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A note on the orbit equivalence of injective actions

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Abstract. We characterise the groupoid C^*-algebras associated to the transformation groupoids of injective actions of discrete countable Ore semi-groups on compact topological spaces in terms of the reduced crossed product from the dual actions, and characterise the continuous orbit equivalence for injective actions by means of the transformation groupoids, as well as their reduced groupoid C^*-algebras. Finally, we characterize the injective action of semi-group on its compactifications.

1. Introduction

There are a large number of significant and interesting research dealing with the interplay between the orbit equivalence of topological dynamical systems and the classification of C^*-algebras. Groupoid theory and the crossed product construction play a very important role for these results. At the very beginning, Giordano, Putnam and Skau studied the relationship between orbit equivalence and C^*-crossed products for minimal homeomorphisms of Cantor sets in [5]. Their studies have been generalized to many different directions, including Tomiyama's results on topologically free homeomorphisms on compact Hausdorff spaces ([17]), Matsumoto et al.'s classification results of irreducible topological Markov shifts in terms of Cuntz-Krieger algebras ([10, 11]), and Li's characterization of group actions and partial dynamical systems by transformation groupoids and their reduced crossed product C^*-algebras ([8, 9]), and so on ([2, 6, 12]).

In [15], Renault and Sundar studied actions of locally compact Ore semi-groups on compact topological spaces. They gave the construction of the semi-direct product $X \rtimes P$, where X is the order compactification of locally compact semi-group P, and turned out that it is a locally compact groupoid and has a continuous Haar system. They also proved that this groupoid is a reduction of a semi-direct product by a group and the Wiener-Hopf C^*-algebra of P is isomorphic to the reduced C^*-algebra of the semi-direct product groupoid $X \rtimes P$. In [4], Ge studied the compactification of natural numbers and characterized the associated compact Hausdorff spaces. Inspired by their beautiful results, in this paper, we will consider injective actions of discrete countable Ore semi-groups on compact topological spaces, and study the relationship between the orbit structure of these actions and algebraic structure of the associated groupoids and their C^*-algebras. We prove that two topologically free injective actions are continuously orbit equivalent if and only if their transformation groupoids are isomorphic as étale groupoids, if and only if there is a C^*-isomorphism

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preserving the canonical Cartan subalgebras between the corresponding groupoid reduced C^*-algebras. We show that an injective action can be dilated as a group action on a quotient space by homeomorphisms, and our groupoid turns out to be a reduction of the dilation. The multiplication operation on the group can naturally give rise to an action of the group on itself, thus we consider the induced action of the semi-group on its compactifications. We also show that the injective action of a semi-group on its one-point compactification is determined uniquely up to conjugacy by two conditions.

The paper is organized as follows. In Section 2, we characterize the reduced groupoid C^*-algebras associated to the transformation groupoids of injective actions of discrete countable Ore semi-groups on compact topological spaces in terms of the reduced crossed product from the dual actions. In Section 3, we introduce the notion of continuous orbit equivalence for injective actions, and characterize them in terms of the associated transformation groupoids, as well as their reduced groupoid C^*-algebras with canonical Cartan subalgebras. In Section 4, we study the action of semi-group on its compactifications.

Throughout this paper, we will use the following notions. For a topological groupoid $\mathcal{G}$, let $\mathcal{G}^{(0)}$ be the unit space. The range and source maps r, s from $\mathcal{G}$ onto $\mathcal{G}^{(0)}$ are defined by $r(g) = gg^{-1}$ and $s(g) = g^{-1}g$, respectively. If r and s are local homeomorphisms, then $\mathcal{G}$ is called to be étale. We say that $\mathcal{G}$ is *topologically principal* if $\left\{u \in \mathcal{G}^{(0)} : \mathcal{G}_u^u = \{u\}\right\}$ is dense in $\mathcal{G}^{(0)}$, where $\mathcal{G}_u^u = \{\gamma \in \mathcal{G} : r(\gamma) = s(\gamma) = u\}$ is the isotropy group at a unit $u \in \mathcal{G}^{(0)}$. We refer to [13, 16] for more details on topological groupoids and their C^*-algebras.

2. The transformation groupoid of injective action

Let X be a second-countable compact Hausdorff space, G a countable discrete group and P a right Ore sub-semigroup of G, i.e., $e \in P$ and $G = PP^{-1}$, where e is the identity element in G. By a right action θ of P on X we mean that θ_a is a continuous and injective map from X into itself and satisfies that $\theta_a\theta_b = \theta_{ba}$ for every $a, b \in P$, and $\theta_e = id_X$, the identity map on X. We use the symbol $P \curvearrowright_\theta X$ to denote such an injective action.

Each injective action $P \curvearrowright_\theta X$ induces a dual action, $P \curvearrowright_\alpha C(X)$, of P on the abelian C^*-algebra $C(X)$ by surjective $*$-homomorphisms, where each α_m is defined by $\alpha_m(f) = f\theta_m$ for $f \in C(X)$ and $\alpha_{ab} = \alpha_a\alpha_b$ for all a and b in P.

Remark 2.1. *Given an injective action $P \curvearrowright_\theta X$, for $x \in X$, let*

$$Q_x := \{g \in G : \exists a, b \in P,\ y \in X \text{ such that } g = ab^{-1} \text{ and } \theta_a(x) = \theta_b(y)\}.$$

For $x, y \in X$ and $g \in G$, one can check that, if $g = ab^{-1} = mn^{-1}$ for $a, b, m, n \in P$, then $\theta_a(x) = \theta_b(y)$ if and only if $\theta_m(x) = \theta_n(y)$. It follows that $g \in Q_x$ if and only if there exists a unique element, denoted by $u(x, g)$, in X such that if $g = ab^{-1}$ for $a, b \in P$, then $\theta_a(x) = \theta_b(u(x, g))$.

Let

$$X \rtimes P = \{(x, g) \in X \times G : g \in Q_x\}.$$

Then, under the following operations

$$(x, g)(y, h) = (x, gh) \text{ only if } y = u(x, g),$$

$$(x, g)^{-1} = (u(x, g), g^{-1}),$$

$X \rtimes P$ is a groupoid with the unit space $(X \rtimes P)^{(0)} = X \times \{e\}$, the range map, $r(x, g) = (x, e)$ and the source map $s(x, g) = (u(x, g), e)$ for $(x, g) \in X \rtimes P$.

Lemma 2.2. *The map $u : (x, g) \in X \rtimes P \to u(x, g) \in X$ is continuous, where $u(x, g)$ is defined as in Remark 2.1. Moreover, $u(x, m) = \theta_m(x)$ for $x \in X$ and $m \in P$, and $u(u(x, g), h) = u(x, gh)$ when $(x, g), (u(x, g), h) \in X \rtimes P$.*

Thus, under the relative product topology on $X \times G$, $X \rtimes P$ is a second-countable locally compact Hausdorff groupoid. Furthermore, $X \rtimes P$ is étale if and only if $\theta_a(X)$ is open in X for each $a \in P$.

Proof. Suppose that $(x_n, g_n) \to (x, g) \in X \rtimes P$. Then $g_n = g$ for large n, so we can assume that $g_n = g$ for all n. Let $y_n = u(x_n, g)$ and suppose that $y_n \to y$. Choose $a, b \in P$ such that $g = ab^{-1}$ and $\theta_a(x_n) = \theta_b(y_n)$. Then $\theta_a(x) = \theta_b(y)$, it follows that $y = u(x, g)$, proving that u is continuous.

Given (x, g) and $(u(x, g), h)$ in $X \rtimes P$, choose a, b, c, d, m and n in P such that $g = ab^{-1}$, $h = cd^{-1}$ and $b^{-1}c = mn^{-1}$. Then $bm = cn$ and $gh = am(dn)^{-1}$. It follows from $\theta_a(x) = \theta_b(u(x, g))$ and $\theta_c(u(x, g)) = \theta_d(u(u(x, g), h))$ that $\theta_{am}(x) = \theta_{dn}(u(u(x, g), h))$. Since moreover $\theta_{am}(x) = \theta_{dn}(u(x, gh))$, we have $u(u(x, g), h) = u(x, gh)$.

By the continuity of u, one checks that $X \rtimes P$ is a second-countable locally compact Hausdorff groupoid. If $X \rtimes P$ is étale, then the source map $s : X \rtimes P \to (X \rtimes P)^{(0)}, (x, g) \mapsto (u(x, g), e)$ is open. It follows that, for each $a \in P$, $s(X \times \{a\}) = \theta_a(X) \times \{e\}$ is open in $X \rtimes P$. Thus $\theta_a(X)$ is open in X for each $a \in P$.

For the converse, assume that $\theta_a(X)$ is open in X for each $a \in P$. Then $\theta_a : X \to \theta_a(X)$ is a homeomorphism. For any $(x_0, g_0) \in X \rtimes P$, there exist $m, n \in P$, $y_0 \in X$ such that $g_0 = mn^{-1}$ and $\theta_m(x_0) = \theta_n(y_0)$. Since $\theta_m(X)$ and $\theta_n(X)$ are open in X, there exist open subsets $W_1 \subseteq \theta_m(X)$ and $W_2 \subseteq \theta_n(X)$ such that $\theta_m(x_0) \in W_1$, $\theta_n(y_0) \in W_2$. Set $W = W_1 \cap W_2$, $U = \theta_m^{-1}(W)$, and $V = \theta_n^{-1}(W)$. Thus $\theta_m(x_0) = \theta_n(y_0) \in W$, U, V are open in X and $\theta_m(U) = \theta_n(V) = W$. In this case, for each $x \in U$, there exists $y \in V$ such that $\theta_m(x) = \theta_n(y)$, it follows that $(x, g_0) \in X \rtimes P$ for each $x \in U$, which implies $U \times \{g_0\}$ is an open neighbourhood of (x_0, g_0) in $X \rtimes P$. Thus the range map $r|_{U \times \{g_0\}} : U \times \{g_0\} \to U \times \{e\}$ is a homeomorphism. Hence $X \rtimes P$ is étale. □

Remark 2.3. *Throughout this paper, we always assume that $\theta_a(X)$ is open in X for each $a \in P$,and we call $X \rtimes P$ the transformation groupoid attached to $P \curvearrowright_\theta X$. In this case, the unit space $(X \rtimes P)^{(0)}$ identifies with X by identifying (x, e) with e. Thus $r(x, g) = x$ and $s(x, g) = u(x, g)$. From the proof of the last lemma, for each $(x, g) \in X \rtimes P$, there exists an open neighbourhood U of x in X such that $(x, g) \in U \times \{g\} \subseteq X \rtimes P$.*

Let $c : X \rtimes P \to G$ be defined by $c(x, g) = g$ for $(x, g) \in X \rtimes P$. Then c is a continuous homomorphism and its kernel $\ker(c) = X \times \{e\}$ is an amenable étale subgroupoid of $X \rtimes P$. It follows from [16, Proposition 10.1.11] that if G is amenable then $X \rtimes P$ is also amenable.

In [15], the transformation groupoid is isomorphic to a reduction of the Mackey range semi-direct product defined by the canonical cocycle c when P is a locally compact Ore semi-group. For the countable discrete case, in the following we can give a direct construction of this result.

Let $\widetilde{X}$ be the quotient space of $X \times G$ by the following equivalence relation:

$$(x, g) \sim (y, h) \Leftrightarrow \exists a, b \in P \text{ such that } gh^{-1} = ab^{-1} \text{ and } \theta_a(x) = \theta_b(y).$$

One can check that, under the quotient topology of the product topology on $X \times G$, $\widetilde{X}$ is a locally compact and Hausdorff space, and the canonical quotient map $\pi : X \times G \to \widetilde{X}$ is surjective, continuous and open. Denote by $[x, g]$ the equivalence class of (x, g) in the equivalence relation. Then $[x, g] = \{(u(x, k), k^{-1}g)) : k \in Q_x\}$ for $(x, g) \in X \times P$, and $[x, e] = [u, e]$ if and only if $x = u$.

Remark 2.4. *Define the right action $G \curvearrowright_{\tilde{\theta}} \widetilde{X}$ of G on $\widetilde{X}$ by homeomorphisms as follows: for $[x, g] \in \widetilde{X}, h \in G$,*

$$\tilde{\theta}_h([x, g]) = [x, gh].$$

The associated transformation groupoid, $\widetilde{X} \rtimes_{\tilde{\theta}} G := \widetilde{X} \times G$, with the product topology and the following multiplication and inverse:

$$([x, g], h)([x, gh], h') = ([x, g], hh'),\ ([x, g], h)^{-1} = ([x, gh], h^{-1}),$$

is an étale groupoid. The unit space $(\widetilde{X} \rtimes_{\tilde{\theta}} G)^{(0)}$ identifies with $\widetilde{X}$ by identifying $([x, g], e)$ with $[x, g]$. Then $r([x, g], h) = [x, g]$ and $s([x, g], h) = [x, gh]$.

Let $X' = \{[x, e] : x \in X\}$. Then X' is clopen in $\widetilde{X}$ and $\widetilde{X} \rtimes_{\tilde{\theta}} G$-full in the sense that the intersection of X' and each orbit of $G \curvearrowright_{\tilde{\theta}} \widetilde{X}$ is not empty. In fact, for any $[x, g] \in \widetilde{X}$, $([x, g], g^{-1}) \in r^{-1}([x, g]) \cap s^{-1}(X')$. Note that the reduction $\widetilde{X} \rtimes_{\tilde{\theta}} G|_{X'} = r^{-1}(X') \cap s^{-1}(X')$ is an étale subgroupoid of $\widetilde{X} \rtimes_{\tilde{\theta}} G$. Recall that two étale groupoids $\mathcal{G}$ and $\mathcal{H}$ are Kakutani equivalent if there are full clopen subsets $X \subseteq \mathcal{G}^{(0)}$ and $Y \subseteq \mathcal{H}^{(0)}$ such that $\mathcal{G}|_X \cong \mathcal{H}|_Y$ ([3]). The following proposition is based on the ideas of [15] by Renault and Sundar.

Proposition 2.5. *$X \rtimes P$ is isomorphic to the reduction $\widetilde{X} \rtimes_{\tilde{\theta}} G|_{X'}$. Consequently, $X \rtimes P$ is Kakutani equivalent to $\widetilde{X} \rtimes_{\tilde{\theta}} G$.*

Proof. Note that $r([x,e],g) = [x,e] \in X'$ and $s([x,e],g) = [x,g] = [u(x,g),e] \in X'$ for each $(x,g) \in X \rtimes P$. We can therefore define a map

$$\Phi : X \rtimes P \to \widetilde{X} \rtimes_{\tilde{\theta}} G|_{X'},\ (x,g) \mapsto ([x,e],g),$$

and Φ is clearly an injective groupoid homomorphism. For a given $(y,g) \in \widetilde{X} \rtimes_{\tilde{\theta}} G|_{X'}$, since $r(y,g) = y \in X'$, there exists $x \in X$ such that $y = [x,e]$. In this case, $s([x,e],g) = [x,g] \in X'$, which implies $(x,g) \in X \rtimes P$, proving that Φ is surjective.

Let φ be the restriction of Φ to the unit space $(X \rtimes P)^{(0)} = X$, one can check that $\varphi : x \in X \to [x,e] \in X'$ is a homeomorphism. It is then easy to see that Φ is a homeomorphism from $X \rtimes P$ onto $\widetilde{X} \rtimes_{\tilde{\theta}} G|_{X'}$. □

Remark 2.6. *Given an injective right action $P \curvearrowright_\theta X$, we further assume that each map θ_m is a homeomorphism on X. For each $g \in G$, it follows from the assumption that there exist $m,n \in P$ such that $g = mn^{-1}$. Define*

$$\hat{\theta}_g(x) = \theta_n^{-1}(\theta_m(x)) \ \text{for } x \in X.$$

One can check that $\hat{\theta}$ is a right action of G on X by homeomorphisms. In this case, the equivalence relation on $X \times G$ defined before Remark 2.4 can be rewritten as follows: for $(x,g),(y,h) \in X \times G$,

$$(x,g) \sim (y,h) \Leftrightarrow y = \hat{\theta}_{gh^{-1}}(x).$$

Thus $[x,g] = [\hat{\theta}_g(x),e]$ for $[x,g] \in \widetilde{X}$, and $\widetilde{X} = X'$.

The transformation groupoid $X \rtimes_{\hat{\theta}} G$ associated to the above group action $(X,G,\hat{\theta})$ is given by the set $X \times G$ with the product topology, multiplication $(x,g)(y,h) = (x,gh)$ if $y = \hat{\theta}_g(x)$, and inverse $(x,g)^{-1} = (\hat{\theta}_g(x),g^{-1})$.

Remark 2.4 and Remark 2.6 combine to give the following result.

Corollary 2.7. *If $P \curvearrowright_\theta X$ is a right action by homeomorphisms, then $X \rtimes P$ is isomorphic to $\widetilde{X} \rtimes_{\tilde{\theta}} G$, and both of them are isomorphic to $X \rtimes_{\hat{\theta}} G$.*

From Proposition 2.5 and Corollary 2.7, the reduced groupoid C^*-algebra $C_r^*(X \rtimes P)$ of $X \rtimes P$ is Morita equivalent to the reduced crossed product C^*-algebra $C(\widetilde{X}) \rtimes_{\tilde{\theta}} G$ associated to $G \curvearrowright_{\tilde{\theta}} \widetilde{X}$, and these two C^*-algebras are isomorphic when θ is a homeomorphism action. In the rest of this section, we characterize the relationship among $C_r^*(X \rtimes P)$, the C^*-algebra from the dual action $P \curvearrowright_\alpha C(X)$ and partial action of G on $C(X)$ given by $P \curvearrowright_\theta X$.

We define $l^2(G,C(X))$ to be the set of all mapping ξ from G into $C(X)$ such that $\sum_{g\in G} |\xi(g)|^2$ converges in $C(X)$. Then it is a (right) Hilbert $C(X)$-module under the following operations:

$$(\xi f)(g) = \xi(g)f,\ <\xi,\eta>= \sum_{g\in G} \xi(g)^*\eta(g)$$

for $f \in C(X), \xi,\eta \in l^2(G,C(X)), g \in G$. Similarly, we have the Hilbert $C(X)$-module $E := l^2(P,C(X))$ and let $\mathcal{L}(E)$ be the C^*-algebra of all adjointable operators on E.

Define representations $\pi : f \in C(X) \to \pi(f) \in \mathcal{L}(E)$ and $v : m \in P \to v_m \in \mathcal{L}(E)$ by

$$\pi(f)(\xi)(m) := \alpha_m(f)\xi(m),\ \ v_m(\xi)(n) := \widetilde{\xi}(nm^{-1}) \quad \text{for } \xi \in \mathcal{L}(E), m,n \in P,$$

where $\widetilde{\xi} \in l^2(G,C(X))$ is given by $\widetilde{\xi}(g) = \begin{cases} \xi(g), & \text{for } g \in P \\ 0, & \text{for otherwise} \end{cases}$ for $g \in G$. Then $v_m^*\xi(n) = \xi(nm)$ for $m,n \in P$, $\xi \in \mathcal{L}(E)$. One can check that $v_e = I$, v_m is an isometry, $v_m v_n = v_{nm}$ and $\pi(f)v_m = v_m\pi(\alpha_m(f))$ for $f \in C(X), m,n \in P$. The C^*-algebra generated by $\{\pi(f), v_m : f \in C(X), m \in P\}$ in $\mathcal{L}(E)$ is the reduced crossed product associated with $P \curvearrowright_\alpha C(X)$, denoted by $C(X) \rtimes_r P$.

To simplify symbol, write $\mathcal{G} = X \rtimes P$. Let $l^2(\mathcal{G})$ be a Hilbert right $C(X)$-module by the completion of $C_c(\mathcal{G})$ under the following operations:

$$(\xi f)(x,g) = \xi(x,g)f(x) \text{ for } \xi \in C_c(\mathcal{G}), f \in C(X), (x,g) \in \mathcal{G}.$$

$$< \xi, \eta > (x) = \sum_{g \in G, (x,g) \in \mathcal{G}} \overline{\xi(x,g)}\eta(x,g) \text{ for } \xi, \eta \in C_c(\mathcal{G}), x \in X.$$

$$\|\xi\| = \sup_{x \in X}(\sum_{g \in G, (x,g) \in \mathcal{G}} |\xi(x,g)|^2)^{\frac{1}{2}} \text{ for } \xi \in C_c(\mathcal{G}).$$

Let $\mathcal{L}(l^2(\mathcal{G}))$ be the C^*-algebra of all adjointable operators on $l^2(\mathcal{G})$. Define the representation $\widetilde{\pi}$ of $C_c(\mathcal{G})$ into $\mathcal{L}(l^2(\mathcal{G}))$ by $\widetilde{\pi} : f \in C_c(\mathcal{G}) \to \widetilde{\pi}(f) \in \mathcal{L}(l^2(\mathcal{G}))$:

$$(\widetilde{\pi}(f)\xi)(x,g) = \sum_{(x,h) \in \mathcal{G}} f(u(x,g), g^{-1}h)\xi(x,h) \text{ for } (x,g) \in \mathcal{G}.$$

Then the reduced groupoid C^*-algebra, $C_r^*(\mathcal{G})$, of $\mathcal{G}$ is the C^*-algebra generated by $\widetilde{\pi}(C_c(\mathcal{G}))$ in $\mathcal{L}(l^2(\mathcal{G}))$ ([1]).

Lemma 2.8. *Let $P \curvearrowright_\theta X$ be an injective action. For $g \in G$, let $X_g = \{x \in X : (x,g) \in X \rtimes P\}$ and $U_g = X_g \times \{g\}$. Then the characteristic function, denoted by u_g, on U_g is in $C_c(X \rtimes P)$. Moreover, the following statements hold:*

(i) u_e is the identity element in $C_c(X \rtimes P)$, u_a^* is an isometry and $u_a u_b = u_{ab}$ for $a, b \in P$;
(ii) for $g \in G$, if $g = ab^{-1}$ for $a, b \in P$, then $u_g = u_a u_b^*$ and $u_g^* = u_{g^{-1}}$;
(iii) for $f \in C(X)$ and $g \in G$, we have $u_g f = V_g(f)u_g$, where for $x \in X$,
$$V_g(f)(x) = \begin{cases} f(u(x,g)), & \text{if } x \in X_g \\ 0, & \text{for otherwise.} \end{cases}$$
(iv) $u_g u_{g^{-1}} = \chi_{X_g} \in C(X)$ and $u_g f u_{g^{-1}} = V_g(f)u_g u_{g^{-1}}$.

Hence $C_c(X \rtimes P) = span\{fu_g : f \in C(X), g \in G\}$.

Proof. Note that for $g \in G$, U_g is an open and compact subset of $X \rtimes P$, thus $u_g \in C_c(X \rtimes P)$. By calculation, we can check the properties stated in the lemma. For $\xi \in C_c(X \rtimes P)$, there exist $g_1, g_2, \cdots, g_n \in G$ such that the support $supp(\xi)$ of ξ is contained in $\cup_{i=1}^n U_{g_i}$. By a partition of unity, we have $\xi = \sum_{i=1}^n \xi_i$ for $\xi_i \in C_c(X \rtimes P)$ and $supp(\xi_i) \subseteq U_{g_i}$. Let

$$f_i(x) = \begin{cases} \xi_i(x, g_i), & \text{if } x \in X_{g_i} \\ 0, & \text{for otherwise} \end{cases}$$

for $x \in X$. Then $f_i \in C(X)$ and $\xi_i = f_i u_{g_i}$ for each i. Thus $C_c(X \rtimes P) = span\{fu_g : f \in C(X), g \in G\}$. □

Theorem 2.9. *Let $\mathcal{M}$ be the closure of the set $\{\xi \in C_c(X \rtimes P) : \xi(x,g) = 0 \text{ if } g \notin P\}$ in $l^2(X \rtimes P)$ and Q be the projection from $l^2(X \rtimes P)$ onto $\mathcal{M}$. Then $\mathcal{M}$ is isomorphic to $l^2(P, C(X))$, and $C(X) \rtimes_r P$ is isomorphic to the C^*-algebra generated by $QC_r^*(X \rtimes P)Q$ in $\mathcal{L}(l^2(X \rtimes P))$.*

Proof. We use the notation in Lemma 2.8 and before. Obviously, $\mathcal{M}$ is a (right) $C(X)$-submodule of $l^2(\mathcal{G})$. Define

$$\Lambda : \mathcal{M} \to l^2(P, C(X)), \quad \Lambda(\zeta)(m)(x) = \zeta(x,m),$$

for $\zeta \in \mathcal{M}$, $m \in P$ and $x \in X$. Then Λ is a bijective bounded $C(X)$-linear mapping with inverse

$$\Lambda^{-1}(\varepsilon)(x,g) = \begin{cases} \varepsilon(g)(x), & \text{if } g \in P, \\ 0, & \text{for otherwise,} \end{cases} \text{ for } \varepsilon \in l^2(P, C(X)), (x,g) \in \mathcal{G}.$$

Moreover, for $\zeta_1, \zeta_2 \in \mathcal{M}$, one can check that $< \zeta_1, \zeta_2 > (x) =< \Lambda\zeta_1, \Lambda\zeta_2 > (x)$ for each $x \in X$, then $\mathcal{M}$ is isomorphic to $l^2(P, C(X))$.

Define

$$W(\eta)(m)(x) = \eta(x, m), \text{ for } \eta \in C_c(\mathcal{G}), m \in P, x \in X,$$

and

$$U(\xi)(x, g) = \begin{cases} \xi(g)(x), & \text{if } g \in P, \\ 0, & \text{for otherwise,} \end{cases} \text{ for } \xi \in C_c(P, C(X)), (x, g) \in \mathcal{G}.$$

Then W and U can be extended to operators in $\mathcal{L}(l^2(\mathcal{G}), l^2(P, C(X)))$ and $\mathcal{L}(l^2(P, C(X)), l^2(\mathcal{G}))$, and if we use the same symbols to denote their extensions then $U^* = W$ and $U^*U = id$, the identity element in $\mathcal{L}(l^2(P, C(X)))$. An easy calculation confirms that $U\pi(f) = \widetilde{\pi}(f)U$ for $f \in C(X)$, and $Uv_m = \widetilde{\pi}(u_m^*)U$ for $m \in P$.

Write $Q = UU^*$. Define the map $\Phi : \mathcal{L}(l^2(P, C(X))) \to \mathcal{L}(l^2(\mathcal{G}))$ by

$$\Phi(T) = UTU^*.$$

Then Φ is an injective $*$-homomorphism, and $\Phi(\pi(f)) = Q\widetilde{\pi}(f)Q$, $\Phi(v_m) = Q\widetilde{\pi}(u_m^*)Q$ for $f \in C(X), m \in P$. One can check that $Q\widetilde{\pi}(u_m)\widetilde{\pi}(u_n^*)Q = Q\widetilde{\pi}(u_m)Q\widetilde{\pi}(u_n^*)Q$, $Q\widetilde{\pi}(f)\widetilde{\pi}(u_g)Q = Q\widetilde{\pi}(f)Q\widetilde{\pi}(u_g)Q$ for each $f \in C(X)$, $m, n \in P$ and $g \in G$. Thus it follows from Lemma 2.8 that $\Phi(C(X) \rtimes P)$ is just the C^*-algebra generated by $QC_r^*(X \rtimes P)Q$ in $\mathcal{L}(l^2(\mathcal{G}))$. □

We adopt notations in Lemma 2.8 and define a mapping $\hat{\alpha}_g : C(X_{g^{-1}}) \to C(X_g)$ as follows:

$$\hat{\alpha}_g(f)(x) = f(u(x, g)), \text{ for } f \in C(X_{g^{-1}}), x \in X_g.$$

The preceding descriptions imply that $\hat{\alpha}_g$ is an isomorphism. Then $\{\hat{\alpha}_g\}_{g \in G}$ defines a partial action of G by partial isomorphism of $C(X)$, and $(C(X), G, \hat{\alpha})$ is a partial C^*-dynamical system in the sense of [7].

Consider now the Hilbert (right) $C(X)$-submodule $F = \{\xi \in l^2(G, C(X)) : \xi(g) \in C(X_g)\}$. Denoted by $\mathcal{L}(F)$ be the C^*-algebra of all adjointable operators on F. Define the representation $\tau : f \in C(X) \to \tau(f) \in \mathcal{L}(F)$ and $v : g \in G \to v_g \in \mathcal{L}(F)$ by

$$\tau(f)\xi(g) = \hat{\alpha}_g(f)\xi(g)|_{X_g}, \ v_h\xi(g) = \xi(gh)|_{X_g \cap X_{gh}}, \text{ for } \xi \in \mathcal{L}(F), g, h \in G.$$

Then v_g is a partial isometry on F with initial space $[\tau(C(X_{g^{-1}}))F]$ and final space $[\tau(C(X_g))F]$ such that

(i) $v_g\tau(f)v_{g^{-1}} = \tau(\hat{\alpha}_g(f))$ for $f \in C(X_{g^{-1}})$;
(ii) $\tau(f)[v_gv_h - v_{gh}] = 0$ for $f \in C(X_g) \cap C(X_{gh})$;
(iii) $v_g^* = v_{g^{-1}}$.

Hence (τ, v, F) is a covariant representation of $(C(X), G, \hat{\alpha})$.

The reduced partial crossed product, denoted by $C(X) \rtimes_r^{\widehat{\alpha}} G$, associated with $(C(X), G, \hat{\alpha})$ is defined as the C^*-algebra generated by $\{\tau(f), v_g : f \in C(X), g \in G\}$ in $\mathcal{L}(F)$.

Theorem 2.10. *$C_r^*(X \rtimes P)$ is isomorphic to $C(X) \rtimes_r^{\widehat{\alpha}} G$.*

Proof. Define

$$\Phi(\eta)(g)(x) = \begin{cases} \eta(x, g) & \text{if } x \in X_g, \\ 0, & \text{for otherwise,} \end{cases} \text{ for } \eta \in C_c(\mathcal{G}), g \in G, \text{and } x \in X.$$

Then Φ can be extended to operator in $\mathcal{L}(l^2(\mathcal{G}), F)$, and we use the same symbol to denote its extension. Moreover, Φ is an adjointable unitary operator in $\mathcal{L}(l^2(\mathcal{G}), F)$ with $\Phi^*(\xi)(x, g) = \xi(g)(x)$ for $\xi \in F, (x, g) \in \mathcal{G}$.

Define $\Psi : T \in \mathcal{L}(l^2(\mathcal{G})) \to \Psi(T) \in \mathcal{L}(F)$ by

$$\Psi(T) = \Phi T \Phi^*.$$

Then Ψ is an $*$-isomorphism, and by calculation, $\Psi(\widetilde{\pi}(f)) = \tau(f)$, $\Psi(\widetilde{\pi}(u_g)) = v_g$, thus $C_r^*(X \rtimes P)$ is isomorphic to $C(X) \rtimes_r^{\widehat{\alpha}} G$. □

3. Continuous orbit equivalence

Let $P \curvearrowright_\theta X$ be an injective action and G a countable group containing P as in Section 2. Define

$$x \sim_\theta y \Leftrightarrow \exists g \in Q_x \text{ such that } y = u(x, g).$$

Then $\sim_\theta$ is an equivalent relation on X. We denote by $[x]_\theta$ the equivalence class of x, i.e., $[x]_\theta := \{u(x, g) : g \in Q_x\}$.

Given two injective actions $P \curvearrowright_\theta X$ and $S \curvearrowright_\rho Y$, we let G and H be two related discrete groups satisfying that $P \subseteq G$, $S \subseteq H$ and the assumption in Section 2.

Definition 3.1. *Let $P \curvearrowright_\theta X$ and $S \curvearrowright_\rho Y$ be two injective actions.*

(i) We say they are conjugate *if there exist a homeomorphism $\varphi : X \to Y$ and a semi-group isomorphism $\alpha : P \to S$ such that $\varphi\theta_m = \rho_{\alpha(m)}\varphi$ for each $m \in P$.*

(ii) We say they are orbit equivalent *if there exists a homeomorphism $\varphi : X \to Y$ such that $\varphi([x]_\theta) = [\varphi(x)]_\rho$ for $x \in X$.*

If $P \curvearrowright_\theta X$ and $S \curvearrowright_\rho Y$ are orbit equivalent via a homeomorphism φ, then for each $x \in X, g \in Q_x$, there exists $h \in Q_{\varphi(x)}$ (depending on x and g) such that $\varphi(u(x, g)) = u(\varphi(x), h)$. Symmetrically, for each $y \in Y, h \in Q_y$, there exists $g \in Q_{\varphi^{-1}(y)}$ (depending on y and h) such that $\varphi^{-1}(u(y, h)) = u(\varphi^{-1}(y), g)$. Note that $\cup_{x\in X}\{x\} \times Q_x = X \rtimes P$, we have the following continuous version of orbit equivalence.

Definition 3.2. *We say two injective actions $P \curvearrowright_\theta X$ and $S \curvearrowright_\rho Y$ are* continuously orbit equivalent *if there exist a homeomorphism $\varphi : X \to Y$, continuous mappings $a : X \rtimes P \to H$ and $b : Y \rtimes S \to G$ such that*

$$\varphi(u(x, g)) = u(\varphi(x), a(x, g)) \text{ for } (x, g) \in X \rtimes P, \tag{1}$$

$$\varphi^{-1}(u(y, h)) = u(\varphi^{-1}(y), b(y, h)) \text{ for } (y, h) \in Y \rtimes S. \tag{2}$$

Proposition 3.3. *If two injective actions $P \curvearrowright_\theta X$ and $S \curvearrowright_\rho Y$ are conjugate, then they are continuously orbit equivalent.*

Proof. Assume that $P \curvearrowright_\theta X$ and $S \curvearrowright_\rho Y$ are conjugate by maps φ and α. For $g \in G$, there exist $a, b \in P$ such that $g = ab^{-1}$. One can check that $\beta : g \in G \to \alpha(a)\alpha(b)^{-1} \in H$ is a well-defined group isomorphism. Define $a(x, g) = \beta(g), b(y, h) = \beta^{-1}(h)$ for $(x, g) \in X \rtimes P$, $(y, h) \in Y \rtimes S$.

For each $(x, g) \in X \rtimes P$, there exist $m, n \in P$ such that $g = mn^{-1}$ and $\theta_m(x) = \theta_n(u(x, g))$. Then $\rho_{\alpha(m)}(\varphi(x)) = \varphi(\theta_m(x)) = \varphi(\theta_n(u(x, g))) = \rho_{\alpha(n)}(\varphi(u(x, g)))$, which implies $\varphi(u(x, g)) = u(\varphi(x), a(x, g))$. In a similar way, we can show that $\varphi^{-1}(u(y, h)) = u(\varphi^{-1}(y), b(y, h))$ for $(y, h) \in Y \rtimes S$. Thus $P \curvearrowright_\theta X$ and $S \curvearrowright_\rho Y$ are continuously orbit equivalent. □

Following [8], we define the topological freeness as follows:

Definition 3.4. *An injective action $P \curvearrowright_\theta X$ is called topologically free if $\{x \in X : x \neq u(x, g)$ for all $g \in Q_x$ with $g \neq e\}$ is dense in X.*

One can check that an injective action $P \curvearrowright_\theta X$ is topologically free if and only if the groupoid $X \rtimes P$ is topologically principal.

Lemma 3.5. *In Definition 3.2, if injective actions $P \curvearrowright_\theta X$ and $S \curvearrowright_\rho Y$ are topologically free, then*

(i) mappings a and b are continuous cocycles;

(ii) $b(\varphi(x), a(x, g)) = g$, $a(\varphi^{-1}(y), b(y, h)) = h$ for $(x, g) \in X \rtimes P$ and $(y, h) \in Y \rtimes S$.

Proof. (i) For $(x_0, g_1), (u(x_0, g_1), g_2)$ in $X \rtimes P$, $(x_0, g_1g_2) \in X \rtimes P$. Choose $s_i, t_i \in S, h_i \in H$, $i = 1, 2, 3$, such that $h_1 = a(x_0, g_1) = s_1t_1^{-1}$, $h_2 = a(u(x_0, g_1), g_2) = s_2t_2^{-1}$ and $h_3 = a(x_0, g_1g_2) = s_3t_3^{-1}$. From the continuity of a and u, there exists an open neighbourhood U of x_0 such that $(u(x, g_1), g_2) \in X \rtimes P$ when $(x, g_1) \in X \rtimes P$, $a(x, g_1) = a(x_0, g_1)$ and $a(u(x, g_1), g_2) = a(u(x_0, g_1), g_2)$ for $x \in U$. Then $\varphi(u(x, g_1)) = u(\varphi(x), h_1)$, $\varphi(u(u(x, g_1), g_2)) = u(\varphi(u(x, g_1)), h_2)$ and $\varphi(u(x, g_1g_2)) = u(\varphi(x), h_3)$, these follow that $\rho_{s_1}(\varphi(x)) = \rho_{t_1}(\varphi(u(x, g_1)))$, $\rho_{s_2}(\varphi(u(x, g_1))) = \rho_{t_2}(\varphi(u(u(x, g_1), g_2)))$ and $\rho_{s_3}(\varphi(x)) = \rho_{t_3}(\varphi(u(x, g_1g_2)))$ for each $x \in U$. Let $t_1^{-1}s_2 = mn^{-1}$, $m, n \in S$. Then $h_1h_2 = (s_1m)(t_2n)^{-1}$ and $\rho_{s_1m}(\varphi(x)) = \rho_{t_2n}(\varphi(u(u(x, g_1)), g_2))$. Since $\rho_{s_1m}(\varphi(x)) = \rho_{t_2n}(u(\varphi(x)), h_1h_2)$, we see that $u(\varphi(x), h_1h_2) = u(\varphi(x), h_3)$ for each $x \in U$. Topological freeness of $S \curvearrowright_\rho Y$ implies $h_3 = h_1h_2$, i.e., $a(x_0, g_1g_2) = a(x_0, g_1)a(u(x_0, g_1), g_2)$. In the same way as above, we can show that b is a cocycle.

(ii) From equations (1) and (2), one sees that $u(x, g) = u(x, b(\varphi(x), a(x, g)))$ for $(x, g) \in X \rtimes P$. By the continuity of a and b, this equation holds for some open neighbourhood U of x. Topological freeness of $P \curvearrowright_\theta X$ implies $b(\varphi(x), a(x, g)) = g$. In the same way, we can show that $a(\varphi^{-1}(y), b(y, h)) = h$ for $(y, h) \in Y \rtimes S$. □

Theorem 3.6. *Let $P \curvearrowright_\theta X$ and $S \curvearrowright_\rho Y$ be two topologically free injective actions. Then the followings are equivalent:*

- *(i) $P \curvearrowright_\theta X$ and $S \curvearrowright_\rho Y$ are continuously orbit equivalent;*
- *(ii) $X \rtimes P$ and $Y \rtimes S$ are isomorphic as étale groupoids;*
- *(iii) There is a C^*-isomorphism $\Phi : C_r^*(X \rtimes P) \to C_r^*(Y \rtimes S)$ such that $\Phi(C(X)) = C(Y)$;*
- *(iv) There is a C^*-isomorphism $\Psi : C(X) \rtimes_r^{\widehat{\alpha}} G \to C(Y) \rtimes_r^{\widehat{\beta}} H$ such that $\Psi(C(X)) = C(Y)$, where $\widehat{\beta}$ is the partial action of H on $C(Y)$.*

Moreover, (ii) ⇒ (i) holds without the assumption of topological freeness.

Proof. (i) ⇒ (ii) Let φ, a and b be three maps implementing the continuous orbit equivalence of $P \curvearrowright_\theta X$ and $S \curvearrowright_\rho Y$. By Lemma 3.5, maps $X \rtimes P \to Y \rtimes S, (x, g) \mapsto (\varphi(x), a(x, g))$ and $Y \rtimes S \to X \rtimes P, (y, h) \mapsto (\varphi^{-1}(y), b(y, h))$ are continuous groupoid homomorphisms, and they are inverse to each other. Hence $X \rtimes P$ and $Y \rtimes S$ are isomorphic as étale groupoids.

(ii) ⇒ (i) Assume that $\Lambda : X \rtimes P \to Y \rtimes S$ is an isomorphism. Let φ be the restriction of Λ to the unit space X, and let $a = c\Lambda$, $b = c\Lambda^{-1}$. Then $\varphi : X \to Y$ is a homeomorphism, and $a : X \rtimes P \to H, b : Y \rtimes S \to G$ are continuous cocycles. Moreover, $\Lambda(x, g) = (\varphi(x), a(x, g))$, $\Lambda^{-1}(y, h) = (\varphi^{-1}(y), b(y, h))$. Then

$$\varphi(u(x, g)) = \Lambda(s(x, g)) = s(\Lambda(x, g)) = u(\varphi(x), a(x, g)),$$

$$\varphi^{-1}(u(y, h)) = \Lambda^{-1}(s(y, h)) = s(\Lambda^{-1}(y, h)) = u(\varphi^{-1}(y), b(y, h)).$$

Thus $P \curvearrowright_\theta X$ and $S \curvearrowright_\rho Y$ are continuously orbit equivalent. This does not use topological freeness.

The equivalence of (ii), (iii) and (iv) follows from [14] and Theorem 2.10. □

The following example comes from [15].

Example 3.7. *Denote by $\mathbb{Q}_+ = \{x \in \mathbb{Q} : x \geq 0\}$, $\mathbb{Q}_+^* = \{x \in \mathbb{Q} : x > 0\}$, $\mathbb{Q}_{\geq 1} = \{x \in \mathbb{Q} : x \geq 1\}$, and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. Let*

$$G = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a \in \mathbb{Q}_+^*, b \in \mathbb{Q} \right\}$$

be the semi-direct of the additive group $\mathbb{Q}$ by the multiplication action of $\mathbb{Q}_+^*$. Let

$$P_1 = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a \in \mathbb{Q}_{\geq 1}, b \in \mathbb{Q}_+ \right\},$$

$$P_2 = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a \in \mathbb{N}^*, b \in \mathbb{Q}_+ \right\}.$$

Then P_1, P_2 are unital semi-groups of G and $G = P_iP_i^{-1}$, $i = 1, 2$.

Let $X = [-\infty, 0] \times [0, 1]$, where $[-\infty, 0]$ is the one-point compactification of $(-\infty, 0]$. For $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \in G$ and $(x, y) \in X$, define

$$(x, y) *_\theta \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} = (\frac{x-b}{a}, \frac{y}{a}).$$

Then θ is not an action of G on X, but $P_1 \curvearrowright_\theta X$ and $P_2 \curvearrowright_\theta X$ are right injective actions. Note that both for $P_1 \curvearrowright_\theta X$ and $P_2 \curvearrowright_\theta X$, $Q_{(x,y)} = \left\{\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \in G : a \geq y, b \geq x\right\}$ for $(x, y) \in X$, and for all $g \in Q_{(x,y)}$ with g is not the identity matrix, $(x, y) \neq u((x, y), g)$. Then $P_1 \curvearrowright_\theta X$ and $P_2 \curvearrowright_\theta X$ are all topologically free.

Let $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ be in P_1 or P_2 arbitrary, observe that $X *_\theta \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} = \frac{1}{a}([-\infty, -b] \times [0, 1])$ is open in X. Then it follows that transformation groupoids $X \rtimes P_1$ and $X \rtimes P_2$ are all étale.

Proposition 3.8. *$P_1 \curvearrowright_\theta X$ and $P_2 \curvearrowright_\theta X$ are continuously orbit equivalent, but they are not conjugate. Moreover, $X \rtimes P_1$ and $X \rtimes P_2$ are isomorphic as étale groupoids.*

Proof. Since $G = P_1P_1^{-1} = P_2P_2^{-1}$, one can check that $(x, g) \in X \rtimes P_1$ if and only if $(x, g) \in X \rtimes P_2$. It is then easy to see that $X \rtimes P_1$ and $X \rtimes P_2$ are étale groupoid isomorphic and thus $P_1 \curvearrowright_\theta X$ and $P_2 \curvearrowright_\theta X$ are continuously orbit equivalent.

An easy check shows that if injective actions $P \curvearrowright_\theta X$ and $S \curvearrowright_\rho Y$ are conjugate by homeomorphism φ and semi-group isomorphism α, assume that there exists $x_0 \in X$ such that $Q_{x_0} = P$. Then $Q_{\varphi(x_0)} = S$. In this case, for $(0, 1) \in X$, note that $Q_{(0,1)} = P_1 \neq P_2$, then $P_1 \curvearrowright_\theta X$ and $P_2 \curvearrowright_\theta X$ are not conjugate. □

In the rest of this section, we discuss the classification of the injective actions and the associated group actions in Section 2 up to conjugacy, no further results have been obtained for their continuous orbit equivalence.

Proposition 3.9. *If injective actions $P \curvearrowright_\theta X$ and $S \curvearrowright_\rho Y$ are conjugate, then $G \curvearrowright_{\tilde{\theta}} \widetilde{X}$ and $H \curvearrowright_{\tilde{\rho}} \widetilde{Y}$ are conjugate.*

Proof. Let $\varphi : X \to Y$ be a homeomorphism and $\alpha : P \to S$ be a semi-group isomorphism such that $\varphi(\theta_m(x)) = \rho_{\alpha(m)}(\varphi(x))$ for $x \in X$ and $m \in P$. For $g \in G$, there exist $a, b \in P$ such that $g = ab^{-1}$. Define $\beta : g \in G \to \alpha(a)\alpha(b)^{-1} \in H$. One can check that β is a well-defined group isomorphism and $(x, g) \sim (y, h)$ in $\widetilde{X}$ if and only if $(\varphi(x), \beta(g)) \sim (\varphi(y), \beta(h))$ in $\widetilde{Y}$. We can therefore define a map

$$\widetilde{\varphi} : \widetilde{X} \to \widetilde{Y},\ [x, g] \mapsto [\varphi(x), \beta(g)]$$

and $\widetilde{\varphi}$ is bijective with inverse $\widetilde{\varphi}^{-1}$, defined by $\widetilde{\varphi}^{-1}([y, h]) = [\varphi^{-1}(y), \beta^{-1}(h)]$.

To see that $\widetilde{\varphi}$ is continuous, it suffices to show that $\widetilde{\varphi} \circ \pi$ is continuous, where $\pi : X \times G \to \widetilde{X}$ is the quotient map. Suppose $(x_n, g_n) \to (x, g)$ in $X \times G$. Then $g_n = g$ for large n, so we can assume that $g_n = g$. Hence $(\varphi(x_n), \beta(g)) \to (\varphi(x), \beta(g))$ in $Y \times H$. Since map $(y, h) \in Y \times H \to [y, h] \in \widetilde{Y}$ is continuous, we have $[\varphi(x_n), \beta(g)] \to [\varphi(x), \beta(g)]$. Thus $\widetilde{\varphi}$ is continuous. In a similar way, we can show that $\widetilde{\varphi}^{-1}$ is continuous and thus $\widetilde{\varphi}$ is a homeomorphism. Finally, for $[x, g] \in \widetilde{X}$ and $h \in G$,

$$\widetilde{\varphi}(\tilde{\theta}_h[x, g]) = \widetilde{\varphi}([x, gh]) = [\varphi(x), \beta(gh)] = \tilde{\rho}_{\beta(h)}([\varphi(x), \beta(g)]) = \tilde{\rho}_{\beta(h)}(\widetilde{\varphi}[x, g]).$$

Hence $G \curvearrowright_{\tilde{\theta}} \widetilde{X}$ and $H \curvearrowright_{\tilde{\rho}} \widetilde{Y}$ are conjugate. □

Proposition 3.10. *Let $P \curvearrowright_\theta X$ be a right action by homeomorphisms as in Remark 2.6, then $G \curvearrowright_{\hat{\theta}} X$ and $G \curvearrowright_{\tilde{\theta}} \widetilde{X}$ are conjugate.*

Proof. Define $\phi : (x, g) \in X \times G \to \hat{\theta}_g(x) \in X$ and $\varphi : x \in X \to [x, e] \in \widetilde{X}$. Then ϕ is continuous and φ is bijective. Moreover, $\varphi(\phi(x, g)) = \varphi(\hat{\theta}_g(x)) = [\hat{\theta}_g(x), e] = [x, g] = \pi(x, g)$ for $(x, g) \in X \times G$. Let U be an open subset in $\widetilde{X}$. Since $\pi^{-1}(U) = \phi^{-1}(\varphi^{-1}(U))$ is open in $X \times G$, the continuity of ϕ implies that $\varphi^{-1}(U)$ is open in X, it follows that φ is continuous. By compactness of X, we conclude that φ is a homeomorphism. Furthermore, for $x \in X, g \in G$, we have

$$\varphi(\hat{\theta}_g(x)) = [\hat{\theta}_g(x), e] = [x, g] = \widetilde{\theta}_g([x, e]) = \widetilde{\theta}_g(\varphi(x)).$$

Therefore $G \curvearrowright_{\hat{\theta}} X$ and $G \curvearrowright_{\tilde{\theta}} \widetilde{X}$ are conjugate. □

4. Injective actions on compactifications of semi-groups

In Section 2, we see that each injective right action of a semi-group can be dilated to be a group action. Recall that a compactification of a locally compact Hausdorff space Z is a compact Hausdorff space containing a dense continuous image of Z. In this section, we consider the right injective action of a semi-group on its compactifications.

Let G be a countable group, P be a right Ore sub-semigroup of G and $G = PP^{-1}$. Denote by $l^\infty(G)$ the set of all bounded complex valued functions on G. It is a unital abelian C^*-algebra. Let ρ_g be the operator on $l^\infty(G)$ such that $\rho_g(\xi)(h) = \xi(hg)$ for $\xi \in l^\infty(G)$ and $g, h \in G$. Then $G \curvearrowright_\rho l^\infty(G)$ is a group action by $*$-isomorphisms.

For any unial C^*-subalgebra $\mathcal{A}$ of $l^\infty(G)$, let $\Sigma_{\mathcal{A}}$ be the maximal ideal space of $\mathcal{A}$, or, equivalently the pure (or, multiplicative) state space of $\mathcal{A}$. Then $\Sigma_{\mathcal{A}}$ is compact Hausdorff and $\mathcal{A}$ is isomorphic to $C(\Sigma_{\mathcal{A}})$ by Gelfand-Naimark theory. Moreover, we have a map $g \in G \to \hat{g} \in \Sigma_{\mathcal{A}}$, where $\hat{g}(\xi) = \xi(g)$ for $\xi \in \mathcal{A}$, whose range is dense in $\Sigma_{\mathcal{A}}$, i.e., $\Sigma_{\mathcal{A}}$ is a compactification of G (with discrete topology). Assume that $\mathcal{A}$ is invariant under ρ, i.e. $\rho_g(\mathcal{A}) = \mathcal{A}$ for each $g \in G$. Then the automorphism action of G on $\mathcal{A}$ induces an action θ of G on $\Sigma_{\mathcal{A}}$ by homeomorphisms, defined by:

$$\theta_g(\hat{h}) = \widehat{hg} \text{ for } g \in G \text{ and } \hat{h} \in \Sigma_{\mathcal{A}}.$$

Let X be the closure of $\{\hat{a} : a \in P\}$ in $\Sigma_{\mathcal{A}}$. Then X is a compactification of P (with discrete topology). Moreover, $\theta_a(X) \subseteq X$ and the map $\theta_a : X \to X$ is injective for each $a \in P$, and $\theta_a\theta_b = \theta_{ba}$ for all $a, b \in P$, i.e., $P \curvearrowright_\theta X$ is a right injective action of P.

We know that if the above C^*-algebra $\mathcal{A}$ is countably generated then $\Sigma_{\mathcal{A}}$ is second-countable and metrizable. For $S \subset G$, let $f = \chi_S \in l^\infty(G)$ be the characteristic function on S and $\mathcal{A}_f$ be the unital C^*-algebra generated by $\{I, \rho_g(f) : g \in G\}$ in $l^\infty(G)$, where I is the unit of $l^\infty(G)$. Then $\mathcal{A}_f$ is invariant under ρ. Let $\Sigma_{\mathcal{A}_f}$, X, and $P \curvearrowright_\theta X$ be as in the above paragraph. Let Y be the closure of $\{\widehat{g} : g \in S\}$ in $\Sigma_{\mathcal{A}_f}$. For $\gamma \in \Sigma_{\mathcal{A}_f}$, we let

$$A_\gamma = \{h \in G : \gamma(\rho_{h^{-1}}(f)) = 1\} = \{h \in G : \gamma(\chi_{Sh}) = 1\}.$$

In particular, $A_{\widehat{g}} = S^{-1}g$ for $g \in G$. We consider the shift action β of G on $\{0, 1\}^G$ by

$$\beta_g(\xi)(h) = \xi(hg^{-1}), \text{ for } g, h \in G, \xi \in \{0, 1\}^G.$$

One can check the map $\pi : \gamma \in \Sigma_{\mathcal{A}_f} \to \chi_{A_\gamma} \in \{0, 1\}^G$ is continuous, injective and G-equivariant, i.e., $\beta_g\pi = \pi\theta_g$ for each $g \in G$. Put

$$\widetilde{\Sigma} = \pi(\Sigma_{\mathcal{A}_f}),\ \widetilde{X} = \pi(X), \widetilde{Y} = \pi(Y).$$

Remark that $\widetilde{Y}$ is the closure of $\{\chi_{S^{-1}h} : h \in S\}$ in $\{0, 1\}^G$.

Theorem 4.1. *Assume that $P \subseteq S \subseteq G$ and $SP \subseteq S$. Then $P \curvearrowright_\theta Y$ is an injective right action of P on Y whose transformation groupoid $Y \rtimes P$ is étale. In particular, when $S = P$, $P \curvearrowright_\theta X$ is conjugate to the right action of P on the order compactification of P.*

Proof. Since $\theta_a(\widehat{h}) = \widehat{ha} \in Y$ for $a \in P$ and $h \in S$, it follows from the assumption that $\theta_a(Y) \subseteq Y$. Thus $P \curvearrowright_\theta Y$ is an injective right action, and $P \curvearrowright_\theta Y$ and $P \curvearrowright_\beta \widetilde{Y}$ are conjugate. Next we show that $\theta_a(Y)$ is open in Y for each $a \in P$.

We claim that $\beta_a(\widetilde{Y}) = \{\xi \in \widetilde{Y} : \xi(a) = 1\}$ for $a \in P$.

In fact, fix $a \in P$, for $\xi \in \beta_a(\widetilde{Y}) \subseteq \widetilde{Y}$, choose $\eta \in \widetilde{Y}$ with $\xi = \beta_a(\eta)$. Since $e \in S^{-1}h$ for each $h \in S$, we have $\chi_{S^{-1}h}(e) = 1$ for each $h \in S$, thus $\eta(e) = 1$. Hence $\xi(a) = \beta_a(\eta)(a) = \eta(e) = 1$.

On the other hand, for $\xi \in \widetilde{Y}$ with $\xi(a) = 1$, we choose $\{a_n\} \subset S$ with $\chi_{S^{-1}a_n} \to \xi$ in $\{0,1\}^G$. So $\chi_{S^{-1}a_n}(a) \to \xi(a) = 1$. It follows that there exists N such that $a \in S^{-1}a_n$ for every $n \geq N$. Choose $b_n \in S$ such that $a = b_n^{-1}a_n$ for $n \geq N$. Also since $\widetilde{Y}$ is compact, we can assume that $\{\chi_{S^{-1}b_n}\}$ converges to ς in $\widetilde{Y}$. It follows from the continuity of the action β that $\beta_a(\chi_{S^{-1}b_n}) \to \beta_a(\varsigma)$, which implies that $\chi_{S^{-1}b_n a} \to \beta_a(\varsigma)$. Thus $\xi = \beta_a(\varsigma)$. We finish the claim.

From the claim, we have $\beta_a(\widetilde{Y}) = \widetilde{Y} \cap \{\xi \in \{0,1\}^G : \xi(a) = 1\}$, thus $\beta_a(\widetilde{Y})$ is open in $\widetilde{Y}$. Consequently, $\theta_a(Y)$ is open in Y for each $a \in P$, and transformation groupoid $Y \rtimes P$ is étale. From the above proof and the argument to the order compactification of P in [15], when $S = P$, $P \curvearrowright_\theta X$ is conjugate to the right action of P on the order compactification of P. □

Let $P_\infty = P \cup \{\infty\}$ be the one-point compactification of P. We consider the right action $P \curvearrowright_\sigma P_\infty$, defined by

$$\sigma_a(b) = ba, \ \sigma_a(\infty) = \infty, \quad \text{for } a, b \in P.$$

Remark that $\sigma_a(P_\infty) = Pa \cup \{\infty\}$ is open in P_∞ when $P \setminus Pa$ is finite for each $a \in P$, in this case the transformation gropoid associated to the injective action is étale. For a complex-valued function ξ on G, we say that $\lim_{g\to\infty} \xi(g)$ exists if there exists a complex number λ such that, for any $\varepsilon > 0$, there exists a finite subset E of G such that $|\xi(g) - \lambda| < \varepsilon$ for all $g \notin E$. In this case, we write $\lim_{g\to\infty} \xi(g) = \lambda$.

Proposition 4.2. *Assume that $S = \{e\}$ and $f = \delta_e \in l^\infty(G)$ is the characteristic function on $\{e\}$. Then $\mathcal{A}_f = \{\xi \in l^\infty(G) : \lim_{g\to\infty} \xi(g) \text{ exists}\}$, and $\Sigma_{\mathcal{A}_f}$ and X are homeomorphic to the one-point compactifications of G and P, respectively. Moreover, the injective action $P \curvearrowright_\theta X$ is conjugate to $P \curvearrowright_\sigma P_\infty$. Thus the transformation groupoid $X \rtimes P$ is étale if and only if $P \setminus Pa$ is finite for each $a \in P$.*

Proof. Remark that $\rho_g(f) = \delta_{g^{-1}}$ for each $g \in G$. Thus $\mathcal{A}_f$ is the closure of $\mathbb{C}G$ under the supremum norm, so $\mathcal{A}_f = \{\xi \in l^\infty(G) : \lim_{g\to\infty} \xi(g) \text{ exists}\}$. One can check the map $g \in G \to \widehat{g} \in \Sigma_{\mathcal{A}_f}$ is injective. Define $\gamma_0(\xi) = \lim_{g\to\infty} \xi(g)$ for $\xi \in \mathcal{A}_f$. We have that $\gamma_0 \in \Sigma_{\mathcal{A}_f}$, $\Sigma_{\mathcal{A}_f} = \{\widehat{g} : g \in G\} \cup \{\gamma_0\}$. Thus $\Sigma_{\mathcal{A}_f}$ and X are homeomorphic to the one-point compactifications of G and P, respectively. By the action of θ, we have $P \curvearrowright_\theta X$ is conjugate to $P \curvearrowright_\sigma P_\infty$. □

From [15, Proposition 5.1], the injective action of a semi-group on its ordered compactification is determined uniquely up to conjugacy by three conditions. For infinite countable right Ore sub-semigroups P and S, respectively, of two groups G and H, one can check that, two actions $P \curvearrowright_\sigma P_\infty$ and $S \curvearrowright_\rho S_\infty$ are orbit equivalent, and they are conjugate if and only if P and S are semi-group isomorphic. Moreover, if two actions are continuously orbit equivalent then G and H are isomorphic.

Theorem 4.3. *Assume that $P \setminus Pa$ is finite for each $a \in P$. Let $P \curvearrowright_\rho X$ be an injective right action of P on a compact Hausdorff space X. If*

(i) there exists a unique x_∞ in X such that $Q_{x_\infty} = G$, and
(ii) there exists x_0 in X such that the map $a \in P \to \rho_a(x_0) \in X$ is injective and has dense range in X,

then $P \curvearrowright_\sigma P_\infty$ and $P \curvearrowright_\rho X$ are conjugate.

Proof. Define $\Lambda : P_\infty \to X$ by $\Lambda(a) = \rho_a(x_0)$ for $a \in P$, and $\Lambda(\infty) = x_\infty$. Then Λ is continuous on P. It suffices therefore to show that Λ is continuous at ∞. For an arbitrary open neighbourhood U of x_∞ in X, we only need to show the set $F = \{a \in P : \rho_a(x_0) \notin U\}$ is finite. For otherwise, if F is infinite, we can choose a sequence $\{a_n\}$ in F, where $x_n \neq x_m$ for $n \neq m$, such that $\{a_n\}$ converges to ∞ in P. By [15, Remark 4.5], the map

$z \in P_\infty \to Q_z \in \mathcal{C}(G)$ is continuous where $\mathcal{C}(G)$ is the space of all subsets of G with the Vietoris topology. Thus $Q_{a_n} = a_n^{-1}P \to Q_\infty = G$. Note that X is compact, so we can assume that $\rho_{a_n}(x_0) \to x$ in X, it follows from the continuity of the map $x \in X \to Q_x \in \mathcal{C}(G)$ that $Q_{\rho_{a_n}(x_0)} \to Q_x$. Since $\rho_{a_n}(x_0) \notin U$, we have $x \neq x_\infty$. Also since $Q_{\rho_{a_n}(x_0)} = a_n^{-1}Q_{x_0} = a_n^{-1}P \to G$, we see that $Q_x = G$, in contradiction with condition (ii). Thus F is finite. Hence Λ is continuous and thus is a homeomorphism.

Moreover, for each $a, m \in P$,

$$\rho_a\Lambda(m) = \rho_a\rho_m(x_0) = \rho_{ma}(x_0) = \Lambda(ma) = \Lambda(\sigma_a(m)).$$

We claim that $\rho_a(x_\infty) = x_\infty$ for each $a \in P$. In fact, write $\rho_a(x_\infty) = u$. Then $Q_u = Q_{\rho_a(x_\infty)} = a^{-1}Q_{x_\infty} = G$, it follows that $u = x_\infty$. Hence $\rho_a\Lambda(\infty) = \rho_a(x_\infty) = x_\infty = \Lambda(\sigma_a(\infty))$ for each $a \in P$, so we conclude that $P \curvearrowright_\sigma P_\infty$ and $P \curvearrowright_\rho X$ are conjugate. □

Example 4.4. *Let $\mathbb{Z}$ and $\mathbb{N}$ denote the (additive) group of integer numbers and its sub-semigroup of natural numbers respectively. Let $\mathbb{Q}^*_{>0}$ and $\mathbb{N}^*_{>0}$ denote the (multiplication) group of positive rational numbers and its sub-semigroup of positive integer numbers respectively. Let $\mathbb{N}^*_\infty$ and $\mathbb{N}_\infty$ denote the one-point compactification of $\mathbb{N}^*_{>0}$ and $\mathbb{N}$ respectively.*

Consider the right injective actions $\mathbb{N} \curvearrowright_\theta \mathbb{N}^*_\infty$ and $\mathbb{N}^*_{>0} \curvearrowright_\rho \mathbb{N}^*_\infty$, defined by

$$\theta_m(n) = n + m, \theta_m(\infty) = \infty, \text{ for } m \in \mathbb{N}, n \in \mathbb{N}^*_{>0}.$$

$$\rho_m(n) = nm, \rho_m(\infty) = \infty \text{ for } m, n \in \mathbb{N}^*_{>0}.$$

Then both of them are topologically free, and for $\mathbb{N} \curvearrowright_\theta \mathbb{N}^*_\infty$,

$$Q_k = \{-k+1, -k+2, -k+3, \cdots\},\ [k]_\theta = \mathbb{N}^*_{>0} \text{ for } k \in \mathbb{N}^*_{>0},$$

$$Q_\infty = \mathbb{Z},\ [\infty]_\theta = \{\infty\}.$$

And for $\mathbb{N}^*_{>0} \curvearrowright_\rho \mathbb{N}^*_\infty$,

$$Q_k = \{1/k, 2/k, 3/k, \cdots\},\ [k]_\rho = \mathbb{N}^*_{>0} \text{ for } k \in \mathbb{N}^*_{>0},$$

$$Q_\infty = \mathbb{Q}^*_{>0},\ [\infty]_\rho = \{\infty\}.$$

Similarly, we can define right injective actions $\mathbb{N} \curvearrowright_\theta \mathbb{N}_\infty$ and $\mathbb{N}^*_{>0} \curvearrowright_\rho \mathbb{N}_\infty$. Note that $\mathbb{N}^* \setminus (\mathbb{N}^* + m)$ and $\mathbb{N} \setminus (\mathbb{N} + m)$ are finite for each $m \in \mathbb{N}$, but $\mathbb{N}^* \setminus \mathbb{N}^* m$ and $\mathbb{N} \setminus \mathbb{N}m$ are not finite for each $m \in \mathbb{N}^*_{>0}$. Theoretically, we discuss the orbit equivalence of injective actions under the conditions of Remark 2.3, but for this example we only consider it in terms of the definition of orbit equivalence, and we give the following result.

Proposition 4.5. *(i) Injective actions $\mathbb{N} \curvearrowright_\theta \mathbb{N}^*_\infty$ and $\mathbb{N}^*_{>0} \curvearrowright_\rho \mathbb{N}^*_\infty$ are orbit equivalent, but they are not continuously orbit equivalent;*

(ii) Injective actions $\mathbb{N} \curvearrowright_\theta \mathbb{N}_\infty$ and $\mathbb{N}^*_{>0} \curvearrowright_\rho \mathbb{N}_\infty$ are not orbit equivalent.

Proof. (i) The identity mapping φ on $\mathbb{N}^*_\infty$ implements the orbit equivalence of $\mathbb{N} \curvearrowright_\theta \mathbb{N}^*_\infty$ and $\mathbb{N}^*_{>0} \curvearrowright_\rho \mathbb{N}^*_\infty$. From Example 4.4, we see that

$$\mathbb{N}^*_\infty \rtimes \mathbb{N} = \{(\infty, g), (k, -k+l) \mid g \in \mathbb{Z}, k, l \in \mathbb{N}^*_{>0}\},$$

$$\mathbb{N}^*_\infty \rtimes \mathbb{N}^*_{>0} = \{(\infty, g), (k, l/k) \mid g \in \mathbb{Q}^*_{>0}, k, l \in \mathbb{N}^*_{>0}\}.$$

Assume that $\mathbb{N} \curvearrowright_\theta \mathbb{N}^*_\infty$ and $\mathbb{N}^*_{>0} \curvearrowright_\rho \mathbb{N}^*_\infty$ are continuously orbit equivalent. Then there exist a homeomorphism $\varphi : \mathbb{N}^*_\infty \to \mathbb{N}^*_\infty$ and continuous maps $a : \mathbb{N}^*_\infty \rtimes \mathbb{N} \to \mathbb{Q}^*_{>0}$, $b : \mathbb{N}^*_\infty \rtimes \mathbb{N}^*_{>0} \to \mathbb{Z}$ satisfy equations (1) and (2). In this case, $\varphi(\infty) = \infty$.

For $k, l \in \mathbb{N}^*_{>0}$, since $\varphi(l) = \varphi(u(k, -k+l)) = u(\varphi(k), a(k, -k+l)) = \varphi(k)a(k, -k+l)$, we see that $a(k, -k+l) = \varphi(l)/\varphi(k)$. For any $g \in G$, by the continuity of a at (∞, g), there exists $m_g \in \mathbb{N}^*_{>0}$ with $m_g + g \geq 1$, such that $a(k, g) = \psi(g)$ for all $k \geq m_g$. Then $\psi(g) = \varphi(k+g)/\varphi(k)$. Calculation gives $\psi(n) = \psi(1)^n$ for $n \in \mathbb{Z}$, it follows that φ cannot be a homeomorphism on $\mathbb{N}^*_\infty$. Hence $\mathbb{N} \curvearrowright_\theta \mathbb{N}^*_\infty$ and $\mathbb{N}^*_{>0} \curvearrowright_\rho \mathbb{N}^*_\infty$ are not continuously orbit equivalent.

(ii) For $\mathbb{N} \curvearrowright_\theta \mathbb{N}_\infty$, note that only the orbit at ∞ is a single point set, but for $\mathbb{N}^*_{>0} \curvearrowright_\rho \mathbb{N}_\infty$, the orbits at 0 and ∞ are all sets of single point. Therefore $\mathbb{N} \curvearrowright_\theta \mathbb{N}_\infty$ and $\mathbb{N}^*_{>0} \curvearrowright_\rho \mathbb{N}_\infty$ are not orbit equivalent. □

References

[1] C. Anantharaman-Delaroche, *Purely infinite C*-algebras arising from dynamical systems*, Bull. Soc. Math. **125** (1997), 199–225.
[2] T. M. Carlsen, S. Eilers, E. Ortega, G. Restorff, *Flow equivalence and orbit equivalence for shifts of finite type and isomorphism of their groupoids*, J. Math. Anal. Appl. **469** (2019), 1088–1110.
[3] C. Farsi, A. Kumjian, D. Pask, A. Sims, *Ample groupoids: Equivalence, homology, and Matui's HK conjecture*, Münster J. of Math. **12** (2019), 411–451.
[4] L. M. Ge, *Topology of natural numbers and entropy of arithmetic functions*, in Operator Algebras and Their Applications: A Tribute to Richard V. Kadison, Contemp. Math. **671** (2016), 127–144.
[5] T. Giordano, I. F. Putnam, C. F. Skau, *Topological orbit equivalence and C*-crossed products*, J. reine angew. Math. **469** (1995), 51–111.
[6] C. J. Hou, X. Q. Qiang, *Asymptotic continuous orbit equivalence of expansive systems*, Studia Math. **259** (2021), 201–224.
[7] A. V. Lebedev, *Topologically Free Partial Actions and Faithful Representations of Crossed Products*, Funct. Anal. Appl. **39** (2003), 207–214.
[8] X. Li, *Continuous orbit equivalence rigidity*, Ergodic Theory Dynam. Systems. **38** (2018), 1543–1563.
[9] X. Li, *Partial transformation groupoids attached to graphs and semigroups*, Int. Math. Res. Not. **2017** , (2017), 5233–5259.
[10] K. Matsumoto, *Orbit equivalence of topological Markov shifts and Cuntz-Krieger algebras*, Pacific J. Math. **246**, (2010), 199–225.
[11] K. Matsumoto, H. Matui, *Continuous orbit equivalence of topological Markov shifts and Cuntz-Krieger algebras*, Kyoto J. Math. **54** (2014), 863–877.
[12] X. Q. Qiang, C. J. Hou, *Continuous orbit equivalence of semigroup actions*, Acta Math. Sin. (Engl. Ser.) **40** (2024), 1581–1598.
[13] J. Renault, *A groupoid approach to C*-algebras*, Lecture Notes in Math. 793, Springer, Berlin, 1980.
[14] J. Renault, *Cartan subalgebras in C*-algebras*, Irish Math. Soc. Bull. **61** (2008), 29–63.
[15] J. Renault, S. Sundar, *Groupoids associated to Ore semigroup actions*, J. Operator Theory. **73** (2015), 491–514.
[16] A. Sims, G. Szabó, D Willianms, Operator algebras and dynamics: groupoids, crossed products and Rokhlin dimension, Switzerland, Birkhauser, 2020.
[17] J. Tomiyama, *Topological full groups and structure of normalizers in transformation group C*-algebras*, Pacific J. Math. **173** (1996), 571–583.