Filomat 38:26 (2024), 9055–9065
https://doi.org/10.2298/FIL2426055E

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On constant angle surfaces constructed by polynomial space curves with Frenet-like curve frame

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Abstract. In this study, the constant angle ruled surfaces constructed by polynomial space curves using the Frenet-like curve (Flc) frame in Euclidean 3-space are examined. We find the conditions for the ruled surfaces to be constant angle surfaces. In this context, we obtain characterizations of minimal, developable, and Weingarten-ruled surfaces. Finally, examples of the constant angle surfaces are given, and their graphics are illustrated.

1. Introduction

Constant angle ruled surfaces constitute a significant mathematical structure that can be employed in fabricating sheet metal, woodworking, plastic forming, textile patterns or clothing, sculpture, art, and architectural or automotive designs. These types of surfaces exhibit a unique mathematical property at each point, wherein the tangent lines maintain a constant angle with the normal vector. This characteristic distinguishes constant angle ruled surfaces as fundamental entities within the broader category of surfaces. In this regard, the comprehension of these surfaces holds relevance across various fields, such as mathematical analysis, physics, engineering, and architecture. The properties of constant angle ruled surfaces play a crucial role in determining the geometric features of designs and structures. Constant angle ruled surfaces serve as a vital link between theoretical mathematics and practical applications, facilitating the resolution of complex problems. In recent years, numerous authors have extensively explored special surfaces, investigating their potential applications in both mathematical and physical contexts. Cermelli and Scala delved into the properties of constant angle surfaces through the lens of the Hamilton-Jacobi equation, focusing on understanding their behavior in situations where the direction field becomes singular along a specific line or point [4]. Munteanu and Nistor contributed to this field by classifying surfaces where the unit normal vector maintains a constant angle with a fixed direction vector, representing the tangent direction in Euclidean 3-space [16]. Various studies were conducted on constant angle surfaces and developable

2020 *Mathematics Subject Classification*. Primary 53A04; Secondary 53A05.
Keywords. Constant angle ruled surfaces, Frenet-like curve frame, Weingarten surfaces, Polynomial curve.
Received: 23 April 2024; Accepted: 12 June 2024
Communicated by Dragan S. Djordjević
The third author acknowledge the grant of the Ministry of Science, Technological Development and Innovation of Serbia 451-03-68/2024-14/200124 for carrying out this research.
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surfaces, shedding light on their properties and characteristics [17, 18]. The constant angle ruled surfaces generated by the Frenet frame vectors were explored in [1]. Recent extensions of the theory of constant angle surfaces encompass other ambient spaces, as seen in [14] and [11], where researchers examined these surfaces in the context of the Lorentzian metric. Furthermore, from [15] to [12], various authors introduced alternative approaches and perspectives to the concept of constant angle surfaces within Lorentzian ambient spaces. In our study, we pay attention to the constant angle ruled surfaces constructed with polynomial curves with the aid of the Frenet-like curve (Flc) frame, defined along these moving polynomial curves. This Flc-frame serves as an alternative to the Frenet frame, which faces limitations at the points where the second and higher-order derivatives of a curve are zero. To address this issue, Dede et al. introduced the Flc-frame for moving polynomial curves [5, 6]. These advantages have attracted the attention of numerous researchers, and they have explored investigations on curves [3, 7, 10, 19] and surfaces [8, 13] using the Flc-frame of polynomial curves. Given the recent developments outlined above, this study focuses on constant angle ruled surfaces with the Flc-frame of polynomial curves in Euclidean 3-space. We determine the necessary and sufficient requirements for any ruled surfaces of polynomial curves to be constant-angle, and we give characterizations for being minimal, developable, and Weingarten ruled surfaces. Finally, we present examples of constant-angle surfaces and illustrate their graphics.

2. Preliminaries

Let $\alpha = \alpha(s)$ be an n. order polynomial space curve in Euclidean 3-space The tangent vector, the binormal-like vector, and the normal-like vector of the Flc-frame along the curve α are defined by

$$T(s) = \frac{\alpha(s)}{\|\alpha'(s)\|},\ D_1(s) = \frac{\alpha'(s) \times \alpha^{(n)}(s)}{\left\|\alpha'(s) \times \alpha^{(n)}(s)\right\|},\ D_2(s) = D_1(s) \times T(s),$$

respectively, where "$'$" and "$^{(n)}$" denote the first and the n. order derivatives of the curve in terms of s. This allows us to define a frame even if $\alpha'(s) \times \alpha''(s) = 0$ where the second or higher order derivatives of the space curve are zero. The new Frenet-like vectors D_1 and D_2 are called binormal-like vectors and normal-like vectors, respectively. The curvatures of the Flc-frame d_1, d_2 and d_3 are given by

$$d_1 = \frac{\langle T', D_2\rangle}{v},\ d_2 = \frac{\langle T', D_1\rangle}{v},\ d_3 = \frac{\langle D_2{}', D_1\rangle}{v},$$

where $\|\alpha'\| = v$. The local rate of change of the Flc-frame, called the Frenet-like formulas, can be expressed in the following form:

$$\begin{bmatrix} T' \\ D'_2 \\ D'_1 \end{bmatrix} = v \begin{bmatrix} 0 & d_1 & d_2 \\ -d_1 & 0 & d_3 \\ -d_2 & -d_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ D_2 \\ D_1 \end{bmatrix},$$

where d_1, d_2 and d_3 are the curvatures of α with the arc-length s (see for more details [5, 6]), respectively. Let Ψ_s and Ψ_u be tangent vectors of a surface $\Psi(s,u)$, then the normal vector field of the surface is calculated by

$$\Omega = \frac{\Psi_s \times \Psi_u}{\|\Psi_s \times \Psi_u\|}, \tag{1}$$

where $\Psi_s = \frac{\partial \Psi}{\partial s}$ and $\Psi_u = \frac{\partial \Psi}{\partial u}$. The coefficients of the first and second fundamental forms of the surface $\Psi(s,u)$ are, respectively, given by

$$E = \left\langle \frac{\partial \Psi}{\partial s}, \frac{\partial \Psi}{\partial s} \right\rangle,\ F = \left\langle \frac{\partial \Psi}{\partial s}, \frac{\partial \Psi}{\partial u} \right\rangle,\ G = \left\langle \frac{\partial \Psi}{\partial u}, \frac{\partial \Psi}{\partial u} \right\rangle \tag{2}$$

and

$$k=\left\langle \frac{\partial^2\Psi}{\partial s^2},\Omega\right\rangle,\ l=\left\langle \frac{\partial^2\Psi}{\partial s\partial u},\Omega\right\rangle,\ m=\left\langle \frac{\partial^2\Psi}{\partial u^2},\Omega\right\rangle. \tag{3}$$

Moreover, the Gaussian curvature and the mean curvature of the surface are defined by

$$K=\frac{km-l^2}{EG-F^2} \text{ and } H=\frac{1}{2}\frac{Em-2El+Gk}{EG-F^2}, \tag{4}$$

respectively. Also, the surface $\Psi(s,u)$ is

- developable surface if and only if the Gaussian curvature vanishes at all points,
- minimal surface if and only if the mean curvature vanishes at all points,
- Weingarten surface if and only if $K_sH_u-K_uH_s=0$ at all points.

3. Constant angle ruled surface constructed by polynomial space curve

In this section, the constant angle surfaces are characterized separately using the elements of the Flc-frame of polynomial curves. A ruled surface is constructed by one parameter family of straight lines, and its parametric equation is given by

$$\Psi(s,u)=\sigma(s)+u\Upsilon(s). \tag{5}$$

Here, the polynomial curve $\sigma(s)$ is the base curve and $\Upsilon(s)=fT+gD_2+hD_1$ is a director curve where f, g and h are smooth function of s. The partial derivatives of the surfaces using the Flc-frame are

$$\Psi_s=(-uf'+v(1-ud_1g-ud_2h))T+u(g'+v(d_1f-d_3h))D_2+u(h'+v(d_2f+d_3g))D_1$$

and

$$\Psi_u=fT+gD_2+hD_1.$$

The cross-product of these partial derivatives is found by

$$\begin{aligned}\Psi_s\times\Psi_u=&-u\left(\left(d_2fg+d_3\left(g^2+h^2\right)-hd_1f\right)v+gh'-hg'\right)T\\&-\left(hv-u\left(\left(d_3fg+d_1gh+d_2\left(f^2+h^2\right)\right)v+fh'-hf'\right)\right)D_2\\&+\left(gv+u\left(\left(-d_1\left(f^2+g^2\right)-gd_2h+fd_3h\right)v+gf'-fg'\right)\right)D_1.\end{aligned}$$

By a straightforward computation, the normal vector field of the surface is obtained as follows:

$$\Omega=\Omega_1T+\Omega_2D_2+\Omega_3D_1,$$

where

$$\Omega_1=\Omega_{11}+u\Omega_{12},\ \Omega_2=\Omega_{21}+u\Omega_{22},\ \Omega_3=\Omega_{31}+u\Omega_{32},$$

and

$$\begin{cases}\Omega_{11}=0,\ \Omega_{12}=-\left(\left(d_2fg+d_3\left(g^2+h^2\right)-hd_1f\right)v+gh'-hg'\right),\\\Omega_{21}=-h,\ \Omega_{22}=\left(d_3fg+d_1gh+d_2\left(f^2+h^2\right)\right)v+fh'-hf',\\\Omega_{31}=g,\ \Omega_{32}=\left(-d_1\left(f^2+g^2\right)-gd_2h+fd_3h\right)v+gf'-fg'.\end{cases} \tag{6}$$

3.1. Constant angle ruled surface parallel to tangent vector

In this subsection, let the normal vector field Ω of the surface $\Psi(s,u)$ be parallel to the tangent vector T of the polynomial curve $\sigma(s)$ with Flc-frame, then we have the following conditions:

$$\Omega_1 \neq 0,\ \Omega_2 = \Omega_3 = 0. \tag{7}$$

Considering Eq. (6) and $\Omega_2 = \Omega_3 = 0$ together, we have $g = h = 0$. This situation contradicts the fact that $\Upsilon(s) \neq 0$ and $\Omega_1 \neq 0$. So, we can give the following theorem:

Theorem 3.1. *Let the normal vector field Ω of a surface $\Psi(s,u)$ be parallel to the tangent vector of the Flc-frame of a polynomial curve, then there is no constant angle ruled surface parallel to the tangent vector.*

3.2. Constant angle ruled surface parallel to the principal normal-like vector

Let the normal vector field Ω of a surface $\Psi(s,u)$ be parallel to the principal normal-like vector D_2 of a polynomial curve $\sigma(s)$ according to Flc-frame, then we have the following conditions:

$$\Omega_2 \neq 0,\ \Omega_1 = \Omega_3 = 0. \tag{8}$$

Since $\Omega_{31} = 0$, g vanishes. In that case, from Eq. (6), we get the following equations:

$$\begin{cases} \Omega_{11} = 0,\ \Omega_{12} = vh\left(d_1 f - d_3 h\right), \\ \Omega_{21} = -h,\ \Omega_{22} = vd_2\left(f^2 + h^2\right) + fh' - hf', \\ \Omega_{31} = 0,\ \Omega_{32} = -vf\left(d_1 f - d_3 h\right). \end{cases}$$

Then, there exist three cases satisfying the condition Eq. (8) as follows:

- $d_1 f = d_3 h$, $f, h \neq 0$ and $g = 0$,
- $g = h = d_1 = 0$ and $f, d_2 \neq 0$,
- $f = g = d_3 = 0$ and $h \neq 0$.

Now, let us investigate these cases separately.

Case 3.2.1. Let $d_1 f = d_3 h$, $f, h \neq 0$ and $g = 0$. In this case, we have obtained a constant ruled surface which takes the form

$$\Psi_1^{D2}(s,u) = \sigma(s) + u\left(fT + hD_1\right),$$

where $d_1 f = d_3 h$. The partial derivatives of the surfaces $\Psi_1^{D2}(s,u)$ using the Flc-frame are found as

$$\left(\Psi_1^{D2}\right)_s = \left(-uf' + v\left(1 - ud_2 h\right)\right)T + uv\left(d_1 f - d_3 h\right)D_2 + u\left(h' + vd_2 f\right)D_1$$

and

$$\left(\Psi_1^{D2}\right)_u = \left(fT + hD_1\right).$$

By a straightforward computation from the last equations and Eq. (1), the normal vector field of the surface $\Psi_1^{D2}(s,u)$ satisfies the relation:

$$\Omega_1^{D2}(s,u) = N.$$

Theorem 3.2. *Let $\Psi_1^{D2}(s,u)$ be a constant ruled surface constructed by a polynomial space curve with Flc-frame, then the Gaussian and mean curvatures are*

$$K_1^{D2}(s,u) = 0$$

and

$$H_1^{D2}(s,u) = \frac{-vd_3\left({d_1}^2 + {d_3}^2\right)}{2\left(u{d_1}^2 d_2 fv + ud_3 f\left(d_2d_3v + f'\right) - f\left(d_3v + ufd_3'\right)\right)},$$

respectively, where $d_1 f = d_3 h$.

Proof. Let $\Psi_1^{D2}(s,u)$ be a constant ruled surface constructed by a polynomial space curve with the Flc-frame. From Eqs. (2) and (3), the coefficients of the first and second fundamental forms are

$$E_1^{D2} = u^2(d_2fv + h')^2 + (v - ud_2hv + uf')^2,$$
$$F_1^{D2} = fv + u(fh)',$$
$$G_1^{D2} = f^2 + h^2$$

and

$$k_1^{D2} = -v\left(u\left(d_2d_3fv - fd_1{}' + hd_3{}' + 2d_3h'\right) + d_1\left((-1 + ud_2h)v - 2uf'\right)\right),$$
$$l_1^{D2} = 0,\ m_1^{D2} = 0,$$

respectively, such that

$$\begin{aligned}
\left(\Psi_1^{D2}\right)_{ss} &= \begin{pmatrix} v' - u\left(v'd_2h - f''\right) - uv\left(hd_2{}' + 2d_2h'\right) \\ +uv^2\left(d_1d_3h - {d_1}^2 f - {d_2}^2 f\right) \end{pmatrix} T \\
&+ \begin{pmatrix} \left(d_1 - u\left(d_2d_3f + d_1d_2h\right)\right)v^2 + uv\left(ff' - hd_3{}' + 2d_1f' - 2d_3h'\right) \\ +u\left(d_1fv' - d_3hv'\right) \end{pmatrix} D_2 + \\
&+ \begin{pmatrix} \left(d_2 + \left(d_1d_3f - {d_2}^2h - {d_3}^2h\right)u\right)v^2 + uv\left(fd_2{}' + 2d_2f'\right) \\ +u\left(d_2fv' + h''\right) \end{pmatrix} D_1, \\
\left(\Psi_1^{D2}\right)_{su} &= \left(-d_2vh + f'\right)T + \left(d_2vf + h'\right)D_1, \\
\left(\Psi_1^{D2}\right)_{uu} &= 0.
\end{aligned}$$

For $d_1 f = d_3 h$, if the coefficients of the first and second fundamental forms of $\Psi_1^{D2}(s,u)$ are substituted in Eq. (4), the Gaussian and mean curvatures of $\Psi_1^{D2}(s,u)$ are found as in the hypothesis. □

Corollary 3.3. *Let $\Psi_1^{D2}(s,u)$ be a constant ruled surface constructed by polynomial space curve with Flc-frame, then the constant angle surface $\Psi_1^{D2}(s,u)$ is*

- *a developable surface,*
- *not a minimal surface.*

Theorem 3.4. *Let $\Psi_1^{D2}(s,u)$ be a constant ruled surface constructed by a polynomial space curve with Flc-frame, then the constant angle surface $\Psi_1^{D2}(s,u)$ is a Weingarten surface.*

Proof. Let $\Psi_1^{D2}(s,u)$ be a constant ruled surface constructed by a polynomial space curve with Flc-frame. If the equations of the Gaussian and mean curvatures given in Theorem 3.2 are differentiated in terms of s and u, we get that $\left(K_1^{D2}\right)_s\left(H_1^{D2}\right)_u - \left(K_1^{D2}\right)_u\left(H_1^{D2}\right)_s = 0$. So, we can say that the constant angle surface $\Psi_1^{D2}(s,u)$ is a Weingarten surface. □

Case 3.2.2. Let $g = h = d_1 = 0$ and $f, d_2 \neq 0$. In this case, a constant ruled surface is formed as

$$\Psi_2^{D2}(s,u) = \sigma(s) + ufT.$$

Theorem 3.5. *Let $\Psi_2^{D2}(s,u)$ be a constant ruled surface constructed by a polynomial space curve with Flc-frame, then the Gaussian and mean curvatures of this constant ruled surface are*

$$K_2^{D2}(s,u) = 0$$

and

$$H_2^{D2}(s,u) = -\frac{d_3}{2ud_2 f},$$

respectively.

The proof is very similar to the proof of Theorem 3.2, it is omitted.

Corollary 3.6. *Let $\Psi_2^{D2}(s,u)$ be a constant ruled surface constructed by a polynomial space curve with Flc-frame, then the constant angle surface $\Psi_2^{D2}(s,u)$ is*

- *a developable surface,*
- *a minimal surface if and only if $d_3 = 0$,*
- *a Weingarten surface.*

Case 3.2.3 Let $f = g = d_3 = 0$ and $h \neq 0$. Then we construct a constant ruled surface presented by

$$\Psi_3^{D2}(s,u) = \sigma(s) + uhD_1.$$

Theorem 3.7. *Let $\Psi_3^{D2}(s,u)$ be a constant ruled surface constructed by a polynomial space curve with Flc-frame, then the Gaussian curvature and mean curvature of $\Psi_3^{D2}(s,u)$ are*

$$K_3^{D2}(s,u) = 0$$

and

$$H_3^{D2}(s,u) = \frac{d_1}{2 - 2ud_2 h},$$

respectively, where $d_2 \neq 0$.

Since the proof of this theorem is also similar to the proof of Theorem 3.2, it is omitted, too.

Corollary 3.8. *Let $\Psi_3^{D2}(s,u)$ be a constant ruled surface constructed by a polynomial space curve with Flc-frame, then the constant angle surface $\Psi_3^{D2}(s,u)$ is*

- *developable surface,*
- *minimal surface if and only if $d_1 = 0$,*
- *Weingarten surface.*

3.3. Constant angle ruled surface parallel to binormal-like vector

Let the normal vector Ω of a surface $\Psi(s,u)$ be parallel to the binormal-like vector D_1 of a polynomial curve $\sigma(s)$ according to the Flc-frame, then we have the following conditions:

$$\Omega_3 \neq 0,\ \Omega_1 = \Omega_2 = 0. \tag{9}$$

Since $\Omega_{21} = 0$, h vanishes. In that case, from Eq. (6), we get the following equations:

$$\begin{cases} \Omega_{11} = 0,\ \Omega_{12} = -vg\left(d_2 f + d_3 g\right), \\ \Omega_{21} = 0,\ \Omega_{22} = vf\left(d_3 g + d_2 f\right), \\ \Omega_{31} = g,\ \Omega_{32} = -d_1 v\left(f^2 + g^2\right) + gf' - fg'. \end{cases}$$

Then, there exist the following cases that satisfy the conditions in Eq. (9).

- $h = 0$, $d_2 f + d_3 g = 0$ and $f, g \neq 0$,
- $f = h = d_3 = 0$ and $g, d_1 \neq 0$,
- $g = h = d_2 = 0$ and $f, d_1 \neq 0$.

Case 3.3.1. Let $h = 0$, $d_2 f + d_3 g = 0$ and $f, g \neq 0$. In this case, we have obtained a constant ruled surface represented by

$$\Psi_1^{D1}(s,u) = \sigma(s) + u\left(fT + gD_2\right).$$

By a straightforward computation, the normal vector field of the surface $\Psi_1^{D1}(s,u)$ is obtained as follows:

$$\Omega_1^{D1}(s,u) = -D_1$$

such that

$$\left(\Psi_1^{D1}\right)_s = \left(v - ud_1 gv + uf'\right)T + \left(d_1 fv + g'\right)D_2 \text{ and } \left(\Psi_1^{D1}\right)_u = fT + gD_2.$$

Theorem 3.9. *Let $\Psi_1^{D1}(s,u)$ be a constant ruled surface constructed by a polynomial curve with Flc-frame, then the Gaussian curvature and mean curvature is*

$$K_1^{D1}(s,u) = 0$$

and

$$H_1^{D1}(s,u) = \frac{v\left(g^2 + f^2\right)\left(-u\left(f\left(d_1 d_3 v + d_2{}'\right) + gd_3{}' + 2d_3 g'\right) + d_2\left(\left(-1 + ud_1 g\right)v - 2uf'\right)\right)}{2\left(ud_1 g^2 v + f\left(d_1 fv + g'\right) - g\left(v + uf'\right)\right)^2},$$

respectively.

Proof. This theorem's proof is similar to the proof of Theorem 3.2. □

Corollary 3.10. *Let $\Psi_1^{D1}(s,u)$ be a constant ruled surface constructed by a polynomial curve with Flc-frame, then the constant angle surface $\Psi_1^{D1}(s,u)$ is*

- *a developable surface,*
- *not a minimal surface,*
- *a Weingarten surface.*

Case 3.3.2. Let $f = h = d_3 = 0$ and $g, d_1 \neq 0$. Then, we have obtained a constant ruled surface, which takes the form

$$\Psi_2^{D1}(s,u) = \sigma(s) + ugD_2.$$

By a straightforward computation, the normal vector field of the surface $\Psi_2^{D1}(s,u)$ is obtained as:

$$\Omega_2^{D1}(s,u) = -D_1$$

such that

$$\left(\Omega_2^{D1}\right)_s = (v - ud_1gv + uf')\,T + (d_1fv + g')\,D_2 \text{ and } \left(\Omega_2^{D1}\right)_u = fT + gD_2.$$

Theorem 3.11. *Let $\Psi_2^{D1}(s,u)$ be a constant ruled surface constructed by a polynomial curve with Flc-frame, then the Gaussian curvature and mean curvature of $\Psi_2^{D1}(s,u)$ are*

$$K_2^{D1}(s,u) = 0 \text{ and } H_2^{D1}(s,u) = \frac{d_2v^2}{2 - 2ud_1g},$$

respectively.

Proof. This theorem is proved in a similar manner to the proof of Theorem 3.2. □

Corollary 3.12. *Let $\Psi_2^{D1}(s,u)$ be a constant ruled surface constructed by a polynomial curve with Flc-frame, then the constant angle surface is*

- *a developable surface,*
- *a minimal surface if and only if $d_2 = 0$.*
- *a Weingarten surface.*

Case 3.3.3. Let $g = h = d_2 = 0$ and $f, d_1 \neq 0$. Then, the constant ruled surface is represented in the form

$$\Psi_3^{D1}(s,u) = \sigma(s) + uf(s)\,T(s).$$

By a straightforward computation, the normal vector field of the surface $\Psi_3^{D1}(s,u)$ is obtained as:

$$\Omega_3^{D1}(s,u) = -D_1$$

such that

$$\left(\Omega_3^{D1}\right)_s = (v + uf')\,T + uvd_1fD_2 \text{ and } \left(\Omega_3^{D1}\right)_u = fT.$$

Theorem 3.13. *Let $\Psi_3^{D1}(s,u)$ be a constant ruled surface constructed by a polynomial curve with Flc-frame, then the Gaussian curvature and mean curvature of $\Psi_3^{D1}(s,u)$ are*

$$K_3^{D1}(s,u) = 0 \text{ and } H_3^{D1}(s,u) = -\frac{d_3}{2ud_1f},$$

respectively.

Proof. The proof of this theorem is similar to the proof of Theorem 3.2. □

Corollary 3.14. *Let $\Psi_3^{D1}(s,u)$ be a constant ruled surface constructed by a polynomial curve with Flc-frame, then the constant angle surface is*

- *a developable surface,*
- *a minimal surface if and only if $d_3 = 0$.*
- *a Weingarten surface.*

Example 3.15. *Let $\sigma(s)$ be a polynomial curve represented by*

$$\sigma(s) = \left(s, \frac{s^2}{2}, \frac{s^3}{6}\right).$$

The tangent, principal normal-like, binormal-like vector field, and the curvatures of the polynomial curve are found as:

$$T = \left(\frac{2}{2+s^2}, \frac{2s}{2+s^2}, \frac{s^2}{2+s^2}\right),$$
$$D_2 = \left(-\frac{s^2}{\sqrt{1+s^2}\,(2+s^2)}, -\frac{s^3}{\sqrt{1+s^2}\,(2+s^2)}, \frac{2\sqrt{1+s^2}}{2+s^2}\right),$$
$$D_1 = \left(\frac{s}{\sqrt{1+s^2}}, -\frac{1}{\sqrt{1+s^2}}, 0\right)$$

and

$$d_1 = \frac{4s}{\sqrt{1+s^2}(2+s^2)^2},\ d_2 = -\frac{4}{\sqrt{1+s^2}(2+s^2)^2},\ d_3 = \frac{2s^2}{(1+s^2)\,(2+s^2)^2}.$$

For $f = s^2$ and $h = 2s$, the equation of the constant angle ruled surface parallel to normal-like vector satisfying Case 3.2.1 with the Flc-frame is

$$\Psi_1^{D_2}(s,u) = \left(s + \left(\frac{2us^2}{\sqrt{1+s^2}} + \frac{2us^2}{2+s^2}\right), \frac{s}{2}\left(s - \frac{4u}{\sqrt{1+s^2}} + \frac{4s^2u}{2+s^2}\right), \frac{s^3}{6} + \frac{s^4u}{2+s^2}\right),$$

see Figure 1.

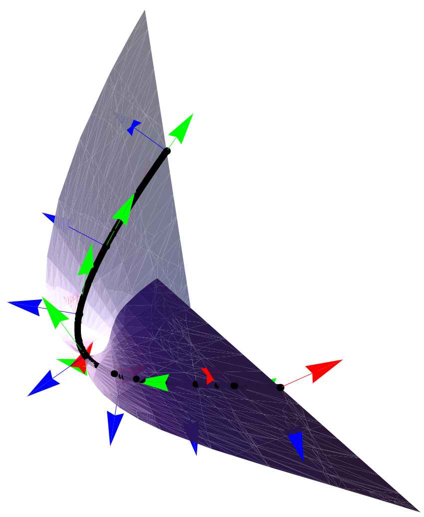

Figure 1: $\Psi_1^{D_2}(s,u)$ with the Flc-frame $\{T, D_2, D_1\}$ (green, red, blue) where $s \in (-3,3)$ and $u \in (-0.5, 0.5)$.

For $f = s^3$ and $g = 2s$, the equation of the constant angle ruled surface parallel to binormal-like vector satisfying Case 3.3.1 with the Flc-frame is

$$\Psi_1^{D1}(s,u) = \left(\frac{2s + s^3\left(1 + \left(2 - \frac{2}{\sqrt{1+s^2}}\right)u\right)}{2+s^2}, \frac{2s^2 + s^4\left(1 + \left(4 - \frac{4}{\sqrt{1+s^2}}\right)u\right)}{2\left(2+s^2\right)}, \frac{2s^3 + s^5 + 6s^5u + 24s\sqrt{1+s^2}u}{12+6s^2} \right),$$

see Figure 2.

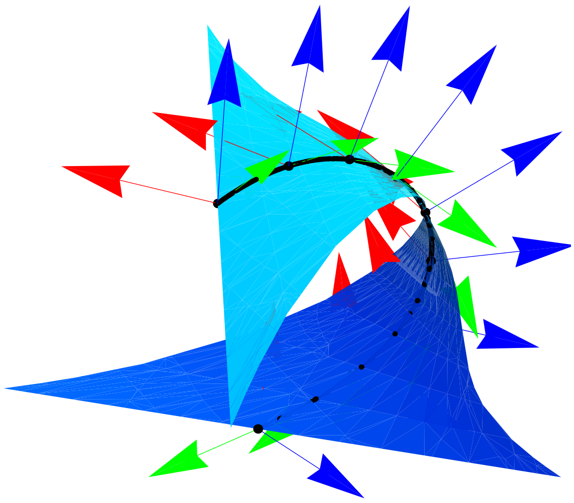

Figure 2: $\Psi_1^{D_1}(s,u)$ with the Flc-frame $\{T, D_2, D_1\}$ (green, red, blue) where $s \in (-2, 2)$ and $u \in (-0.5, 0.5)$.

4. Conclusion

This study is devoted to exploring constant angle ruled surfaces utilizing polynomial space curves in Euclidean 3-space via the advantages of Flc-frame. The investigation involves calculating the necessary conditions based on the Flc-frame for these ruled surfaces to maintain a constant angle. Within this context, we also derive conditions for them to be minimal, developable, and Weingarten surfaces. Notably, this study presents novel insights as constant angle ruled surfaces have not been previously examined in the context of polynomial curves. Finally, the study provides examples of constant angle surfaces with accompanying graphic representations and stands as an original contribution, offering a valuable foundation for future research in this field.

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

References

[1] A. T. Ali, *A constant angle ruled surfaces*, Int. J. Geom. **7** (2018), 69–80.
[2] A. T. Ali, *Non-lightlike constant angle ruled surfaces in Minkowski 3-space*, J. Geom. Phys. **157** (2020), 103833.
[3] K. H. Ayvacı, S. Şenyurt, D. Canlı, *Some characterizations of spherical indicatrix curves generated by Flc-frame*, Turk. J. Math. Comput. Sci. **13** (2021), 379–387.
[4] P. Cermelli, A. J. Di Scala, *Constant angle surfaces in liquid crystals*, Philos. Mag. **87** (2007), 1871–1888.
[5] M. Dede, *A new representation of tubular surfaces*, Houston J. Math. **45** (2019), 707-–720.
[6] M. Dede, *Why Flc-frame is better than Frenet frame on polynomial space curves?*, Math. Sci. Appl. E-Notes **10** (2022), 190–198.
[7] K. Eren, K. H. Ayvacı, S. Şenyurt, *On characterızatıons of sphercal curves using Frenet lıke curve frame*, Honam Math. J. **44** (2022), 391-–401.
[8] K. Eren, K. H. Ayvacı, S. Şenyurt, *On ruled surfaces constructed by the evolution of a polynomial space curve*, J. Sci. Arts **23** (2023), 77–90.
[9] F. Güler, G. Şaffak, E. Kasap, *Timelike constant angle surfaces in Minkowski space R^3_1*, Int. J. Contemp. Math. Sci. **6** (2011), 2189–2200.
[10] Í. A. Güven, *Some integral curves with a new frame*, Open Math. **18** (2020), 1332–1341.
[11] S.Ö. Karakuş, *Certain constant angle surfaces constructed on curves in Minkowski 3-space*, Int. Electron. J. Geom. **11** (2018), 37–47.
[12] G. U. Kaymanlı, C. Ekici, Y. Ünlütürk, *Constant angle ruled surfaces due to the Bishop frame in Minkowski 3-space*, J. Sci. Arts **22** (2022), 105–114.
[13] Y. Li, K. Eren, K. H. Ayvacı, S. Ersoy, *Simultaneous characterizations of partner ruled surfaces using Flc-frame*, AIMS Math. **7** (2022), 20213–20229.
[14] C. Y. Li, C. G. Zhu, *Construction of the spacelike constant angle surface family in Minkowski 3-space*, AIMS Math. **5** (2020), 6341–6354.
[15] R. Lopez, M. I. Munteanu, *Constant angle surfaces in Minkowski space*, Bull. Belg. Math. Soc. Simon Stevin **18** (2011), 271–286.
[16] M. I. Munteanu, A. I. Nistor, *A new approach on constant angle surfaces in E^3*, Turkish J. Math. **33** (2009), 1–10.
[17] A. I. Nistor, *Certain constant angle surfaces constructed on curves*, Int. Electron. J. Geom. **4** (2011), 79–87.
[18] S. Özkaldı, Y. Yaylı, *Constant angle surfaces and curves in E^3*, Int. Electron. J. Geom. **4** (2011), 70-78.
[19] S. Şenyurt, K. Eren, K. H. Ayvacı, *A study on inextensible flows of polynomial curves with Flc-frame*, Appl. Appl. Math. **17** (2022), 123-133.