end{cases}$$

$$\varphi_{Sum} = \begin{cases} 1, & -\sum_{i=1}^{n} P_i > c \\ 0, & ow, \end{cases}$$

where Φ is the cdf of standard normal distribution.

3. Definitions and preliminaries

In this section we will state some definitions and preliminaries that will be used

Definition 3.1. (Bahadur efficiency and exact Bahadur slope (EBS)) *Let $X_1, \dots, X_n$ be i.i.d. from a distribution with a probability density function $f(x,\theta)$, and we want to test $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \in \Theta - \{\theta_0\}$. Let $\left\{T_n^{(1)}\right\}$ and $\left\{T_n^{(2)}\right\}$ be two sequences of test statistics for testing H_0. Let the significance attained by $T_n^{(i)}$ be $L_n^{(i)} = 1 - F_i\left(T_n^{(i)}\right)$, where $F_i\left(T_n^{(i)}\right) = P_{H_0}\left(T_n^{(i)} \le t_i\right)$, $i = 1, 2$. Then there exists a positive valued function $C_i(\theta)$ called the exact Bahadur slope of the sequence $\{T_n^{(i)}\}$ such that*

$$C_i(\theta) = \lim_{\theta\to\infty} -2n^{-1} \ln\left(L_n^i\right)$$

with probability 1 (w.p.1) under θ and the Bahadur efficiency of $\left\{T_n^{(1)}\right\}$ relative to $\left\{T_n^{(2)}\right\}$ is given by $e_B(T_1, T_2) = C_1(\theta)/C_2(\theta)$.

Theorem 3.2. (Large deviation theorem) *Let $X_1, X_2, \dots, X_n$ be i.i.d., with distribution F and put $S_n = \sum_{i=1}^{n} X_i$. Assume existence of the moment generating function (mgf) $M(z) = E_F\left(e^{zX}\right)$, z real, and put $m(t) = \inf_z E_F\left(e^{-z(X-t)}\right) = \inf_z e^{-zt}M(z)$. The behavior of large deviation probabilities $P(S_n \ge t_n)$, where $t_n \to \infty$ at rates slower than $O(n)$. The case $t_n = tn$, if $-\infty < t \le EY$, then $P(S_n \le nt) \le [m(t)]^n$, the*

$$-2n^{-1} \ln P_F(S_n \ge nt) \to -2\ln m(t) \quad a.s. \quad (F_\theta).$$

Theorem 3.3. (Bahadur theorem) *Let $\{T_n\}$ be a sequence of test statistics which satisfies the following:*

1. *Under $H_1 : \theta \in \Theta - \{\theta_0\}$:*
$$n^{-\frac{1}{2}} T_n \to b(\theta) \quad a.s. \quad (F_\theta),$$
where $b(\theta) \in \mathbb{R}$.

2. *There exists an open interval I containing $\{b(\theta) : \theta \in \Theta - \{\theta_0\}\}$, and a function g continuous on I, such that*

$$\lim_n -2n^{-1} \log \sup_{\theta \in \Theta_0} \left[1 - F_{\theta_n}(n^{\frac{1}{2}} t)\right] = \lim_n -2n^{-1} \log \left[1 - F_{\theta_n}(n^{\frac{1}{2}} t)\right] = g(t), \quad t \in I.$$

If $\{T_n\}$ satisfied (1)-(2), then for $\theta \in \Theta - \{\theta_0\}$

$$-2n^{-1} \log \sup_{\theta \in \Theta_0} [1 - F_{\theta_n}(T_n)] \to C(\theta) \quad a.s. \quad (F_\theta).$$

Theorem 3.4. *Let $X_1, \ldots, X_n$ be i.i.d. with probability density function $f(x, \theta)$, and we want to test $H_0 : \theta = 0$ vs. $H_1 : \theta > 0$. For $j = 1, 2$, let $T_{n,j} = \sum_{i=1}^{n} f_i(x_i)/\sqrt{n}$ be a sequence of statistics such that H_0 will be rejected for large values of $T_{n,j}$ and let φ_j be the test based on $T_{n,j}$. Assume $\mathbb{E}_\theta(f_i(x)) > 0, \forall \theta \in \Theta$, $\mathbb{E}_0(f_i(x)) = 0$, $Var(f_i(x)) > 0$ for $j = 1, 2$. Then*
1. If the derivative $b'_j(0)$ is finite for $j = 1, 2$, then

$$\lim_{\theta \to 0} \frac{C_1(\theta)}{C_2(\theta)} = \frac{Var_{\theta=0}(f_2(x))}{Var_{\theta=0}(f_1(x))} \left[\frac{b'_1(0)}{b'_2(0)}\right]^2,$$

where $b_i(\theta) = \mathbb{E}_\theta(f_j(x))$, and $C_j(\theta)$ is the EBS of test φ_j at θ.
2. If the derivative $b'_j(0)$ is infinite for $j = 1, 2$, then

$$\lim_{\theta \to 0} \frac{C_1(\theta)}{C_2(\theta)} = \frac{Var_{\theta=0}(f_2(x))}{Var_{\theta=0}(f_1(x))} \left[\lim_{\theta \to 0} \frac{b'_1(\theta)}{b'_2(\theta)}\right]^2.$$

Theorem 3.5. *If $T_n^{(1)}$ and $T_n^{(2)}$ are two test statistics for testing $H_0 : \theta = 0$ vs. $H_1 : \theta > 0$ with distribution functions $F_0^{(1)}$ and $F_0^{(2)}$ under H_0, respectively, and that $T_n^{(1)}$ is at least as powerful as $T_n^{(2)}$ at θ for any α, then if φ_j is the test based on $T_n^{(j)}$, $j = 1, 2$, then*

$$C_{\varphi_1}^{(1)}(\theta) \geq C_{\varphi_2}^{(2)}(\theta)$$

.

Corollary 3.6. *If T_n is the uniformly most powerful test for all α, then it is the best via EBS.*

Theorem 3.7.

$$2t \leq m_S(t) \leq et, \ \ \forall : 0 \leq t \leq 0.5,$$

where

$$m_S(t) = \inf_{z>0} e^{-zt} \frac{e^z - 1}{z}.$$

Theorem 3.8. 1. $m_L(t) \geq 2te^{-t}, \quad \forall t \geq 0,$

2. $m_L(t) \leq te^{1-t}, \quad \forall t \geq 0.852,$
3. $m_L(t) \leq t\left(\frac{t^2}{1+t^2}\right)^3 e^{1-t}, \quad \forall t \geq 4,$
where $m_L(t) = \inf_{z \in (0,1)} e^{-zt} \pi z \, csc(\pi z)$ and csc is an abbreviation for cosecant function.

Theorem 3.9. *For $x > 0$,*

$$\phi(x)\left[\frac{1}{x} - \frac{1}{x^3}\right] \leq 1 - \Phi(x) \leq \frac{\phi(x)}{x}.$$

Where ϕ is the pdf of standard normal distribution.

Theorem 3.10. *For $x > 0$,*

$$1 - \Phi(x) > \frac{\phi(x)}{x + \sqrt{\frac{\pi}{2}}}.$$

Lemma 3.11. 1. $m_L(t) \geq \inf_{0<z<1} e^{-zt} = e^{-t}$

2. $m_L(t) \leq \dfrac{e^{-t^2/(t+1)}\left(\frac{\pi t}{t+1}\right)}{sin\left(\frac{\pi t}{t+1}\right)}$

3. $\begin{cases} m_s(t) = \inf_{z>0} \frac{e^{-zt}(1-e^{-z})}{z} \leq \inf_{z>0} \frac{e^{-zt}}{z} \leq -et, & t<0 \\ m_s(t) \geq -2t, & -\frac{1}{2} \leq t \leq 0. \end{cases}$

4. $\dfrac{x-1}{x} \leq \ln x \leq x-1, \;\; x>0$

Theorem 3.12. *For any integrable function f and any η in the interior of Θ, the integral*

$$\int f(x)e^{\sum \eta_i T_i(x)} h(x) d\mu(x)$$

is continuous and has derivatives of all orders with respect to the η's, and these can be obtained by differentiating under the integral sign.

4. Derivation of the EBS with general DF G_θ

In this section we will study testing problem (6). We will compare the four methods viz. Fisher, logistic, sum of P-values and the inverse normal method via EBS.
Let $X_1, \dots, X_n$ be i.i.d. with probability density function $\mathscr{L}(\gamma\theta, 1)$, and we want to test (6). The P-value in this case is given by

$$P_n(X_n) = 1 - F_{H_0}(X_n) = 1 - F_0(x) = (1+e^x)^{-1} \tag{7}$$

The next four lemmas give the EBS for Fisher (C_F), logistic (C_L), inverse normal (C_N), and sum of p-values (C_S) methods.

Lemma 4.1. *The exact Bahadur's slope (EBS's) result for the tests, which is given in Section 2, are as follows:*

B1. Fisher method. $C_F(\gamma) = b_F(\gamma) - 2\ln(b_F(\gamma)) + 2\ln(2) - 2$,
where

$$b_F(\gamma) = 2\gamma\, \mathbb{E}_{G_\theta}\left\{\theta\left(1 - e^{-\gamma\theta}\right)^{-1}\right\}.$$

B2. Logistic method. $C_L(\gamma) = -2\ln(m(b_L(\gamma)))$, *where*

$$m_L(t) = \inf_{z\in(0,1)} e^{-zt}\pi z\, csc(\pi z)$$

and

$$b_L(\gamma) = \mathbb{E}_{G_\theta}\left[\int_{\mathbb{R}} x\mathscr{F}_{\mathscr{L}}(x-\gamma\theta)\,dx\right] = \gamma\, \mathbb{E}_{G_\theta}(\theta).$$

B3. Sum of p-values method. $C_S(\gamma) = -2\ln(m(b_S(\gamma)))$, *where*

$$m_S(t) = \inf_{z>0} e^{-zt}\frac{1-e^{-z}}{z}$$

and

$$b_S(\gamma) = \mathbb{E}_{G_\theta}\left\{\frac{(1-\gamma\theta)e^{-\gamma\theta} - e^{-2\gamma\theta}}{(1-e^{-\gamma\theta})^2}\right\}.$$

B4. Inverse Normal method. $C_N(\gamma) = -2\ln(m(b_N(\gamma))) = b_N^2(\gamma)$ *where*

$$b_N(\gamma) = -\,\mathbb{E}_{G_\theta}\,\mathbb{E}_{\mathscr{L}(\gamma\theta,1)}\left\{\Phi^{-1}((1+e^x)^{-1})|\theta\right\}$$

Proof. [Proof of B1]

$$T_F = 2\sum_{i=1}^{n} \frac{\ln(1+e^{x_i})}{\sqrt{n}}.$$

By the strong law of large number (SLLN)

$$\frac{T_F}{\sqrt{n}} \xrightarrow{\text{w.p.1}} b_F(\gamma) = 2\,\mathbb{E}^{H_1}\left[\ln(1+e^x)\right]$$

then

$$b_F(\gamma) = 2\,\mathbb{E}_{G_\theta}\,\mathbb{E}_{\mathscr{L}(\gamma\theta,1)}\{\ln(1+e^x)\,|\theta\} = 2\iint_{\mathbb{R}} \ln(1+e^x)\,\mathscr{F}_{\mathscr{L}}(x-\gamma\theta)\,dx\,dG_\theta = 2\gamma\,\mathbb{E}_{G_\theta}\left\{\theta\left(e^{\gamma\theta}-1\right)^{-1}\right\}.$$

Now under H_0, then by Theorem (1):
$M(z) = \mathbb{E}_{\mathscr{L}(0,1)}\left\{e^{\ln(1+e^x)^{2z}}\right\} = \mathbb{E}_{\mathscr{L}(0,1)}\left\{(1+e^x)^{2z}\right\}$, then let $t = 1+e^x$, $\dfrac{dt}{dx} = e^x$, and $x = \ln(t-1)$, thus
$M(z) = \displaystyle\int_1^\infty \frac{t^{-2z}}{t-1}\mathscr{F}_{\mathscr{L}}(\ln(t-1))\,dt = (1-2z)^{-1},\ \ z < \frac{1}{2}$. Then,

$$m_F(t) = \inf_{z>0} e^{-zt}(1-2z)^{-1} = \frac{t}{2}e^{1-\frac{t}{2}}.$$

Then by Theorem 2

$$C_F(\gamma) = -2\ln(m_F(b_F(\gamma))) = -2\ln\left(\frac{b_F(\gamma)}{2}e^{1-\frac{b_F(\gamma)}{2}}\right) = b_F(\gamma) - 2\ln(b_F(\gamma)) + 2\ln(2) - 2$$

. □

Proof. [Proof of B3]

$$T_S = -\sum_{i=1}^{n} \frac{(1+e^{x_i})^{-1}}{\sqrt{n}}.$$

By the strong law of large number (SLLN)

$$\frac{T_S}{\sqrt{n}} \xrightarrow{\text{w.p.1}} b_S(\gamma) = -\,\mathbb{E}^{H_1}\left[(1+e^x)^{-1}\right]$$

then

$$b_S(\gamma) = -\,\mathbb{E}_{G_\theta}\,\mathbb{E}_{\mathscr{L}(\gamma\theta,1)}\left\{(1+e^x)^{-1}\,|\theta\right\} = -\iint_{\mathbb{R}} (1+e^x)^{-1}\,\mathscr{F}_{\mathscr{L}}(x-\gamma\theta)\,dx\,dG_\theta = \mathbb{E}_{G_\theta}\left\{\frac{(1-\gamma\theta)e^{\gamma\theta}-1}{(e^{\gamma\theta}-1)^2}\right\}.$$

Now, by Theorem 1, we have $m_S(t) = \inf_{z>0} e^{-zt}M_S(z)$, where $M_S(z) = \mathbb{E}_F(e^{zX})$. Under $H_0 : -(1+e^{x_i})^{-1} \sim U(-1,0)$, so $M_S(z) = \frac{1-e^{-z}}{z}$, by part (2) of Theorem 2 we complete the proof, that is $C_S(\gamma) = -2\ln(m_S(b_S(\gamma)))$. □

Proof. [Proof of B4]

$$T_N = -\sum_{i=1}^{n} \frac{\Phi^{-1}((1+e^{x_i})^{-1})}{\sqrt{n}}.$$

By the strong law of large number (SLLN)

$$\frac{T_N}{\sqrt{n}} \xrightarrow{\text{w.p.1}} b_N(\gamma) = -\,\mathbb{E}^{H_1}\left\{\Phi^{-1}((1+e^x)^{-1})\right\}$$

where

$$b_N(\gamma) = -\,\mathbb{E}_{G_\theta}\,\mathbb{E}_{\mathscr{L}(\gamma\theta,1)}\left\{\Phi^{-1}((1+e^x)^{-1})|\theta\right\} = -\,\mathbb{E}_{G_\theta}\left\{\int_{\mathbb{R}}\Phi^{-1}((1+e^x)^{-1})\mathscr{F}_{\mathscr{L}}(x-\gamma\theta)\,dx\right\}.$$

Now, by Theorem 1, we have $m_N(t) = \inf_{z>0} e^{-zt}M_N(z)$, where $M_N(z) = \mathbb{E}_F(e^{zX})$. Under $H_0 : -\Phi^{-1}((1+e^{x_i})^{-1}) \sim N(0,1)$, so $M_N(z) = e^{z^2/2}$, by part (2) of Theorem 2, $C_N(\gamma) = -2\ln(m_N(b_N(\gamma))) = b_N^2(\gamma)$. □

4.1. The Limiting ratio of the EBS for different tests when $\gamma \to 0$

Corollary 4.2. *The limits of ratios for different tests are as follows:*

A1. $\lim\limits_{\gamma\to 0}\dfrac{C_S(\gamma)}{C_F(\gamma)} = 1.333333$

A2. $\lim\limits_{\gamma\to 0}\dfrac{C_L(\gamma)}{C_F(\gamma)} = 1.215854$

A3. $\lim\limits_{\gamma\to 0}\dfrac{C_N(\gamma)}{C_F(\gamma)} = 1.273239$

A4. $\lim\limits_{\gamma\to 0}\dfrac{C_N(\gamma)}{C_L(\gamma)} = 1.0472$

A5. $\lim\limits_{\gamma\to 0}\dfrac{C_N(\gamma)}{C_S(\gamma)} = 0.954929$

A6. $\lim\limits_{\gamma\to 0}\dfrac{C_S(\gamma)}{C_L(\gamma)} = 1.09662$

Proof. [Proof of A1]

$$b_F(\gamma) = 2\gamma\,\mathbb{E}_{G_\theta}\left\{\theta\left(1-e^{-\gamma\theta}\right)^{-1}\right\}.$$

Therefore

$$b_F'(\gamma) = 2\,\mathbb{E}_{G_\theta}\left\{\theta\frac{1-(1+\gamma\theta)e^{-\gamma\theta}}{(1-e^{-\gamma\theta})^2}\right\} = \mathbb{E}_{G_\theta}\left(\theta\frac{e^{\gamma\theta}-\gamma\theta-1}{\cosh(\gamma\theta)-1}\right).$$

By using L'Hopital's rule twice, we get

$$\lim_{\gamma\to 0} b_F'(\gamma) = -\lim_{\gamma\to 0}\mathbb{E}_{G_\theta}\left[\theta e^{-\gamma\theta}\right] = -\,\mathbb{E}_{G_\theta}(\theta) < \infty.$$

Also

$$b_S(\gamma) = \mathbb{E}_{G_\theta}\left\{\frac{(1-\gamma\theta)e^{\gamma\theta}-1}{(e^{\gamma\theta}-1)^2}\right\}.$$

By using L'Hopital's rule three times, we get

$$\lim_{\gamma\to 0} b_S'(\gamma) = \frac{1}{4}\lim_{\gamma\to 0}\mathbb{E}_{G_\theta}\left\{\theta\left[-2+\gamma\theta\coth\left(\frac{\gamma\theta}{2}\right)\right]\operatorname{csch}^2\left(\frac{\gamma\theta}{2}\right)\right\} = \lim_{\gamma\to 0}\mathbb{E}_{G_\theta}\left\{\frac{\theta\left[5+\gamma\theta+e^{\gamma\theta}\left(-4+8\gamma\theta\right)\right]}{3\left[1-8e^{\gamma\theta}+9e^{2\gamma\theta}\right]}\right\}$$
$$= \frac{1}{6}\,\mathbb{E}_{G_\theta}(\theta) < \infty.$$

Now under $H_0 : h_F(x) = 2\ln(1+e^x) \sim \chi_2^2$ and $h_S(x) = -(1+e^x)^{-1} \sim U(-1,0)$, so $Var_{\gamma=0}(h_F(x)) = 4$ and $Var_{\gamma=0}(h_S(x)) = \frac{1}{12}$, also, $\dfrac{b_S'(0)}{b_F'(0)} = \dfrac{1}{6}$. By applying Theorem 3 we can get $\lim\limits_{\gamma\to 0}\dfrac{C_S(\gamma)}{C_F(\gamma)} = 1.33333$. Similarly we can prove the other parts. □

4.2. The Limiting ratio of the EBS for different tests when $\gamma \to \infty$

Corollary 4.3. *The limits of ratios for different tests are as follows:*

D1. $\lim\limits_{\gamma\to\infty} \dfrac{C_L(\gamma)}{C_F(\gamma)} = 1$

D2. $\lim\limits_{\gamma\to\infty} \dfrac{C_N(\gamma)}{C_S(\gamma)} = 0$

D3. $\lim\limits_{\gamma\to\infty} \dfrac{C_N(\gamma)}{C_L(\gamma)} = 0$

D4. $\lim\limits_{\gamma\to\infty} \dfrac{C_N(\gamma)}{C_F(\gamma)} = 0$

D5. $\lim\limits_{\gamma\to\infty} \dfrac{C_L(\gamma)}{C_S(\gamma)} = 0$

D6. $\lim\limits_{\gamma\to\infty} \dfrac{C_F(\gamma)}{C_S(\gamma)} = 0$

D7. $\lim\limits_{\gamma\to\infty} (C_F(\gamma) - C_L(\gamma)) \leq 0$

Proof. [Proof of D1] By Lemma 1 part (1) $C_L(\gamma) \leq 2\gamma\, \mathbb{E}_{G_\theta}(\theta)$. So

$$\lim_{\gamma\to\infty} \frac{C_L(\gamma)}{C_F(\gamma)} \leq \lim_{\gamma\to\infty} \frac{2\gamma\, \mathbb{E}_{G_\theta}(\theta)}{b_F(\gamma) - 2\ln(b_F(\gamma)) + 2\ln(2) - 2}$$

where

$$b_F(\gamma) = 2\gamma\, \mathbb{E}_{G_\theta}\left\{\theta\left(1 - e^{-\gamma\theta}\right)^{-1}\right\}.$$

It is sufficient to obtain the limit of $\lim\limits_{\gamma\to\infty} \dfrac{2\gamma\, \mathbb{E}_{G_\theta}(\theta)}{b_F(\gamma)}$. Then

$$\lim_{\gamma\to\infty} \frac{2\gamma\, \mathbb{E}_{G_\theta}(\theta)}{2\gamma\, \mathbb{E}_{G_\theta}\left\{\theta\,(1 - e^{-\gamma\theta})^{-1}\right\}} = \lim_{\gamma\to\infty} \frac{\mathbb{E}_{G_\theta}(\theta)}{\mathbb{E}_{G_\theta}\left\{\theta\,(1 - e^{-\gamma\theta})^{-1}\right\}} = 1.$$

Then

$$\lim_{\gamma\to\infty} \frac{C_L(\gamma)}{C_F(\gamma)} \leq 1.$$

Also, by Lemma 1 part (2), we have

$$C_L(\gamma) \geq 2\frac{\gamma^2\, \mathbb{E}^2_{G_\theta}(\theta)}{1 + \gamma\, \mathbb{E}_{G_\theta}(\theta)} - 2\ln\left[\frac{\pi\gamma\, \mathbb{E}_{G_\theta}(\theta)}{1 + \gamma\, \mathbb{E}_{G_\theta}(\theta)}\right] + 2\ln\left[\sin\left(\frac{\pi\gamma\, \mathbb{E}_{G_\theta}(\theta)}{1 + \gamma\, \mathbb{E}_{G_\theta}(\theta)}\right)\right].$$

So,

$$\lim_{\gamma\to\infty} \frac{C_L(\gamma)}{C_F(\gamma)} \geq \lim_{\gamma\to\infty} \frac{2\frac{\gamma^2\, \mathbb{E}^2_{G_\theta}(\theta)}{1+\gamma\, \mathbb{E}_{G_\theta}(\theta)} - 2\ln\left[\frac{\pi\gamma\, \mathbb{E}_{G_\theta}(\theta)}{1+\gamma\, \mathbb{E}_{G_\theta}(\theta)}\right] + 2\ln\left[\sin\left(\frac{\pi\gamma\, \mathbb{E}_{G_\theta}(\theta)}{1+\gamma\, \mathbb{E}_{G_\theta}(\theta)}\right)\right]}{b_F(\gamma) - 2\ln(b_F(\gamma)) + 2\ln(2) - 2}.$$

It is sufficient to obtain the limit of $\lim\limits_{\gamma\to\infty} \dfrac{2\frac{\gamma^2\, \mathbb{E}^2_{G_\theta}(\theta)}{1+\gamma\, \mathbb{E}_{G_\theta}(\theta)}}{b_F(\gamma)}$. Then

$$\lim_{\gamma\to\infty} \frac{2\frac{\gamma^2\, \mathbb{E}^2_{G_\theta}(\theta)}{1+\gamma\, \mathbb{E}_{G_\theta}(\theta)}}{2\gamma\, \mathbb{E}_{G_\theta}\left\{\theta\,(1 - e^{-\gamma\theta})^{-1}\right\}} = \lim_{\gamma\to\infty} \frac{\frac{\gamma\, \mathbb{E}^2_{G_\theta}(\theta)}{1+\gamma\, \mathbb{E}_{G_\theta}(\theta)}}{\mathbb{E}_{G_\theta}(\theta)} = 1.$$

Then $\lim_{\gamma\to\infty} \frac{C_L(\gamma)}{C_F(\gamma)} \geq 1$.

By pinching theorem, we have $\lim_{\gamma\to\infty} \frac{C_L(\gamma)}{C_F(\gamma)} = 1$. □

Proof. [Proof of D2] We have

$$C_N(\gamma) = b_N^2(\gamma)$$

where

$$b_N(\gamma) = -\mathbb{E}_{G_\theta}\,\mathbb{E}_{\mathscr{L}(\gamma\theta,1)}\left\{\Phi^{-1}((1+e^x)^{-1})|\theta\right\}.$$

By lemma 1 part (3) $C_S(\gamma) \geq -2 - 2\ln(-b_S(\gamma))$, where $b_S(\gamma) = \mathbb{E}_{G_\theta}\left\{\frac{(1-\gamma\theta)e^{-\gamma\theta} - e^{-2\gamma\theta}}{(1-e^{-\gamma\theta})^2}\right\}$.

So

$$\lim_{\gamma\to\infty} \frac{C_N(\gamma)}{C_S(\gamma)} \leq \lim_{\gamma\to\infty} \frac{\left\{\mathbb{E}_{G_\theta}\,\mathbb{E}_{\mathscr{L}(\gamma\theta,1)}\left\{\Phi^{-1}((1+e^x)^{-1})|\theta\right\}\right\}^2}{-2-2\ln(-b_S(\gamma))}.$$

Putting $u = x - \gamma\theta$, we get

$$\lim_{\gamma\to\infty} \frac{\left\{\mathbb{E}_{G_\theta}\,\mathbb{E}_{\mathscr{L}(\gamma\theta,1)}\left\{\Phi^{-1}((1+e^x)^{-1})|\theta\right\}\right\}^2}{-2-2\ln(-b_S(\gamma))} = \lim_{\gamma\to\infty} \frac{\left\{\mathbb{E}_{G_\theta}\,\mathbb{E}_{\mathscr{L}(0,1)}\left\{\Phi^{-1}(\left(1+e^{u+\gamma\theta}\right)^{-1})|\theta\right\}\right\}^2}{-2-2\ln(-b_S(\gamma))}.$$

Again, let $t = \Phi^{-1}(\left(1+e^{u+\gamma\theta}\right)^{-1}$, $\Phi(t) = \left(1+e^{u+\gamma\theta}\right)^{-1}$, $\phi(t)\frac{dt}{du} = -e^{u+\gamma\theta}\left(1+e^{u+\gamma\theta}\right)^{-2} = -\frac{1}{2(1+\cosh(u+\gamma\theta))}$, we get

$$\lim_{\gamma\to\infty} \frac{\left\{\mathbb{E}_{G_\theta}\,\mathbb{E}_{\mathscr{L}(0,1)}\left\{\Phi^{-1}(\left(1+e^{u+\gamma\theta}\right)^{-1})|\theta\right\}\right\}^2}{-2-2\ln(-b_S(\gamma))} = \lim_{\gamma\to\infty} \frac{\left\{\mathbb{E}_{G_\theta}\,\mathbb{E}_{\mathscr{L}(0,1)}\left\{e^{\gamma\theta}t\phi(t)\left[\left(e^{\gamma\theta}-1\right)\Phi(t)+1\right]^{-2}|\theta\right\}\right\}^2}{-2-2\ln(-b_S(\gamma))}.$$

Now, $b_S(\gamma) \to 0$ as $\gamma \to \infty$, and $e^{\gamma\theta}t\phi(t)\left[\left(e^{\gamma\theta}-1\right)\Phi(t)+1\right]^{-2} \to 0$ as $\gamma \to \infty$. Then

$$\lim_{\gamma\to\infty} \frac{C_N(\gamma)}{C_S(\gamma)} \leq 0.$$

So

$$\lim_{\gamma\to\infty} \frac{C_N(\gamma)}{C_S(\gamma)} = 0.$$

□

Proof. [Proof of D3] Same as the proof for D1 and D2
by Lemma 1 part (2), we have

$$C_L(\gamma) \geq 2\frac{\gamma^2\,\mathbb{E}^2_{G_\theta}(\theta)}{1+\gamma\,\mathbb{E}_{G_\theta}(\theta)} - 2\ln\left[\frac{\pi\gamma\,\mathbb{E}_{G_\theta}(\theta)}{1+\gamma\,\mathbb{E}_{G_\theta}(\theta)}\right] + 2\ln\left[\sin\left(\frac{\pi\gamma\,\mathbb{E}_{G_\theta}(\theta)}{1+\gamma\,\mathbb{E}_{G_\theta}(\theta)}\right)\right],$$

and

$$C_N(\gamma) = b_N^2(\gamma)$$

where

$$b_N(\gamma) = -\mathbb{E}_{G_\theta}\,\mathbb{E}_{\mathscr{L}(\gamma\theta,1)}\left\{\Phi^{-1}((1+e^x)^{-1})|\theta\right\}.$$

$$\lim_{\gamma\to\infty}\frac{C_N(\gamma)}{C_L(\gamma)} \le \lim_{\gamma\to\infty}\frac{\left\{\mathbb{E}_{G_\theta}\,\mathbb{E}_{\mathscr{L}(0,1)}\left\{e^{\gamma\theta}t\phi(t)\left[\left(e^{\gamma\theta}-1\right)\Phi(t)+1\right]^{-2}|\theta\right\}\right\}^2}{2\frac{\gamma^2\,\mathbb{E}^2_{G_\theta}(\theta)}{1+\gamma\,\mathbb{E}_{G_\theta}(\theta)}-2\ln\left[\frac{\pi\gamma\,\mathbb{E}_{G_\theta}(\theta)}{1+\gamma\,\mathbb{E}_{G_\theta}(\theta)}\right]+2\ln\left[\sin\left(\frac{\pi\gamma\,\mathbb{E}_{G_\theta}(\theta)}{1+\gamma\,\mathbb{E}_{G_\theta}(\theta)}\right)\right]}.$$

Now, as $\gamma\to\infty$

$$\frac{\gamma^2\,\mathbb{E}^2_{G_\theta}(\theta)}{1+\gamma\,\mathbb{E}_{G_\theta}(\theta)}\to\infty$$

$$-2\ln\left[\frac{\pi\gamma\,\mathbb{E}_{G_\theta}(\theta)}{1+\gamma\,\mathbb{E}_{G_\theta}(\theta)}\right]\to-2\ln(\pi)$$

$$2\ln\left[\sin\left(\frac{\pi\gamma\,\mathbb{E}_{G_\theta}(\theta)}{1+\gamma\,\mathbb{E}_{G_\theta}(\theta)}\right)\right]\to\sin(\pi)=0.$$

Then we have

$$\lim_{\gamma\to\infty}\frac{C_N(\gamma)}{C_L(\gamma)}\le 0$$

so

$$\lim_{\gamma\to\infty}\frac{C_N(\gamma)}{C_L(\gamma)}=0.$$

□

Proof. [Proof of D7] By Theorem 6 part (3)

$$C_L(\gamma)\ge -2+2\gamma\,\mathbb{E}_{G_\theta}(\theta)-2\ln\left(\mathbb{E}_{G_\theta}(\theta)\right)-2\ln(\gamma)-6\ln\left[\frac{\gamma^2\,\mathbb{E}_{G_\theta}(\theta)}{1+\gamma^2\,\mathbb{E}_{G_\theta}(\theta)}\right].$$

So

$$C_F(\gamma)-C_L(\gamma)\le b_F(\gamma)-2\ln(b_F(\gamma))+2\ln(2)-2\gamma\,\mathbb{E}_{G_\theta}(\theta)+2\ln\left(\mathbb{E}_{G_\theta}(\theta)\right)+2\ln(\gamma)+6\ln\left[\frac{\gamma^2\,\mathbb{E}_{G_\theta}(\theta)}{1+\gamma^2\,\mathbb{E}_{G_\theta}(\theta)}\right]$$

$$=2\gamma\,\mathbb{E}_{G_\theta}\left\{\theta\left(1-e^{-\gamma\theta}\right)^{-1}\right\}-2\ln\left[\mathbb{E}_{G_\theta}\left\{\theta\left(1-e^{-\gamma\theta}\right)^{-1}\right\}\right]-2\gamma\,\mathbb{E}_{G_\theta}(\theta)+2\ln\left(\mathbb{E}_{G_\theta}(\theta)\right)+6\ln\left[\frac{\gamma^2\,\mathbb{E}_{G_\theta}(\theta)}{1+\gamma^2\,\mathbb{E}_{G_\theta}(\theta)}\right].$$

Now

$$6\ln\left[\frac{\gamma^2\,\mathbb{E}_{G_\theta}(\theta)}{1+\gamma^2\,\mathbb{E}_{G_\theta}(\theta)}\right]=6\ln\left[\frac{\mathbb{E}_{G_\theta}(\theta)}{\frac{1}{\gamma^2}+\mathbb{E}_{G_\theta}(\theta)}\right]\to\ln(1)=0\quad\text{as}\quad\gamma\to\infty$$

and

$$-2\ln\left[\mathbb{E}_{G_\theta}\left\{\theta\left(1-e^{-\gamma\theta}\right)^{-1}\right\}\right]\to-2\ln\left[\mathbb{E}_{G_\theta}(\theta)\right]\quad\text{as}\quad\gamma\to\infty$$

$$2\gamma\,\mathbb{E}_{G_\theta}\left\{\theta\left(1-e^{-\gamma\theta}\right)^{-1}\right\}-2\gamma\,\mathbb{E}_{G_\theta}(\theta)=2\gamma\left[\mathbb{E}_{G_\theta}\left\{\theta\left(1-e^{-\gamma\theta}\right)^{-1}\right\}-\mathbb{E}_{G_\theta}(\theta)\right]=2\frac{\mathbb{E}_{G_\theta}\left\{\theta\left(1-e^{-\gamma\theta}\right)^{-1}\right\}-\mathbb{E}_{G_\theta}(\theta)}{\frac{1}{\gamma}}$$

and by using L'Hopital's rule three times, we get

$$2\gamma\,\mathbb{E}_{G_\theta}\left\{\theta\left(1-e^{-\gamma\theta}\right)^{-1}\right\}-2\gamma\,\mathbb{E}_{G_\theta}(\theta)\to0\quad\text{as}\quad\gamma\to\infty.$$

Then,

$$C_F(\gamma)-C_L(\gamma)\le0\quad\text{as}\quad\gamma\to\infty$$

□

4.3. *Comparison of the EBS for the four combination procedures*

From the relations in section (4.1) we conclude that locally as $\gamma \to 0$, the sum of p-values procedure is better than all other procedures since it has the highest EBS, followed in decreasing order by the logistic and inverse normal procedure. The worst is the Fisher's procedure, i.e,

$$C_S(\gamma) > C_L(\gamma) > C_N(\gamma) > C_F(\gamma).$$

Whereas, from result of Section (4.2) as $\gamma \to \infty$ the sum of p-values procedure is better than the other procedures, followed in decreasing order by the logistic and Fisher's procedure. The worst is the inverse normal procedure, i.e,

$$C_S(\gamma) > C_L(\gamma) > C_F(\gamma) > C_N(\gamma)$$

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