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Sufficiency and duality for complex multiobjective fractional programming involving cone constraints

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Abstract. This paper aims to investigate sufficiency and duality of a multiobjective fractional programming problem in complex space. The functions involved are ratio of two functions and the constraints are defined in cones. Firstly, the result asserting the ratio invexity in complex space has been discussed and the same has been supported by examples. This is followed by sufficiency conditions for the problem under consideration. Further, to illustrate this result, an example has been provided. It is worth mentioning that an efficient solution of the problem considered in the example is also obtained using Multiobjective Genetic Algorithm (MOGA) which alongwith the sufficiency conditions become more reliable. In the literature, various forms of duals for a fractional programming problem have been studied. Here, we have formulated a Bector type dual corresponding to a fractional programming problem in complex space and important duality results relating the solutions of primal and dual problems have been proved under generalized convexity assumptions which widen its application in diverse fields.

1. Introduction

Recently, there has been a lot of focus on the development of complex programming problems as these are formed while dealing with earthly situations in several branches of engineering and science including signal processing, electromagnetism, vibration analysis, control theory, etc. For example, in electromagnetism, electric and magnetic field can be represented as real and imaginary parts of complex numbers, respectively.

Levinson [1] developed the idea of complex programming problem by extending the linear programming in complex space. In [2], Bhatia and Kaul derived the duality results for a complex nonlinear programming problem. The duality theory for linear problem defined on polyhedral cones was discussed by Ben-Israel [3]. Subsequently, the duality theorems were demonstrated for a wider range of nonlinear functions satisfying the linear constraints in [4].

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In nonlinear programming, Abrams and Ben-Israel [5] and Abrams [6] generalized the renowned Kuhn-Tucker optimality conditions and associated duality theorems to complex space. The complex variant of the Fritz John necessary conditions were presented in [7]. The optimality conditions and duality theory were developed by Smart and Mond [8] for nonlinear programming problems using invexity of involved functions in complex space. There is more analysis on complex programming problems which can be seen in [9–11].

The vector valued linear programming problem in complex space was put forth by Duca [12]. Elbrolosy [13] took into account multi-objective programming problems in complex space in generalized form in which the objective function consists of both the components of complex number. By introducing the generalized Charnes-Cooper transformation, an analogy between fractional and non-fractional programming problems was discovered by Chen et al [14] in complex space. The parametric duality and the important duality theorems in this framework were developed by Lai et al. [15]. More work on complex fractional programming problems can be seen in [16, 17].

Furthermore, Huang and Ho [18] used generalized convexity assumptions to develop the optimality criteria for a multiobjective fractional problem and the duality theorems were established by constructing parametric type dual. The duality results for the symmetric duals of first and second order over general polyhedral cones were developed by Ahmad et al. [19]. For a multiobjective problem in complex space, Huang and Tanaka [20] presented characteristics of the efficient solution by developing a scalarization method to determine the optimality criteria, and established the important duality theory. Chen et al. [21] obtained Fritz John-type and Karush-Kuhn-Tucker optimality conditions by incorporating the robust counterpart of complex non-differentiable fractional problem by considering uncertainty in the objective function.

Motivated by the work of Bector et al. [22] in which a lagrangian approach to duality is provided to a nonlinear fractional programming problem in complex space, here a novel complex multiobjective fractional problem is presented which seeks wider application in real life situations. Moreover, the dual studied by handful authors, that is, Bector type dual is formulated and the sufficiency duality results are established. While deriving the results, we extend the concept of ratio of invexity presented in [23] to complex space by proving that ratio of real part of invex functions in complex space is invex. Additionally some results obtained are verified through examples which depicts the existence of the problem under study.

This paper comprises of diverse sections as different components of the work. In Section 2, preliminaries and basic notations are presented. In Section 3, the concept of ratio invexity is extended to complex space from real space. Then a complex multiobjective fractional programming problem is considered and the sufficiency conditions are established using the generalized convexity of the functions involved. The illustration for the same is also provided. In Section 4, the Bector type dual is formulated for the given programming problem and the fundamental duality results namely weak, strong and strict converse duality theorems are developed. In the last section, the conclusion of the work done in this paper and its future scope is given.

2. Preliminaries and Basic Notations

Let C^p represents the complex space having dimension p, $C^{m\times p}$ be the space of $m \times p$ matrices having complex entries, R^q_+ be the nonnegative orthant of R^q defined by

$$R^q_+ = \{x \in R^q : x_j \geqq 0,\ j = 1, 2, \dots, q\}.$$

For $x, y \in R^q$, $x \geqq y \Leftrightarrow x_j \geqq y_j$ $(j = 1, 2, \dots, q)$; $x \geq y \Leftrightarrow x \geqq y$ and $x \neq y$; $x > y \Leftrightarrow x_j > y_j$ $(j = 1, 2, \dots, q)$.
The real component of $\varrho \in C^p$ is denoted by $Re(\varrho)$ and the imaginary component by $Im(\varrho)$. Denote the conjugate of ϱ as $\bar{\varrho} = Re(\varrho) - i\,Im(\varrho)$. For the given matrix $A \in C^{m\times p}$, A^T, $\bar{A}$, and A^H are respectively the transpose, conjugate, and conjugate transpose of A.

Consider $\langle \varrho_1, \varrho_2 \rangle = \varrho_1^H \varrho_2$ as the inner product of $\varrho_1, \varrho_2 \in C^p$. Define the polyhedral cone with matrix $N \in C^{k\times p}$ as $S = \{\varrho \in C^p : Re(N\varrho) \geqq 0\}$, k being a positive integer. The dual cone S^* of S will be

$$S^* = \{\vartheta \in C^p : Re\langle \varrho, \vartheta \rangle \geqq 0, \ \forall\ \varrho \in C\}$$

and the linear manifold M is defined as

$$M = \{(\varrho, \bar{\varrho}) : \varrho \in C^p\} \subset C^{2p}.$$

The following complex multiobjective fractional programming problem is examined in this work :

$$\textbf{(MOFP)} \quad \text{Minimize } \varphi(\varrho, \bar{\varrho}) = \left(\frac{Re\, f_1(\varrho, \bar{\varrho})}{Re\, h_1(\varrho, \bar{\varrho})}, \ldots, \frac{Re\, f_m(\varrho, \bar{\varrho})}{Re\, h_m(\varrho, \bar{\varrho})} \right)$$
$$\text{subject to } \chi = \left\{ (\varrho, \bar{\varrho}) \in K : \frac{g(\varrho, \bar{\varrho})}{Re\, h_i(\varrho, \bar{\varrho})} \in S \right\}$$

where $f_i(\cdot,\cdot)$, $h_i(\cdot,\cdot) : C^{2p} \to C$ for $i = 1, 2, \ldots, m$ and $g(\cdot,\cdot) : C^{2p} \to C^p$ are analytic functions defined on the linear manifold $M \subset C^{2p}$, K being a convex subset of M. Also, $Re\, h_i(\cdot,\cdot) \neq 0$ and is bounded for all $(\varrho, \bar{\varrho}) \in C^{2p}$.

Now, we recall some definitions of efficient solution and generalized convex functions in complex space as follows which is crucial for establishing the results in the subsequent sections :

Definition 2.1. *A feasible point* $(\varrho_0, \bar{\varrho}_0)$ *is called an efficient solution of* (*MOFP*), *if* $\exists$ *no other* $(\varrho, \bar{\varrho}) \in \chi$ *s.t.*

$$\varphi(\varrho_0, \bar{\varrho}_0) \geq \varphi(\varrho, \bar{\varrho}).$$

Definition 2.2. *[8] Consider the function* $f_i : C^{2p} \to C$. *Then the real part of* f_i *is defined to be, respectively,*

(i) convex w.r.t. R_+, *if* $\forall\ \varrho, \varrho_0 \in C^p$,

$$Re[f_i(\varrho, \bar{\varrho}) - f_i(\varrho_0, \bar{\varrho}_0) - (\varrho - \varrho_0)^T \nabla_\varrho f_i(\varrho_0, \bar{\varrho}_0) - (\varrho - \varrho_0)^H \nabla_{\bar{\varrho}} f_i(\varrho_0, \bar{\varrho}_0)] \geqq 0.$$

(ii) pseudoconvex w.r.t. R_+, *if* $\forall\ \varrho, \varrho_0 \in C^p$,

$$Re[(\varrho - \varrho_0)^T \nabla_\varrho f_i(\varrho_0, \bar{\varrho}_0) + (\varrho - \varrho_0)^H \nabla_{\bar{\varrho}} f_i(\varrho_0, \bar{\varrho}_0)] \geqq 0,$$

implies

$$Re[f_i(\varrho, \bar{\varrho}) - f_i(\varrho_0, \bar{\varrho}_0)] \geqq 0.$$

(iii) invex w.r.t. R_+ *if* $\exists$ *a function* $\eta : C^{2p} \to C^p$ *s.t.*

$$Re[f_i(\varrho, \bar{\varrho}) - f_i(\varrho_0, \bar{\varrho}_0) - \eta^T(\varrho, \varrho_0) \nabla_\varrho f_i(\varrho_0, \bar{\varrho}_0) - \eta^H(\varrho, \varrho_0) \nabla_{\bar{\varrho}} f_i(\varrho_0, \bar{\varrho}_0)] \geqq 0.$$

(iv) pseudoinvex w.r.t. R_+, *if* $\exists$ *a function* $\eta : C^{2p} \to C^p$ *s.t.*

$$Re[\eta^T(\varrho, \varrho_0) \nabla_\varrho f_i(\varrho_0, \bar{\varrho}_0) + \eta^H(\varrho, \varrho_0) \nabla_{\bar{\varrho}} f_i(\varrho_0, \bar{\varrho}_0)] \geqq 0,$$

implies

$$Re[f_i(\varrho, \bar{\varrho}) - f_i(\varrho_0, \bar{\varrho}_0)] \geqq 0.$$

Definition 2.3. *[8] The analytic function* $-g : C^{2p} \to C^p$ *is called, respectively* *(i) convex w.r.t. S, if for all* $\nu \in S^*$ *and* $\varrho, \varrho_0 \in C^p$

$$Re\langle \nu, -g(\varrho, \bar{\varrho}) + g(\varrho_o, \bar{\varrho}_0) + \nabla_{\varrho} g(\varrho_0, \bar{\varrho}_0)(\varrho - \varrho_0) + \nabla_{\bar{\varrho}} g(\varrho_0, \bar{\varrho}_0)\overline{(\varrho - \varrho_0)}\rangle \geqq 0.$$

(ii) quasiconvex w.r.t. S, if for all $\nu \in S^*$ *and* $\varrho, \varrho_0 \in C^p$,

$$Re\langle \nu, -g(\varrho, \bar{\varrho}) + g(\varrho_0, \bar{\varrho}_0)\rangle \leqq 0.$$

implies

$$Re\langle \nu, \nabla_{\varrho} g(\varrho_0, \bar{\varrho}_0)(\varrho - \varrho_0) + \nabla_{\bar{\varrho}} g(\varrho_0, \bar{\varrho}_0)\overline{(\varrho - \varrho_0)}\rangle \geqq 0.$$

(iii) invex w.r.t. S, for all $\nu \in S^*$ *and* $\varrho, \varrho_0 \in C^p$, $\exists$ *a function* $\eta : C^{2p} \to C^p$ *s.t.*

$$Re\langle \nu, -g(\varrho, \bar{\varrho}) + g(\varrho_0, \bar{\varrho}_0) + \nabla_{\varrho} g(\varrho_0, \bar{\varrho}_0)\eta(\varrho, \varrho_0) + \nabla_{\bar{\varrho}} g(\varrho_0, \bar{\varrho}_0)\overline{\eta(\varrho, \varrho_0)}\rangle \geqq 0.$$

(iv) quasiinvex w.r.t. S, if $\exists$ *a function* $\eta : C^{2p} \to C^p$ *s.t.*

$$Re\langle \nu, -g(\varrho, \bar{\varrho}) + g(\varrho_0, \bar{\varrho}_0)\rangle \leqq 0,$$

implies

$$Re\langle \nu, \nabla_{\varrho} g(\varrho_0, \bar{\varrho}_0)\eta(\varrho_0, \varrho_0) + \nabla_{\bar{\varrho}} g(\varrho_0, \bar{\varrho}_0)\overline{\eta(\varrho, \varrho_0)}\rangle \geqq 0.$$

3. Necessary and Sufficiency Conditions

Now we present necessary and sufficiency conditions for the problem (*MOFP*) under generalized convexity of the functions involved as follows:

Theorem 3.1 (Necessary Conditions). *For the given problem (MOFP), consider* $(\varrho_0, \bar{\varrho}_0)$ *to be an efficient solution and suppose that the constraint qualification is satisfied at* $(\varrho_0, \bar{\varrho}_0)$*. Then* $\exists\ 0 < \gamma_i \in R$*, and* $Y_i \in S^* \subset C^p$ *for* $i = 1, 2, \ldots, m$*, s.t.*

$$\sum_{i=1}^{m} \gamma_i \left(\overline{\nabla_{\varrho} \frac{f_i(\varrho_0, \bar{\varrho}_0)}{Re\, h_i(\varrho_0, \bar{\varrho}_0)}} + \nabla_{\bar{\varrho}} \frac{f_i(\varrho_0, \bar{\varrho}_0)}{Re\, h_i(\varrho_0, \bar{\varrho}_0)} - Y_i^T \overline{\nabla_{\varrho} \frac{g(\varrho_0, \bar{\varrho}_0)}{Re\, h_i(\varrho_0, \bar{\varrho}_0)}} - Y_i^H \nabla_{\bar{\varrho}} \frac{g(\varrho_0, \bar{\varrho}_0)}{Re\, h_i(\varrho_0, \bar{\varrho}_0)} \right) = 0, \tag{1}$$

$$Re \left\langle Y_i, \frac{g(\varrho_0, \bar{\varrho}_0)}{Re\, h_i(\varrho_0, \bar{\varrho}_0)} \right\rangle = 0,\ i = 1, 2, \ldots, m, \tag{2}$$

where

$$Y = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1m} \\ y_{21} & y_{22} & \cdots & y_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ y_{p1} & y_{p2} & \cdots & y_{pm} \end{pmatrix}$$

$Y \in C^{p\times m}$ *is the matrix which contains Lagrange multipliers for the ith set of constraints as its* i^{th} *column.*

At the moment, some results in the form of lemmas are given, which play a significant role in deriving sufficiency conditions and important duality theorems. The proofs of the following Lemmas 3.2-3.4 can be easily derived on the lines of [22, 24].

Lemma 3.2. *[24] The problem (MOFP) has a constraint qualification at $(\varrho_0, \bar{\varrho}_0)$, if for any $\Lambda\ (\neq 0) \in S^* \subset C^p$,*

$$\Lambda^T \overline{\nabla_\varrho \frac{g(\varrho_0, \bar{\varrho}_0)}{Re\, h_i(\varrho_0, \bar{\varrho}_0)}} + \Lambda^H \nabla_{\bar{\varrho}} \frac{g(\varrho_0, \bar{\varrho}_0)}{Re\, h_i(\varrho_0, \bar{\varrho}_0)} \neq 0.$$

Lemma 3.3. *[22] If $f_i(\varrho, \bar{\varrho})$ and $h_i(\varrho, \bar{\varrho})$ have convex and concave real part, respectively, where $Re\, f_i(\varrho, \bar{\varrho}) \geqq 0$, $Re\, h_i(\varrho, \bar{\varrho}) > 0$, then the real part of function $f_i(\varrho, \bar{\varrho}) / Re\, h_i(\varrho, \bar{\varrho})$ is pseudoconvex w.r.t. R_+ on K.*

Lemma 3.4. *[22] If $(\varrho_0, \bar{\varrho}_0) \in K$, $Re\, h_i(\cdot, \cdot)$ and $g(\cdot, \cdot)$ are concave with respect to R^+ and S on K, respectively, $Y_i \in S^*$ and $Re\, \langle Y_i, g(\varrho_0, \bar{\varrho}_0)\rangle \leqq 0$. Then the real part of $Y_i^H \dfrac{g(\varrho, \bar{\varrho})}{Re\, h_i(\varrho, \bar{\varrho})}$ is quasiconcave w.r.t. R_+ at $(\varrho_0, \bar{\varrho}_0)$ on K.*

Lemma 3.5. *For $Re\, f_i(\varrho, \bar{\varrho}) \leqq 0$, $Re\, h_i(\varrho, \bar{\varrho}) > 0$, if $Re\, f_i(\varrho, \bar{\varrho})$ and $-Re\, h_i(\varrho, \bar{\varrho})$ are invex w.r.t. $\eta(\varrho, \varrho_0)$, then $\dfrac{f_i(\varrho, \bar{\varrho})}{Re\, h_i(\varrho, \bar{\varrho})}$ has invex real part w.r.t.*

$$\widetilde{\eta}(\varrho, \varrho_0) = \frac{Re\, h_i(\varrho_0, \bar{\varrho}_0)}{Re\, h_i(\varrho, \bar{\varrho})} \eta(\varrho, \varrho_0)$$

Proof. Taking $(\varrho, \bar{\varrho}) = \varrho$ and $(\varrho_0, \bar{\varrho}_0) = \varrho_0$ for the sake of simplicity. By differential calculus, we have

$$\nabla_\varrho \frac{f_i(\varrho)}{h_i(\varrho)} = \frac{1}{h_i(\varrho)} (\nabla_\varrho f_i(\varrho) + \nabla_{\bar{\varrho}} f_i(\varrho)) - \frac{f_i(\varrho)}{(h_i(\varrho))^2} (\nabla_\varrho h_i(\varrho) + \nabla_{\bar{\varrho}} h_i(\varrho))$$

and

$$\frac{f_i(\varrho)}{h_i(\varrho)} - \frac{f_i(\varrho_0)}{h_i(\varrho_0)} = \frac{1}{h_i(\varrho)} (f_i(\varrho) - f_i(\varrho_0)) - f_i(\varrho_0) \frac{(h_i(\varrho) - h_i(\varrho_0))}{h_i(\varrho) h_i(\varrho_0)}.$$

Since $Re\, f_i(\varrho)$ and $-Re\, h_i(\varrho)$ are invex functions with respect to $\eta(\varrho, \varrho_0)$, we get the following :

$$\frac{Re\, f_i(\varrho)}{Re\, h_i(\varrho)} - \frac{Re\, f_i(\varrho_0)}{Re\, h_i(\varrho_0)} = \frac{Re(f_i(\varrho) - f_i(\varrho_0))}{Re\, h_i(\varrho)} - \frac{Re\, f_i(\varrho_0)\, Re(h_i(\varrho) - h_i(\varrho_0))}{Re\, h_i(\varrho)\, Re\, h_i(\varrho_0)}$$

$$\geqq \frac{1}{Re\, h_i(\varrho)} Re[\eta^T(\varrho, \varrho_0) \nabla_\varrho f_i(\varrho_0) + \eta^H(\varrho, \varrho_0) \nabla_{\bar{\varrho}} f(\varrho_0)]$$

$$- \frac{Re\, f_i(\varrho_0)}{Re\, h_i(\varrho)\, Re\, h_i(\varrho_0)} Re[\eta^T(\varrho, \varrho_0) \nabla_\varrho h_i(\varrho_0) + \eta^H(\varrho, \varrho_0) \nabla_{\bar{\varrho}} h_i(\varrho_0)]$$

$$= \frac{Re\, h_i(\varrho_0)}{Re\, h_i(\varrho)} Re\Big(\eta^T(\varrho, \varrho_0)\big(\frac{1}{Re\, h_i(\varrho_0)} \nabla_\varrho f_i(\varrho_0) - \frac{Re\, f_i(\varrho_0)}{(Re\, h_i(\varrho_0))^2} \nabla_\varrho h_i(\varrho_0)\big)$$

$$+ \eta^H(\varrho, \varrho_0)\big(\frac{1}{Re\, h_i(\varrho_0)} \nabla_{\bar{\varrho}} f_i(\varrho_0) - \frac{Re\, f_i(\varrho_0)}{(Re\, h_i(\varrho_0))^2} \nabla_{\bar{\varrho}} h_i(\varrho_0)\big)\Big)$$

$$+ \qquad = \frac{Re\, h_i(\varrho_0)}{Re\, h_i(\varrho)} Re\Big(\eta^T(\varrho, \varrho_0) \nabla_\varrho \frac{f_i(\varrho_0)}{Re\, h_i(\varrho_0)} + \eta^H(\varrho, \varrho_0) \nabla_{\bar{\varrho}} \frac{f_i(\varrho_0)}{Re\, h_i(\varrho_0)}\Big)$$

Thus, $\dfrac{f_i(\varrho)}{Re\, h(\varrho)}$ has invex real part with respect to $\widetilde{\eta}(\varrho, \varrho_0) = \dfrac{Re\, h_i(\varrho_0)}{Re\, h_i(\varrho)} \eta(\varrho, \varrho_0)$. □

Here we give two illustrations which provides the validation of Lemma 3.5 .

Example 3.6. *Let $f_1, h_1 : C^{2p} \to C$ be defined by*

$$f_1(\varrho, \bar{\varrho}) = -\varrho\bar{\varrho}, \quad h_1(\varrho, \bar{\varrho}) = ln\Big(\frac{\varrho + \bar{\varrho}}{2}\Big)$$

on the convex set $K = \{(\varrho, \bar{\varrho}) : \varrho \in C, 2 \leq Re\,\varrho \leq 3, 4 \leq Im\,\varrho \leq 5\} \subset M = \{(\varrho, \bar{\varrho}) : \varrho \in C, Re\,\varrho \neq \{0, 1\}, Im\,\varrho \neq 0\}$.

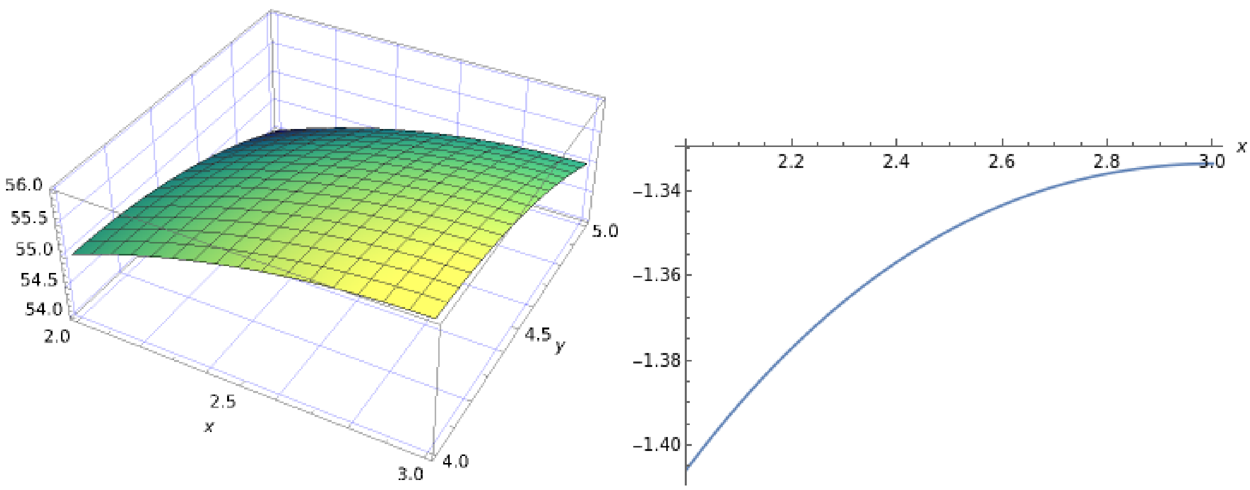
Clearly, $Re\, f_1(\varrho, \bar{\varrho}) < 0$ and $Re\, h_1(\varrho, \bar{\varrho}) > 0$ for $(\varrho, \bar{\varrho}) \in M$. Firstly using Definition 2.2 (iii), we show that $Re\, f_1(.,.)$ and $-Re\, h_1(.,.)$ are invex w.r.t. the function $\eta : C^{2p} \to C^p$ defined by $\eta(\varrho, \varrho_0) = (1 + \varrho)$ at $(\varrho_0, \bar{\varrho}_0) = (x_0 + iy_0, x_0 - iy_0) = (3 + 4i, 3 - 4i)$. So, consider

$$\begin{aligned} F_1 &= Re[f_1(\varrho, \bar{\varrho}) - f_1(\varrho_0, \bar{\varrho}_0) - \eta^T(\varrho, \varrho_0)\nabla_\varrho f_1(\varrho_0, \bar{\varrho}_0) - \eta^H(\varrho, \varrho_0)\nabla_{\bar{\varrho}} f_1(\varrho_0, \bar{\varrho}_0)] \\ &= Re\Big[- \varrho\bar{\varrho} + \varrho_0\bar{\varrho}_0 - (1 + \varrho)(-\bar{\varrho}_0) - (1 + \bar{\varrho})(-\varrho_0)\Big] \\ &= -(x^2 + y^2) + 25 + 6(1 + x) + 8y \qquad \textit{(by taking } \varrho = x + iy) \\ &> 0 \;\; (\textit{From Figure } 1(a)) \end{aligned}$$

and

$$\begin{aligned} F_2 &= Re[h_1(\varrho, \bar{\varrho}) - h_1(\varrho_0, \bar{\varrho}_0) - \eta^T(\varrho, \varrho_0)\nabla_\varrho h_1(\varrho_0, \bar{\varrho}_0) - \eta^H(\varrho, \varrho_0)\nabla_{\bar{\varrho}} h_1(\varrho_0, \bar{\varrho}_0)] \\ &= Re\Big[ln\Big(\frac{\varrho + \bar{\varrho}}{2}\Big) - ln\Big(\frac{\varrho_0 + \bar{\varrho}_0}{2}\Big) - (1 + \varrho)\frac{1}{2x_0} - (1 + \bar{\varrho})\frac{1}{2x_0}\Big] \\ &= lnx - ln3 - \frac{(1 + x)}{3} \\ &< 0 \;\; (\textit{From Figure } 1(b)) \end{aligned}$$

As, $F_1 > 0$ and $F_2 < 0$, the real parts of both $f_1(\varrho, \bar{\varrho})$ and $-h_1(\varrho, \bar{\varrho})$ are invex, thus by Lemma 3.5, we must have that



(a) (b)

Figure 1: Plot of F_1 and F_2 in (a) and (b) respectively

$\dfrac{Re\, f_1(\varrho, \bar{\varrho})}{Re\, h_1(\varrho, \bar{\varrho})}$ *is invex.*

Therefore next, we show that $\dfrac{Re\, f_1(\varrho, \bar{\varrho})}{Re\, h_1(\varrho, \bar{\varrho})}$ is invex at $(\varrho_0, \bar{\varrho}_0)$ with respect to

$$\widetilde{\eta_1}(\varrho, \bar{\varrho}_0) = \frac{(1 + \varrho) ln\Big(\frac{\varrho_0 + \bar{\varrho}_0}{2}\Big)}{ln\Big(\frac{\varrho + \bar{\varrho}}{2}\Big)}.$$

Taking

$$\begin{aligned}
F_3 &= Re\left[\frac{f_1(\varrho,\bar{\varrho})}{h_1(\varrho,\bar{\varrho})} - \frac{f_1(\varrho_0,\bar{\varrho}_0)}{h_1(\varrho_0,\bar{\varrho}_0)} - \widetilde{\eta}_1^T(\varrho,\varrho_0)\nabla_\varrho \frac{f_1(\varrho_0,\bar{\varrho}_0)}{h_1(\varrho_0,\bar{\varrho}_0)} - \widetilde{\eta}_1^H(\varrho,\varrho_0)\nabla_{\bar{\varrho}} \frac{f_1(\varrho_0,\bar{\varrho}_0)}{h_1(\varrho_0,\bar{\varrho}_0)}\right] \\
&= Re\left[\frac{-\varrho\bar{\varrho}}{ln\left(\frac{\varrho+\bar{\varrho}}{2}\right)} + \frac{\varrho_0\bar{\varrho}_0}{ln\left(\frac{\varrho_0+\bar{\varrho}_0}{2}\right)} - \frac{(1+\varrho)}{lnx}\left(\frac{-2x_0lnx_0\bar{\varrho}_0+\varrho_0\bar{\varrho}_0}{2x_0lnx_0}\right) - \frac{(1+\bar{\varrho})}{lnx}\left(\frac{-2x_0lnx_0\varrho_0+\varrho_0\bar{\varrho}_0}{2x_0lnx_0}\right)\right] \\
&= \frac{-(x^2+y^2)}{lnx} + \frac{x_0^2+y_0^2}{lnx_0} + \frac{36(lnx_0)+(36x+48y)(lnx_0)-50-50x}{6(lnx)(lnx_0)} \\
&> 0 \ \ (\textit{From Figure } 2)
\end{aligned}$$

Thus, $F_3 > 0$, which implies the invexity of $\frac{Re\, f_1(\varrho,\bar{\varrho})}{Re\, h_1(\varrho,\bar{\varrho})}$ at $(\varrho_0,\bar{\varrho}_0)$.

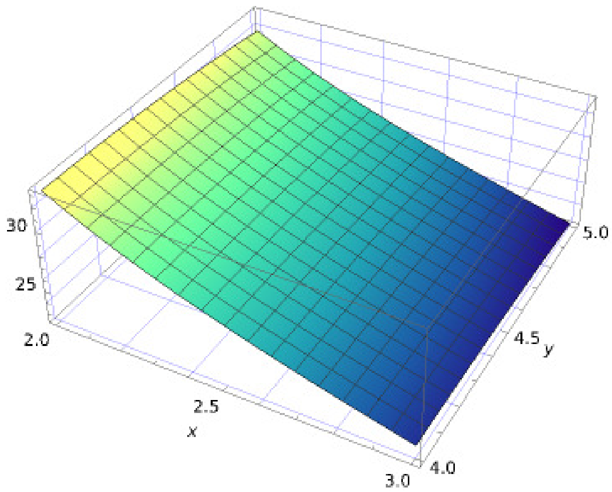


Figure 2: Plot of F_3 against x and y

Example 3.7. *Let f_2 and $h_2 : C^{2p} \to C$ be defined by*

$$f_2(\varrho,\bar{\varrho}) = (1-\varrho) \text{ and } h_2(\varrho,\bar{\varrho}) = i\bar{\varrho}^2$$

on the same K and M as in Example 3.6. We first show that Re $f_2(\varrho,\bar{\varrho})$ and Re $h_2(\varrho,\bar{\varrho})$ are invex w.r.t. $\eta(\varrho,\varrho_0) = (1+\varrho)$ at $(\varrho_0,\bar{\varrho}_0) = (x_0+iy_0, x_0-iy_0) = (3+4i, 3-4i)$.
It may be noted that Re $f_2(\varrho,\bar{\varrho}) < 0$ and Re $h_2(\varrho,\bar{\varrho}) > 0$ for $(\varrho,\bar{\varrho}) \in M$. Consider

$$\begin{aligned}
F_4 &= Re[f_2(\varrho,\bar{\varrho}) - f_2(\varrho_0,\bar{\varrho}_0) - \eta^T(\varrho,\varrho_0)\nabla_\varrho f_2(\varrho_0,\bar{\varrho}_0) - \eta^H(\varrho,\varrho_0)\nabla_{\bar{\varrho}} f_2(\varrho_0,\bar{\varrho}_0)] \\
&= Re[1-\varrho-(1-\varrho_0)-(1+\varrho)(-1)-(1+\bar{\varrho}_0)(0)] \\
&= Re[x_0+iy_0+1] \\
&= 4 > 0
\end{aligned}$$

and

$$F_5 = Re[h_2(\varrho,\bar{\varrho}) - h_2(\varrho_0,\bar{\varrho}_0) - \eta^T(\varrho,\varrho_0)\nabla_\varrho h_2(\varrho_0,\bar{\varrho}_0) - \eta^H(\varrho,\varrho_0)\nabla_{\bar{\varrho}} h_2(\varrho_0,\bar{\varrho}_0)]$$

$$
\begin{aligned}
&= Re[i\bar{\varrho}^2 - i\bar{\varrho_0}^2 - (1+\varrho)(0) - (1+\bar{\varrho})2i\varrho_0] \\
&= Re[2xy - 2x_0y_0 - 2i(1+x)x_0 - 2(1+x)y_0 - 2x_0y + 2iyy_0] \\
&= 2xy - 24 - 8(1+x) - 6y \\
&< 0 \ \ (\textit{From Figure } 3(a))
\end{aligned}
$$

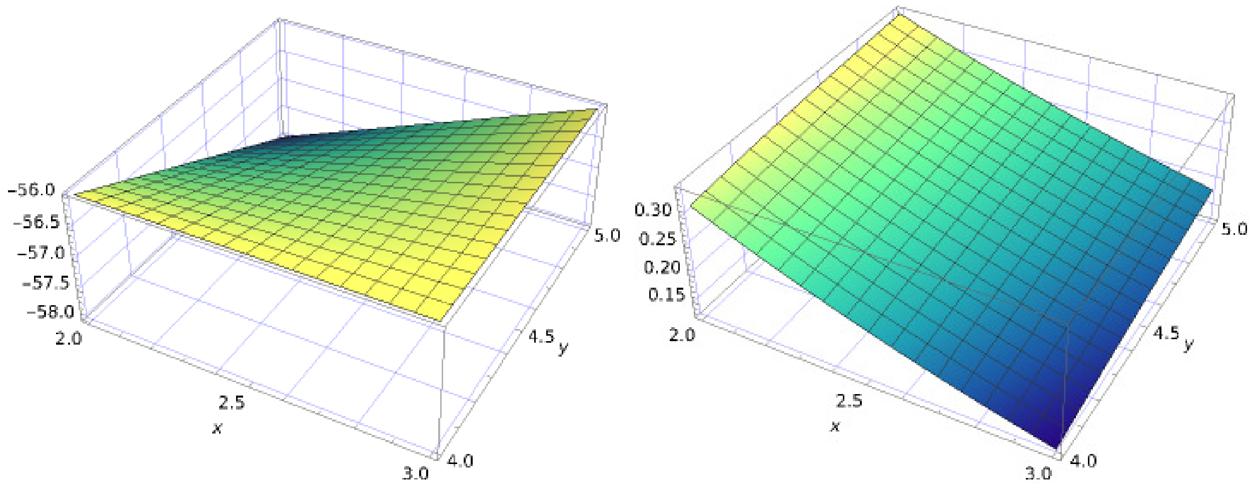
Hence, real part of $f_2(\varrho, \bar{\varrho})$ *and* $-h_2(\varrho, \bar{\varrho})$ *are invex.*

Lastly, we need to check the invexity of $\dfrac{Re\, f_2(\varrho, \bar{\varrho})}{Re\, h_2(\varrho, \bar{\varrho})} = \dfrac{Re(1-\varrho)}{Re(i\bar{\varrho}^2)}$ *at* $(\varrho_0, \bar{\varrho}_0)$ *w.r.t.*

$$
\widetilde{\eta_2}(\varrho, \bar{\varrho}_0) = \frac{(1+\varrho)i\varrho_0^2}{i\varrho^2} = \frac{(1+\varrho)12}{xy}.
$$

Therefore, taking

$$
\begin{aligned}
F_6 &= Re\left[\frac{1-\varrho}{i\bar{\varrho}^2} - \frac{1-\varrho_0}{i\bar{\varrho_0}^2} - \frac{(1+\varrho)12}{xy}\left(\frac{-1}{i\bar{\varrho_0}^2}\right) - \frac{(1+\bar{\varrho})12}{xy}\left(\frac{-2(1-\varrho_0)}{i\bar{\varrho}_0^3}\right)\right] \\
&= Re\left[\frac{1-x-iy}{2xy + i(x^2-y^2)} - \frac{1-x_0-y_0}{2x_0y_0 + i(x_0^2-y_0^2)} - \frac{(1+x+iy)12(i)}{xy(x_0-iy_0)^2} - \left(\frac{(1+x-iy)12}{xy}\right)\left(\frac{(1-x_0-iy_0)2i}{(x_0-iy_0)^3}\right)\right] \\
&= \frac{(1-x)2xy}{4x^2y^2 + (x^2-y^2)^2} - \frac{y(x^2-y^2)}{4x^2y^2 + (x^2-y^2)^2} - \frac{(1-x_0)2x_0y_0}{4x_0^2y_0^2 + (x_0^2-y_0^2)^2} - \frac{y_0(x_0^2-y_0^2)}{4x_0^2y_0^2 + (x_0^2-y_0^2)^2} \\
&\qquad + \frac{12(-7y + 24(1+x))}{625xy} - \frac{0.58(1+x)}{xy} + \frac{0.63}{x} \\
&> 0 \ \ (\textit{As from Figure } 3(b))
\end{aligned}
$$



(a) (b)

Figure 3: Plot of F_5 and F_6 in (a) and (b) respectively

Thus, $\dfrac{Re\, f_2(\varrho, \bar{\varrho})}{Re\, h_2(\varrho, \bar{\varrho})}$ *is invex at* $(\varrho_0, \bar{\varrho}_0)$.

Therefore, by the above illustrations it can be seen that the ratio of two invex functions is invex and thus Lemma 3.5 *holds.*

Theorem 3.8 (Sufficiency Conditions)**.** *Let $f_i(\cdot,\cdot), h_i(\cdot,\cdot) : C^{2p} \to C$, $i = 1, 2, \ldots, m$ and $g(\cdot,\cdot) : C^{2p} \to C^p$ be analytic functions on χ. Let there exists $0 < \gamma_i \in R$ and $Y_i \in S^* \subset C^p$ such that conditions* (1) *and* (2) *are satisfied at a feasible point $(\varrho_0, \bar{\varrho}_0)$. Moreover, if any of the following conditions are true :*

(i) *for $Re\, f_i(\cdot,\cdot) \geqq 0$ and $Re\, h_i(\cdot,\cdot) > 0$; the functions $Re\, f_i(\cdot,\cdot)$ and $Re\, h_i(\cdot,\cdot)$ are convex and concave, respectively w.r.t. R_+ on K for each $i = 1, 2, \ldots, m$. Also, consider $g(\cdot,\cdot)$ to be convex w.r.t. S on K at $(\varrho_0, \bar{\varrho}_0) \in K$;*

(ii) *for $Re\, f_i(\cdot,\cdot) \leqq 0$ and $Re\, h_i(\cdot,\cdot) > 0$; the functions $Re\, f_i(\cdot,\cdot), -Re\, h_i(\cdot,\cdot)$ for $i = 1, 2, \ldots, m$, and $g(\cdot,\cdot)$ are invex w.r.t. $\eta(\varrho, \varrho_0)$ at $(\varrho_0, \bar{\varrho}_0) \in K$;*

then, $(\varrho_0, \bar{\varrho}_0)$ is an efficient solution of (*MOFP*).

Proof. Assuming that $(\varrho_0, \bar{\varrho}_0)$ is not an efficient solution of the primal problem (*MOFP*). Then, $\exists\ (\varrho, \bar{\varrho}) \in \chi$ such that

$$\left(\frac{Re\, f_1(\varrho,\bar{\varrho})}{Re\, h_1(\varrho,\bar{\varrho})}, \ldots, \frac{Re\, f_m(\varrho,\bar{\varrho})}{Re\, h_m(\varrho,\bar{\varrho})}\right) \leq \left(\frac{Re\, f_1(\varrho_0,\bar{\varrho}_0)}{Re\, h_1(\varrho_0,\bar{\varrho}_0)}, \ldots, \frac{Re\, f_m(\varrho_0,\bar{\varrho}_0)}{Re\, h_m(\varrho_0,\bar{\varrho}_0)}\right)$$

As $\gamma_i > 0$ for $i = 1, 2, \ldots, m$, we have

$$\sum_{i=1}^{m} \gamma_i \left(\frac{Re\, f_i(\varrho,\bar{\varrho})}{Re\, h_i(\varrho,\bar{\varrho})} - \frac{Re\, f_i(\varrho_0,\bar{\varrho}_0)}{Re\, h_i(\varrho_0,\bar{\varrho}_0)}\right) < 0 \tag{3}$$

Assume that hypothesis (i) holds true. Then by Lemma 3.3, $\dfrac{f_i(\cdot,\cdot)}{Re\, h_i(\cdot,\cdot)}$ has pseudoconvex real part w.r.t. R_+ on K. Therefore by definition of pseudoconvexity and (3), we get

$$Re\left[\sum_{i=1}^{m} \gamma_i \left((\varrho - \varrho_0)^T \nabla_\varrho \frac{f_i(\varrho_0,\bar{\varrho}_0)}{Re\, h_i(\varrho_0,\bar{\varrho}_0)} + (\varrho - \varrho_0)^H \nabla_{\bar{\varrho}} \frac{f_i(\varrho_0,\bar{\varrho}_0)}{Re\, h_i(\varrho_0,\bar{\varrho}_0)}\right)\right] < 0$$

or

$$Re\left\langle \varrho - \varrho_0, \sum_{i=1}^{m} \gamma_i \left(\overline{\nabla_\varrho \frac{f_i(\varrho_0,\bar{\varrho}_0)}{Re\, h_i(\varrho_0,\bar{\varrho}_0)}} + \nabla_{\bar{\varrho}} \frac{f_i(\varrho_0,\bar{\varrho}_0)}{Re\, h_i(\varrho_0,\bar{\varrho}_0)}\right)\right\rangle < 0 \tag{4}$$

For $Y_i \in S^* \subset C^p$, $i = 1, 2, \ldots, m$,

$$Re\left\langle Y_i, \frac{-g(\varrho,\bar{\varrho})}{Re\, h_i(\varrho,\bar{\varrho})}\right\rangle \leqq 0 = Re\left\langle Y_i, \frac{-g(\varrho_0,\bar{\varrho}_0)}{Re\, h_i(\varrho_0,\bar{\varrho}_0)}\right\rangle \tag{5}$$

using the definition of dual cone and (2).

By hypothesis (i) and Lemma 3.4, $-Y_i^H \dfrac{g(\cdot,\cdot)}{Reh_i(\cdot,\cdot)}$ has quasiconcave real part at $(\varrho_0, \bar{\varrho}_0)$ which on using the above inequality and $\gamma_i > 0$ yields,

$$Re\left\langle \varrho - \varrho_0, \sum_{i=1}^{m} \gamma_i \left(-\overline{Y_i^T \nabla_\varrho \frac{g(\varrho_0,\bar{\varrho}_0)}{Re\, h_i(\varrho_0,\bar{\varrho}_0)}} - Y_i^H \nabla_{\bar{\varrho}} \frac{g(\varrho_0,\bar{\varrho}_0)}{Re\, h_i(\varrho_0,\bar{\varrho}_0)}\right)\right\rangle \leqq 0 \tag{6}$$

Adding (4) and (6), we obtain

$$Re\left\langle \varrho - \varrho_0, \sum_{i=1}^{m} \gamma_i \left(\overline{\nabla_\varrho \frac{f(\varrho_0,\bar{\varrho}_0)}{Re\, h_i(\varrho_0,\bar{\varrho}_0)}} + \nabla_{\bar{\varrho}} \frac{f(\varrho_0,\bar{\varrho}_0)}{Re\, h_i(\varrho_0,\bar{\varrho}_0)} - \right.\right.$$
$$\left.\left.\overline{Y_i^T \nabla_\varrho \frac{g(\varrho_0,\bar{\varrho}_0)}{Re\, h_i(\varrho_0,\bar{\varrho}_0)}} - Y_i^H \nabla_{\bar{\varrho}} \frac{g(\varrho_0,\bar{\varrho}_0)}{Re\, h_i(\varrho_0,\bar{\varrho}_0)}\right)\right\rangle < 0$$

which contradicts equation (1). Hence, $(\varrho_0, \bar{\varrho}_0)$ is an efficient solution of (*MOFP*).

Now, assume that hypothesis (*ii*) holds true. Then by Lemma 3.5 , we have that real part of $\frac{f_i(\cdot,\cdot)}{Re\, h_i(\cdot,\cdot)}$ is also an invex function w.r.t. $\widetilde{\eta}(\varrho, \varrho_0)$, i.e.,

$$Re\left(\frac{f_i(\varrho,\bar{\varrho})}{Re\, h_i(\varrho,\bar{\varrho})} - \frac{f_i(\varrho_0,\bar{\varrho}_0)}{Re\, h_i(\varrho_0,\bar{\varrho}_0)}\right) \geqq Re\left\langle \widetilde{\eta}(\varrho,\varrho_0), \overline{\nabla_\varrho \frac{f_i(\varrho_0,\bar{\varrho}_0)}{Re\, h_i(\varrho_0,\bar{\varrho}_0)}} + \nabla_{\bar{\varrho}} \frac{f_i(\varrho_0,\bar{\varrho}_0)}{Re\, h_i(\varrho_0,\bar{\varrho}_0)}\right\rangle \tag{7}$$

Since $g(\cdot,\cdot)$ is also an invex function, then again by using Lemma 3.5 , we have that $-Y_i^H \frac{g(\varrho_0,\bar{\varrho}_0)}{Re\, h_i(\varrho_0,\bar{\varrho}_0)}$ is an invex function w.r.t $\widetilde{\eta}(\varrho, \varrho_0)$. Thus, by (5) we get

$$\begin{aligned}
0 &\geqq Re\left\langle Y_i, \frac{-g(\varrho,\bar{\varrho})}{Re\, h_i(\varrho,\bar{\varrho})}\right\rangle - Re\left\langle Y_i, \frac{-g(\varrho_0,\bar{\varrho}_0)}{Re\, h_i(\varrho_0,\bar{\varrho}_0)}\right\rangle \\
&\geqq Re\left\langle Y_i, \widetilde{\eta}^T(\varrho,\varrho_0)\nabla_\varrho \frac{-g(\varrho_0,\bar{\varrho}_0)}{Re\, h_i(\varrho_0,\bar{\varrho}_0)} + \widetilde{\eta}^H(\varrho,\varrho_0)\nabla_{\bar{\varrho}} \frac{-g(\varrho_0,\bar{\varrho}_0)}{Re\, h_i(\varrho_0,\bar{\varrho}_0)}\right\rangle \\
&= Re\left\langle \widetilde{\eta}(\varrho,\varrho_0), -Y_i^T\nabla_\varrho \frac{g(\varrho_0,\bar{\varrho}_0)}{Re\, h_i(\varrho_0,\bar{\varrho}_0)} - Y_i^H\nabla_{\bar{\varrho}} \frac{g(\varrho_0,\bar{\varrho}_0)}{Re\, h_i(\varrho_0,\bar{\varrho}_0)}\right\rangle
\end{aligned}$$

Adding this inequality to (7) and using the fact that $\gamma_i > 0$ for $i = 1, 2, \ldots, m$, we have

$$\begin{aligned}
\sum_{i=1}^{m} \gamma_i\left(\frac{Re\, f_i(\varrho,\bar{\varrho})}{Re\, h_i(\varrho,\bar{\varrho})} - \frac{Re\, f_i(\varrho_0,\bar{\varrho}_0)}{Re\, h_i(\varrho_0,\bar{\varrho}_0)}\right) \geqq Re\Bigg\langle \widetilde{\eta}(\varrho,\bar{\varrho}_0), \sum_{i=1}^{m}\gamma_i\Bigg(&\overline{\nabla_\varrho \frac{f(\varrho_0,\bar{\varrho}_0)}{Re\, h_i(\varrho_0,\bar{\varrho}_0)}} \\
&+ \nabla_{\bar{\varrho}} \frac{f(\varrho_0,\bar{\varrho}_0)}{Re\, h_i(\varrho_0,\bar{\varrho}_0)} - Y_i^T \overline{\nabla_z \frac{g(\varrho_0,\bar{\varrho}_0)}{Re\, h_i(\varrho_0,\bar{\varrho}_0)}} - Y_i^H \nabla_{\bar{\varrho}} \frac{g(\varrho_0,\bar{\varrho}_0)}{Re\, h_i(\varrho_0,\bar{\varrho}_0)}\Bigg)\Bigg\rangle
\end{aligned}$$

= 0 (using (1)), which is a contradiction to (3).
Therefore, $(\varrho_0, \bar{\varrho}_0)$ is an efficient solution for (*MOFP*). □

We now proceed to present the example which demonstrates the sufficiency conditions derived above.

Example 3.8. Consider $S = \{\varrho \in C : Re(\varrho) \geqq 0\}$ be the polyhedral cone, K and M be defined as in Example 3.6. Let the multiobjective fractional programming problem (*MOFPP*) be as below:

$$\textbf{(MOFPP)}\ \ \textit{Minimize}\ \varphi(\varrho,\bar{\varrho}) = \left(\frac{Re -\varrho\bar{\varrho}}{Re\, ln\left(\frac{\varrho+\bar{\varrho}}{2}\right)}, \frac{Re(1-\varrho)}{Re\, i\bar{\varrho}^2}\right)$$

$$\text{subject to } \chi = \left\{(\varrho,\bar{\varrho}) \in K : \frac{\varrho}{Re\, ln\left(\frac{\varrho+\bar{\varrho}}{2}\right)} \in S, \frac{\varrho}{Re\, i\bar{\varrho}^2} \in S\right\}.$$

To check the validity of sufficiency theorem 3.8 we need to have $0 < \gamma_i \in R$ and $Y_i \in S^* \subset C^p$ such that (1) and (2) are satisfied at a feasible point $(\varrho_0, \bar{\varrho}_0) = (x_0 + iy_0, x_0 - iy_0) = (3 + 4i, 3 - 4i)$. Taking $Y_1 = (a_1 + ib_1)$ and using condition (2), we obtain

$$Re\left\langle a_1 + ib_1, \frac{x_0}{ln(x_0)} + i\frac{y_0}{ln(x_0)}\right\rangle = 0$$
$$\Rightarrow a_1x_0 - b_1y_0 = 0$$

$\Rightarrow b_1 = 0.75a_1$

If $a_1 = -2$, then $b_1 = -1.5$. Thus, we have $Y_1 = -2 - 1.5i \in S^*$.

Similarly, taking $Y_2 = (a_2 + ib_2)$ and using condition (2), we have

$$Re\left\langle a_2 + ib_2, \frac{\varrho}{Re\, i\bar{\varrho}^2}\right\rangle = 0$$

we obtain $b_2 = 0.75a_2$. If $a_2 = -3$, we get $b_2 = -2.25$. Thus, $Y_2 = -3 - 2.25i \in S^*$.

Now using Y_1, Y_2 obtained above and equation (1), we have

$$\sum_{i=1}^{2}\gamma_i\left(\overline{\nabla_{\varrho}\frac{f_i(\varrho_0,\bar{\varrho}_0)}{Re\, h_i(\varrho_0,\bar{\varrho}_0)}} + \nabla_{\bar{\varrho}}\frac{f_i(\varrho_0,\bar{\varrho}_0)}{Re\, h_i(\varrho_0,\bar{\varrho}_0)} - \overline{Y_i^T\nabla_{\varrho}\frac{g(\varrho_0,\bar{\varrho}_0)}{Re\, h_i(\varrho_0,\bar{\varrho}_0)}} - Y_i^H\nabla_{\bar{\varrho}}\frac{g(\varrho_0,\bar{\varrho}_0)}{Re\, h_i(\varrho_0,\bar{\varrho}_0)}\right) = 0,$$

$$\Rightarrow \gamma_1\Big(\frac{-2\varrho_0}{ln3} - \frac{Y_1}{ln3}\Big) + \gamma_2\Big(\frac{-1}{24} - \frac{Y_2}{24}\Big) = 0$$

$\Rightarrow \gamma_1 = 0.02\gamma_2$
If $\gamma_2 = 1 > 0$, then $\gamma_1 = 0.02 > 0$.

Hence, there exists $0 < \gamma_i \in R$ and $Y_i \in S^* \subset C^p$ such that (1) and (2) are satisfied at a feasible point $(\varrho_0, \bar{\varrho}_0)$. Also, note that both the objective functions are invex as proved earlier in examples 3.6 and 3.7. Consequently, $(\varrho_0, \bar{\varrho}_0) = (3 + i4, 3 - i4)$ is an efficient solution of (*MOFPP*) by Theorem 3.8.

Many algorithms are presented by authors in literature to find the optimal solution of an optimization problem where the objective functions are highly nonlinear. But the multiobjective programming problems fabricated to study the real life model are mostly solved using Multiobjective Genetic Algorithm (MOGA) in MATLAB. It was introduced by Fonseca and Fleming [25] and is based on the concepts of biological evolution via natural selection and genetics. It can explore various areas of the solution space, making it useful for solving multiobjective optimization problems. To see its application in various fields refer [26–28]. It is noteworthy that we attained $(\varrho_0, \bar{\varrho}_0) = (3 + i4, 3 - i4)$ as an efficient solution of (*MOFPP*) using MOGA. However in this study, as the sufficiency conditions are taken into account and verified under generalized convexity assumptions, the solution obtained using MOGA is efficient undoubtedly.

4. Bector Type Dual Formulation and Duality Results

In the literature, various types of duals have been discussed. Here we have studied the Bector type dual in order to explore its application to various mathematical programming problem. Acknowledging that the development of conceptual and computational elements of real mathematical programming problems has benefited by the use of Lagrangian functions, we formulate Bector type dual for the primal problem (*MOFP*) and then establish important duality results. These theorems relate the optimal solutions of both the problems.

The Bector type dual to the primal (*MOFP*) is formulated as follows :

(BD) Maximize $B(\vartheta, \bar{\vartheta}, \gamma, Y) =$

$$\Big(\frac{Re(f_1(\vartheta,\bar{\vartheta}) - \langle Y_1, g(\vartheta,\bar{\vartheta})\rangle)}{Re\, h_1(\vartheta,\bar{\vartheta})}, \ldots, \frac{Re(f_m(\vartheta,\bar{\vartheta}) - \langle Y_m, g(\vartheta,\bar{\vartheta})\rangle)}{Re\, h_m(\vartheta,\bar{\vartheta})}\Big)$$

subject to

$$\sum_{i=1}^{m}\gamma_i\left(\nabla_{\varrho}\overline{\frac{f_i(\vartheta,\bar{\vartheta})}{Re\,h_i(\vartheta,\bar{\vartheta})}}+\nabla_{\bar{\varrho}}\frac{f_i(\vartheta,\bar{\vartheta})}{Re\,h_i(\vartheta,\bar{\vartheta})}-Y_i^T\overline{\nabla_{\varrho}\frac{g(\vartheta,\bar{\vartheta})}{Re\,h_i(\vartheta,\bar{\vartheta})}}-Y_i^H\nabla_{\bar{\varrho}}\frac{g(\vartheta,\bar{\vartheta})}{Re\,h_i(\vartheta,\bar{\vartheta})}\right)=0, \tag{8}$$

$(\vartheta,\bar{\vartheta})\in K,\ 0<\gamma_i\in R,\ Y_i\in S^*\subset C^p$ for $i=1,2,\ldots,m$.

We proceed to the major component of this section, where the duality results are developed for the aforesaid Bector type dual (BD).

Theorem 4.1 (Weak Duality Theorem). *Consider $(\varrho,\bar{\varrho})\in\chi$ i.e., a feasible solution of (MOFP) and $(\vartheta,\bar{\vartheta},\gamma,Y)$ to be a feasible solution of (BD). In addition, suppose that any of the conditions (i) or (ii) of Theorem 3.8 holds, then $\phi(\varrho,\bar{\varrho})\geqq B(\vartheta,\bar{\vartheta},\gamma,Y)$.*

Proof. Consider $\phi(\varrho,\bar{\varrho})-B(\vartheta,\overline{\vartheta},\gamma,Y)$, i.e., for each $i=1,2,\ldots,m$,

$$\frac{Re\,f_i(\varrho,\bar{\varrho})}{Re\,h_i(\varrho,\bar{\varrho})}-\frac{Re(f_i(\vartheta,\bar{\vartheta})-\langle Y_i,g(\vartheta,\bar{\vartheta})\rangle)}{Re\,h_i(\vartheta,\bar{\vartheta})}\geqq\frac{Re\,f_i(\varrho,\bar{\varrho})-Re\,\langle Y_i,g(\varrho,\bar{\varrho})\rangle}{Re\,h_i(\varrho,\bar{\varrho})}-\frac{Re\,f_i(\vartheta,\bar{\vartheta})-Re\,\langle Y_i,g(\vartheta,\bar{\vartheta})\rangle}{Re\,h_i(\vartheta,\bar{\vartheta})}$$

$$\left(\left\langle Y_i,\frac{g(\varrho,\bar{\varrho})}{Re\,h_i(\varrho,\bar{\varrho})}\right\rangle\text{ being non-negative.}\right)$$

(i) Consider

$$\frac{Re\,f_i(\varrho,\bar{\varrho})-Re\,\langle Y_i,g(\varrho,\bar{\varrho})\rangle}{Re\,h_i(\varrho,\bar{\varrho})}-\frac{Re\,f_i(\vartheta,\bar{\vartheta})-Re\,\langle Y_i,g(\vartheta,\bar{\vartheta})\rangle}{Re\,h_i(\vartheta,\bar{\vartheta})}$$
$$=\frac{1}{Re\,h_i(\varrho,\bar{\varrho})\,Re\,h_i(\vartheta,\bar{\vartheta})}\Big[Re\,h_i(\vartheta,\bar{\vartheta})\,Re\Big(f_i(\varrho,\bar{\varrho})-\langle Y_i,g(\varrho,\bar{\varrho})\rangle\Big)-Re\,h_i(\varrho,\bar{\varrho})\,Re\Big(f_i(\vartheta,\bar{\vartheta})-\langle Y_i,g(\vartheta,\bar{\vartheta})\rangle\Big)\Big]$$

$$\geqq\frac{1}{Re\,h_i(\varrho,\bar{\varrho})\,Re\,h_i(\vartheta,\bar{\vartheta})}\Big[Re\,h_i(\vartheta,\bar{\vartheta})\,Re\{\nabla_{\varrho}(f_i(\vartheta,\bar{\vartheta})-\langle Y_i,g(\vartheta,\bar{\vartheta})\rangle)(\varrho-\vartheta)+\nabla_{\bar{\varrho}}(f_i(\vartheta,\bar{\vartheta})-\langle Y_i,g(\vartheta,\bar{\vartheta})\rangle)(\bar{\varrho}-\bar{\vartheta})\}-$$
$$Re(f_i(\vartheta,\bar{\vartheta})-\langle Y_i,g(\vartheta,\bar{\vartheta})\rangle)\,Re\{\nabla_{\varrho}h_i(\vartheta,\bar{\vartheta})(\varrho-\vartheta)+\nabla_{\bar{\varrho}}h_i(\vartheta,\bar{\vartheta})(\bar{\varrho}-\bar{\vartheta})\}\Big]\ \text{(using hypothesis } (i) \text{ of Theorem 3.8)}$$
$$\geqq\frac{1}{Re\,h_i(\varrho,\bar{\varrho})\,Re\,h_i(\vartheta,\bar{\vartheta})}\,Re\Big[\{Re\,h_i(\vartheta,\bar{\vartheta})\Big(\nabla_{\varrho}(f_i(\vartheta,\bar{\vartheta})-\langle Y_i,g(\vartheta,\bar{\vartheta})\rangle\Big)-(f_i(\vartheta,\bar{\vartheta})\,\langle Y_i,g(\vartheta,\bar{\vartheta})\rangle)$$
$$\nabla_{\varrho}\,Re\,h_i(\vartheta,\bar{\vartheta})\}(\varrho-\vartheta)+\{Re\,h_i(\vartheta,\bar{\vartheta})\nabla_{\bar{\varrho}}(f_i(\vartheta,\bar{\vartheta})-\langle Y_i,g(\vartheta,\bar{\vartheta})\rangle)-(f_i(\vartheta,\bar{\vartheta})-\langle Y_i,g(\vartheta,\bar{\vartheta})\rangle)\nabla_{\bar{\varrho}}\,Re\,h_i(\vartheta,\bar{\vartheta})\}(\bar{\varrho}-\bar{\vartheta})\Big]$$
$$\geqq\frac{Re\,h_i(\vartheta,\bar{\vartheta})}{Re\,h_i(\varrho,\bar{\varrho})}\,Re\left[\nabla_{\varrho}\left(\frac{f_i(\vartheta,\bar{\vartheta})-Y_i^Hg(\vartheta,\bar{\vartheta})}{Re\,h_i(\vartheta,\bar{\vartheta})}\right)(\varrho-\vartheta)+\nabla_{\bar{\varrho}}\left(\frac{f_i(\vartheta,\bar{\vartheta})-Y_i^Hg(\vartheta,\bar{\vartheta})}{Re\,h_i(\vartheta,\bar{\vartheta})}\right)(\bar{\varrho}-\bar{\vartheta})\right]$$
$$=\frac{Re\,h_i(\vartheta,\bar{\vartheta})}{Re\,h_i(\varrho,\bar{\varrho})}\,Re\left\langle\varrho-\vartheta,\overline{\nabla_{\varrho}\left(\frac{f_i(\vartheta,\bar{\vartheta})-Y_i^Hg(\vartheta,\bar{\vartheta})}{Re\,h_i(\vartheta,\bar{\vartheta})}\right)}+\nabla_{\bar{\varrho}}\left(\frac{f_i(\vartheta,\bar{\vartheta})-Y_i^Hg(\vartheta,\bar{\vartheta})}{Re\,h_i(\vartheta,\bar{\vartheta})}\right)\right\rangle$$
$=0$ (by dual constraint (8) and $\gamma_i>0,\ i=1,2,\ldots,m$)
Therefore, $\varphi(\varrho,\bar{\varrho})\geqq B(\vartheta,\bar{\vartheta},\gamma,Y)$.

(ii) By invexity of $\dfrac{f_i(\cdot,\cdot)-\langle Y_i,g(\cdot,\cdot)\rangle}{Re\,h_i(\cdot,\cdot)}$, the result can be obtained by replacing $\varrho-\vartheta$ with $\widetilde{\eta}(\varrho,\varrho_0)$ in the proof of part (i) above. □

Corollary 4.2. *Consider $(\varrho_0,\bar{\varrho}_0)$ to be a feasible solution of (MOFP) and $(\vartheta_0,\bar{\vartheta}_0,\gamma,Y)$ be a feasible solution of (BD) with $\varphi(\varrho_0,\bar{\varrho}_0)=B(\vartheta_0,\bar{\vartheta}_0,\gamma,Y)$. Further, assume that the assumptions of Theorem 4.1 holds true. Then $(\varrho_0,\bar{\varrho}_0)$ and $(\vartheta_0,\bar{\vartheta}_0,\gamma,Y)$ are the efficient solutions of (MOFP) and (BD) respectively.*

Proof. Assuming that $(\varrho_0,\bar{\varrho}_0)$ is not an efficient solution of (MOFP). Then there exists $(\hat{\varrho}_0,\bar{\hat{\varrho}}_0)\in\chi$ such that

$$\left(\frac{Re\,f_1(\hat{\varrho},\bar{\hat{\varrho}})}{Re\,h_1(\hat{\varrho},\bar{\hat{\varrho}})},\ldots,\frac{Re\,f_m(\hat{\varrho},\bar{\hat{\varrho}})}{Re\,h_m(\hat{\varrho},\bar{\hat{\varrho}})}\right)\leq\left(\frac{Re\,f_1(\varrho_0,\bar{\varrho}_0)}{Re\,h_1(\varrho_0,\bar{\varrho}_0)},\ldots,\frac{Re\,f_m(\varrho_0,\bar{\varrho}_0)}{Re\,h_m(\varrho_0,\bar{\varrho}_0)}\right)$$

$$= B(\vartheta_0, \bar{\vartheta}_0, \gamma, Y),$$
$$\text{or} \quad \varphi(\hat{\varrho}, \bar{\hat{\varrho}}) \leq \varphi(\varrho_0, \bar{\varrho}_0) = B(\vartheta_0, \bar{\vartheta}_0, \gamma, Y)$$

which contradicts Theorem 4.1.
Thus, $(\varrho_0, \bar{\varrho}_0)$ *is an efficient solution of* (*MOFP*)*. The efficiency of* $(\vartheta_0, \bar{\vartheta}_0, \gamma, Y)$ *follows similarly.* □

Theorem 4.3 (Strong Duality Theorem)**.** *Consider* $(\varrho_0, \bar{\varrho}_0)$ *to be an efficient solution of* (*MOFP*) *and the constraint qualification hold at* $(\varrho_0, \bar{\varrho}_0)$*. Then* $\exists\, \hat{\gamma}_i > 0, \hat{Y}_i \in S^*$, $i = 1, 2, \ldots, m$*, such that* $(\varrho_0, \bar{\varrho}_0, \hat{\gamma}, \hat{Y})$ *is a feasible solution of* (*BD*)*. If in addition hypotheses of Theorem* 4.1 *holds true, then* $(\varrho_0, \bar{\varrho}_0, \hat{\gamma}, \hat{Y})$ *is an efficient solution of* (*BD*)*.*

Proof. Assume $(\varrho_0, \bar{\varrho}_0)$ is an efficient solution of (*MOFP*). Then by Theorem 3.1, $\exists\, \hat{\gamma}_i > 0, \hat{Y}_i \in S^*, i = 1, 2, \ldots, m$ such that $(\varrho_0, \bar{\varrho}_0, \hat{\gamma}_i, \hat{Y}_i)$ satisfies (8). Hence, $(\varrho_0, \bar{\varrho}_0, \hat{\gamma}, \hat{Y})$ is a feasible solution of (*BD*).
As

$$Re\left\langle \hat{Y}_i, \frac{g(\varrho_0, \bar{\varrho}_0)}{Re\, h_i(\varrho_0, \bar{\varrho}_0)} \right\rangle = 0, \quad i = 1, 2, \ldots, m,$$

and so, the optimal value of the primal (*MOFP*) and dual (*BD*) problems are equal at $(\varrho_0, \bar{\varrho}_0, \hat{\gamma}, \hat{Y})$, i.e., $\varphi(\varrho_0, \bar{\varrho}_0) = B(\varrho_0, \bar{\varrho}_0, \hat{\gamma}, \hat{Y})$. Thus, by Corollary 4.2, $(\varrho_0, \bar{\varrho}_0, \hat{\gamma}, \hat{Y})$ is an efficient solution of (*BD*). □

Theorem 4.4 (Strict Converse Duality Theorem)**.** *Consider* $(\varrho_0, \bar{\varrho}_0)$ *to be an efficient solution of* (*MOFP*) *and* $(\vartheta_0, \bar{\vartheta}_0, \hat{\gamma}, \hat{Y})$ *be an efficient solution of* (*BD*) *and suppose that any of the following condition holds true*

(i) *for* $Re\, f_i(\cdot,\cdot) \geqq 0, Re\, h_i(\cdot,\cdot) > 0$, $\dfrac{f_i(\cdot,\cdot) - \langle Y_i, g(\cdot,\cdot)\rangle}{Re\, h_i(\cdot,\cdot)}$, $i = 1, 2, \ldots, m$*, has strictly convex real part at* $(\vartheta_0, \bar{\vartheta}_0)$*,*

(ii) *for* $Re\, f_i(\cdot,\cdot) \leqq 0, Re\, h_i(\cdot,\cdot) > 0$, $\dfrac{f_i(\cdot,\cdot) - \langle Y_i, g(\cdot,\cdot)\rangle}{Re\, h_i(\cdot,\cdot)}$, $i = 1, 2, \ldots, m$*, has strictly invex real part w.r.t.* $\widetilde{\eta}(\varrho, \varrho_0)$ *at* $(\vartheta_0, \bar{\vartheta}_0)$*,*

Then $(\varrho_0, \bar{\varrho}_0) = (\vartheta_0, \bar{\vartheta}_0)$*, i.e.,* $(\vartheta_0, \bar{\vartheta}_0)$ *is an efficient solution of* (*MOFP*) *and* $\varphi(\varrho_0, \bar{\varrho}_0) = B(\vartheta_0, \bar{\vartheta}_0, \hat{\gamma}, \hat{Y})$*.*

Proof. Let us assume that $(\varrho_0, \bar{\varrho}_0) \neq (\vartheta_0, \bar{\vartheta}_0)$. Now since $(\varrho_0, \bar{\varrho}_0)$ is an efficient solution for the problem (*MOFP*), then by Theorem 4.3, $\exists\, \widetilde{\gamma}$ and $\widetilde{Y} > 0$ s.t. $(\varrho_0, \bar{\varrho}_0, \widetilde{\gamma}, \widetilde{Y})$ is an efficient solution of (*BD*).
Also by the assumption of the theorem, $(\vartheta_0, \bar{\vartheta}_0, \hat{\gamma}, \hat{Y})$ is also an efficient solution of (*BD*) and hence for each $i = 1, 2, \ldots, m$,

$$\frac{Re\left(f_i(\varrho_0, \bar{\varrho}_0) - \left\langle \widetilde{Y}_i, g(\varrho_0, \bar{\varrho}_0)\right\rangle\right)}{Re\, h_i(\varrho_0, \bar{\varrho}_0)} = \frac{Re\left(f_i(\vartheta_0, \bar{\vartheta}_0) - \left\langle \hat{Y}_i, g(\vartheta_0, \bar{\vartheta}_0)\right\rangle\right)}{Re\, h_i(\vartheta_0, \bar{\vartheta}_0)} \tag{9}$$

By strictly convexity of real part of $\dfrac{f_i(\cdot,\cdot) - \langle Y_i, g(\cdot,\cdot)\rangle}{Re\, h_i(\cdot,\cdot)}$ for each $i = 1, 2, \ldots, m$ at $(\vartheta_0, \bar{\vartheta}_0)$, we obtain,

$$\begin{aligned} &\frac{Re\left(f_i(\varrho_0, \bar{\varrho}_0) - \left\langle \hat{Y}_i, g(\varrho_0, \bar{\varrho}_0)\right\rangle\right)}{Re\, h_i(\varrho_0, \bar{\varrho}_0)} - \frac{Re\left(f_i(\vartheta_0, \bar{\vartheta}_0) - \left\langle \hat{Y}_i, g(\vartheta_0, \bar{\vartheta}_0)\right\rangle\right)}{Re\, h_i(\vartheta_0, \bar{\vartheta}_0)} \\ &> Re\left\langle \varrho_0 - \vartheta_0, \overline{\nabla_{\varrho} \frac{f_i(\vartheta_0, \bar{\vartheta}_0)}{Re\, h_i(\vartheta_0, \bar{\vartheta}_0)}} + \nabla_{\bar{\varrho}} \frac{f_i(\vartheta_0, \bar{\vartheta}_0)}{Re\, h_i(\vartheta_0, \bar{\vartheta}_0)} - \overline{\hat{Y}_i^T \nabla_{\varrho} \frac{g(\vartheta_0, \bar{\vartheta}_0)}{Re\, h_i(\vartheta_0, \bar{\vartheta}_0)}} - \hat{Y}_i^H \nabla_{\bar{\varrho}} \frac{g(\vartheta_0, \bar{\vartheta}_0)}{Re\, h_i(\vartheta_0, \bar{\vartheta}_0)} \right\rangle \\ &= 0 \text{ (by (8) and } \gamma_i > 0, i = 1, 2, \ldots, m) \end{aligned} \tag{10}$$

which contradicts (9). Thus, $(\varrho_0, \bar{\varrho}_0) = (\vartheta_0, \bar{\vartheta}_0)$ or $(\vartheta_0, \bar{\vartheta}_0)$ is an efficient solution of (*MOFP*).
Using (2), (9) becomes

$$\frac{Re\, f_i(\varrho_0, \bar{\varrho}_0)}{Re\, h_i(\varrho_0, \bar{\varrho}_0)} = \frac{Re\left(f_i(\vartheta_0, \bar{\vartheta}_0) - \left\langle \hat{Y}_i, g(\vartheta_0, \bar{\vartheta}_0)\right\rangle\right)}{Re\, h_i(\vartheta_0, \bar{\vartheta}_0)}$$

i.e., $\phi(\varrho_0, \bar{\varrho}_0) = B(\vartheta_0, \bar{\vartheta}_0, \hat{\gamma}, \hat{Y})$. □

5. Conclusion

In this work, we have considered a complex multiobjective fractional programming problem (*MOFP*). The complex version of ratio invexity result is provided. The sufficient optimality conditions have been developed by considering the generalized convexity of functions. Illustrative application is also provided for the better insight of efficiency conditions for the given problem. Further, a Bector type dual corresponding to the (*MOFP*) is formulated and the fundamental duality results namely weak duality, strong duality and strict converse duality theorems are established under the consideration of invexity of involved functions. The developed results can be extended to the case where the involved functions are non-analytic. Moreover, the robustness in the model under study can also be explored in future.

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