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Optimality and duality for interval-valued vector problems

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Abstract. Our goal in this paper is to retain the ideology of convexificators to construct adequate Fritz John and KKT optimality conditions for nonsmooth programming problems with local weakly LU-Pareto solutions having inequality, equality, and set constraints in Banach space where the involved functions admit convexificators. Sufficiency criteria for local weakly LU-Pareto solutions have been formulated under suitable conditions on the generalized convexity. The desired duality theorems have been proposed for both Mond-Weir dual problems (MDCIMP) and Wolfe-type dual problems (WDCIMP). Numerical examples are constructed to justify the methodology adopted in the paper. Our paper extends some of the recently published articles to a great extent.

1. Introduction

The vast panorama of linear and nonlinear programming uses the fact that the coefficients appearing in objective functions as well as constraints are constant. In recent times, optimization models have become essential as businesses become larger and more complicated and as engineering design becomes more ambitious. A small change in operations may cause a tremendous increase in profit. The last few decades have witnessed astonishing improvements for which this presumption is not fulfilled. In most cases, the accuracy and certainty of input data were not always ensured. Hence, considering uncertainty in the data becomes essential to model real-world problems. Uncertainty can be tackled in numerous ways, such as using fuzzy numbers, stochastic processes, and interval-valued problems. Interval-valued programming problems address uncertainty, making optimization models closer to real-world applications. In interval-valued problems, coefficients appearing in the objective function, as well as constraints, vary over closed intervals. For solving interval-valued programming problems, a wide range of solution procedures have been developed. Wu [21] had introduced four kinds of interval-valued problems and given the concept of no duality gap for such problems. Jayswal et al. [9] proposed sufficient optimality results for interval-valued programs and framed duality criteria for Wolfe as well as Mond-Weir type dual problems. Ahmad et al. [1] focused on interval-valued programming problems and established sufficient optimality conditions along with various duality results. Bhurjee and Panda [2] worked on multiobjective fractional interval-valued optimization problems, ensuring the existence of an efficient solution. Later on, Bhurjee and Panda [3] also worked on nonlinear programming problems in which both the objective functions and constraints are

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considered interval-valued. Jayswal et al. [10] elaborated the concept of convexificator with the interval-valued optimization problem using LU-optimal solution and deduced the optimality conditions and duality results.

In nonsmooth optimization, the concept of a convexificator has shown to be a useful tool for deriving optimality criteria and duality results. The term compact convex convexificator was developed specifically by Demyanov [5] (1994). Jeyakumar and Luc [11] introduced the notion of approximate Jacobian matrices for continuous vector-valued functions using the idea of convexificators of the real-valued functions. Later, Jeyakumar and Luc [12] worked on noncompact convexificators and presented various calculus rules and properties for convexificators. Convexificator is a generalization of various well-known subdifferential concepts, such as Clarke's subdifferentials given by Clarke [4]. Luu [14],[15] derived the necessary conditions of multiobjective optimization problems using equality, inequality, and constraints set in Banach spaces for local Pareto solutions and weak minimal solutions using convexificators. Luu and Mai [16] worked on interval-valued programming problems and established the necessary conditions of Fritz John and KKT-type for local LU-optimal solutions using the concept of convexificators as well as derived the duality results for the Mond-Weir type dual problems and Wolfe-type dual problems both. Recently, Rayanki et al. [18] focused on a specific type of interval-valued problem and derived the KKT-type necessary and sufficient optimality conditions using the LU-optimal solutions. They considered the Wolfe-type dual variant and derived the appropriate duality theorems. Mohapatra [17] worked on mathematical problems with vanishing constraints using directional convexificators and derived KKT-type necessary and sufficient optimality conditions using convexity. Further, they introduced a Wolfe-type dual model in terms of directional convexificators and obtained duality results. Gadhi and Ohda [6] applied convexificators on a nonsmooth, nonconvex multiobjective robust optimization problem and established the robust necessary optimality conditions for weakly robust efficient solutions in terms of convexificators. Upadhyay [20] focused on nonsmooth semidefinite multiobjective programming problems with equilibrium constraints using convexificators and derived weak, strong, and strict converse duality theorems for both the Mond-Weir and Wolfe-type dual models.

The present paper elaborates the way we construct a multiobjective interval-valued programming problem and formulate the Fritz John and KKT-type sufficiency criteria for constrained interval-valued multiobjective programs with equality, inequality, and set constraints in Banach spaces using convexificators that are regular as defined by Ioffe [8]. The weak and strong duality results have been established for Mond-Weir as well as Wolfe-type dual formulations. The paper is structured as per the following scheme: In Section 2, we recall some elementary ideas and definitions exercised in the paper. In Section 3, we establish the Fritz John-type necessary criteria for constrained interval-valued multiobjective programming problems applying the concept of convexificators. Section 4 is focused on framing the Karush-Kuhn-Tucker conditions for local weakly LU-Pareto solutions to the problems. In Section 5, we establish sufficiency criteria for local weakly LU-Pareto solutions with the help of asymptotic pseudoconvexity, asymptotic quasiconvexity, and asymptotic quasilinearity. In Section 6, we derive relevant duality statements for the Mond-Weir and Wolfe-type dual problems for the multiobjective interval-valued problem projected in the paper.

2. Preliminaries

In the current article, we begin with the following convention for inequalities and equalities, which is utilized in the later part of this paper. For any $p = (p_1, p_2, \dots, p_m)$, $q = (q_1, q_2, \dots, q_m)$ in $\mathfrak{R}^m$, we have

(i) $p = q \Leftrightarrow p_i = q_i, \forall i = 1, \dots, m;$

(ii) $p > q \Leftrightarrow p_i > q_i, \forall i = 1, \dots, m;$

(iii) $p \geqq q \Leftrightarrow p_i \geqq q_i, \forall i = 1, \dots, m;$

(iv) $p \geq q \Leftrightarrow p \geqq q, p \neq q.$

In this sequel, we cast some elementary definitions that will be used throughout the entire article. First of all, recall the definition of convexificators given by Jeyakumar and Luc [12]. Let X^* represent the topological dual space of a real Banach space X.

Definition 2.1. *A function $\Xi : X \to \bar{\mathbb{R}}$ ($\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$) has an upper and lower Dini directional derivatives at $\bar{\pi} \in X$ along the specified direction $\varrho \in X$, respectively, provided*

$$\Xi^+(\bar{\pi}; \varrho) = \lim_{\gamma \downarrow 0} \sup \frac{\Xi(\bar{\pi} + \gamma\varrho) - \Xi(\bar{\pi})}{\gamma}$$

$$\Xi^-(\bar{\pi}; \varrho) = \lim_{\gamma \downarrow 0} \inf \frac{\Xi(\bar{\pi} + \gamma\varrho) - \Xi(\bar{\pi})}{\gamma}.$$

Moreover, if $\Xi^+(\bar{\pi}; \varrho) = \Xi^-(\bar{\pi}; \varrho)$, then the common value is denoted by $\Xi'(\bar{\pi}; \varrho)$ and known as the Dini derivative of the function Ξ at a point $\bar{\pi}$ along the specified direction ϱ.

Definition 2.2. *A function $\Xi : X \to \bar{\mathbb{R}}$ ($\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$) have an upper and lower convexificators denoted by $\partial^*\Xi(\bar{\pi})$ and $\partial_*\Xi(\bar{\pi})$ at a point $\bar{\pi}$, respectively if $\partial^*\Xi(\bar{\pi}) \subseteq X^*$ and $\partial_*\Xi(\bar{\pi}) \subseteq X^*$ are weakly* closed subset of X^* and*

$$\Xi^-(\bar{\pi}; \varrho) \leqq \sup_{\xi \in \partial^*\Xi(\bar{\pi})} \langle \xi, \varrho \rangle, \ \forall\ \varrho \in X,$$

$$\Xi^+(\bar{\pi}; \varrho) \geqq \inf_{\xi \in \partial_*\Xi(\bar{\pi})} \langle \xi, \varrho \rangle, \ \forall\ \varrho \in X.$$

A weakly* closed subset of X^* is called convexificator of Ξ at a point $\bar{\pi}$ and is denoted by $\partial\Xi(\bar{\pi})$ whenever $\partial\Xi(\bar{\pi}) = \partial^*\Xi(\bar{\pi}) = \partial_*\Xi(\bar{\pi})$.

Definition 2.3. *A function $\Xi : X \to \bar{\mathbb{R}}$ ($\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$) have an upper and lower semi-regular convexificators denoted by $\partial^*\Xi(\bar{\pi})$ and $\partial_*\Xi(\bar{\pi})$ at a point $\bar{\pi}$, respectively if $\partial^*\Xi(\bar{\pi})$ and $\partial_*\Xi(\bar{\pi})$ are weakly* closed and*

$$\Xi^+(\bar{\pi}; \varrho) \leqq \sup_{\xi \in \partial^*\Xi(\bar{\pi})} \langle \xi, \varrho \rangle, \ \forall\ \varrho \in X,$$

$$\Xi^-(\bar{\pi}; \varrho) \geqq \inf_{\xi \in \partial_*\Xi(\bar{\pi})} \langle \xi, \varrho \rangle, \ \forall\ \varrho \in X.$$

Similarly, a function Ξ has upper and lower regular convexificators denoted by $\partial^*\Xi(\bar{\pi})$ and $\partial_*\Xi(\bar{\pi})$ at a point $\bar{\pi}$, respectively, if $\partial^*\Xi(\bar{\pi})$ and $\partial_*\Xi(\bar{\pi})$ are weakly* closed and

$$\Xi^+(\bar{\pi}; \varrho) = \sup_{\xi \in \partial^*\Xi(\bar{\pi})} \langle \xi, \varrho \rangle, \ \forall\ \varrho \in X,$$

$$\Xi^-(\bar{\pi}; \varrho) = \inf_{\xi \in \partial_*\Xi(\bar{\pi})} \langle \xi, \varrho \rangle, \ \forall\ \varrho \in X.$$

Definition 2.4. C*larke [4] The Clarke directional derivative of function $\Xi : X \to \mathbb{R}$ at point $\bar{\pi}$ with respect to the direction ϱ is expressed as*

$$\Xi(\bar{\pi}; \varrho) := \lim_{\pi \to \bar{\pi}} \sup_{\gamma \downarrow 0} \frac{\Xi(\pi + \gamma\varrho) - \Xi(\pi)}{\gamma}. \tag{1}$$

The subdifferential of Ξ due to Clarke at $\bar{\pi}$ can be expressed mathematically as

$$\partial_\circ\Xi(\bar{\pi}) := \Big\{\xi \in X^* : \langle \xi, \varrho \rangle \leqq\ \Xi(\bar{\pi}; \varrho)\ \forall\ \varrho \in X\Big\}.$$

Remark 2.5. *If the function Ξ is locally Lipschitz, then the Clarke subdifferential turns to the convexificator of Ξ at point $\bar{\pi}$. A locally Lipschitz function Ξ is called regular at point $\bar{\pi}$ provided there exist $\Xi'(\bar{\pi};\varrho)$ for each $\varrho \in X$ having the same value as that of $\Xi(\bar{\pi};\varrho)$. The Clarke subdifferential of a regular function Ξ is the upper regular convexificator, and the convexificator mapping $\partial\Xi$ is bounded at a point $\bar{\pi}$ locally. Furthermore, if the dimension of X is finite, the mapping $\partial\Xi$ becomes upper semicontinuous at a point $\bar{\pi}$.*
If the function Ξ is convex corresponding to X, then its subdifferential is specified as

$$\partial_C\Xi(\bar{\pi}) := \Big\{\xi \in X^* : \langle \xi, \pi - \bar{\pi}\rangle \leqq \Xi(\pi) - \Xi(\bar{\pi});\ \forall\ \pi \in X\Big\}.$$

If the function Ξ is locally Lipschitz and convex, corresponding to X at a point $\bar{\pi} \in X$, then

$$\partial_C\Xi(\bar{\pi}) = \partial_\circ\Xi(\bar{\pi}).$$

For a set C, the Clarke tangent cone at $\bar{\pi} \in C$ can be rephrased mathematically as

$$\mathbb{T}(C;\bar{\pi}) := \Big\{\varrho \in X : \forall\ \pi_n \in C,\ \pi_n \to \bar{\pi},\ \forall\ t_n \downarrow 0,\ \exists\ \varrho_n \to \varrho \text{ with } \pi_n + t_n\varrho_n \in C,\ \forall\ n\Big\},$$

whereas the Clarke normal cone at a point $\bar{\pi}$ can be recast as

$$\mathbb{N}_C(\bar{\pi}) := \Big\{\xi \in X^* : \langle \xi, \varrho\rangle \leqq 0,\ \forall\ \varrho \in \mathbb{T}(C;\bar{\pi})\Big\}.$$

3. Necessary criteria: Fritz-John type

Let the set of all bounded and closed intervals in $\mathbb{R}$ be denoted by $\mathbb{S}$, and the partial ordering of intervals for $£ = [\xi^L, \xi^U] \in \mathbb{S}$, $ð = [\eta^L, \eta^U] \in \mathbb{S}$ can be defined as follows:

$$£ \leqq_{LU} ð \text{ indicate } \xi^L \leqq \eta^L \text{ and } \xi^U \leqq \eta^U,$$

$$£ <_{LU} ð \text{ indicate } £ \leqq_{LU} ð,\ £ \neq ð.$$

Equivalently, $£ <_{LU} ð$ if any one of the following three conditions is satisfied:

$$\begin{aligned} &\xi^L < \eta^L,\ \ \xi^U < \eta^U \\ &\xi^L \leqq \eta^L,\ \ \xi^U < \eta^U \\ &\xi^L < \eta^L,\ \ \xi^U \leqq \eta^U \end{aligned}$$

A function $\aleph : X \to \mathbb{S}$ is said to be an interval-valued function. In other words, for all $\pi \in X$, the interval-valued function $\aleph(\pi)$ is defined by $\aleph(\pi) = [\aleph^L(\pi), \aleph^U(\pi)]$, where $\aleph^L(\pi)$ and $\aleph^U(\pi)$ are functions defined on X with the condition $\aleph^L(\pi) \leqq \aleph^U(\pi)$. We note that $[\aleph(\pi)]^L = \aleph^L(\pi)$ and $[\aleph(\pi)]^U = \aleph^U(\pi)$.
We contemplate the following constrained multiobjective interval-valued programs:

$$\begin{aligned} \text{(CIMP)} \qquad \text{minimize} \quad \aleph(\pi) &= \Big(\aleph_1(\pi), \aleph_2(\pi), \ldots, \aleph_p(\pi)\Big) \\ &= \Big([\aleph_1^L(\pi), \aleph_1^U(\pi)], [\aleph_2^L(\pi), \aleph_2^U(\pi)], \ldots, [\aleph_p^L(\pi), \aleph_p^U(\pi)]\Big) \\ \text{s.t.} \qquad \pi \in M &:= \{\pi \in C : \phi_i(\pi) \leqq 0\ (i \in \mathfrak{I}_m), \psi_j(\pi) = 0\ (j \in \mathfrak{L}_l)\}. \end{aligned}$$

For $\bar{\pi} \in M$, define $\mathfrak{I}_m(\bar{\pi}) := \{i \in \mathfrak{I}_m : \phi_i(\bar{\pi}) = 0\}$ where the function $\aleph_k$ maps from X to $\mathbb{S}$ for each $k \in \mathfrak{I}_p := \{1, 2, \ldots, p\}$. $\aleph_k(\pi)$ is a bounded and closed interval in $\mathbb{R}$ such that $\aleph_k(\pi) = [\aleph_k^L(\pi), \aleph_k^U(\pi)]$ where $\aleph_k^L(\pi), \aleph_k^U(\pi)$ are real-valued functions defined on X. Let $C \subseteq X$ be a closed set, and ϕ and ψ defined on X map to $\mathbb{R}^m$ and $\mathbb{R}^l$, respectively. It may be noted that $\phi_i,\ i \in \mathfrak{I}_m : \{1, \ldots, m\}$ and $\psi_j,\ j \in \mathfrak{L}_l : \{1, \ldots, l\}$ are functions defined on X to $\mathbb{R} \cup \{-\infty, +\infty\}$.
Note If we take $p = 1$, then the above problem reduces to that considered in Luu and Mai [16].

Definition 3.1. *A feasible point $\bar{\pi} \in M$ is known as a local LU-Pareto (local LU-efficient) solution of (CIMP) if there is no feasible point $\pi \in M \cap B(\bar{\pi};\delta)$, $\delta > 0$ satisfying*

$$\aleph_k(\pi) \leqq_{LU} \aleph_k(\bar{\pi}),\ \forall\, k \in \mathfrak{J}_p,$$

and

$$\aleph_k(\pi) <_{LU} \aleph_k(\bar{\pi}) \text{ for atleast one } k \in \mathfrak{J}_p.$$

Definition 3.2. *A feasible point $\bar{\pi} \in M$ is known as a local weakly LU-Pareto (local weakly LU-efficient) solution of (CIMP) if there is no feasible point $\pi \in M \cap B(\bar{\pi};\delta)$, $\delta > 0$ satisfying*

$$\aleph_k(\pi) <_{LU} \aleph_k(\bar{\pi}),\ \forall\, k \in \mathfrak{J}_p,$$

which means that $\aleph_k(\pi) \leqq_{LU} \aleph_k(\bar{\pi})$, $\aleph_k(\pi) \neq \aleph_k(\bar{\pi})$.
Equivalently,

$$\begin{aligned} &\aleph_k^L(\pi) < \aleph_k^L(\bar{\pi}), \qquad \aleph_k^U(\pi) < \aleph_k^U(\bar{\pi}),\\ &\aleph_k^L(\pi) \leqq \aleph_k^L(\bar{\pi}), \qquad \aleph_k^U(\pi) < \aleph_k^U(\bar{\pi}),\\ &\aleph_k^L(\pi) < \aleph_k^L(\bar{\pi}), \qquad \aleph_k^U(\pi) \leqq \aleph_k^U(\bar{\pi}).\end{aligned}$$

Example 3.3. *Let $X = \mathbb{R}$, $C = [-3,5]$, $\bar{\pi} = 0$. Let $\aleph_k$ $(k = 1,2)$ be a mapping from X into $\mathbb{S}$ defined by $\aleph_k(\pi) = [\aleph_k^L(\pi), \aleph_k^U(\pi)]$ $(\forall \pi \in \mathbb{R})$ such that*

$$\aleph_1^L(\pi) = \begin{cases} -\pi - 1, & \pi \geqq 0 \\ \pi - 1, & \pi < 0 \end{cases}$$

$$\aleph_1^U(\pi) = \begin{cases} \pi + 5, & \pi \geqq 0 \\ -\pi + 5, & \pi < 0 \end{cases}$$

$$\aleph_2^L(\pi) = \begin{cases} -\pi^2 - 1, & \pi < 0 \\ -\pi - 3, & \pi \geqq 0 \end{cases}$$

$$\aleph_2^U(\pi) = \begin{cases} \pi^2 + \pi + 1, & \pi \geqq 0 \\ \pi^2 - \pi + 1, & \pi < 0 \end{cases}$$

and $\phi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\phi(\pi) := \pi^2 - 5\pi.$$

Then $\aleph_k^L(\pi) \leqq \aleph_k^U(\pi)$ $(k = 1,2)$ which implies that $\aleph_k(\pi) \in \mathbb{S}$ for each $\pi \in C$. It is clear that the point $\bar{\pi} = 0$ is a weakly LU-Pareto solution to the multiobjective programming problem:

$$\begin{aligned} \text{minimize}\quad & \aleph(\pi) = (\aleph_1(\pi), \aleph_2(\pi)) \\ & \qquad = \left([\aleph_1^L(\pi), \aleph_1^U(\pi)], [\aleph_2^L(\pi), \aleph_2^U(\pi)]\right) \\ \text{s.t.}\quad & \pi \in M_1 := \{\pi \in C : \phi(\pi) \leqq 0\} = [0,5]. \end{aligned}$$

Remark 3.4. *The local weakly LU-Pareto solution of the problem (CIMP) may not be a locally efficient solution to the multiobjective bi-criteria programming problem (MP):*

$$\begin{aligned} \text{minimize}\quad & \widetilde{\aleph}_k(\pi) = (\aleph_k^L(\pi), \aleph_k^U(\pi)) \\ \text{s.t.}\quad & \pi \in M := \{\pi \in C : \phi_i(\pi) \leqq 0, i \in \mathfrak{J}_m, \psi_j(\pi) = 0, j \in \mathfrak{L}_l\}. \end{aligned}$$

Remark 3.5. *If a feasible point $\bar{\pi} \in M$ is a weakly LU-Pareto solution of the problem (CIMP), then it is a locally weak minimal solution of the multiobjective bicriteria programming problems (MP), but the converse may not be true.*

Remark 3.6. *If a feasible point $\bar{\pi}$ is a locally Pareto minimal of the multiobjective bi-criteria programming problems (MP), and $\aleph_k^L(\pi) \leqq \aleph_k^U(\bar{\pi})$ ($\pi \in C \cap B(\bar{\pi};\delta)$ for some $\delta > 0$). In case this condition is not satisfied, then the feasible point $\bar{\pi}$ may not be Pareto minimal to the problem (MP).*

Definition 3.7. *A point $\bar{\pi}$ is known as a regular for ψ corresponding to C iff there exist $\Upsilon > 0$ and $\delta > 0$ satisfying*

$$d_Q(\pi) \leqq \Upsilon \parallel \psi(\pi) - \psi(\bar{\pi}) \parallel, \ \forall\, \pi \in C \cap B(\bar{\pi};\delta),$$

where $Q := \{\pi \in C : \psi(\pi) = \psi(\bar{\pi})\}$, $d_Q(\pi)$ signify the distance between π and Q. If the problem (CIMP) has only inequality constraints, then any $\pi \in C$ is regular.

Now let us make the following assumptions, which will be used to obtain the necessary criteria of Fritz John-type for local weakly LU-Pareto solution at point $\bar{\pi}$ of (CIMP).

Assumption 3.8.

(i) The functions $\aleph_k^L, \aleph_k^U, \psi_1, \dots, \psi_l$, ($k \in \Im_p := \{1, 2, \dots, p\}$) are locally Lipschitz at a point $\bar{\pi}$ whereas ϕ_i ($i \in \Im_m(\bar{\pi})$) are continuous and C is convex.

(ii) $\aleph_k^L$ and $\aleph_k^U$ are functions that admit bounded convexificators $\partial\aleph_k^L(\bar{\pi})$, $\partial\aleph_k^U(\bar{\pi})$ ($k \in \Im_p := \{1, 2, \dots, p\}$) and ϕ_i admit upper convexificators $\partial^*\phi_i(\bar{\pi})$ ($i \in \Im_m(\bar{\pi})$) at a point $\bar{\pi}$.

(iii) The functions $\mid \psi_j \mid$, $j \in \mathfrak{L}_l$ are regular at point $\bar{\pi}$ as proposed by Clarke [4].

Theorem 3.9. *Let the solution $\bar{\pi}$ be locally weak LU-Pareto of* (CIMP). *Suppose that $\bar{\pi}$ is regular corresponding to C and it satisfies all the criteria mentioned in Assumption* 3.8. *Furthermore, ψ_j ($\forall\, j \in \mathfrak{L}_l$) admits a convexificator $\partial\psi_j(\pi)$ at a point π in the neighborhood of $\bar{\pi}$, and the convexificator maps $\partial\aleph_k^L$, $\partial\aleph_k^U$, $\partial\psi_j$ are upper semicontinuous at a point $\bar{\pi}$. Then there exist $\bar{\alpha}^L$, $\bar{\alpha}^U \geqq 0$, $\bar{\beta} \geqq 0$, where $\bar{\alpha}^L = (\bar{\alpha}_1^L, \dots, \bar{\alpha}_p^L)$, $\bar{\alpha}^U = (\bar{\alpha}_1^U, \dots, \bar{\alpha}_p^U)$, and $\bar{\beta} = (\bar{\beta}_1, \dots, \bar{\beta}_m)$, and $\bar{\rho}_j \in \mathbb{R}$ satisfying $\sum_{k\in\Im_p} \bar{\alpha}_k^L + \bar{\alpha}_k^U + \sum_{i\in\Im_m(\bar{\pi})} \bar{\beta}_i = 1$, along with*

$$0 \ \in \ \mathrm{cl}\Big(\sum_{k\in\Im_p} \bar{\alpha}_k^L \ \mathrm{conv}\ \partial\aleph_k^L(\bar{\pi}) + \bar{\alpha}_k^U \ \mathrm{conv}\ \partial\aleph_k^U(\bar{\pi}) + \sum_{i\in\Im_m(\bar{\pi})} \bar{\beta}_i \ \mathrm{conv}\ \partial^*\phi_i(\bar{\pi}) + \sum_{j\in\mathfrak{L}_l} \bar{\rho}_j \ \mathrm{conv}\ \partial\psi_j(\bar{\pi}) + \mathbb{N}_C(\bar{\pi})\Big). \quad (2)$$

Proof. Let $\widetilde{\aleph}_k(\pi)$ represents the open interval $(\aleph_k^L(\pi), \aleph_k^U(\pi))$, where $k \in \Im_p := \{1, 2, \dots, p\}$. Since the vector $\bar{\pi}$ is a locally weak LU-Pareto solution to the problem (CIMP), there does not exist any $\pi \in M \cap B(\bar{\pi};\delta)$, where $\delta > 0$ satisfying

$$\aleph_k^L(\pi) < \aleph_k^L(\bar{\pi}),\ \aleph_k^U(\pi) < \aleph_k^U(\bar{\pi}),$$
$$\aleph_k^L(\pi) \leqq \aleph_k^L(\bar{\pi}),\ \aleph_k^U(\pi) < \aleph_k^U(\bar{\pi}),$$
$$\aleph_k^L(\pi) < \aleph_k^L(\bar{\pi}),\ \aleph_k^U(\pi) \leqq \aleph_k^U(\bar{\pi}).$$

Therefore,

$$\widetilde{\aleph}_k(\pi) - \widetilde{\aleph}_k(\bar{\pi}) \notin -\mathrm{int}\mathbb{R}_+^2 \ (\forall\, \pi \in \ M \cap B(\bar{\pi};\delta)),$$

and hence $\bar{\pi}$ is a locally weak minimal solution of the problem (MP):

$$\text{minimize} \quad \widetilde{\aleph}_k(\pi) = (\aleph_k^L(\pi), \aleph_k^U(\pi))$$
$$\text{s.t.} \qquad \pi \in M := \{\pi \in C : \phi_i(\pi) \leqq 0, i \in \Im_m, \psi_j(\pi) = 0, j \in \mathfrak{L}_l\}.$$

With the help of the scalarization theorem, one can see that there exists a subadditive function Λ defined on $\mathbb{R}^2$ that is continuous, homogeneous, and positive such that

$$\vartheta_2 - \vartheta_1 \in \mathrm{int}\mathbb{R}_+^2 \Rightarrow \Lambda(\vartheta_2) < \Lambda(\vartheta_1),$$

and

$$\Lambda \circ (\widetilde{\aleph}_k(\pi) - \widetilde{\aleph}_k(\bar{\pi})) \geqq 0 \ (\forall\, \pi \in\ M \cap B(\bar{\pi};\delta)). \tag{3}$$

Substituting $\widehat{\aleph}_{k,\bar{\pi}}(\pi) := \widehat{\aleph}_k(\bar{\pi},\pi) := \widetilde{\aleph}_k(\pi) - \widetilde{\aleph}_k(\bar{\pi})$ in above

$$\widehat{\aleph}_{k,\bar{\pi}}(\pi) = (\widehat{\aleph}^L_{k,\bar{\pi}}(\pi), \widehat{\aleph}^U_{k,\bar{\pi}}(\pi)) = (\aleph^L_k(\pi) - \aleph^L_k(\bar{\pi}), \aleph^U_k(\pi) - \aleph^U_k(\bar{\pi})).$$

Since the function Λ is continuous as well as convex, using Proposition 2.2.6 (Clarke [4]), it becomes locally Lipschitz. Hence, its subdifferential $\partial_C\Lambda(\widehat{\aleph}_{k,\bar{\pi}}(\pi))$ is a bounded convexificator of Λ at $\widehat{\aleph}_{k,\bar{\pi}}(\bar{\pi}) = 0$. Schirotzek [19] in Proposition 7.3.9 pointed out that if the function Λ is convex and locally Lipschitz, then

$$\partial_C\Lambda(\widehat{\aleph}_{k,\bar{\pi}}(\bar{\pi})) = \partial_\circ\Lambda(\widehat{\aleph}_{k,\bar{\pi}}(\bar{\pi})),\ \forall\, k \in \mathfrak{I}_p.$$

Furthermore, the mapping $\partial_C\Lambda$ is upper semicontinuous at $\widehat{\aleph}_{k,\bar{\pi}}(\bar{\pi})$. On the flip side, it can be seen that

$$\partial\widehat{\aleph}_{k,\bar{\pi}}(\bar{\pi}) = \partial\aleph_k(\bar{\pi}),\ \forall\, k \in \mathfrak{I}_p.$$

Using Assumption 3.8, we can conclude that $\partial\aleph^L_k(\bar{\pi})$ and $\partial\aleph^U_k(\bar{\pi})$ for each $k \in \mathfrak{I}_p$ are bounded convexificators of $\aleph^L_k$ and $\aleph^U_k$ at a point $\bar{\pi}$. Furthermore, the mappings $\partial\aleph^L_k$ and $\partial\aleph^U_k$ are upper semicontinuous at a point $\bar{\pi}$. Consequently, $\partial_C\Lambda(\widehat{\aleph}_{k,\bar{\pi}}(\bar{\pi}))(\partial\aleph^L_k(\pi), \partial\aleph^U_k(\pi))$ is convexificators of $\Lambda \circ \widehat{\aleph}_{k,\bar{\pi}}(\pi)$ at a point $\bar{\pi}$. Since all assumptions of Theorem 3.1 of Gong [7] are satisfied for the equilibrium problem: Finding $\bar{\pi} \in M$ such as $\forall \pi \in M$,

$$\Lambda \circ \widehat{\aleph}(\bar{\pi},\pi) \geqq 0.$$

Making use of Theorem 3.2 of Luu [13] for the vector equilibrium problem to estimate that $\bar{\tau} \geqq 0,\ \bar{\beta} \geqq 0$, where $\bar{\tau} = (\bar{\tau}_1,\ldots,\bar{\tau}_p),\ \rho_j \in \mathbb{R}$ satisfying $\sum_{k\in\mathfrak{I}_p}\bar{\tau}_k + \sum_{i\in\mathfrak{I}_m(\bar{\pi})}\bar{\beta}_i = 1$, along with

$$\begin{aligned} 0 \ \in\ \mathrm{cl}\Big(&\sum_{k\in\mathfrak{I}_p}\bar{\tau}_k\partial_C\Lambda(\widehat{\aleph}_{k,\bar{\pi}}(\bar{\pi}))(\mathrm{conv}\ \partial\aleph^L_{k,\bar{\pi}}(\bar{\pi}), \mathrm{conv}\ \partial\aleph^U_{k,\bar{\pi}}(\bar{\pi})) \\ &+ \sum_{i\in\mathfrak{I}_m(\bar{\pi})}\bar{\beta}_i\ \mathrm{conv}\ \partial^*\phi_i(\bar{\pi}) + \sum_{j\in\mathfrak{L}_l}\bar{\rho}_j\ \mathrm{conv}\ \partial\psi_j(\bar{\pi}) + \mathbb{N}_C(\bar{\pi})\Big). \end{aligned} \tag{4}$$

As $\partial\widehat{\aleph}_{k,\bar{\pi}}(\bar{\pi}) = \partial\aleph_k(\bar{\pi})$ ($\forall\, k \in \mathfrak{I}_p$) and $\widehat{\aleph}_{k,\bar{\pi}}(\bar{\pi}) = 0$, it follows from (4) that one can get a sequence

$$\begin{aligned} \chi_n \in &\sum_{k\in\mathfrak{I}_p}\bar{\tau}_k\partial_C\Lambda(0)(\mathrm{conv}\ \partial\aleph^L_k(\bar{\pi}), \mathrm{conv}\ \partial\aleph^U_k(\bar{\pi})) \\ &+ \sum_{i\in\mathfrak{I}_m(\bar{\pi})}\bar{\beta}_i\ \mathrm{conv}\ \partial^*\phi_i(\bar{\pi}) + \sum_{j\in\mathfrak{L}_l}\bar{\rho}_j\ \mathrm{conv}\ \partial\psi_j(\bar{\pi}) + \mathbb{N}_C(\bar{\pi}), \end{aligned} \tag{5}$$

such as $\lim_{n\to\infty}\chi_n = 0$. Therefore, a sequence $\{\varkappa_n\} \subset \partial_C\Lambda(0) \subset \mathbb{R}^2$ exists satisfying

$$\begin{aligned} \chi_n \in &\sum_{k\in\mathfrak{I}_p}\bar{\tau}_k\varkappa_n(\mathrm{conv}\ \partial\aleph^L_k(\bar{\pi}), \mathrm{conv}\ \partial\aleph^U_k(\bar{\pi})) \\ &+ \sum_{i\in\mathfrak{I}_m(\bar{\pi})}\bar{\beta}_i\ \mathrm{conv}\ \partial^*\phi_i(\bar{\pi}) + \sum_{j\in\mathfrak{L}_l}\bar{\rho}_j\ \mathrm{conv}\ \partial\psi_j(\bar{\pi}) + \mathbb{N}_C(\bar{\pi}). \end{aligned} \tag{6}$$

Since $\partial_C\Lambda(\widehat{\aleph}_{k,\bar{\pi}}(\bar{\pi}))$ is a compact set in $\mathbb{R}^2$, suppose that $\varkappa_n \to \bar{\varkappa} \in \partial_C\Lambda(\widehat{\aleph}_{k,\bar{\pi}}(\bar{\pi}))$. Substituting $\bar{\alpha}_k = \bar{\tau}_k\bar{\varkappa}$, where $\bar{\alpha}_k = (\bar{\alpha}^L_k, \bar{\alpha}^U_k) \in \mathbb{R}^2$. From (6) we can conclude that

$$0 \in \sum_{k\in\mathfrak{I}_p}\bar{\alpha}_k(\mathrm{conv}\ \partial\aleph^L_k(\bar{\pi}), \mathrm{conv}\ \partial\aleph^U_k(\bar{\pi}))$$

$$+ \sum_{i \in \Im_m(\bar{\pi})} \bar{\beta}_i \operatorname{conv} \partial^* \phi_i(\bar{\pi}) + \sum_{j \in \mathfrak{L}_l} \bar{\rho}_j \operatorname{conv} \partial \psi_j(\bar{\pi}) + \mathbb{N}_C(\bar{\pi}),$$

which is equal to (2). Let us observe that $\bar{\varkappa} \in \mathbb{R}^2_+ \backslash \{0\}$. Therefore, for any $\vartheta \in \operatorname{int}\mathbb{R}^2_+$, we can write $0 - (-\vartheta) \in \operatorname{int}\mathbb{R}^2_+$. Hence,

$$\begin{aligned} \langle \bar{\varkappa}, -\vartheta \rangle &= \langle \bar{\varkappa}, (\widehat{\aleph}_{k,\bar{\pi}}(\bar{\pi}) - \vartheta) - \widehat{\aleph}_{k,\bar{\pi}}(\bar{\pi}) \rangle \\ &\leqq \Lambda(\widehat{\aleph}_{k,\bar{\pi}}(\bar{\pi}) - \vartheta) - \Lambda(\widehat{\aleph}_{k,\bar{\pi}}(\bar{\pi})) \\ &= \Lambda(-\vartheta) < \Lambda(0) = 0. \end{aligned}$$

Therefore, $\bar{\varkappa} \in \mathbb{R}^2_+ \backslash \{0\}$. and suppose $\sum_{k \in \Im_p} \bar{\varkappa}^L_k + \bar{\varkappa}^U_k = 1$, which implies that $\sum_{k \in \Im_p} \bar{\alpha}^L_k + \bar{\alpha}^U_k + \sum_{i \in \Im_m(\bar{\pi})} \bar{\beta}_i = 1$, which makes the proof complete. □

Corollary 3.10. *Let the solution $\bar{\pi}$ be locally weak LU-Pareto to* (CIMP) *having only inequality constraints. Suppose that the convexificator maps $\partial\aleph^L_k$, $\partial\aleph^U_k$ are upper semicontinuous at a point $\bar{\pi}$ which satisfies all the criteria mentioned in Assumptions 3.8 (without equality constraint). Then there exist $\bar{\alpha}^L$, $\bar{\alpha}^U \geqq 0$, $\bar{\beta} \geqq 0$, where $\bar{\alpha}^L = (\bar{\alpha}^L_1, \ldots, \bar{\alpha}^L_p)$, $\bar{\alpha}^U = (\bar{\alpha}^U_1, \ldots, \bar{\alpha}^U_p)$, and $\bar{\beta} = (\bar{\beta}_1, \ldots, \bar{\beta}_m)$, satisfying $\sum_{k \in \Im_p} \bar{\alpha}^L_k + \bar{\alpha}^U_k + \sum_{i \in \Im_m(\bar{\pi})} \bar{\beta}_i = 1$, along with*

$$0 \in \operatorname{cl}\Big(\sum_{k \in \Im_p} \Big(\bar{\alpha}^L_k \operatorname{conv} \partial\aleph^L_k(\bar{\pi}) + \bar{\alpha}^U_k \operatorname{conv} \partial\aleph^U_k(\bar{\pi}) \Big) + \sum_{i \in \Im_m(\bar{\pi})} \bar{\beta}_i \operatorname{conv} \partial^* \phi_i(\bar{\pi}) + \mathbb{N}_C(\bar{\pi}) \Big).$$

Corollary 3.11. *Let the solution $\bar{\pi}$ be locally weak LU-Pareto to* (CIMP) *with $X = \mathbb{R}^n$ and C be convex. Furthermore, suppose that $\aleph^L_k$, $\aleph^U_k$ ($\forall k \in \Im_p$), ϕ_i ($\forall\, i \in \Im_m(\bar{\pi})$) and ψ_j ($\forall\, j \in \mathfrak{L}_l$) are locally Lipschitz at a point $\bar{\pi}$; $\partial^*\phi_i$ ($\forall\, i \in \Im_m(\bar{\pi})$) is the upper convexificator of the function ϕ_i at a point $\bar{\pi}$; and the functions $|\ \psi_j(\bar{\pi})\ |$ ($\forall\ j \in \mathfrak{L}_l$) are regular at a point $\bar{\pi}$. Then the mapping $\partial^*\phi_i$ ($\forall\, i \in \Im_m(\bar{\pi})$) are bounded locally convexificators of ϕ_i at a point $\bar{\pi}$, and there exist $\bar{\alpha}^L$, $\bar{\alpha}^U \geqq 0$, $\bar{\beta} \geqq 0$, where $\bar{\alpha}^L = (\bar{\alpha}^L_1, \ldots, \bar{\alpha}^L_p)$, $\bar{\alpha}^U = (\bar{\alpha}^U_1, \ldots, \bar{\alpha}^U_p)$, and $\bar{\beta} = (\bar{\beta}_1, \ldots, \bar{\beta}_m)$, and $\bar{\rho}_j \in \mathbb{R}$ for all j in $\mathfrak{L}_l$ satisfying $\sum_{k \in \Im_p} \bar{\alpha}^L_k + \bar{\alpha}^U_k + \sum_{i \in \Im_m(\bar{\pi})} \bar{\beta}_i = 1$, along with*

$$0 \in \sum_{k \in \Im_p} (\bar{\alpha}^L_k \operatorname{conv} \partial\aleph^L_k(\bar{\pi}) + \bar{\alpha}^U_k \operatorname{conv} \partial\aleph^U_k(\bar{\pi})) + \sum_{i \in \Im_m(\bar{\pi})} \bar{\beta}_i \operatorname{conv} \partial^* \phi_i(\bar{\pi}) + \sum_{j \in \mathfrak{L}_l} \bar{\rho}_j \operatorname{conv} \partial \psi_j(\bar{\pi}) + \mathbb{N}_C(\bar{\pi}).$$

where $\aleph^L_k$, $\aleph^U_k$ and ψ_j have the Clarke subdifferentials $\partial_\circ\aleph^L_k(\bar{\pi})$, $\partial_\circ\aleph^U_k(\bar{\pi})$ and $\partial_\circ\psi_j(\bar{\pi})$ at a point $\bar{\pi}$, respectively.

4. Necessary conditions of KKT-type

This section deals with how to deduce KKT-type necessary conditions using constraint qualification given by Mangasarian-Fromovitz (CQ) for local weakly LU-Pareto solutions to the problem (CIMP). Here, we intend to show that there exist $\omega_0 \in \mathbb{T}_C(\bar{\pi})$ and $a_i > 0$ ($i \in \Im_m(\bar{\pi})$), so that

(*i*) $\langle \zeta_i, \omega_0 \rangle \leqq -a_i$ ($\forall\ \zeta_i \in \partial^*\phi_i(\bar{\pi})$, $\forall i \in \Im_m(\bar{\pi})$),

(*ii*) $\langle \mu_j, \omega_0 \rangle = 0$ ($\forall\ \mu_j \in \partial^*\psi_j(\bar{\pi})$, $\forall j \in \mathfrak{L}_l$).

Theorem 4.1. *Suppose the solution $\bar{\pi}$ is local weakly LU-Pareto to* (CIMP) *and satisfies the constraint qualification* (CQ). *Under the presumptions of Theorem 3.9, there exist $\bar{\alpha}^L$, $\bar{\alpha}^U \geq 0$, where $\bar{\alpha}^L = (\bar{\alpha}^L_1, \ldots, \bar{\alpha}^L_p)$, $\bar{\alpha}^U = (\bar{\alpha}^U_1, \ldots, \bar{\alpha}^U_p)$, such that $\sum_{k \in \Im_p} (\bar{\alpha}^L_k + \bar{\alpha}^U_k) = 1$; $\bar{\beta}_i \geqq 0$ for all i in $\Im_m(\bar{\pi})$, and $\bar{\rho}_j \in \mathbb{R}$ for all j in $\mathfrak{L}_l$ in such a way*

$$0 \in \operatorname{cl}\Big(\sum_{k \in \Im_p} \bar{\alpha}^L_k \operatorname{conv} \partial\aleph^L_k(\bar{\pi}) + \bar{\alpha}^U_k \operatorname{conv} \partial\aleph^U_k(\bar{\pi}) + \sum_{i \in \Im_m(\bar{\pi})} \bar{\beta}_i \operatorname{conv} \partial^* \phi_i(\bar{\pi}) + \sum_{j \in \mathfrak{L}_l} \bar{\rho}_j \operatorname{conv} \partial \psi_j(\bar{\pi}) + \mathbb{N}_C(\bar{\pi}) \Big). \quad (7)$$

Proof. As the hypothesis of Theorem 3.9 are satisfied, so we infer that there exist $\bar{\alpha}^L,\ \bar{\alpha}^U \geqq 0,\ \bar{\beta}_i \geqq 0$ for all i in $\Im_m(\bar{\pi})$, and $\bar{\rho}_j \in \mathbb{R}$ for all j in $\mathfrak{L}_l$ satisfying $\sum_{k\in\Im_p} \bar{\alpha}^L_k + \bar{\alpha}^U_k + \sum_{i\in\Im_m(\bar{\pi})} \bar{\beta}_i = 1$, and

$$0 \ \in \ \mathrm{cl}\Big(\sum_{k\in\Im_p} \bar{\alpha}^L_k \ \mathrm{conv}\ \partial\aleph^L_k(\bar{\pi}) + \bar{\alpha}^U_k \ \mathrm{conv}\ \partial\aleph^U_k(\bar{\pi}) + \sum_{i\in\Im_m(\bar{\pi})} \bar{\beta}_i \ \mathrm{conv}\ \partial^*\phi_i(\bar{\pi}) + \sum_{j\in\mathfrak{L}_l} \bar{\rho}_j \ \mathrm{conv}\ \partial\psi_j(\bar{\pi}) + \mathbb{N}_C(\bar{\pi})\Big).$$

If $\bar{\alpha}^L_k = 0,\ \bar{\alpha}^U_k = 0$ for each k in $\Im_p$, with $\sum_{i\in\Im_m(\bar{\pi})} \bar{\beta}_i = 1$, there exist $\zeta^{(n)}_i \in \mathrm{conv}\ \partial^*\phi_i(\bar{\pi})$ for all i in $\Im_m(\bar{\pi}),\ \mu^{(n)}_j \in \mathrm{conv}\ \partial\psi_j(\bar{\pi})$ for all j in $\mathfrak{L}_l$ and $\xi^{(n)} \in \mathbb{N}_C(\bar{\pi})$ satisfying

$$0 = \lim_{n\to\infty} \Big[\sum_{i\in\Im_m(\bar{\pi})} \bar{\beta}_i \zeta^{(n)}_i + \sum_{j\in\Im_p} \bar{\rho}_j \mu^{(n)}_j + \xi^{(n)} \Big],$$

which indicates that

$$0 = \lim_{n\to\infty} \Big[\sum_{i\in\Im_m(\bar{\pi})} \bar{\beta}_i \langle \zeta^{(n)}_i, \omega_0\rangle + \sum_{j\in\Im_p} \bar{\rho}_j \langle \mu^{(n)}_j, \omega_0\rangle + \langle \xi^{(n)}, \omega_0\rangle \Big];\ (\forall \vartheta \in X). \tag{8}$$

Using the constraint qualification (CQ) and the fact that $\sum_{i\in\Im_m(\bar{\pi})} \bar{\beta}_i = 1$, we get

$$\begin{aligned}
&\lim_{n\to\infty} \Big[\sum_{i\in\Im_m(\bar{\pi})} \bar{\beta}_i \langle \zeta^{(n)}_i, \omega_0\rangle + \sum_{j\in\Im_p} \bar{\rho}_j \langle \mu^{(n)}_j, \omega_0\rangle + \langle \xi^{(n)}, \omega_0\rangle \Big] \\
&\quad \leqq \lim_{n\to\infty} \Big[\sum_{i\in\Im_m(\bar{\pi})} \bar{\beta}_i \langle \zeta^{(n)}_i, \omega_0\rangle + \sum_{j\in\Im_p} \bar{\rho}_j \langle \mu^{(n)}_j, \omega_0\rangle \Big] \\
&\quad \leqq - \sum_{i\in\Im_m(\bar{\pi})} \bar{\beta}_i a_i < 0,
\end{aligned}$$

which contradicts (8). Hence we can conclude that $\sum_{k\in\Im_p} \bar{\alpha}^L_k + \bar{\alpha}^U_k \neq 0$. □

To determine the component (nonzero) of Lagrange multipliers relative to the objective function, one has to consider the following constraint qualification (stronger Mangasarian- Fromovitz-type) (SCQ): there exist $s \in \Im_p,\ \omega_0 \in \mathbb{T}_C(\bar{\pi})$ and numbers $a_i > 0$ $(i \in \Im_m(\bar{\pi}))$, $b_k > 0$ $(s \neq k \in \Im_p)$ satisfying the following conditions

(*i*) $\langle \zeta_i, \omega_0\rangle \leqq -a_i$ $(\forall\ \zeta_i \in \partial^*\phi_i(\bar{\pi}),\ \forall i \in \Im_m(\bar{\pi}))$, $\langle \varkappa^L_k, \omega_0\rangle \leqq -b^L_k$ $(\forall\ \varkappa^L_k \in \partial^*\aleph^L_{k,\bar{\pi}}(\bar{\pi}))$, $\langle \varkappa^U_k, \omega_0\rangle \leqq -b^U_k$ $(\forall\ \varkappa^U_k \in \partial^*\aleph^U_{k,\bar{\pi}}(\bar{\pi})$, $\forall s \neq k \in \Im_p)$, and

(*ii*) $\langle \mu_j, \omega_0\rangle = 0$ $(\forall\ \mu_j \in \partial^*\psi_j(\bar{\pi}),\ \forall j \in \mathfrak{L}_l)$.

Remark 4.2.

(a) (SCQ) $\Longrightarrow$ (CQ).

(b) If (SCQ) holds, then for some element $s \in \Im_p$, there exist $\bar{\alpha}^L_s,\ \bar{\alpha}^U_s > 0$, and $\bar{\alpha}^L_k,\ \bar{\alpha}^U_k \geqq 0$ for $k \neq s$ in $\Im_p,\ \bar{\beta}_i \geqq 0$ for all $i \in \Im_m(\bar{\pi})$, and $\rho_j \in \mathbb{R}$ for all j in $\mathfrak{L}_l$ with

$$\begin{aligned}
0 \ \in \ \mathrm{cl}\Big(&\sum_{k\in\Im_p} \Big(\bar{\alpha}^L_k \ \mathrm{conv}\ \partial\aleph^L_k(\bar{\pi}) + \bar{\alpha}^U_k \ \mathrm{conv}\ \partial\aleph^U_k(\bar{\pi})\Big) + \sum_{i\in\Im_m(\bar{\pi})} \bar{\beta}_i \ \mathrm{conv}\ \partial^*\phi_i(\bar{\pi}) \\
&+ \sum_{j\in\mathfrak{L}_l} \bar{\rho}_j \ \mathrm{conv}\ \partial\psi_j(\bar{\pi}) + \mathbb{N}_C(\bar{\pi})\Big).
\end{aligned} \tag{9}$$

Theorem 4.3. *Suppose the solution $\bar{\pi}$ is a local weakly LU-Pareto to the problem* (CIMP) *and the presumptions of Theorem 3.9 are satisfied. Under the given constraint qualification (SCQ), there exist $\bar{\alpha}^L$, $\bar{\alpha}^U \geq 0$, where $\bar{\alpha}^L = (\bar{\alpha}_1^L, \dots, \bar{\alpha}_p^L)$, $\bar{\alpha}^U = (\bar{\alpha}_1^U, \dots, \bar{\alpha}_p^U)$, $\bar{\beta}_i \geqq 0$ for all i in $\mathfrak{I}_m(\bar{\pi})$, and $\bar{\rho}_j \in \mathbb{R}$ for all j in $\mathfrak{L}_l$ satisfying*

$$0 \in \mathrm{cl}\Big(\sum_{k \in \mathfrak{I}_p} \big(\bar{\alpha}_k^L \operatorname{conv} \partial \aleph_k^L(\bar{\pi}) + \bar{\alpha}_k^U \operatorname{conv} \partial \aleph_k^U(\bar{\pi}) \big) + \sum_{i \in \mathfrak{I}_m(\bar{\pi})} \bar{\beta}_i \operatorname{conv} \partial^* \phi_i(\bar{\pi}) + \sum_{j \in \mathfrak{L}_l} \bar{\rho}_j \operatorname{conv} \partial \psi_j(\bar{\pi}) + \mathbb{N}_C(\bar{\pi}) \Big). \tag{10}$$

Proof. In view of Remark 4.2, for some s in $\mathfrak{I}_p$, there exist $\alpha_s^{(s)L}$, $\alpha_s^{(s)U} > 0$, $\alpha_k^{(s)L}$, $\alpha_k^{(s)U} \geqq 0$ for all ($s \neq k \in \mathfrak{I}_p$), $\beta_i^{(s)} \geqq 0$ for all i in $\mathfrak{I}_m(\bar{\pi})$, $\rho_j^{(s)} \in \mathbb{R}$ for all j in $\mathfrak{L}_l$ such that

$$0 \in \mathrm{cl}\Big(\sum_{k \in \mathfrak{I}_p} \big(\alpha_k^{(s)L} \operatorname{conv} \partial \aleph_k^L(\bar{\pi}) + \alpha_k^{(s)U} \operatorname{conv} \partial \aleph_k^U(\bar{\pi}) \big) + \sum_{i \in \mathfrak{I}_m(\bar{\pi})} \beta_i^{(s)} \operatorname{conv} \partial^* \phi_i(\bar{\pi}) + \sum_{j \in \mathfrak{L}_l} \rho_j^{(s)} \operatorname{conv} \partial \psi_j(\bar{\pi}) + \mathbb{N}_C(\bar{\pi}) \Big). \tag{11}$$

Putting $s = 1, \dots, p$ and adding both sides of equation (11), we get

$$\begin{aligned} 0 \in & \sum_{s \in \mathfrak{I}_p} \mathrm{cl}\Big(\sum_{k \in \mathfrak{I}_p} \big(\alpha_k^{(s)L} \operatorname{conv} \partial \aleph_k^L(\bar{\pi}) + \alpha_k^{(s)U} \operatorname{conv} \partial \aleph_k^U(\bar{\pi}) \big) \\ & + \sum_{i \in \mathfrak{I}_m(\bar{\pi})} \beta_i^{(s)} \operatorname{conv} \partial^* \phi_i(\bar{\pi}) + \sum_{j \in \mathfrak{L}_l} \rho_j^{(s)} \operatorname{conv} \partial \psi_j(\bar{\pi}) + \mathbb{N}_C(\bar{\pi}) \Big). \\ & \subseteq \mathrm{cl}\Big(\sum_{k \in \mathfrak{I}_p} \big(\bar{\alpha}_k^L \operatorname{conv} \partial \aleph_k^L(\bar{\pi}) + \bar{\alpha}_k^U \operatorname{conv} \partial \aleph_k^U(\bar{\pi}) \big) + \sum_{i \in \mathfrak{I}_m(\bar{\pi})} \bar{\beta}_i \operatorname{conv} \partial^* \phi_i(\bar{\pi}) \\ & + \sum_{j \in \mathfrak{L}_l} \bar{\rho}_j \operatorname{conv} \partial \psi_j(\bar{\pi}) + \mathbb{N}_C(\bar{\pi}) \Big), \end{aligned}$$

where $\bar{\alpha}_k^L = \alpha_s^{(s)L} + \sum_{s \in \mathfrak{I}_p, s \neq k} \alpha_k^{(s)L} > 0$, $\bar{\alpha}_k^U = \alpha_s^{(s)U} + \sum_{s \in \mathfrak{I}_p, s \neq k} \alpha_k^{(s)U} > 0$, for some s in $\mathfrak{I}_p$, and $\bar{\alpha}_k^L$, $\bar{\alpha}_k^U \geqq 0$ for $k \neq s \in \mathfrak{I}_p$, $\bar{\beta}_i = \sum_{s \in \mathfrak{I}_p} \beta_i^{(s)} \geqq 0$ for all i in $\mathfrak{I}_m(\bar{\pi})$, $\bar{\rho}_j = \sum_{s \in \mathfrak{I}_p} \rho_j{}^{(s)} \in \mathbb{R}$ for all j in $\mathfrak{L}_l$. Thus, the proof is complete. □

5. Sufficiency criteria of local weakly LU-Pareto solutions

Definition 5.1. *A function Ξ defined on X having upper convexificator $\partial^* \Xi(\bar{\pi})$ is known as asymptotic pseudoconvex at a point $\bar{\pi}$ corresponding to C if for some $\pi_n^* \in \operatorname{conv} \partial^* \Xi(\bar{\pi})$, the following condition is satisfied*

$$\lim_{n \to \infty} \langle \pi_n^*, \pi - \bar{\pi} \rangle \geqq 0 \implies \Xi(\pi) \geqq \Xi(\bar{\pi}), \ \forall \, \pi \in C.$$

Definition 5.2. *A function Ξ defined on X having upper convexificator $\partial^* \Xi(\bar{\pi})$ is known as asymptotic quasiconvex at a point $\bar{\pi}$ corresponding to C if for some $\pi_n^* \in \operatorname{conv} \partial^* \Xi(\bar{\pi})$, the following condition is satisfied*

$$\Xi(\pi) \leqq \Xi(\bar{\pi}) \implies \lim_{n \to \infty} \langle \pi_n^*, \pi - \bar{\pi} \rangle \leqq 0, \ \forall \, \pi \in C.$$

A function Ξ is known as asymptotic quasiconcave at a point $\bar{\pi}$ corresponding to C if $-\Xi$ is asymptotic quasiconvex at a point $\bar{\pi}$ corresponding to C.

Definition 5.3. *A asymptotic quasilinear function Ξ at a point $\bar{\pi}$ is characterized by the fact that it is both asymptotic quasiconcave as well as asymptotic quasiconvex at a point $\bar{\pi}$ corresponding to C.*

Theorem 5.4. *If the solution $\bar{\pi} \in M$ is local weakly LU-Pareto to the problem* (CIMP)*, it holds the following two conditions*

(*i*) *there exist $\bar{\alpha}^L$, $\bar{\alpha}^U \geq 0$, where $\bar{\alpha}^L = (\bar{\alpha}_1^L, \ldots, \bar{\alpha}_p^L)$, $\bar{\alpha}^U = (\bar{\alpha}_1^U, \ldots, \bar{\alpha}_p^U)$, $\bar{\beta}_i \geqq 0$ for all i in $\Im_m(\bar{\pi})$, and $\bar{\rho}_j \in \mathbb{R}$ for all j in $\mathfrak{L}_l$ satisfying*

$$0 \in \operatorname{cl}\Big(\sum_{k\in\Im_p} \left(\bar{\alpha}_k^L \operatorname{conv} \partial^* \aleph_k^L(\bar{\pi}) + \bar{\alpha}_k^U \operatorname{conv} \partial^* \aleph_k^U(\bar{\pi})\right) + \sum_{i\in\Im_m(\bar{\pi})} \bar{\beta}_i \operatorname{conv} \partial^* \phi_i(\bar{\pi}) + \sum_{j\in\mathfrak{L}_l} \bar{\rho}_j \operatorname{conv} \partial^* \psi_j(\bar{\pi}) + \mathbb{N}_C(\bar{\pi})\Big). \tag{12}$$

(*ii*) *$\partial^* \aleph_k^L(\bar{\pi})$ and $\partial^* \aleph_k^U(\bar{\pi})$ ($k \in \Im_p$) are upper regular at the point $\bar{\pi}$ for at most one of the upper convexificators, the function $\bar{\alpha}_k \widetilde{\aleph}_k := \sum_{k\in\Im_p} \left(\bar{\alpha}_k^L \aleph_k^L + \bar{\alpha}_k^U \aleph_k^U\right)$ is asymptotic pseudoconvex at the point $\bar{\pi}$ corresponding to M, ϕ_i are asymptotic quasiconvex at the point $\bar{\pi}$ corresponding to M ($\forall\ i \in \Im_m(\bar{\pi})$), ψ_j are asymptotic quasilinear at point $\bar{\pi}$ corresponding to M ($\forall\ j \in \mathfrak{L}_l$) and C is convex.*

Proof. From (12), we may conclude that $\varkappa_k^{(n)L} \in \operatorname{conv} \partial^* \aleph_k^L(\bar{\pi})$, $\varkappa_k^{(n)U} \in \operatorname{conv} \partial^* \aleph_k^U(\bar{\pi})$, $\zeta_i^{(n)} \in \operatorname{conv} \partial^* \phi_i(\bar{\pi})$, $\mu_j^{(n)} \in \operatorname{conv} \partial^* \psi_j(\bar{\pi})$, $\xi^{(n)} \in \mathbb{N}_C(\bar{\pi})$ satisfy

$$0 = \lim_{n\to\infty} \Big[\sum_{k\in\Im_p} \left(\bar{\alpha}_k^L \varkappa_k^{(n)L} + \bar{\alpha}_k^U \varkappa_k^{(n)U}\right) + \sum_{i\in\Im_m(\bar{\pi})} \bar{\beta}_i \zeta_i^{(n)} + \sum_{j\in\mathfrak{L}_l} \bar{\rho}_j \mu_j^{(n)} + \xi^{(n)} \Big],$$

which ensures that

$$\lim_{n\to\infty} \Big[\Big\langle \sum_{k\in\Im_p} (\bar{\alpha}_k^L \varkappa_k^{(n)L} + \bar{\alpha}_k^U \langle \varkappa_k^{(n)U}), \pi - \bar{\pi} \Big\rangle + \sum_{i\in\Im_m(\bar{\pi})} \bar{\beta}_i \langle \zeta_i^{(n)}, \pi - \bar{\pi}\rangle + \sum_{j\in\mathfrak{L}_l} \bar{\rho}_j \langle \mu_j^{(n)}, \pi - \bar{\pi}\rangle + \langle \xi^{(n)}, \pi - \bar{\pi}\rangle \Big] = 0;\ \forall \pi \in M. \tag{13}$$

Since for all $\pi \in M$, $\phi_i(\pi) \leqq 0 = \phi_i(\bar{\pi})$, $\forall\ i \in \Im_m(\bar{\pi})$, therefore by applying asymptotic quasiconvexity to function ϕ_i at point $\bar{\pi}$, we get

$$\lim_{n\to\infty} \langle \zeta_i^{(n)}, \pi - \bar{\pi}\rangle \leqq 0,\ \forall \pi \in M. \tag{14}$$

Moreover, since $\psi_j(\pi) = 0 = \psi_j(\bar{\pi})$, so using the definition of asymptotic quasilinearity, we obtain

$$\lim_{n\to\infty} \langle \mu_i^{(n)}, \pi - \bar{\pi}\rangle = 0,\ \forall \pi \in M. \tag{15}$$

Due to the fact that C is convex, $\pi - \bar{\pi} \in \mathbb{T}_C(\bar{\pi})$, $\forall\ \pi \in C$, and, hence

$$\lim_{n\to\infty} \langle \xi_i^{(n)}, \pi - \bar{\pi}\rangle \leqq 0,\ \forall \pi \in M. \tag{16}$$

By linking up (13)-(16), we have

$$\lim_{n\to\infty} \Big\langle \sum_{k\in\Im_p} \bar{\alpha}_k^L \varkappa_k^{(n)L}, \pi - \bar{\pi} + \sum_{k\in\Im_p} \bar{\alpha}_k^U \varkappa_k^{(n)U}, \pi - \bar{\pi} \Big\rangle \geqq 0.$$

As $\partial^*\aleph_k^L(\bar{\pi})$ and $\partial^*\aleph_k^U(\bar{\pi})$ $(k \in \mathfrak{I}_p)$ are upper regular at point $\bar{\pi}$ for at most one of the upper convexificators, therefore the function $\sum_{k\in\mathfrak{I}_p}(\bar{\alpha}_k^L\aleph_k^L + \bar{\alpha}_k^U\aleph_k^U)$ has an upper convexificator $\sum_{k\in\mathfrak{I}_p}(\bar{\alpha}_k^L\partial^*\aleph_k^L + \bar{\alpha}_k^U\partial^*\aleph_k^U)$ at a point $\bar{\pi}$. Now, using asymptotic pseudoconvexity of $\sum_{k\in\mathfrak{I}_p}(\bar{\alpha}_k^L\aleph_{\bar{\pi}}^L + \bar{\alpha}_k^U\aleph_{\bar{\pi}}^U)$, we get

$$\sum_{k\in\mathfrak{I}_p}(\bar{\alpha}_k^L\aleph_k^L(\pi) + \bar{\alpha}_k^U\aleph_k^U(\pi)) \geqq \sum_{k\in\mathfrak{I}_p}(\bar{\alpha}_k^L\aleph_k^L(\bar{\pi}) + \bar{\alpha}_k^U\aleph_k^U(\bar{\pi})),\ (\forall \pi \in M),$$

which gives that

$$\bar{\alpha}_k\widetilde{\aleph}_k(\pi) \geqq \bar{\alpha}_k\widetilde{\aleph}_k(\bar{\pi}).$$

Since $\bar{\alpha}^L \geq 0$ and $\bar{\alpha}^U \geq 0$, there is no $\pi \in M$ satisfying

$$\aleph_k^L(\pi) < \aleph_k^L(\bar{\pi}), \qquad \aleph_k^U(\pi) < \aleph_k^U(\bar{\pi}),$$

$$\aleph_k^L(\pi) \leqq \aleph_k^L(\bar{\pi}), \qquad \aleph_k^U(\pi) < \aleph_k^U(\bar{\pi}),$$

$$\aleph_k^L(\pi) < \aleph_k^L(\bar{\pi}), \qquad \aleph_k^U(\pi) \leqq \aleph_k^U(\bar{\pi}).$$

Hence, the solution $\bar{\pi}$ is a weakly LU-Pareto to (CIMP). □

6. Duality

Mond-Weir type dual formulation for (CIMP):

$$\text{(MDCIMP)} \qquad \text{maximize} \qquad \aleph(\sigma) = \Big(\aleph_1(\sigma), \aleph_2(\sigma), \ldots, \aleph_p(\sigma)\Big)$$

$$= \Big([\aleph_1^L(\sigma), \aleph_1^U(\sigma)], [\aleph_2^L(\sigma), \aleph_2^U(\sigma)], \ldots, [\aleph_p^L(\sigma), \aleph_p^U(\sigma)]\Big),$$

$$\text{s.t.} \qquad 0 \in \operatorname{cl}\Big(\sum_{k\in\mathfrak{I}_p}\alpha_k^L \operatorname{conv}\partial^*\aleph_k^L(\sigma) + \alpha_k^U \operatorname{conv}\partial^*\aleph_k^U(\sigma)$$

$$+ \sum_{i\in\mathfrak{I}_m(\sigma)}\beta_i \operatorname{conv}\partial^*\phi_i(\sigma) + \sum_{j\in\mathfrak{L}_l}\rho_j \operatorname{conv}\partial^*\psi_j(\sigma) + \mathbb{N}_C(\sigma)\Big),$$

$$\beta_i\phi_i(\sigma) \geqq 0\ (i \in \mathfrak{I}_m(\sigma)),$$

$$\rho_j\psi_j(\sigma) = 0\ (j = \{1, 2, \ldots, l\}),$$

$$\bar{\alpha}^L \geq 0;\ \bar{\alpha}^L = (\bar{\alpha}_1^L, \ldots, \bar{\alpha}_p^L),$$

$$\bar{\alpha}^U \geq 0;\ \bar{\alpha}^U = (\bar{\alpha}_1^U, \ldots, \bar{\alpha}_p^U),$$

$$\beta_i \geqq 0\ (i \in \mathfrak{I}_m(\sigma)),$$

$$\beta_r = 0\ (r \notin \mathfrak{I}_m(\sigma)),$$

$$\rho_j \in \mathbb{R}\ (j \in \mathfrak{L}_l),$$

$$\sigma \in C.$$

Let M_1 be the set of all feasible solutions to the problem (MDCIMP). Define $\alpha^L := (\alpha_k^L)_{k\in\mathfrak{I}_p}$, $\alpha^U := (\alpha_k^U)_{k\in\mathfrak{I}_p}$, $\beta := (\beta_i)_{i\in\mathfrak{I}_m}$, and $\rho := (\rho_j)_{j\in\mathfrak{L}_l}$.

We construct a Mond-Weir type dual model in the following example for our discussions.

Example 6.1. *Let* $X = \mathbb{R}^2$, $C = [0,2] \times [0,2]$. *Let* $\aleph_k$ $(k = 1,2)$ *be mapping from* $\mathbb{R}^2$ *into* $\mathbb{S}$ *defined by* $\aleph_k(\pi) = [\aleph_k^L(\pi), \aleph_k^U(\pi)]$ *such that*

$$\aleph_1^L(\pi) = \begin{cases} \pi_1^4 \cos\frac{1}{\pi_1} - \pi_1^5 - \pi_1, & \pi_1 \neq 0 \\ -2, & \pi_1 = 0 \end{cases}$$

$$\aleph_1^U(\pi) = \begin{cases} \pi_2^3 \sin\frac{1}{\pi_2} + \pi_2^4 + \pi_2, & \pi_2 \neq 0 \\ -\sin\pi_2 + 1, & \pi_2 = 0 \end{cases}$$

$$\aleph_2^L(\pi) = \begin{cases} -\pi_1^3 + \pi_2^2 - 3, & \pi_1 \geqq 0 \\ \pi_1^3 + \pi_2^2 + \pi_1 - 3, & \pi_1 < 0 \end{cases}$$

$$\aleph_2^U(\pi) = \begin{cases} \pi_1 + \pi_2^5 + 5, & \pi_1 \geqq 0 \\ -\pi_1 + \pi_2^5 + 5, & \pi_1 < 0 \end{cases}$$

for $\pi = (\pi_1, \pi_2) \in \mathbb{R}^2$. *Define functions* $\phi, \psi : \mathbb{R}^2 \to \mathbb{R}$ *by*

$$\phi(\pi) = 1 - e^{\pi_1},$$

$$\psi(\pi) = \pi_1 - \pi_2.$$

Then, $\aleph_k^L(\pi) \leqq \aleph_k^U(\pi)$ *for* $k = 1,2$. *It is clear that the point* $\bar{\pi} = (0,0)$ *is a weakly LU-Pareto solution of the interval-valued multiobjective programming problem given by*

$$\begin{aligned} \text{minimize} \quad & \aleph(\pi) = (\aleph_1(\pi), \aleph_2(\pi)) \\ & \quad = \left([\aleph_1^L(\pi), \aleph_1^U(\pi)], [\aleph_2^L(\pi), \aleph_2^U(\pi)]\right), \\ \text{s.t.} \quad & \pi \in M := \{\pi \in [0,2] \times [0,2] : \phi(\pi) \leqq 0, \psi(\pi) = 0\}. \end{aligned}$$

We have $M = \{(\pi_1, \pi_2) \in \mathbb{R}^2 : \pi_1 = \pi_2,\ 0 \leqq \pi_2 \leqq 2\}$. *The Mond-Weir dual problem of (MDCIMP) where* $\sigma = (\sigma_1, \sigma_2) \in [0,2] \times [0,2]$, $\mathbb{T}_C(\sigma) = \mathbb{R}^2_+$, $\mathbb{N}_C(\sigma) = \mathbb{R}^2_-$, *and*

$$\partial^*\aleph_1^L(\sigma) = \begin{cases} 4\sigma_1^3 \cos\frac{1}{\sigma_1} + \sigma_1^2 \sin\frac{1}{\sigma_1} - 5\sigma_1^4 - 1, & \sigma_1 > 0 \\ (-1,0), (0,0), & \sigma_1 = 0 \end{cases}$$

$$\partial^*\aleph_1^U(\sigma) = \begin{cases} (0,1), & \sigma_2 > 0 \\ (0,-1), (0,1), & \sigma_2 = 0 \end{cases}$$

$$\partial^*\aleph_2^L(\sigma) = \begin{cases} -3\sigma_1^2,\ 2\sigma_2, & \sigma_1 > 0 \\ (0,0), (1,0), & \sigma_1 = 0 \end{cases}$$

$$\partial^*\aleph_2^U(\sigma) = \begin{cases} (1,0), & \sigma_1 > 0 \\ (1,0), (-1,0), & \sigma_1 = 0 \end{cases}$$

$$\partial^*\phi(\sigma) = \{(-1,0)\}$$

$$\partial^*\psi(\sigma) = \{(1,-1)\}.$$

Theorem 6.2. (Weak duality) *Let* π *be any feasible point to the problem (CIMP) and* $(\sigma, \alpha^L, \alpha^U, \beta, \rho)$ *be the feasible point to its Mond-Weir dual (MDCIMP), respectively. Furthermore, suppose that* (*i*) $\partial^*\aleph_k^L(\sigma)$, $\partial^*\aleph_k^U(\sigma)$ $(k \in \mathfrak{J}_p)$, $\partial^*\beta_1\phi_1(\sigma), \ldots, \partial^*\beta_m\phi_m(\sigma), \partial^*\rho_1\psi_1(\sigma), \ldots, \partial^*\rho_l\psi_l(\sigma)$ *are the upper convexificator of the functions* $\aleph_k^L$, $\aleph_k^U$ $(k \in \mathfrak{J}_p)$, $\beta_1\phi_1, \ldots, \beta_m\phi_m, \rho_1\psi_1, \ldots, \rho_l\psi_l$ *at a point* σ*, respectively, and one of the upper convexificators of* $\partial^*\aleph_k^L(\sigma)$, $\partial^*\aleph_k^U(\sigma)$ *is upper regular,*

(*ii*) *the function* $\sum_{k\in\mathfrak{J}_p}\left(\alpha_k^L\aleph_k^L + \alpha_k^U\aleph_k^U\right)$ *is asymptotic pseudoconvex at point* σ *on* C, $\beta_i\phi_i$ *is asymptotic quasiconvex at point* σ *on* C $(\forall\ i \in \mathfrak{J}_m(\sigma))$, $\rho_j\psi_j$ *are asymptotic quasilinear at point* σ *on* C *for all* j *in* $\mathfrak{L}_l$, C *is convex.*

Then $\aleph_k(\pi) \nless_{LU} \aleph_k(\sigma)$.

Proof. Since $(\sigma, \alpha^L, \alpha^U, \beta, \rho) \in M_1$, there exist $\varkappa_k^{(n)L} \in \text{conv}\ \partial^*\aleph_k^L(\sigma)$, $\varkappa_k^{(n)U} \in \text{conv}\partial^*\aleph_k^U(\sigma)$ $(k \in \mathfrak{J}_p)$, $\zeta_i^{(n)} \in \text{conv}\partial^*(\beta_i\phi_i)(\sigma)$ $(i \in \mathfrak{J}_m(\sigma))$, $\mu_j^{(n)} \in \text{conv}\partial^*(\rho_j\psi_j(\sigma))$ $(j \in \mathfrak{L}_l)$, $\xi^{(n)} \in \mathbb{N}_C(\sigma)$ satisfying

$$0 = \lim_{n\to\infty}\Bigg[\sum_{k\in\mathfrak{J}_p}\left(\alpha_k^L\varkappa_k^{(n)L} + \alpha_k^U\varkappa_k^{(n)U}\right) + \sum_{i\in\mathfrak{J}_m(\sigma)}\zeta_i^{(n)} + \sum_{j\in\mathfrak{L}_l}\mu_j^{(n)} + \xi^{(n)}\Bigg],$$

therefore, for $\pi \in M$, we obtain

$$\lim_{n\to\infty}\left\langle\sum_{k\in\mathfrak{J}_p}(\alpha_k^L\varkappa_k^{(n)L} + \alpha_k^U\varkappa_k^{(n)U}), \pi - \sigma\right\rangle + \sum_{i\in\mathfrak{J}_m(\sigma)}\lim_{n\to\infty}\langle\zeta_i^{(n)}, \pi - \sigma\rangle$$
$$+ \sum_{j\in\mathfrak{L}_l}\lim_{n\to\infty}\langle\mu_j^{(n)}, \pi - \sigma\rangle + \lim_{n\to\infty}\langle\xi^{(n)}, \pi - \sigma\rangle = 0. \tag{17}$$

For any $\pi \in M$, we have $\beta_i\phi_i(\pi) \leqq 0 \leqq \beta_i\phi_i(\sigma)$ $(\forall\ i \in \mathfrak{J}_m(\sigma))$. Applying asymptotic quasiconvexity to functions $\beta_i\phi_i$ at point σ $(\forall i \in \mathfrak{J}_m(\sigma))$, we get

$$\lim_{n\to\infty}\langle\zeta_i^{(n)}, \pi - \sigma\rangle \leqq 0,\ \forall\pi \in M. \tag{18}$$

Moreover, we have $\rho_j\psi_j(\pi) = 0 = \rho_j\psi_j(\sigma)$, and therefore, using the definition of asymptotic quasilinearity, we obtain

$$\lim_{n\to\infty}\langle\mu_i^{(n)}, \pi - \sigma\rangle = 0,\ \forall\pi \in M. \tag{19}$$

Due to the convexity of C, we get

$$\lim_{n\to\infty}\langle\xi_i^{(n)}, \pi - \sigma\rangle \leqq 0,\ \forall\pi \in M. \tag{20}$$

By linking up (17)-(20), we have

$$\lim_{n\to\infty}\left\langle\sum_{k\in\mathfrak{J}_p}(\alpha_k^L\varkappa_k^{(n)L} + \alpha_k^U\varkappa_k^{(n)U}), \pi - \sigma\right\rangle \geqq 0.$$

As $\partial^*\aleph_k^L(\sigma)$ and $\partial^*\aleph_k^U(\sigma)$ $(k \in \mathfrak{J}_p)$ are upper regular at point σ for at most one of the upper convexificators, it is clear that the function $\sum_{k\in\mathfrak{J}_p}(\alpha_k^L\aleph_k^L(\sigma) + \alpha_k^U\aleph_k^U(\sigma))$ has an upper convexificator $\sum_{k\in\mathfrak{J}_p}(\alpha_k^L\partial^*\aleph_k^L + \alpha_k^U\partial^*\aleph_k^U)$ at a point σ, and therefore, using asymptotic pseudoconvexity of $\sum_{k\in\mathfrak{J}_p}(\alpha_k^L\aleph_k^L + \alpha_k^U\aleph_k^U)$, we obtain

$$\sum_{k\in\mathfrak{J}_p}(\alpha_k^L\aleph_k^L(\pi) + \alpha_k^U\aleph_k^U(\pi)) \geqq \sum_{k\in\mathfrak{J}_p}(\alpha_k^L\aleph_k^L(\sigma) + \alpha_k^U\aleph_k^U(\sigma)),\ \forall\pi \in M, \tag{21}$$

which gives

$$\aleph_k(\pi) \nless_{LU} \aleph_k(\sigma). \tag{22}$$

In case it is not so, then

$$\aleph_k(\pi) <_{LU} \aleph_k(\sigma),$$

which is equivalent to any one of the three given relations

$$\aleph_k^L(\pi) < \aleph_k^L(\sigma), \qquad \aleph_k^U(\pi) < \aleph_k^U(\sigma),$$

$$\aleph_k^L(\pi) \leqq \aleph_k^L(\sigma), \qquad \aleph_k^U(\pi) < \aleph_k^U(\sigma),$$

$$\aleph_k^L(\pi) < \aleph_k^L(\sigma), \qquad \aleph_k^U(\pi) \leqq \aleph_k^U(\sigma).$$

Since $\alpha^L,\ \alpha^U \geq 0$ and from the above inequalities, we get the contradiction to inequality (21), and therefore, we conclude that the inequality (22) holds. □

Theorem 6.3. (Strong duality) *Let the solution $\bar{\pi}$ be local weakly LU-Pareto to* (CIMP). *Suppose that all the assumptions of* Theorem 4.3 *are satisfied. Then there exist $\bar{\alpha}^L,\ \bar{\alpha}^U \geq 0$, where $\bar{\alpha}^L = (\bar{\alpha}_1^L, \ldots, \bar{\alpha}_p^L)$, $\bar{\alpha}^U = (\bar{\alpha}_1^U, \ldots, \bar{\alpha}_p^U)$, $\bar{\beta}_i \geqq 0$ for all i in $\Im_m(\bar{\pi})$, and $\bar{\rho}_j \in \mathbb{R}$ for all j in $\mathfrak{L}_l$ such as $(\bar{\pi}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\beta}, \bar{\rho})$ is a feasible solution to the dual problem (MDCIMP) giving the same objective value as that of the problem (CIMP) at their respective feasible point. Furthermore, if the second assumption of Theorem 5.4 is satisfied, then $(\bar{\pi}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\beta}, \bar{\rho})$ is a weakly LU-Pareto solution to (MDCIMP).*

Proof. As the solution $\bar{\pi}$ is local weakly LU-Pareto to (CIMP), therefore, by Theorem 4.3, we conclude that $\exists\ \bar{\alpha}^L \geq 0,\ \bar{\alpha}^U \geq 0,\ \bar{\beta}_i \geqq 0\ (\forall\ i \in \Im_m(\bar{\pi})),\ \bar{\rho}_j \in \mathbb{R}\ (\forall\ j \in \mathfrak{L}_l)$ satisfying

$$0 \ \in \ \mathrm{cl}\Big(\sum_{k \in \Im_p} \bar{\alpha}_k^L \ \mathrm{conv}\ \partial^* \aleph_k^L(\bar{\pi}) + \bar{\alpha}_k^U \ \mathrm{conv}\ \partial^* \aleph_k^U(\bar{\pi}) + \sum_{i \in \Im_m(\bar{\pi})} \bar{\beta}_i \ \mathrm{conv}\ \partial^* \phi_i(\bar{\pi}) + \sum_{j \in \mathfrak{L}_l} \bar{\rho}_j \ \mathrm{conv}\ \partial^* \psi_j(\bar{\pi}) + \mathbb{N}_C(\bar{\pi}) \Big).$$

If $i \notin \Im_m(\bar{\pi})$, and let $\bar{\beta}_i = 0$, then $(\bar{\pi}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\beta}, \bar{\rho})$ is a feasible point of the dual problem (MDCIMP), and the value of the objective functions of the problem (CIMP) at the feasible point $\bar{\pi}$ is the same as that of the value of the objective function of its corresponding dual problem (MDCIMP) at $(\bar{\pi}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\beta}, \bar{\rho})$. Furthermore, if assumption (ii) of Theorem 5.4 is satisfied, then all the hypotheses of Theorem 6.2 are also satisfied. Hence,

$$\aleph_k(\bar{\pi}) \not<_{LU} \aleph_k(\sigma),$$

for every feasible point $(\pi, \alpha^L, \alpha^U, \beta, \rho)$ of (MDCIMP). Consequently, there does not exist any feasible point $(\pi, \alpha^L, \alpha^U, \beta, \rho) \in M_1$ so that $\aleph_k(\bar{\pi}) <_{LU} \aleph_k(\sigma)$. Therefore, $(\bar{\pi}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\beta}, \bar{\rho})$ is a weakly LU-Pareto solution of the dual problem (MDCIMP). □

Wolfe-type dual formulation for (CIMP):

$$\text{(WDCIMP)} \qquad \text{maximize} \qquad \aleph(\sigma) + \sum_{i=1}^{m} \beta_i\ \phi_i(\sigma) + \sum_{j=1}^{l} \rho_j\ \psi_j(\sigma),$$

$$\text{s.t.} \qquad 0 \ \in \ \mathrm{cl}\Big(\sum_{k \in \Im_p} \alpha_k^L \ \mathrm{conv}\ \partial^* \aleph_k^L(\sigma) + \alpha_k^U \ \mathrm{conv}\ \partial^* \aleph_k^U(\sigma) + \sum_{i \in \Im_m(\sigma)} \beta_i \ \mathrm{conv}\ \partial^* \phi_i(\sigma) + \sum_{j \in \mathfrak{L}_l} \rho_j \ \mathrm{conv}\ \partial^* \psi_j(\sigma) + \mathbb{N}_C(\sigma) \Big),$$

$$\bar{\alpha}^L \geq 0;\ \bar{\alpha}^L = (\bar{\alpha}_1^L, \ldots, \bar{\alpha}_p^L),$$

$$\bar{\alpha}^U \geq 0;\ \bar{\alpha}^U = (\bar{\alpha}_1^U, \dots, \bar{\alpha}_p^U),$$
$$\alpha_k^L + \alpha_k^U = 1,$$
$$\beta_i \geqq 0\ (i \in \Im_m(\sigma)),$$
$$\beta_r = 0\ (r \notin \Im_m(\sigma)),$$
$$\rho_j \in \mathbb{R}\ (j \in \mathfrak{L}_l),$$
$$\sigma \in C.$$

Let M_2 denote the set to all feasible solutions of the problem (WDCIMP). We construct a Wolfe-type dual model in the following example for our discussions.

Example 6.4. *Let $X = \mathbb{R}^2$, $C = [0,2] \times [0,2]$, $\bar{\pi} = (0,0)$. Let $\aleph_k$ $(k = 1,2)$ be mapping from $\mathbb{R}^2$ into $\mathbb{S}$ defined by $\aleph_k(\pi) = [\aleph_k^L(\pi), \aleph_k^U(\pi)]$ such that*

$$\aleph_1^L(\pi) = \begin{cases} -5\pi_1^2 - \pi_2^2 - 4, & \pi_1 \geqq 0 \\ 5\pi_1 + \pi_2^2 - 2, & \pi_1 < 0 \end{cases}$$

$$\aleph_1^U(\pi) = \begin{cases} \pi_1 + 3\pi_2 + 8, & \pi_2 \geqq 0 \\ \pi_1 - 3\pi_2, & \pi_2 < 0 \end{cases}$$

$$\aleph_2^L(\pi) = \begin{cases} \pi_1^2 + \pi_2^2 - 9, & \pi_1 \geqq 0 \\ -\pi_1^2 + \pi_2 - 9, & \pi_1 < 0 \end{cases}$$

$$\aleph_2^U(\pi) = \begin{cases} \pi_1 + 3\pi_2 + 5, & \pi_1 \geqq 0 \\ -\pi_1 + 3\pi_2 + 5 & \pi_1 < 0 \end{cases}$$

for $\pi = (\pi_1, \pi_2) \in \mathbb{R}^2$. The functions $\phi_i : \mathbb{R}^2 \to \mathbb{R}$ $(i = 1,2)$ and $\psi : \mathbb{R}^2 \to \mathbb{R}$ are defined by

$$\phi_1(\pi) = \pi_2^2 + \pi_2$$
$$\phi_2(\pi) = \pi_1^2 + 2\pi_1$$
$$\psi(\pi) = \pi_1^2 - \pi_2.$$

It is clear that the point $\bar{\pi} = (0,0)$ is a weakly LU-Pareto solution to the multiobjective interval-valued programming problem:

$$\begin{aligned} \text{minimize} \quad & \aleph(\pi) = \big(\aleph_1(\pi), \aleph_2(\pi)\big) \\ & = \big([\aleph_1^L(\pi), \aleph_1^U(\pi)], [\aleph_2^L(\pi), \aleph_2^U(\pi)]\big) \\ \text{s.t.} \quad & \pi \in M := \{\pi \in [0,2] \times [0,2] : \pi_1^2 = \pi_2\}. \end{aligned}$$

The Wolfe-type dual formulation is identical to (WDCIMP), where $\sigma = (\sigma_1, \sigma_2) \in [0,2] \times [0,2]$, and $\mathbb{T}_C(\sigma) = \mathbb{R}_+^2$, $\mathbb{N}_C(\sigma) = \mathbb{R}_-^2$, and

$$\partial^* \aleph_1^L(\sigma) = \begin{cases} (-10\sigma_1, -2\sigma_2), & \sigma_1 > 0 \\ \{(0,0), (5,0)\}, & \sigma_1 = 0 \end{cases}$$

$$\partial^* \aleph_1^U(\sigma) = \begin{cases} (1,3), & \sigma_2 > 0 \\ (1,-3), (1,3), & \sigma_2 = 0 \end{cases}$$

$$\partial^*\aleph_2^L(\sigma) = \begin{cases} (2\sigma_1, 2\sigma_2), & \sigma_1 > 0 \\ (0,0), (0,1), & \sigma_1 = 0 \end{cases}$$

$$\partial^*\aleph_2^U(\sigma) = \begin{cases} (1,3), & \sigma_1 > 0 \\ (1,3), (-1,3), & \sigma_1 = 0 \end{cases}$$

$$\partial^*\phi_1(\sigma) = \{(0,1)\}$$

$$\partial^*\phi_2(\sigma) = \{(2,0)\}$$

$$\partial^*\psi(\sigma) = \{(0,-1)\}.$$

Theorem 6.5. (Weak duality) *Let the feasible solutions π to the problem (CIMP) and $(\sigma, \alpha^L, \alpha^U, \beta, \rho)$ to its Wolfe-type dual (WDCIMP) satisfy the following two conditions:*

(*i*) *$\partial^*\aleph_k^L(\sigma)$, $\partial^*\aleph_k^U(\sigma)$ $(k \in \mathfrak{I}_p)$ and $\partial^*\beta_1\phi_1(\sigma), \dots, \partial^*\beta_m\phi_m(\sigma)$ are the upper convexificators of the functions $\aleph_k^L$, $\aleph_k^U$ $(k \in \mathfrak{I}_p)$ and $\beta_1\phi_1, \dots, \beta_m\phi_m$ at a point σ, respectively, and at most one of them must be upper regular. The functions $\psi_1, \dots, \psi_l$ are Gâteaux differentiable at a point σ with corresponding Gâteaux derivatives $\nabla_G\psi_1(\sigma), \dots, \nabla_G\psi_l(\sigma)$,*

(*ii*) *The function $\sum_{k\in\mathfrak{I}_p} \alpha_k^L\aleph_k^L + \alpha_k^U\aleph_k^U + \sum_{i\in\mathfrak{I}_m(\sigma)} \beta_i\phi_i + \sum_{j\in\mathfrak{L}_l} \rho_j\psi_j$ is asymptotic pseudoconvex at point σ with respect to C.*

Then,

$$\aleph_k(\pi) \not<_{LU} \aleph_k(\sigma) + \sum_{i=1}^{m} \beta_i\ \phi_i(\sigma) + \sum_{j=1}^{l} \rho_j\ \psi_j(\sigma).$$

Proof. As per Proposition 3.1 of Jeyakumar and Luc [12], $\partial^*\psi_j(\sigma) := \{\nabla_G\psi_j(\sigma)\}$ is a convexificator that is both lower and upper regular of ψ_j at a point $\bar{\pi}$ $(\forall\ j \in \mathfrak{L}_l)$. Moreover, $(\sigma, \alpha^L, \alpha^U, \beta, \rho) \in M_2$, therefore, there exist $\varkappa_k^{(n)L} \in \text{conv}\ \partial^*\aleph_k^L(\sigma)$, $\varkappa_k^{(n)U} \in \text{conv}\partial^*\aleph_k^U(\sigma)$ $(k \in \mathfrak{I}_p)$, $\zeta_i^{(n)} \in \text{conv}\partial^*\phi_i(\sigma)(i \in \mathfrak{I}_m(\sigma))$, $\xi^{(n)} \in \mathbb{N}_C(\sigma)$ so that

$$0 = \lim_{n\to\infty}\Bigg[\sum_{k\in\mathfrak{I}_p}\left(\alpha_k^L\varkappa_k^{(n)L} + \alpha_k^U\varkappa_k^{(n)U}\right) + \sum_{i\in\mathfrak{I}_m(\sigma)} \beta_i\zeta_i^{(n)} + \sum_{j\in\mathfrak{L}_l} \rho_j\nabla_G\psi_j(\sigma) + \xi^{(n)}\Bigg].$$

Thus, for $\pi \in M$, we have

$$\lim_{n\to\infty}\Bigg[\Big\langle \sum_{k\in\mathfrak{I}_p}(\alpha_k^L\varkappa_k^{(n)L} + \alpha_k^U\varkappa_k^{(n)U}), \pi - \sigma\Big\rangle + \sum_{i\in\mathfrak{I}_m(\sigma)} \beta_j\langle\zeta_i^{(n)}, \pi - \sigma\rangle + \sum_{j\in\mathfrak{L}_l} \rho_j\langle\nabla_G\psi_j(\sigma), \pi - \sigma\rangle + \langle\xi^{(n)}, \pi - \sigma\rangle\Bigg] = 0. \tag{23}$$

Using the fact C is convex, we can conclude that

$$\lim_{n\to\infty}\langle\xi^{(n)}, \pi - \sigma\rangle \leqq 0;\ \forall\pi \in M. \tag{24}$$

Combining (23)-(24), we get

$$\lim_{n\to\infty}\Bigg[\Big\langle \sum_{k\in\mathfrak{I}_p}(\alpha_k^L\varkappa_k^{(n)L} + \alpha_k^U\varkappa_k^{(n)U}), \pi - \sigma\Big\rangle + \sum_{i\in\mathfrak{I}_m(\sigma)} \beta_j\langle\zeta_i^{(n)}, \pi - \sigma\rangle + \sum_{j\in\mathfrak{L}_l} \rho_j\langle\nabla_G\psi_j(\sigma), \pi - \sigma\rangle\Bigg] \geqq 0. \tag{25}$$

Since the function $\sum_{k\in\Im_p}(\alpha_k^L\aleph_k^L+\alpha_k^U\aleph_k^U)+\sum_{i=1}^m\beta_i\,\phi_i+\sum_{j=1}^l\rho_j\,\psi_j$ has an upper convexificator $\sum_{k\in\Im_p}(\alpha_k^L\partial^*\aleph_k^L(\sigma)+\alpha_k^U\partial^*\aleph_k^U(\sigma))+\sum_{i=1}^m\beta_i\,\partial^*\phi_i(\sigma)+\sum_{j=1}^l\rho_j\nabla_G\psi_j(\sigma)$ at a point σ, therefore, using asymptotic pseudoconvexity at $\sum_{k\in\Im_p}(\alpha_k^L\aleph_k^L+\alpha_k^U\aleph_k^U)+\sum_{i=1}^m\beta_i\,\phi_i+\sum_{j=1}^l\rho_j\,\psi_j$, for all $\pi\in M$, we obtain

$$\sum_{k\in\Im_p}(\alpha_k^L\aleph_k^L(\pi)+\alpha_k^U\aleph_k^U(\pi))+\sum_{i=1}^m\beta_i\,\phi_i(\pi)+\sum_{j=1}^l\rho_j\psi_j(\pi)$$

$$\geqq\sum_{k\in\Im_p}(\alpha_k^L\aleph_k^L(\sigma)+\alpha_k^U\aleph_k^U(\sigma))+\sum_{i=1}^m\beta_i\,\phi_i(\sigma)+\sum_{j=1}^l\rho_j\psi_j(\sigma).$$

The fact that $\phi_i(\pi)\leqq 0\ (\forall\ i\in(I)),\ \psi_j(\pi)=0\ (\forall\ j\in(L)),\ \beta_r=0\ (\forall r\notin\Im_m(\sigma))$ implies

$$\sum_{k\in\Im_p}(\alpha_k^L\aleph_k^L(\pi)+\alpha_k^U\aleph_k^U(\pi))\geqq\sum_{k\in\Im_p}(\alpha_k^L\aleph_k^L(\sigma)+\alpha_k^U\aleph_k^U(\sigma))+\sum_{i=1}^m\beta_i\,\phi_i(\sigma)+\sum_{j=1}^l\rho_j\psi_j(\sigma). \tag{26}$$

Thus, we have

$$\aleph_k(\pi)\not<_{LU}\aleph_k(\sigma)+\sum_{i=1}^m\beta_i\,\phi_i(\sigma)+\sum_{j=1}^l\rho_j\psi_j(\sigma). \tag{27}$$

In case it is not so, we get

$$\aleph_k(\pi)<_{LU}\aleph_k(\sigma)+\sum_{i=1}^m\beta_i\,\phi_i(\sigma)+\sum_{j=1}^l\rho_j\psi_j(\sigma),$$

which is equivalent to any one pair of conditions given below

$$\aleph_k^L(\pi)<\aleph_k^L(\sigma)+\sum_{i=1}^m\beta_i\,\phi_i(\sigma)+\sum_{j=1}^l\rho_j\psi_j(\sigma),$$

$$\aleph_k^U(\pi)<\aleph_k^U(\sigma)+\sum_{i=1}^m\beta_i\,\phi_i(\sigma)+\sum_{j=1}^l\rho_j\psi_j(\sigma), \tag{28}$$

or

$$\aleph_k^L(\pi)\leqq\aleph_k^L(\sigma)+\sum_{i=1}^m\beta_i\,\phi_i(\sigma)+\sum_{j=1}^l\rho_j\psi_j(\sigma),$$

$$\aleph_k^U(\pi)<\aleph_k^U(\sigma)+\sum_{i=1}^m\beta_i\,\phi_i(\sigma)+\sum_{j=1}^l\rho_j\psi_j(\sigma), \tag{29}$$

or

$$\aleph_k^L(\pi)<\aleph_k^L(\sigma)+\sum_{i=1}^m\beta_i\,\phi_i(\sigma)+\sum_{j=1}^l\rho_j\psi_j(\sigma),$$

$$\aleph_k^U(\pi)\leqq\aleph_k^U(\sigma)+\sum_{i=1}^m\beta_i\,\phi_i(\sigma)+\sum_{j=1}^l\rho_j\psi_j(\sigma). \tag{30}$$

Since $\alpha^L\geq 0$, $\alpha^U\geq 0$ and $\sum_{k\in\Im_p}\left(\alpha_k^L+\alpha_k^U\right)=1$, we get a contradiction to inequality (26) with the help of any one of the pair of inequalities mentioned above. Hence, inequality (27) holds, and the proof is complete. □

Theorem 6.6. (Strong duality) *Let the local weakly LU-Pareto solution $\bar{\pi}$ to the problem* (CIMP) *satisfies assumptions of Theorem 4.3. Then there exist $\bar{\alpha}^L$, $\bar{\alpha}^U \geq 0$, where $\bar{\alpha}^L = (\bar{\alpha}^L_1, \dots, \bar{\alpha}^L_p)$, $\bar{\alpha}^U = (\bar{\alpha}^U_1, \dots, \bar{\alpha}^U_p)$, $\bar{\beta}_i \geqq 0$ for all i in $\mathfrak{I}_m(\bar{\pi})$, and $\bar{\rho}_j \in \mathbb{R}$ for all j in $\mathfrak{L}_l$ such that $(\bar{\pi}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\beta}, \bar{\rho})$ becomes a feasible point to the dual problem (WDCIMP) and the value of the objective functions of the problem (CIMP) at the feasible point $\bar{\pi}$ and its dual problem (WDCIMP) at $(\bar{\pi}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\beta}, \bar{\rho})$ are equal. Furthermore, if assumptions of Theorem 6.5 are satisfied, then $(\bar{\pi}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\beta}, \bar{\rho})$ is a weakly LU-Pareto solution of (WDCIMP).*

Proof. Since the solution $\bar{\pi}$ is local weakly LU-Pareto to (CIMP), therefore, using Theorem 4.3, we conclude that there exist $\bar{\alpha}^L \geq 0$, $\bar{\alpha}^U \geq 0$, $\bar{\beta}_i \geqq 0$ $(\forall\, i \in \mathfrak{I}_m(\bar{\pi}))$, $\bar{\rho}_j \in \mathbb{R}$ $(\forall\, j \in \mathfrak{L}_l)$ satisfying the condition

$$0 \in \operatorname{cl}\Big(\sum_{k \in \mathfrak{I}_p} \bar{\alpha}^L_k \operatorname{conv} \partial^* \aleph^L_k(\bar{\pi}) + \bar{\alpha}^U_k \operatorname{conv} \partial^* \aleph^U_k(\bar{\pi}) + \sum_{i \in \mathfrak{I}_m(\bar{\pi})} \bar{\beta}_i \operatorname{conv} \partial^* \phi_i(\bar{\pi})$$

$$+ \sum_{j \in \mathfrak{L}_l} \bar{\rho}_j \operatorname{conv} \partial^* \psi_j(\bar{\pi}) + \mathbb{N}_C(\bar{\pi}) \Big).$$

Take $\bar{\beta}_i = 0$ for $i \notin \mathfrak{I}_m(\bar{\pi})$. Consequently, $(\bar{\pi}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\beta}, \bar{\rho})$ becomes a feasible point of the dual problem (WDCIMP), and the value of the objective functions of the problem (CIMP) at the feasible point $\bar{\pi}$ and its dual problem (WDCIMP) at $(\bar{\pi}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\beta}, \bar{\rho})$ are equal. Furthermore, it satisfies all the assumptions of Theorem 6.5; therefore, we can conclude that for every feasible point $(\pi, \alpha^L, \alpha^U, \beta, \rho)$ of (WDCIMP),

$$\aleph_k(\bar{\pi}) \nless_{LU} \aleph_k(\sigma) + \sum_{i=1}^{m} \beta_i\, \phi_i(\sigma) + \sum_{j=1}^{l} \rho_j \psi_j(\sigma). \tag{31}$$

Consequently, there does not exist any feasible point $(\pi, \alpha^L, \alpha^U, \beta, \rho) \in M_2$ satisfying

$$\aleph_k(\bar{\pi}) <_{LU} \aleph_k(\sigma) + \sum_{i=1}^{m} \beta_i\, \phi_i(\sigma) \sum_{j=1}^{l} \rho_j \psi_j(\sigma),$$

which contradicts inequality (31). Therefore, $(\bar{\pi}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\beta}, \bar{\rho})$ is a weakly LU-Pareto solution of the dual problem (WDCIMP). □

7. Conclusions

The objective of this paper is to study a nonsmooth interval-valued multiobjective programming problem for local weakly LU-Pareto solutions. It should be noted that Banach space has previously been used for interval-valued programming problems and that it is characterized by equality, inequality, and set constraints. Under the Mangasarian-Fromovitz-type constraint qualification (CQ), we were able to deduce the Karush-Kuhn-Tucker type necessary conditions for local weakly LU-Pareto solutions of (CIMP). Using the stronger Mangasarian-Fromovitz-type constraint qualification (SCQ), we find the nonzero component of Lagrange multipliers equivalent to the objective functions. The necessary KKT criteria will become sufficiency criteria under adequate assumptions. The Mond-Weir dual problems (MDCIMP) and Wolfe-type dual problems (WDCIMP) have been discussed, and weak and strong duality theorems have been derived.

Declarations

Availability of Data and Materials Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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