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# Sequence space $bv_p^N$ of neutrosophic real numbers

**Arghyadip Debroy[a], Runu Dhar[a]**

[a]*Department of Mathematics, Maharaja BirB ikram University, Agartala-799004, Tripura, India*

**Abstract.** The main purpose of this paper is to introduce the notion of p-bounded variation neutrosophic real number sequences $bv_p^N$, for $1\leq p < \infty$. We shall provide suitable counter examples to justify the sequence space $bv_p^N$ is not symmetric. We shall also prove with suitable examples that the classes of sequences $bv_p^N$ is neither monotone nor symmetric. We shall study some of its properties like completeness, monotonicity, convergence free and symmetricity. Also, we have established some inclusion results.

## 1. Introduction:

We are facing problems in reality due to uncertainties. These uncertainties cannot always be explained by classic methods. Zadeh [19] introduced fuzzy set (FS) theory to overcome such uncertainties. But it is not sufficient to address indeterminancy as it contains membership function. Thereafter, Atanassov [1] talked on intuitionistic fuzzy sets (IFS) which deals with membership and non-membership functions. Finally neutrosophic set theory was revealed by Smarandache [13] and further investigated by Smarandache [14] where truth, falsity and indeterminancy are defined as independent of each other.

The investigation of sequence spaces was successfully carried out with the help of FS. the notion of the fuzzy number sequences $bv_p^F$ was introduced by Tripathy and Das [15]. Beside this, lots of work have been contributed by the researchers [2, 3, 6, 7, 8, 11, 16, 17, 18]. IFS was used in all areas where FS theory was developed. Park [12] defined IF metric space (IFMS) as generalization of FMSs. Kocinac et al [10] studied on some topological properties of intuitionistic 2-fuzzy n-normed linear spaces.

Bera and Mahapatra [4] defined the neutrosophic soft linear spaces (NSLSs). Later, neutrosophic soft normed linear spaces (NSNLS) has been defined by Bera and Mahapatra [5]. In [5], neutrosophic norm, Cauchy sequence in NSNLS, convexity of NSNLS, metric in NSNLS were studied. In this paper, we shall introduce the notion of the class of neutrosophic number sequences $bv_p^N$. We will study some of their basic properties. In section 2 , we will mention some known basic definitions and results for ready reference. In section 3 , we shall introduce and study basic properties of the class of neutrosophical number sequences $bv_p^N$.

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*Email addresses:* dxarghyadip@gmail.com (Arghyadip Debroy), runu.dhar@gmail.com (Runu Dhar)

## 2. Some basic definitions and results:

In this section, some basic definitions and results would be defined.

**Definition 2.1.**

[16] The neutrosophic real number is denoted by $R^N$ such that $R^N = \{(\mathrm{x},\ T_R(x), F_R(x), I_R(x)) : \mathrm{x} \in \mathbb{R}\}$ where $T_R : R \to [0,1], F_R : R \to [0,1], I_R : R \to [0,1]$.

**Definition 2.2**

A neutrosophic real number sequence $(X_k)$ is said to be bounded if $|TX_k| \leq \mu$, $|FX_k| \leq \mu$ and $|IX_k| \leq \mu$, for some $\mu \in R^*(I)$.

**Definition 2.3**

A class of sequences $E^N$ is said to be normal (or solid) if $(Y_k) \in E^N$, whenever $\bar{d}\left(TY_k, \overline{0_N}\right) \leq \bar{d}\left(TX_k, \overline{0_N}\right), \bar{d}\left(FY_k, \overline{0_N}\right) \leq \bar{d}\left(FX_k, \overline{0_N}\right)$ and $\bar{d}\left(IY_k, \overline{0_N}\right) \leq \bar{d}\left(IX_k, \overline{0_N}\right)$ for all $k \in \mathbb{N}$ and $(X_k) \in E^N$, where $\overline{0_N}$ is the real number having 1 at 0 and zero elsewhere.

**Definition 2.4**

. Let $\mathrm{K} = \{\mathrm{k}_1 < \mathrm{k}_2 < \mathrm{k}_3 \ldots\ldots\} \subseteq \mathbb{N}$ and $E^N$ be a class of sequences. A K-step set of $E^N$ is a set of sequences $\lambda_k^{E^N} = \left\{(X_k) \in W^N : (X_n)\right) \in E^N\right\}$.

**Definition 2.5**

A canonical pre-image of a sequence $(X_{k_n}) \in \lambda_k^{E^N}$ is a sequence $(Y_n) \in \omega^N$, defined as follows:

$$Y_n = \begin{cases} \mathrm{X_n}, \mathrm{n} \in \mathrm{k}; \\ \overline{0_N}, \quad \text{otherwise}. \end{cases}$$

**Definition 2.6**

A canonical pre-image of a step set $\lambda_k^{E^N}$ is a set of canonical pre-images of all elements in $\lambda_k^{E^N}$, i.e., $Y$ is in canonical pre-image $\lambda_k^{E^N}$ if and only if Y is canonical pre-image of some $X \in \lambda_k^{E^N}$.

**Definition 2.7**

A class of sequences $E^N$ is said to be monotone if $E^N$ contains the canonical pre-images of all its step sets.

**Remark 2.8**

A class of sequences $E^N$ is solid $\Rightarrow$ $E^N$ is monotone. (One may refer to Kamthan and Gupta [12]).

**Definition 2.9**

A class of sequences $E^N$ is said to be symmetric if $\left(X_{\pi(n)}\right) \in E^N$, whenever $(X_k) \in E^N$, where $\pi$ is a permutation of $\mathbb{N}$.

**Definition 2.10**

A class of sequences $E^N$ is said to be convergence free if $(Y_k) \in E^N$, whenever $(X_k) \in E^N$ and $X_k = \overline{0_N}$ implies $Y_k = \overline{0_N}$.

Throughout the article $\omega^{\mathrm{N}}, \ell_\infty^N$ and $\mathrm{c}^{\mathrm{N}}$ denote the class of all, bounded and convergent sequences of neutrosophic real numbers respectively. The class of sequences $\ell_p^N$, for $1 \leq p < \infty$ of neutrosophic real numbers is introduced as follows:

$l_p^N = \left\{\mathrm{X} = (X_k) \in \omega^{\mathrm{N}} : \sum_{\mathrm{k}=1}^{\infty} \left[\left\{\bar{d}\left(TX_k, \overline{0_N}\right)^{\mathrm{p}}\right\} + \left\{\bar{d}\left(FX_k, \overline{0_N}\right)^{\mathrm{p}}\right\} + \left\{\bar{d}\left(IX_k, \overline{0_N}\right)^{\mathrm{p}}\right\}\right] < \infty\right\}$.

## 3. Main Results:

In this section, we state and prove the results of this article.

**Definition 3.1.**

We introduce the class of $p$-bounded variation sequences of neutrosophic real numbers $bv_p^N$, for $1 \leq p < \infty$ as follows:

$bv_p^N = \left\{X = (X_k) \in \omega^N : \sum_{k=1}^{\infty} \left[\left\{\bar{d}\left(T\Delta X_k, \overline{0_N}\right)\right\}^p + \left\{\bar{d}\left(F\Delta X_k, \overline{0_N}\right)\right\}^p + \left\{\bar{d}\left(I\Delta X_k, \overline{0_N}\right)\right\}^p\right] < \infty\right\}$, where $\Delta X_k = X_k - X_{k+1}$, for all $k \in \mathbb{N}$.

**Lemma 3.2.**

$\mathbb{R}(I)$ is complete metric space with respect to the metric $\rho(X, Y) = \text{Sup}\left\{\left|X_L^\alpha - Y_L^\alpha\right|, \left|X_R^\alpha - Y_R^\alpha\right|\right\}$ for $0 \leq \alpha \leq 1$.

**Theorem 3.3.**

The class of sequences $bv_p^N, 1 \leq p < \infty$ is a complete metric space with the metric

$$p(X, Y) = \bar{d}(X_1, Y_1) + \left[\sum_{k=1}^{\infty} \left\{\bar{d}(T\Delta X_k, T\Delta Y_k)\right\}^p\right]^{1/p} + \left[\sum_{k=1}^{\infty} \left\{\bar{d}(F\Delta X_k, F\Delta Y_k)\right\}^p\right]^{1/p} + \left[\sum_{k=1}^{\infty} \left\{\bar{d}(I\Delta X_k, I\Delta Y_k)\right\}^p\right]^{1/p}$$

where $X = (X_k), Y = (Y_k) \in bv_p^N$.

**Proof.**

Let $\left(X^{(n)}\right)$ be a Cauchy sequence in $bv_p^N$, where $X^{(n)} = \left(X_k^{(n)}\right) = \left(X_1^{(n)}, X_2^{(n)}, X_3^{(n)}, \ldots\right) \in bv_p^N$, for all n $\in \mathbb{N}$.
Then, for each $0 < \varepsilon < 1$, there exists a positive integer $n_0$ such that for all $m, n \geq n_0$,

$$\rho\left(X^{(n)}, X^{(m)}\right) = \bar{d}\left(X_1^{(n)}, X_1^{(m)}\right) + \left[\sum_{k=1}^{\infty} \left\{\bar{d}\left(T\Delta X_k^{(n)}, T\Delta X_k^{(m)}\right)\right\}^p\right]^{\frac{1}{p}} + \left[\sum_{k=1}^{\infty} \left\{\bar{d}\left(F\Delta X_k^{(n)}, F\Delta X_k^{(m)}\right)\right\}^p\right]^{\frac{1}{p}} + \left[\sum_{k=1}^{\infty} \left\{\bar{d}\left(I\Delta X_k^{(n)}, I\Delta X_k^{(m)}\right)\right\}^p\right]^{1/p} < \varepsilon$$

It follows that

$$\bar{d}\left(X_1^{(n)}, X_1^{(m)}\right) < \varepsilon, \text{ for all } m, n \geq n_0 \tag{1}$$

$$\text{and } \left[\sum_{k=1}^{\infty} \left\{\bar{d}\left(T\Delta X_k^{(n)}, T\Delta X_k^{(m)}\right)\right\}^p\right]^{1/p} < \varepsilon, \text{ for all } m, n \geq n_0 \ldots\ldots \tag{2}$$

$$\Rightarrow \bar{d}\left(T\Delta X_k^{(n)}, T\Delta X_k^{(m)}\right) < \varepsilon, \text{ for all } k \in \mathbb{N} \text{ and } m, n \geq n_0 \tag{3}$$

Thus $\left(X_1^{(n)}\right)$ and $\left(\Delta X_k^{(n)}\right)$, belong to $\mathbb{R}(I)$ for all $k \in \mathbb{N}$ are Cauchy sequences in $\mathbb{R}(I)$.
Since, $\mathbb{R}(I)$ is complete, so $\left(X_1^{(n)}\right)$ and $\left(\left(\Delta X_k^{(n)}\right)\right.$, for all $k \in \mathbb{N}$ are convergent in $\mathbb{R}(I)$.

$$\text{Let } \lim_{n\to\infty} X_1^{(n)} = X_1 \tag{4}$$

and

$$\lim_{n\to\infty} \Delta X_k^{(n)} = Z_k, \text{for all k} \in \mathbb{N} \tag{5}$$

From equations (4) and (5), we have

$$\lim_{n\to\infty} X_k^{(n)} = X_k, \text{ for all } k \in \mathbb{N}.$$

Now, fix $n \geq n_0$ and let $m \to \infty$ in equations (1) and (2), we have

$$\bar{d}\left(X_1^{(n)}, X_1\right) < \varepsilon \text{ and } \left[\sum_{k=1}^{\infty} \left\{\bar{d}\left(T\Delta X_k^{(n)}, T\Delta X_k\right)\right\}^p\right]^{1/p} < \varepsilon \text{ for all } n \geq n_0 \tag{6}$$

This implies $\rho\left(X^{(n)}, X\right) < \varepsilon$, for all $n \geq n_0$.

$$\text{i.e., } X^{(n)} \to X, \text{ as } n \to \infty, \text{ where } X = (X_k).$$

Next, we show that $X \in bv_p^N$.
From equation (6), we have for all $n \geq n_0$,

$$\sum_{k=1}^{\infty} \left\{\bar{d}\left(T\Delta X_k^{(n)}, T\Delta X_k\right)\right\}^p < \infty$$

Again, for all $n \in N, X^{(n)} = \left(X_k^{(n)}\right) \in bv_p^N$

$$\Rightarrow \sum_{k=1}^{\infty} \left\{\bar{d}\left(T\Delta X_k^{(n)}, \overline{0_N}\right)\right\}^p < \infty.$$

Now for all $n \geq n_0$, we have

$$\sum_{k=1}^{\infty} \left\{\bar{d}\left(T\Delta X_k, \overline{0_N}\right)\right\}^p = \left[\sum_{k=1}^{\infty} \left\{\bar{d}\left(T\Delta X_k, T\Delta X_k^{(n)}\right)\right\}^p + \sum_{k=1}^{\infty} \left\{\bar{d}\left(T\Delta X_k^{(n)}, \overline{0_N}\right)\right\}^p\right] < \infty$$

Similarly, we can show that

$$\sum_{k=1}^{\infty} \left\{\bar{d}\left(F\Delta X_k, \overline{0_N}\right)\right\}^{\frac{1}{p}} = \left[\sum_{k=1}^{\infty} \left\{\bar{d}\left(F\Delta X_k, F\Delta X_k^{(n)}\right)\right\}^p + \sum_{k=1}^{\infty} \left\{\bar{d}\left(F\Delta X_k^{(n)}, \overline{0_N}\right)\right\}^p\right] < \infty \text{ and}$$

$$\sum_{k=1}^{\infty} \left\{\bar{d}\left(I\Delta X_k, \overline{0_N}\right)\right\}^{\frac{1}{p}} = \left[\sum_{k=1}^{\infty} \left\{\bar{d}\left(I\Delta X_k, I\Delta X_k^{(n)}\right)\right\}^p + \sum_{k=1}^{\infty} \left\{\bar{d}\left(I\Delta X_k^{(n)}, \overline{0_N}\right)\right\}^p\right] < \infty$$

Hence $X \in bv_p^N$. This proves the completeness of $bv_p^N$.

**Theorem 3.4.**

The class of sequences $bv_p^N, p > 1$ is not symmetric.

**Proof.**

The result follows from the following example.

**Example 3.5.**

Consider a sequence $(X_k) \in bv_p^F$ be defined as follows:

$$X_1(t) = \begin{cases} 1, \text{ for } -\frac{1}{2} \le t \le 0, \\ 0, \text{ otherwise} \end{cases}$$

and for $k \ge 2$,

$$X_2(t) = \begin{cases} 1, \text{ for } -\left\{\sum_{r=1}^{k-1}\left(\frac{1}{r}\right) + \frac{1}{2k}\right\} \le t \le -\sum_{r=1}^{k-1}\left(\frac{1}{r}\right) \\ 0, \text{ otherwise.} \end{cases}$$

Then $[X_1]^\alpha = \left[-\frac{1}{2}, 0\right]$
and for $k \ge 2$,

$$[X_k]^\alpha = \left[-\left\{\sum_{r=1}^{k-1}\left(\frac{1}{r}\right) + \frac{1}{2k}\right\}, -\sum_{r=1}^{k-1}\left(\frac{1}{r}\right)\right]$$

Now, for all $k \in \mathbb{N}$, $[\Delta X_k]^\alpha = \left[-\left\{\frac{1}{2k} - \frac{1}{k}\right\}, \left\{\frac{1}{k} + \frac{1}{2(k+1)}\right\}\right] = \left[\frac{1}{2k}, \left\{\frac{1}{k} + \frac{1}{2(k+1)}\right\}\right]$.
For $p > 1$ we have,
$\sum_{k=1}^{\infty}\left\{\bar{d}\left(T\Delta X_k, \overline{0_N}\right)\right\}^p = \sum_{k=1}^{\infty}\left\{\frac{1}{k} + \frac{1}{2(k+1)}\right\}^p \le 2^p \sum_{k=1}^{\infty}\left\{\frac{1}{k^p} + \frac{1}{2^p(k+1)^p}\right\} < \infty$.
Similarly, we can show that $\sum_{k=1}^{\infty}\left\{\bar{d}\left(F\Delta X_k, \overline{0_N}\right)\right\}^p < \infty$ and $\sum_{k=1}^{\infty}\left\{\bar{d}\left(I\Delta X_k, \overline{0_N}\right)\right\}^p < \infty$.
Thus, $(X_k) \in bv_p^N, p > 1$.
Let $(Y_k)$ be a rearrangement of the sequence $(X_k)$, defined by
$(Y_k) = (X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9, X_{10}, \ldots)$
i.e., $Y_K = X_{\left(\frac{k+1}{2}\right)^2}$, for k odd,

$$= X_{\left(n+\frac{k}{2}\right)^2}, \text{ for k even and } n \in N, \text{ satisfying } n(n-1) < \frac{k}{2} \le n(n+1)$$

Then for $k = 1$, we have
$[T\Delta Y_k]^\alpha = [T\Delta Y_1]^\alpha = [X_1]^\alpha - [X_2]^\alpha = [0.5, 1.25]$.
Again
for k odd with $k > 1$ and $n \in \mathbb{N}$, satisfying $n(n-1) < \frac{k+1}{2} \le n(n+1)$,

$$\begin{aligned} [T\Delta Y_k]^\alpha &= \left[X_{\left(\frac{k+1}{2}\right)^2}\right]^\alpha - \left[X_{\left(n+\frac{k+1}{2}\right.}\right]^\alpha \\ &= \left[-\left\{\sum_{r=\left(n+\frac{k+1}{2}\right)}^{\left(\frac{k+1}{2}\right)^2-1}\frac{1}{r} + \frac{1}{2\left(\frac{k+1}{2}\right)^2}\right\}, -\left\{\sum_{r=\left(n+\frac{k+1}{2}\right)}^{\left(\frac{k+1}{2}\right)^2-1}\frac{1}{r}\right\} + \frac{1}{2\left(n+\frac{k+1}{2}\right)}\right] \end{aligned}$$

and
for k even and $n \in \mathbb{N}$, satisfying $n(n-1) < \frac{k}{2} \le n(n+1)$,

$$[T\Delta Y_k]^\alpha = \left[X_{\left(n+\frac{k}{2}\right)}\right]^\alpha - \left[X_{\left(\frac{k+2}{2}\right)}\right]^\alpha = \left[\left\{\sum_{r=\left(n+\frac{k}{2}\right)}^{\left(\frac{k+1}{2}\right)^2-1}\frac{1}{r}\right\} - \frac{1}{2\left(n+\frac{k}{2}\right)}, \left\{\sum_{r=\left(n+\frac{k}{2}\right)}^{\left(\frac{k+1}{2}\right)^2-1}\frac{1}{r} + \frac{1}{2\left(\frac{k+2}{2}\right)^2}\right\}\right]$$

It is observed that the distance of $[T\Delta Y_k]^\alpha$ from $[\overline{0_N}]^\alpha$, for all (odd and even) $k \in \mathbb{N}$ is numerically greater than 0.1 .
Therefore, $\sum_{k=1}^{\infty}\left\{\bar{d}\left(T\Delta Y_k, \overline{0_N}\right)\right\}^p$ is unbounded for p > 1.
Thus $(Y_k) \notin bv_p^F, p > 1$.
Hence $bv_p^N, p > 1$ is not symmetric.

**Theorem 3.6.**

(a) $bv_q^N \subset bv_p^N$, for $1 \leq q < p < \infty$ and the inclusion is strict.
(b) $bv^N \subset bv_p^N$, for $1 < p < \infty$ and the inclusion is strict.

**Proof.**

(a) Let $(X_k) \in bv_q^N$. Then $\sum_{k=1}^{\infty}\left[\left\{\bar{d}\left(T\Delta X_k, \overline{0_N}\right)\right\}^q + \left\{\bar{d}\left(F\Delta X_k, \overline{0_N}\right)\right\}^q + \left\{\bar{d}\left(F\Delta X_k, \overline{0_N}\right)\right\}^q\right] < \infty$. Since, $T\Delta X_k \to \overline{0_N}$, as $k \to \infty$, so there exists a positive integer $n_0$ such that

$$\bar{d}\left(T\Delta X_k, \overline{0_N}\right) \leq 1, \text{ for all } k > n_0.$$

We have

$$\sum_{k=1}^{\infty}\left\{\bar{d}\left(T\Delta X_k, \overline{0_N}\right)\right\}^p = \sum_{k=1}^{n-1}\left\{\bar{d}\left(T\Delta X_k, \overline{0_N}\right)\right\}^p + \sum_{k=n_0}^{\infty}\left\{\bar{d}\left(T\Delta X_k, \overline{0_N}\right)\right\}^p \tag{7}$$

Clearly, $\sum_{k=n_0}^{\infty}\left\{\bar{d}\left(T\Delta X_k, \overline{0_N}\right)\right\}^p \leq \sum_{k=n_0}^{\infty}\left\{\bar{d}\left(T\Delta X_k, \overline{0_N}\right)\right\}^q < \infty$, for p > q
and $\sum_{k=1}^{n_0-1}\left\{\bar{d}\left(T\Delta X_k, \overline{0_N}\right)\right\}^p$ is a finite sum.
Hence (7) implies $\sum_{k=n_0}^{\infty}\left\{\bar{d}\left(T\Delta X_k, \overline{0_N}\right)\right\}^p < \infty$.
Similarly, we can show that $\sum_{k=n_0}^{\infty}\left\{\bar{d}\left(F\Delta X_k, \overline{0_N}\right)\right\}^p < \infty$ and $\sum_{k=n_0}^{\infty}\left\{\bar{d}\left(I\Delta X_k, \overline{0_N}\right)\right\}^p < \infty$.
Hence $(X_k) \in bv_p^N$ and thus $bv_q^N \subset bv_p^N$.
The strictness of the inclusion follows from the following example.

**Example 3.7.**

Consider the sequence $(X_k)$ such that $T\Delta X_k = k^{\frac{-1}{q}}$, for all $k \in \mathbb{N}$.
Then,
$\sum_{k=1}^{\infty}\left\{\bar{d}\left(T\Delta X_k, \overline{0_N}\right)\right\}^q = 1 + \left(2^{-\frac{1}{q}}\right)^q + \left(3^{-\frac{1}{q}}\right)^q + \cdots = \sum_{k=1}^{\infty}\frac{1}{k}$, which is unbounded.
Hence, we have $(X_k) \notin bv_q^N$.
But,
$\sum_{k=1}^{\infty}\left\{\bar{d}\left(T\Delta X_k, \overline{0_N}\right)\right\}^p = \left(1^{-\frac{1}{q}}\right)^p + \left(2^{-\frac{1}{q}}\right)^p + \left(3^{-\frac{1}{q}}\right)^p + \cdots = \sum_{k=1}^{\infty}\frac{1}{k^r} < \infty$, where $r = \frac{p}{q} > 1$.
Similarly,
$\sum_{k=1}^{\infty}\left\{\bar{d}\left(F\Delta X_k, \overline{0_N}\right)\right\}^p < \infty$ and $\sum_{k=1}^{\infty}\left\{\bar{d}\left(I\Delta X_k, \overline{0_N}\right)\right\}^p < \infty$.
Hence, we have $(X_k) \in bv_p^N$.
Thus, the inclusion is strict.
(b) Let $(X_k) \in bv^N$. Then $\sum_{k=1}^{\infty}\left\{\bar{d}\left(T\Delta X_k, \overline{0_N}\right)\right\} < \infty$.
Since $T\Delta X_k \to \overline{0_N}$, as k → ∞,
so there exists a positive integer $n_0$ such that

$$\bar{d}\left(T\Delta X_k, \overline{0_N}\right) \leq 1, \text{ for all } k > n_0$$

We have

$$\sum_{k=1}^{\infty}\left\{\bar{d}\left(T\Delta X_k, \overline{0_N}\right)\right\}^p = \sum_{k=1}^{n-1}\left\{\bar{d}\left(T\Delta X_k, \overline{0_N}\right)\right\}^p + \sum_{k=n_0}^{\infty}\left\{\bar{d}\left(T\Delta X_k, \overline{0_N}\right)\right\}^p \tag{8}$$

Clearly,
$\sum_{k=n_0}^{\infty}\left\{\bar{d}\left(T\Delta X_k, \overline{0_N}\right)\right\}^p \leq \sum_{k=n_0}^{\infty}\left\{\bar{d}\left(T\Delta X_k, \overline{0_N}\right)\right\}^p < \infty$
and

$\sum_{k=1}^{n_0-1}\left\{\bar{d}\left(T\Delta X_k,\overline{0_N}\right)\right\}^p$, is a finite sum.
Hence, equation (8) implies $\sum_{k=n_0}^{\infty}\left\{\bar{d}\left(T\Delta X_k,\overline{0_N}\right)\right\}^p<\infty$. This implies $(X_k)\in bv_p^N$ and thus $bv^N\subset bv_p^N$.
The strictness of the inclusion follows from the following example.

**Example 3.8.**
Consider the sequence $(X_k)\in bv_p^N$, where $\mathrm{T}X_k=FX_k=IX_k$ defined by

$$TX_1(t)=\begin{cases}1, \text{ for } -\frac{1}{\mathrm{q}}\leq \mathrm{t}\leq 0\\ 0, \text{ otherwise}\end{cases}$$

and for $k\geq 2$,

$$TX_k(t)=\begin{cases}1, \text{ for } -\left\{\sum_{r=1}^{k-1}\left(\frac{1}{r}\right)+\frac{1}{2k}\right\}\leq t\leq -\sum_{r=1}^{k-1}\left(\frac{1}{r}\right)\\ 0, \text{ otherwise}\end{cases}$$

Then

$$\sum_{k=1}^{\infty}\left\{\bar{d}\left(T\Delta X_k,\overline{0_N}\right)\right\}=\sum_{\mathbf{k=1}}^{\infty}\left\{\frac{\mathbf{1}}{\mathbf{k}}+\frac{\mathbf{1}}{\mathbf{2(k+1)}}\right\}, \text{ which is unbounded.}$$

Hence, $(X_k)\notin bv^N$.
But, $\sum_{k=1}^{\infty}\left\{\bar{d}\left(T\Delta X_k,\overline{0_N}\right)\right\}^p=\sum_{k=1}^{\infty}\left\{\frac{1}{k}+\frac{1}{2(k+1)}\right\}^p\leq 2^p\sum_{k=1}^{\infty}\left\{\frac{1}{k^p}+\frac{1}{2^p(k+1)^p}\right\}<\infty$.
Similarly, we can get
$\sum_{k=1}^{\infty}\left\{\bar{d}\left(F\Delta X_k,\overline{0_N}\right)\right\}^p<\infty$ and $\sum_{k=1}^{\infty}\left\{\bar{d}\left(I\Delta X_k,\overline{0_N}\right)\right\}^p<\infty$.
Thus, $(X_k)\in bv_p^N$.
Hence, the inclusion is proper.

**Theorem 3.9.**
The class of sequences $bv_p^N$ is neither monotone nor solid.

**Proof.**
This result follows from the following example.

**Example 3.10.**
Let us consider the sequence $(X_k)$, defined as follows:
$\mathrm{X_k(t)}=\begin{cases}1-5^{-1}\mathrm{k}^{\frac{2}{p}}(\mathrm{t}-2), \text{ for } 2\leq \mathrm{t}\leq 2+5\mathrm{k}^{-\frac{2}{p}};\\ 0, \text{ otherwise.}\end{cases}$
Then

$$\begin{aligned}[T\Delta X_k]^\alpha&=\left[5(\alpha-1)(k+1)^{-\frac{2}{p}},5(1-\alpha)k^{-\frac{2}{p}}\right]\\&=[X_k-X_{k+1}]^\alpha=\left[a_1^\alpha-b_2^\alpha,b_1^\alpha-a_2^\alpha\right].\\X_2(t)\geq\alpha&\Rightarrow 1-5^{-1}k^{\frac{2}{p}}(t-2)\geq\alpha.\\&\Rightarrow -\frac{1}{5}k^{\frac{2}{p}}(t-2)\geq\alpha-1.\\&\Rightarrow k^{\frac{2}{p}}(t-2)\leq 5(1-\alpha).\\&\Rightarrow t-2\leq 5(1-\alpha)k^{-\frac{2}{p}}.\\&\Rightarrow t\leq 2+5(1-\alpha)k^{-\frac{2}{p}}.\end{aligned}$$

Similarly, we can show that

$$X_{k+1}(t) \geq \alpha$$
$$\Rightarrow t \leq 2 + 5(1-\alpha)(1+k)^{-\frac{2}{p}}$$

Thus,
$\sum_{k=1}^{\infty} \left\{\bar{d}\left(T\Delta X_k, \overline{0_N}\right)\right\}^p = \sum_{k=1}^{\infty} \left\{5(1-\alpha)k^{-\frac{2}{p}}\right\}^p < \infty.$
Similarly it can be shown that $\sum_{k=1}^{\infty} \left\{\bar{d}\left(F\Delta X_k, \overline{0_N}\right)\right\}^p < \infty$ and $\sum_{k=1}^{\infty} \left\{\bar{d}\left(I\Delta X_k, \overline{0_N}\right)\right\}^p < \infty$.
Therefore,

$$(X_k) \in bv_p^N$$

Let $J = \{k \in \mathbb{N} : k = 2i-1, i \in \mathbb{N}\}$ be a subset of $\mathbb{N}$ and let $\overline{\overline{\left(bv_p^N\right)}}$ be the canonical preimage of the J- step set $\left(bv_p^N\right)_J$ of $\left(bv_p^N\right)$, defined as follows:

$$Y_k = \begin{cases} X_k, \text{ for } k \in J \\ \overline{0_N}, \text{ for } k \notin J \end{cases}$$

Then, we have

$$[Y_k]^\alpha = \begin{cases} \left[2, \left\{2 + 5(1-\alpha)k^{-\frac{2}{p}}\right\}\right], \text{ for } k \in J; \\ [0,0] \text{ for } k \notin J. \end{cases}$$

and

$$[T\Delta Y_k]^\alpha = \begin{cases} \left[2, \left\{2 + 5(1-\alpha)k^{-\frac{2}{p}}\right\}\right], \text{ for } k \in J; \\ \left[-\left\{2 + 5(1-\alpha)(k+1)^{-\frac{2}{p}}\right\}, -2\right], \text{ for } k \notin J. \end{cases}$$

Therefore, $\sum_{k=1}^{\infty} \left\{\bar{\mathrm{d}}\left(\mathrm{T}\Delta \mathrm{Y}_\mathrm{k}, \overline{0_N}\right)\right\}^\mathrm{p} = \sum_{k\in \mathrm{J}} \left\{\mathbf{2 + 5(1-\alpha)}\boldsymbol{k}^{-\frac{2}{p}}\right\}^p + \sum_{k\notin J} \{\mathbf{2+5(1-\alpha)(k+1)}\ )^{\frac{-2}{p}}\}^p \geq 2^p \sum_{k\in J} \left\{2^p + 5(1-\alpha)^p k^{-2}\right\}$, which is unbounded.
Thus, $(Y_k) \notin bv_p^N$.
Hence, $bv_p^N$ is not monotone.
The class $bv_p^N$ is not solid which follows from the remark 2.8.

**Theorem 3.11.**
The class of sequences $bv_p^N$ is not convergent free.

**Proof.**
The result follows from the following example.

**Example 3.12.**
Consider the sequence $(X_k) \in bv_p^N$ defined as follows:
For k even,

$$X_k(t) = \begin{cases} 1 + k^{\frac{3}{p}} t, \text{ for } -k^{-\frac{2}{p}} \leq t \leq 0 \\ 1 - k^{\frac{3}{p}} t, \text{ for } 0 < t \leq k^{-\frac{2}{p}} \\ 0, \text{ otherwise} \end{cases}$$

and for k odd, $X_k = \overline{0_N}$.
Then

$$[X_k]^\alpha = \begin{cases} \left[(\alpha-1)\mathrm{k}^{-\frac{3}{\mathrm{p}}}, (1-\alpha)\mathrm{k}^{-\frac{3}{\mathrm{p}}}\right], & \text{for k even} \\ [0,0], & \text{for k odd} \end{cases}$$

and

$$[T\Delta X_k]^\alpha = \begin{cases} \left[(\alpha-1)(k+1)^{-\frac{3}{p}}, (1-\alpha)(k+1)^{-\frac{3}{p}}\right], & \text{for } k \text{ odd };\\ \left[(\alpha-1)k^{-\frac{3}{p}}, (1-\alpha)k^{-\frac{3}{p}}\right], & \text{for } k \text{ even.} \end{cases}$$

Therefore,

$$\sum_{k=1}^{\infty} \left\{\bar{d}\left(T\Delta X_k, \overline{0_N}\right)\right\}^p = 2\sum_{i=1}^{\infty} \left\{\frac{1-\alpha}{(2i)^{\frac{3}{p}}}\right\}^p < \infty$$

Thus, $(X_k) \in bv_p^N$.
Let us define a sequence $(Y_k)$ as follows:
For k odd, $(Y_k) = \overline{0_N}$,
and for $k$ even,

$$Y_k(t) = \begin{cases} 1 + k^{\frac{1}{p}}t, & \text{for } -k^{-\frac{1}{p}} \le t \le 0 \\ 1 - k^{\frac{1}{p}}t, & \text{for } 0 < t \le k^{-\frac{1}{p}} \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$[Y_k]^\alpha = \begin{cases} [0,0], & \text{for k odd;} \\ \left[(\alpha-1)k^{-\frac{1}{p}}, (1-\alpha)k^{-\frac{1}{p}}\right], & \text{for k even} \end{cases}$$

and

$$[T\Delta Y_k]^\alpha = \begin{cases} [\alpha-1)k^{-\frac{1}{p}}, (1-\alpha)k^{-\frac{1}{p}}\right] & \text{for k even,} \\ \left[(\alpha-1)(k+1)^{-\frac{1}{p}}, (1-\alpha)(k+1)^{-\frac{1}{p}}\right] & \text{for odd.} \end{cases}$$

Thus $\sum_{\mathrm{k}=1}^{\infty} \left\{\overline{\mathrm{d}}\left(\mathrm{T}\Delta\mathrm{Y_k}, \overline{0_N}\right)\right\}^{\mathrm{P}} = 2\sum_{\mathrm{i}=1}^{\infty} \left\{\frac{1-\alpha}{(2\mathrm{i})^{\frac{1}{\mathrm{P}}}}\right\}^{\mathrm{P}}$, which is unbounded
i.e., $(Y_k) \notin bv_p^N$.
Hence $bv_p^N$, is not convergent free.

## 4. Conclusions:

This paper has provided the notion of p-bounded variation neutrosophic real number sequences $\mathrm{b}v_p^N$, for $1 \le p < \infty$. We have investigated some of its properties like completeness, monotonicity, convergence free and symmetricity. Some inclusion results have also been provided. This paper will definitely helpful for further investigation on neutrosophic sequence spaces.

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