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Constructing group inverse and MP-inverse of the product of some generalized inverses via w-core inverse

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Abstract. Let R be a ring with an involution. In this paper, by studying the w-core inverses of R, we construct the group inverse and MP-inverse of the product of some generalized inverses in R.

1. Introduction

The study of generalized inverses in rings with involution is an important ingredient in the ring theory. Many researchers have done lots of results in this area. For instances, Mosić, Djordjević and Koliha gave numerous excellent conclusions of generalized inverses in rings with involution in [2–7]. Mosić et al. [3] presented a number of new characterizations of EP elements in purely algebraic terms. Furthermore, in [5], Mosić and Djordjević provided many new characterizations of EP, normal and Hermitian elements. In recent years, the third author in this paper and his cooperators [9, 11–14] investigated generalized inverses by using solutions of certain equations. For example, Shi and Wei studied the equivalent conditions of normal elements by the solutions of related equations [9]. In [13], Zhao and Wei characterized the partial isometry elements by the existence of solutions of equations in rings in a certain set, and also by the form of solutions of given equations. Recently, Zhu et al. defined and studied the w-core inverses and weighted w-core inverses in rings with involution in [10], [15] and [16].

In this paper, we construct the group inverse and MP-inverse of the product of some generalized inverses via w-core inverse. The paper is organized as follows: In Section 2, we recall the basic definitions. In Section 3, we study the w-core inverses and give some new results. In Section 4, based on the results in Section 3, we construct the group inverse and MP-inverse of the product of some generalized inverses via w-core inverse.

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2. Preliminaries

Throughout the paper, the letters $\mathbb{Z}$ and $\mathbb{Z}_+$ stand for the the ring of integers and the ring of positive integers, respectively.

Let R be an associative ring with 1. An element $a \in R$ is called group invertible if there exists $a^\# \in R$ such that

$$a = aa^\# a, \ a^\# = a^\# aa^\#, \ aa^\# = a^\# a.$$

The element $a^\#$ is the group inverse of a, which is unique if it exists. The set of all group invertible elements of R is denoted by $R^\#$. In particular, if $a = a^2b = ca^2$ for some $b, c \in R$, then $a^\# = cab = c^2a = ab^2$.

A map $* : R \to R, a \mapsto a^*$ is called an involution of R if

$$(a^*)^* = a, (ab)^* = b^*a^*, (a + b)^* = a^* + b^*.$$

An element $a \in R$ is said to be an Hermitian element [2] if $a^* = a$. The set of all Hermitian elements of R is denoted by R^{Her}. We say that $a^\dagger \in R$ is the Moore-Penrose inverse (or MP-inverse) of a, if

$$aa^\dagger a = a, \ a^\dagger aa^\dagger = a^\dagger, \ (aa^\dagger)^* = aa^\dagger, \ (a^\dagger a)^* = a^\dagger a.$$

If $a^\dagger$ exists, then it is unique [8]. The set of all MP-invertible elements in R is denoted by $R^\dagger$. We call an element $a \in R^\# \cap R^\dagger$ EP if $a^\# = a^\dagger$. The set of all EP elements of R is denoted by R^{EP}.

For $a, w \in R$, a is called the w-core invertible element of R if there exists $a_w^{\text{\textcircled{\#}}} \in R$ such that

$$a_w^{\text{\textcircled{\#}}} = aw(a_w^{\text{\textcircled{\#}}})^2, \ a = a_w^{\text{\textcircled{\#}}} awa, \ (awa_w^{\text{\textcircled{\#}}})^* = awa_w^{\text{\textcircled{\#}}},$$

where $a_w^{\text{\textcircled{\#}}}$ is called the w-core inverse of a. If $a_w^{\text{\textcircled{\#}}}$ exists, then it is unique. We write $R_w^{\text{\textcircled{\#}}}$ for the set of all w-core invertible elements of R. In particular, a is called a core invertible element of R if

$$a^{\text{\textcircled{\#}}} = a(a^{\text{\textcircled{\#}}})^2, \ a = a^{\text{\textcircled{\#}}} a^2, \ (aa^{\text{\textcircled{\#}}})^* = aa^{\text{\textcircled{\#}}},$$

where $a^{\text{\textcircled{\#}}}$ is said to be the core inverse of a. We denote the set of all core invertible elements of R by $R^{\text{\textcircled{\#}}}$.

3. New results from $a_w^{\text{\textcircled{\#}}}$

In this section, we first give some new results of w-core inverses. Then based on these results, we construct the group inverse and MP-inverse of the product of some generalized inverses.

The following conclusion was proved by Zhu et al., see [15, Theorem 2.10].

Lemma 3.1. *Let $a_w^{\text{\textcircled{\#}}} = x$. Then $(aw)^{\text{\textcircled{\#}}}$ exists, and $(aw)^{\text{\textcircled{\#}}} = x$.*

In [16], Zhu et al. showed the followings:

(1) if $(aw)^{\text{\textcircled{\#}}} = x$, then

$$x(aw)^2 = aw = awxaw, \ awx^2 = x = xawx, \ (awx)^* = awx.$$

(2) if $a_w^{\text{\textcircled{\#}}} = x$, then

$$awx^2 = x = xawx, \ xawa = a = awxa, \ (awx)^* = awx.$$

Lemma 3.2. *Let $a_w^{\circledast} = x$ and $n \in \mathbb{Z}_+$. Then the followings hold.*
(1) $(aw)^n x^n = awx$.
(2) $(awx)^n = awx$.
(3) $x^n(aw)^n = xaw$.
(4) $(xaw)^n = xaw$.
(5) $awx^{n+1} = x^n$.
(6) $x(aw)^{n+1} = (aw)^n$.
(7) $awx(aw)^n = (aw)^n$.
(8) $xawx^n = x^n$.

Proof. (1) By $awx^2 = x$, we have

$$\begin{aligned}(aw)^n x^n &= (aw)^{n-1}(awx^2)x^{n-2}\\ &= (aw)^{n-1}x^{n-1}\\ &= \cdots\\ &= awx.\end{aligned}$$

(2) From $awxa = a$, we have

$$\begin{aligned}(awx)^n &= (awxa)wx(awx)^{n-2}\\ &= (awx)^{n-1}\\ &= \cdots\\ &= awx.\end{aligned}$$

(3) Note that $x(aw)^2 = aw$, so

$$\begin{aligned}x^n(aw)^n &= x^{n-1}(x(aw)^2)(aw)^{n-2}\\ &= x^{n-1}(aw)^{n-1}\\ &= \cdots\\ &= xaw.\end{aligned}$$

(4) Notice that $xawx = x$, so

$$\begin{aligned}(xaw)^n &= (xawx)aw(xaw)^{n-2}\\ &= (xaw)^{n-1}\\ &= \cdots\\ &= xaw.\end{aligned}$$

(5) It follows from $awx^2 = x$ that

$$awx^{n+1} = (awx^2)x^{n-1} = x^n.$$

(6) It is noted that $x(aw)^2 = aw$, hence

$$x(aw)^{n+1} = (x(aw)^2)(aw)^{n-1} = (aw)^n.$$

(7) Since $awxaw = aw$,

$$awx(aw)^n = (awxaw)(aw)^{n-1} = (aw)^n.$$

(8) According to $xawx = x$, we have

$$xawx^n = (xawx)x^{n-1} = x^n. \qquad \square$$

Theorem 3.3. *Let* $a_w^{\textcircled{\#}} = x$ *and* $n, m \in \mathbb{Z}_+$. *Then*

(1) $a_{w(aw)^n}^{\textcircled{\#}}$ *exists, and* $a_{w(aw)^n}^{\textcircled{\#}} = x^{n+1}$.

(2) $[(aw)^n]_{(aw)^m}^{\textcircled{\#}}$ *exists, and* $[(aw)^n]_{(aw)^m}^{\textcircled{\#}} = x^{n+m}$.

(3) $[(aw)^n a]_{w(aw)^m}^{\textcircled{\#}}$ *exists, and* $[(aw)^n a]_{w(aw)^m}^{\textcircled{\#}} = x^{n+m+1}$.

Proof. (1) By Lemma 3.2 (1), (3) and (5),

$$\begin{aligned} a(w(aw)^n)(x^{n+1})^2 &= ((aw)^{n+1}x^{(n+1)})x^{n+1} \\ &= awx^{n+2} \\ &= x^{n+1}, \end{aligned}$$

$$\begin{aligned} x^{n+1}a(w(aw)^n)a &= (x^{n+1}(aw)^{n+1})a \\ &= xawa \\ &= a, \end{aligned}$$

$$\begin{aligned} a(w(aw)^n)x^{n+1} &= (aw)^{n+1}x^{n+1} \\ &= awx \\ &= (awx)^* \\ &= (a(w(aw)^n)x^{n+1})^*. \end{aligned}$$

(2) From Lemma 3.2 (1), (5) and (6), we have

$$\begin{aligned} (aw)^n(aw)^m(x^{n+m})^2 &= ((aw)^{n+m}x^{n+m})x^{n+m} \\ &= awx^{n+m+1} \\ &= x^{n+m}, \end{aligned}$$

$$\begin{aligned} x^{n+m}(aw)^n(aw)^m(aw)^n &= (x^{n+m}(aw)^{n+m})(aw)^n \\ &= x(aw)^{n+1} \\ &= (aw)^n, \end{aligned}$$

$$\begin{aligned} (aw)^n(aw)^m x^{n+m} &= (aw)^{n+m}x^{n+m} \\ &= awx \\ &= (awx)^* \\ &= ((aw)^n(aw)^m x^{n+m})^*. \end{aligned}$$

(3) By Lemma 3.2 (1), (5) and (6),

$$\begin{aligned} ((aw)^n a)(w(aw)^m)(x^{n+m+1})^2 &= ((aw)^{n+m+1}x^{n+m+1})x^{n+m+1} \\ &= awx^{n+m+2} \\ &= x^{n+m+1}, \end{aligned}$$

$$\begin{aligned} x^{n+m+1}((aw)^n a)(w(aw)^m)((aw)^n a) &= (x^{n+m+1}(aw)^{n+m+1})(aw)^n a \\ &= (x(aw)^{n+1})a \\ &= (aw)^n a, \end{aligned}$$

$$\begin{aligned} ((aw)^n a)(w(aw)^m)x^{n+m+1} &= (aw)^{n+m+1}x^{n+m+1} \\ &= awx \\ &= (awx)^* \\ &= (((aw)^n a)(w(aw)^m)x^{n+m+1})^*. \end{aligned}$$

□

Theorem 3.4. *Let* $a_w^{\textcircled{\#}} = x$ *and* $m, n \in \mathbb{Z}_+$. *Then*

(1) $a_{w(aw)^n x}^{\textcircled{\#}}$ *exists, and* $a_{w(aw)^n x}^{\textcircled{\#}} = x^n$.

(2) $[(aw)^n]_{(aw)^m x}^{\textcircled{\#}}$ *exists, and* $[(aw)^n]_{(aw)^m x}^{\textcircled{\#}} = x^{n+m-1}$.

(3) $[(aw)^n a]_{w(aw)^m x}^{\textcircled{\#}}$ *exists, and* $[(aw)^n a]_{w(aw)^m x}^{\textcircled{\#}} = x^{n+m}$.

Proof. (1) By Lemma 3.2 (1) and (5),

$$\begin{aligned} a(w(aw)^n x)(x^n)^2 &= ((aw)^{n+1} x^{n+1}) x^n \\ &= awx^{n+1} \\ &= x^n, \end{aligned}$$

$$\begin{aligned} x^n a(w(aw)^n x)a &= x^n (aw)^n (awxa) \\ &= xawa \\ &= a, \end{aligned}$$

$$\begin{aligned} a(w(aw)^n x)x^n &= (aw)^{n+1} x^{n+1} \\ &= awx \\ &= (awx)^* \\ &= (a(w(aw)^n x)x^n)^*. \end{aligned}$$

(2) It follows from Lemma 3.2 (1), (5), (6) and (7) that

$$\begin{aligned} (aw)^n((aw)^m x)(x^{n+m-1})^2 &= ((aw)^{n+m} x^{n+m}) x^{n+m-1} \\ &= awx^{n+m} \\ &= x^{n+m-1}, \end{aligned}$$

$$\begin{aligned} x^{n+m-1}(aw)^n((aw)^m x)(aw)^n &= x^{n+m-1}(aw)^{n+m-1}(awx(aw)^n) \\ &= xaw(aw)^n \\ &= (aw)^n, \end{aligned}$$

$$\begin{aligned} (aw)^n((aw)^m x)x^{n+m-1} &= (aw)^{n+m} x^{n+m} \\ &= awx \\ &= (awx)^* \\ &= ((aw)^n((aw)^m x)x^{n+m-1})^*. \end{aligned}$$

(3) By Lemma 3.2 (1), (5), (6) and (7),

$$\begin{aligned} ((aw)^n a)(w(aw)^m x)(x^{n+m})^2 &= ((aw)^{n+m+1} x^{n+m+1}) x^{n+m} \\ &= awx^{n+m+1} \\ &= x^{n+m}, \end{aligned}$$

$$\begin{aligned} x^{n+m}((aw)^n a)(w(aw)^m x)((aw)^n a) &= (x^{n+m}(aw)^{n+m})(awx(aw)^n)a \\ &= (xaw(aw)^n)a \\ &= (aw)^n a, \end{aligned}$$

$$
\begin{aligned}
((aw)^n a)(w(aw)^m x)x^{n+m} &= (aw)^{n+m+1}x^{n+m+1} \\
&= awx \\
&= (awx)^* \\
&= (((aw)^n a)(w(aw)^m x)x^{n+m})^*.
\end{aligned}
$$

□

Corollary 3.5. *Let $a_w^{\circledast} = x$ and $n, m \in \mathbb{Z}_+$. Then*

(1) $a^{\circledast}_{w(aw)^n x^l}$ exists, and $a^{\circledast}_{w(aw)^n x^l} = x^{n-l+1}$, where $1 \leq l \leq n$.

(2) $[(aw)^n]^{\circledast}_{(aw)^m x^l}$ exists, and $[(aw)^n]^{\circledast}_{(aw)^m x^l} = x^{n+m-l}$, where $1 \leq l \leq n+m-1$.

(3) $[(aw)^n a]^{\circledast}_{w(aw)^m x^l}$ exists, and $[(aw)^n a]^{\circledast}_{w(aw)^m x^l} = x^{n+m-l+1}$, where $1 \leq l \leq n+m$.

Proof. (1) By Lemma 3.2 (1), (3) and (5),

$$
\begin{aligned}
a(w(aw)^n x^l)(x^{n-l+1})^2 &= ((aw)^{n+1}x^{n+1})x^{n-l+1} \\
&= awx^{n-l+2} \\
&= x^{n-l+1},
\end{aligned}
$$

$$
\begin{aligned}
x^{n-l+1}a(w(aw)^n x^l)a &= (x^{n-l+1}(aw)^{n-l+1})((aw)^l x^l)a \\
&= xaw(awxa) \\
&= xawa \\
&= a,
\end{aligned}
$$

$$
\begin{aligned}
a(w(aw)^n x^l)x^{n-l+1} &= (aw)^{n+1}x^{n+1} \\
&= awx \\
&= (awx)^* \\
&= (a(w(aw)^n x)x^n)^*.
\end{aligned}
$$

(2) According to Lemma 3.2 (1), (5), (6) and (7), we have

$$
\begin{aligned}
(aw)^n((aw)^m x^l)(x^{n+m-l})^2 &= ((aw)^{n+m}x^{n+m})x^{n+m-l} \\
&= awx^{n+m-l+1} \\
&= x^{n+m-l},
\end{aligned}
$$

$$
\begin{aligned}
x^{n+m-l}(aw)^n((aw)^m x^l)(aw)^n &= (x^{n+m-l}(aw)^{n+m-l})((aw)^l x^l)(aw)^n \\
&= (xaw)((awx)(aw)^n) \\
&= xaw(aw)^n \\
&= (aw)^n,
\end{aligned}
$$

$$
\begin{aligned}
(aw)^n((aw)^m x^l)x^{n+m-l} &= (aw)^{n+m}x^{n+m} \\
&= awx \\
&= (awx)^* \\
&= ((aw)^n((aw)^m x^l)x^{n+m-l})^*.
\end{aligned}
$$

(3) By Lemma 3.2 (1), (3), (5), (6) and (7),

$$
\begin{aligned}
((aw)^n a)(w(aw)^m x^l)(x^{n+m-l+1})^2 &= ((aw)^{n+m+1}x^{n+m+1})x^{n+m-l+1} \\
&= awx^{n+m-l+2} \\
&= x^{n+m-l+1},
\end{aligned}
$$

$$\begin{aligned} x^{n+m-l+1}((aw)^n a)(w(aw)^m x^l)((aw)^n a) &= (x^{n+m-l+1}(aw)^{n+m-l+1})((aw)^l x^l)(aw)^n a \\ &= (xaw)((awx)(aw)^n)a \\ &= (xaw(aw)^n)a \\ &= (aw)^n a, \end{aligned}$$

$$\begin{aligned} ((aw)^n a)(w(aw)^m x^l)x^{n+m-l+1} &= (aw)^{n+m+1}x^{n+m+1} \\ &= awx \\ &= (awx)^* \\ &= (((aw)^n a)(w(aw)^m x^l)x^{n+m-l+1})^*. \end{aligned}$$

Remark 3.6. *In Corollary 3.5, let $l = 1$, then one can obtain Theorem 3.4.*

Theorem 3.7. *Let $a_w^{\textcircled{\#}} = x$ and $n \in \mathbb{Z}_+$. Then*
(1) $a_{wx}^{\textcircled{\#}}$ exists, and $a_{wx}^{\textcircled{\#}} = awx$.
(2) $[(aw)^n]_{x^n}^{\textcircled{\#}}$ exists, and $[(aw)^n]_{x^n}^{\textcircled{\#}} = awx$.
(3) $[(aw)^n x]_{x^{n-1}}^{\textcircled{\#}}$ exists, and $[(aw)^n x]_{x^{n-1}}^{\textcircled{\#}} = awx$.
(4) $[(aw)^n]_{x^n aw}^{\textcircled{\#}}$ exists, and $[(aw)^n]_{x^n aw}^{\textcircled{\#}} = x$.
(5) $[(aw)^n x]_{x^{n-1}aw}^{\textcircled{\#}}$ exists, and $[(aw)^n x]_{x^{n-1}aw}^{\textcircled{\#}} = x$.

Proof. (1) By Lemma 3.2 (1),

$$\begin{aligned} a(wx)(awx)^2 &= (awx)^3 \\ &= awx, \end{aligned}$$

$$\begin{aligned} (awx)a(wx)a &= awxa \\ &= a, \end{aligned}$$

$$\begin{aligned} a(wx)(awx) &= (awx)^2 \\ &= awx \\ &= (awx)^* \\ &= (a(wx)(awx))^*. \end{aligned}$$

(2) From Lemma 3.2 (1) and (7), we have

$$\begin{aligned} (aw)^n x^n (awx)^2 &= (awx)^3 \\ &= awx, \end{aligned}$$

$$\begin{aligned} (awx)(aw)^n x^n (aw)^n &= awx(aw)^n \\ &= (aw)^n, \end{aligned}$$

$$\begin{aligned} (aw)^n x^n (awx) &= (awx)^2 \\ &= awx \\ &= (awx)^* \\ &= ((aw)^n x^n (awx))^*. \end{aligned}$$

(3) By (2) and Lemma 3.2 (3) and (7),

$$((aw)^n x)x^{n-1}(awx)^2 = awx,$$

$$\begin{aligned}(awx)((aw)^n x)x^{n-1}((aw)^n x) &= (awx(aw)^n)(x^n(aw)^n)x \\ &= (aw)^n(xawx) \\ &= (aw)^n x,\end{aligned}$$

$$((aw)^n x)x^{n-1}(awx) = (((aw)^n x)x^{n-1}(awx))^*.$$

(4) It follows from Lemma 3.2 (1), (2) and (6) that

$$\begin{aligned}(aw)^n(x^n(aw))x^2 &= ((aw)^n x^n)(awx^2) \\ &= awx^2 \\ &= x,\end{aligned}$$

$$\begin{aligned}x(aw)^n(x^n(aw))(aw)^n &= (aw)^{n-1}(x^n(aw)^n)aw \\ &= (aw)^{n-1}x(aw)^2 \\ &= (aw)^{n-1}aw \\ &= (aw)^n,\end{aligned}$$

$$\begin{aligned}(aw)^n(x^n(aw))x &= ((aw)^n x^n)(awx) \\ &= (awx)^2 \\ &= awx \\ &= (awx)^* \\ &= ((aw)^n(x^n(aw))x)^*.\end{aligned}$$

(5) By (4) and Lemma 3.2 (3) and (6),

$$((aw)^n x)(x^{n-1}aw)x^2 = x,$$

$$\begin{aligned}x((aw)^n x)(x^{n-1}aw)((aw)^n x) &= (aw)^{n-1}(x^n(aw)^n)awx \\ &= (aw)^{n-1}(xawa)wx \\ &= (aw)^{n-1}awx \\ &= (aw)^n x,\end{aligned}$$

$$((aw)^n x)(x^{n-1}aw)x = (((aw)^n x)(x^{n-1}aw)x)^*.$$

□

Remark 3.8. *In fact, Theorem 3.7 (3) can be seen as* $[(aw)^n x^l]^{\circledast}_{x^{n-l}} = awx$, *where* $1 \le l < n$. *This is since* $(aw)^n x^l = (aw)^{n-l}((aw)^l x^l) = (aw)^{n-l}awx = (aw)^{n-l+1}x$, *then* $[(aw)^n x^l]^{\circledast}_{x^{n-l}} = awx$ *becomes* $[(aw)^{n-l+1}x]^{\circledast}_{x^{n-l}} = awx$. *Similarly, Theorem 3.7 (5) can be regarded as* $[(aw)^n x^l]^{\circledast}_{x^{n-l}aw} = x$, *where* $1 \le l < n$.

Theorem 3.9. *Let* $a^{\circledast}_w = x$ *and* $n \in \mathbb{Z}_+$. *Then*

(1) $a^{\circledast}_{wx(aw)^n}$ *exists, and* $a^{\circledast}_{wx(aw)^n} = x^n$.

(2) $(aw)^{\circledast}_{x(aw)^n}$ *exists, and* $(aw)^{\circledast}_{x(aw)^n} = x^n$.

(3) $(awx)^{\circledast}_{(aw)^n}$ *exists, and* $(awx)^{\circledast}_{(aw)^n} = x^n$.

Proof. (1) By Lemma 3.2 (1), (3), (5) and (7),

$$\begin{aligned}a(wx(aw)^n)(x^n)^2 &= (awx(aw)^n)x^{2n} \\ &= ((aw)^n x^n)x^n \\ &= awx^{n+1} \\ &= x^n,\end{aligned}$$

$$\begin{aligned} x^n a(wx(aw)^n)a &= x^n(awx(aw)^n)a \\ &= x^n(aw)^n a \\ &= xawa \\ &= a, \end{aligned}$$

$$\begin{aligned} a(wx(aw)^n)x^n &= (awx)((aw)^n x^n) \\ &= (awx)^2 \\ &= awx \\ &= (awx)^* \\ &= (a(wx(aw)^n)x^n)^*. \end{aligned}$$

(2) From (1) and Lemma 3.2 (3) and (7), we have

$$(aw)(x(aw)^n)(x^n)^2 = x^n,$$

$$\begin{aligned} x^n(aw)(x(aw)^n)(aw) &= x^n(awx(aw)^n)aw \\ &= x(aw)^2 \\ &= aw, \end{aligned}$$

$$(aw)(x(aw)^n)x^n = ((aw)(x(aw)^n)x^n)^*.$$

(3) By (1) and Lemma 3.2 (1), (3) and (7),

$$(awx)(aw)^n(x^n)^2 = x^n,$$

$$\begin{aligned} (x^n(awx)(aw)^n(awx) &= x^n(awx(aw)^n)awx \\ &= x^n(aw)^n awx \\ &= (xawa)wx \\ &= awx, \end{aligned}$$

$$(awx)(aw)^n x^n = ((awx)(aw)^n x^n)^*.$$

□

Theorem 3.10. *Let $a_w^{\textcircled{\#}} = x$ and $n \in \mathbb{Z}_+$. Then*

(1) $(x^n)_{(aw)^n}^{\textcircled{\#}}$ exists, and $(x^n)_{(aw)^n}^{\textcircled{\#}} = awx$.

(2) $(x^n a)_{w(aw)^{n-1}}^{\textcircled{\#}}$ exists, and $(x^n a)_{w(aw)^{n-1}}^{\textcircled{\#}} = awx$.

(3) $(x^n aw)_{(aw)^{n-1}}^{\textcircled{\#}}$ exists, and $(x^n aw)_{(aw)^{n-1}}^{\textcircled{\#}} = awx$.

Proof. (1) By Lemma 3.2 (2), (3), and (5),

$$\begin{aligned} x^n(aw)^n(awx)^2 &= (xawa)wx \\ &= awx, \end{aligned}$$

$$\begin{aligned} (awx)x^n(aw)^n x^n &= (awx^2)(awx^n) \\ &= xx^{n-1} \\ &= x^n, \end{aligned}$$

$$\begin{aligned} x^n(aw)^n(awx) &= (xawa)wx \\ &= awx \\ &= (awx)^* \\ &= (a(wx(aw)^n)x^n)^*. \end{aligned}$$

(2) From (1) and Lemma 3.2 (1) and (5), we have

$$(x^n a)(w(aw)^{n-1})(awx)^2 = awx,$$

$$\begin{aligned}(awx)(x^n a)(w(aw)^{n-1})x^n a &= (awx^{n+1})((aw)^n x^n)a\\ &= x^n(awxa)\\ &= x^n a,\end{aligned}$$

$$(x^n a)(w(aw)^{n-1})(awx) = ((x^n a)(w(aw)^{n-1})(awx))^*.$$

(3) By (1) and Lemma 3.2 (5),

$$(x^n aw)(aw)^{n-1}(awx)^2 = awx,$$

$$\begin{aligned}(awx)(x^n aw)(aw)^{n-1}(x^n aw) &= (awx)(x^n(aw)^n)x^n aw\\ &= (awx^2)(awx^n)aw\\ &= xx^{n-1}aw\\ &= x^n aw,\end{aligned}$$

$$(x^n aw)(aw)^{n-1}(awx) = ((x^n aw)(aw)^{n-1}(awx))^*.$$

□

Remark 3.11. *Theorem 3.10 (2) can be viewed as $[x^n(aw)^l a]^{\textcircled{\#}}_{w(aw)^{n-l-1}} = awx$, where $1 \leq l < n$. This is because $x^n(aw)^l a = x^{n-l}(x^l(aw)^l)a = x^{n-l}(xawa) = x^{n-l}a$. Thus $[x^n(aw)^l a]^{\textcircled{\#}}_{w(aw)^{n-l-1}} = awx$ is reduced as $(x^{n-l}a)^{\textcircled{\#}}_{w(aw)^{n-l-1}} = awx$. Similarly, Theorem 3.10 (3) can be regarded as $[x^n(aw)^l]^{\textcircled{\#}}_{(aw)^{n-l}} = awx$, where $1 \leq l < n$.*

Theorem 3.12. *Let $a^{\textcircled{\#}}_w = x$ and $n \in \mathbb{Z}_+$. Then*

(1) $x^{\textcircled{\#}}_{(aw)^{n+1}}$ exists, and $x^{\textcircled{\#}}_{(aw)^{n+1}} = x^n$.

(2) $(xa)^{\textcircled{\#}}_{w(aw)^n}$ exists, and $(xa)^{\textcircled{\#}}_{w(aw)^n} = x^n$.

(3) $(xaw)^{\textcircled{\#}}_{(aw)^n}$ exists, and $(xaw)^{\textcircled{\#}}_{(aw)^n} = x^n$.

Proof. (1) By Lemma 3.2 (2), (3), and (6),

$$\begin{aligned}x(aw)^{n+1}(x^n)^2 &= x((aw)^{n+1}x^{n+1})x^{n-1}\\ &= (xawx)x^{n-1}\\ &= x^n,\end{aligned}$$

$$\begin{aligned}x^n x(aw)^{n+1}x &= (x^{n+1}(aw)^{n+1})x\\ &= xawx\\ &= x,\end{aligned}$$

$$\begin{aligned}x(aw)^{n+1}x^n &= (aw)^n x^n\\ &= awx\\ &= (awx)^*\\ &= (x(aw)^{n+1}x^n)^*.\end{aligned}$$

(2) According to (1) and Lemma 3.2 (3), we have

$$(xa)(w(aw)^n)(x^n)^2 = x^n,$$

$$\begin{aligned}x^n(xa)(w(aw)^n)xa &= (x^{n+1}(aw)^{n+1})xa\\ &= (xawx)a\\ &= xa,\end{aligned}$$

$$(xa)(w(aw)^n)x^n = ((xa)(w(aw)^n)x^n)^*.$$

(3) By (1) and Lemma 3.2 (3),

$$(xaw)(aw)^n(x^n)^2 = x^n,$$

$$\begin{aligned} x^n(xaw)(aw)^n(xaw) &= (x^{n+1}(aw)^{n+1})xaw \\ &= (xawx)aw \\ &= xaw, \end{aligned}$$

$$(xaw)(aw)^n x^n = ((xaw)(aw)^n x^n)^*.$$

□

Remark 3.13. *From* $[(aw)^n a]^{\text{\textcircled{\#}}}_{w(aw)^m} = x^{n+m+1}$*, see Theorem 3.3 (3), one can obtain that if* $1 \leq l < n$*, then* $[x(aw)^l a]^{\text{\textcircled{\#}}}_{w(aw)^{n-l}} = x^n$*. This is since* $x(aw)^l a = (aw)^{l-1}a$*, then* $[x(aw)^l a]^{\text{\textcircled{\#}}}_{w(aw)^{n-l}} = x^n$ *becomes* $[(aw)^{l-1}a]^{\text{\textcircled{\#}}}_{w(aw)^{n-l}} = x^n$*. Similarly, by* $[(aw)^n]^{\text{\textcircled{\#}}}_{(aw)^m} = x^{n+m}$*, see Theorem 3.3 (2), one can get that in case* $1 \leq l < n$*, then* $[x(aw)^l]^{\text{\textcircled{\#}}}_{(aw)^{n-l}} = x^n$*.*

Theorem 3.14. *Let* $a^{\text{\textcircled{\#}}}_w = x$ *and* $m, n \in \mathbb{Z}_+$*. Then*

(1) $(x^n)^{\text{\textcircled{\#}}}_{(aw)^{n+m}}$ *exists, and* $(x^n)^{\text{\textcircled{\#}}}_{(aw)^{n+m}} = x^m$*.*

(2) $(x^n a)^{\text{\textcircled{\#}}}_{w(aw)^{n+m-1}}$ *exists, and* $(x^n a)^{\text{\textcircled{\#}}}_{w(aw)^{n+m-1}} = x^m$*.*

(3) $(x^n aw)^{\text{\textcircled{\#}}}_{(aw)^{n+m-1}}$ *exists,* $(x^n aw)^{\text{\textcircled{\#}}}_{(aw)^{n+m-1}} = x^m$*.*

Proof. (1) By Lemma 3.2 (1), (3), (5), and (8),

$$\begin{aligned} x^n(aw)^{n+m}(x^m)^2 &= (x^n(aw)^n)((aw)^m x^m)x^m \\ &= (xawa)wxx^m \\ &= awx^{m+1} \\ &= x^m, \end{aligned}$$

$$\begin{aligned} x^m x^n(aw)^{n+m}x^n &= (x^{m+n}(aw)^{n+m})x^n \\ &= xawx^n \\ &= x^n, \end{aligned}$$

$$\begin{aligned} x^n(aw)^{n+m}x^m &= (x^n(aw)^n)((aw)^m x^m) \\ &= (xawa)wx \\ &= awx \\ &= (awx)^* \\ &= (x(aw)^{n+1}x^n)^*. \end{aligned}$$

(2) From (1) and Lemma 3.2 (3) and (8), we have

$$(x^n a)(w(aw)^{n+m-1})(x^m)^2 = x^m,$$

$$\begin{aligned} x^m(x^n a)(w(aw)^{n+m-1})(x^n a) &= (x^{n+m}(aw)^{n+m})(x^n a) \\ &= (xawx^n)a \\ &= x^n a, \end{aligned}$$

$$(x^n a)(w(aw)^{n+m-1})x^m = ((x^n a)(w(aw)^{n+m-1})x^m)^*.$$

(3) By (1) and Lemma 3.2 (3) and (8),

$$(x^n aw)(aw)^{n+m-1}(x^m)^2 = x^m,$$

$$\begin{aligned} x^m(x^n aw)(aw)^{n+m-1}(x^n aw) &= (x^{n+m}(aw)^{n+m})x^n aw \\ &= (xawx^n)aw \\ &= x^n aw, \end{aligned}$$

$$(x^n aw)(aw)^{n+m-1}x^m = ((x^n aw)(aw)^{n+m-1}x^m)^*.$$

□

Remark 3.15. *We claim that* $[x^n(aw)^l a]^{\textcircled{\#}}_{w(aw)^{n+m-l}} = x^m$. *This is since if* $1 \leq l < n$, *then* $x^n(aw)^l a = x^{n-l}(x^l(aw)^l)a = x^{n-l}(xawa) = x^{n-l}a$. *Thus, in this case,* $[x^n(aw)^l a]^{\textcircled{\#}}_{w(aw)^{n+m-l}} = (x^{n-l}a)^{\textcircled{\#}}_{w(aw)^{n+m-l}} = x^m$, *which coincides with Theorem 3.14 (2). The case of* $l = n$ *follows from Theorem 3.3 (1). Provided that* $n < l < n+m$, *then* $x^n(aw)^l a = (x^n(aw)^n)(aw)^{l-n}a = xaw(aw)^{l-n}a = (aw)^{l-n}a$. *Thus, by Theorem 3.3 (3),* $[x^n(aw)^l a]^{\textcircled{\#}}_{w(aw)^{n+m-l}} = [(aw)^{l-n}a]^{\textcircled{\#}}_{w(aw)^{n+m-l}} = x^m$. *Similarly, one can check that* $[x^n(aw)^l]^{\textcircled{\#}}_{(aw)^{n+m-l}} = x^m$ *by Theorems 3.3 (2), 3.12 (3) and 3.14 (3).*

4. Construct group inverse and MP-inverse via w-core inverse

In this section, we will give new results of group inverses and MP-inverses of the product of some generalized inverses.

Theorem 4.1. *Let* $(aw)^{\textcircled{\#}} = x$. *Then* $x \in R^{EP}$ *and* $x^\# = x^+ = awawx$.

Proof. Since $(aw)^{\textcircled{\#}} = x$, $x = awx^2$. Moreover,

$$x = xawx = x(xawaw)x = x^2(awawx),$$

hence $x \in R^\#$ and $x^\# = (aw)x(awawx) = aw(xawaw)x = awawx$. Thus

$$xx^\# = xawawx = (xawaw)x = awx = (awx)^* = (xx^\#)^*.$$

It follows that $x \in R^{EP}$. □

Corollary 4.2. *Let* $(aw)^{\textcircled{\#}} = x$ *and* $n \in \mathbb{Z}_+$. *Then* $(x^\#)^n = (x^+)^n = (aw)^{n+1}x$.

Proof. By $x^\# = x^+ = awawx$, we have

$$\begin{aligned} (x^\#)^n &= (x^+)^n \\ &= (awawx)^n \\ &= aw(awxaw)awx(awawx)^{n-2} \\ &= aw(awawx)^{n-1} \\ &= awaw(awxaw)awx(aw)^{n-3} \\ &= (aw)^2(awawx)^{n-2} \\ &= \cdots \\ &= (aw)^{n-1}awawx \\ &= (aw)^{n+1}x. \end{aligned}$$

□

Theorem 4.3. *Let* $(aw)^{\textcircled{\#}} = x$. *Then* $aw \in R^\#$ *and* $(aw)^\# = x^2 aw$.

Proof. On one hand, by $(aw)^{\textcircled{\#}} = x$, we have $aw = x(aw)^2$. On the other hand,

$$aw = awxaw = awx^{\#}x^2aw = aw(awawx)x^2aw = (aw)^2(awx^3)aw = (aw)^2x^2aw.$$

Hence, $aw \in R^{\#}$ and $(aw)^{\#} = x(aw)x^2aw = x(awx^2)aw = x^2aw$. □

Corollary 4.4. *Let $(aw)^{\textcircled{\#}} = x$ and $n \in \mathbb{Z}_+$. Then $((aw)^{\#})^n = x^{n+1}aw$.*

Proof. By $(aw)^{\#} = x^2aw$, we have

$$\begin{aligned}((aw)^{\#})^n &= (x^2aw)^n \\ &= x^2(awx^2)aw(x^2aw)^{n-2} \\ &= x^3aw(x^2aw)^{n-2} \\ &= x^3(awx^2)aw(x^2aw)^{n-3} \\ &= x^4aw(x^2aw)^{n-3} \\ &= \cdots \\ &= x^n(awx^2)aw \\ &= x^{n+1}aw.\end{aligned}$$

□

Corollary 4.5. *Let $(aw)^{\textcircled{\#}} = x$ and $n \in \mathbb{Z}_+$. Then*
(1) $(x^{\#})^n((aw)^{\#})^n = (x^+)^n((aw)^{\#})^n = xaw$.
(2) $((aw)^{\#})^n(x^{\#})^n = ((aw)^{\#})^n(x^+)^n = awx$.

Proof. (1) By Lemma 3.2 (1) and Corollary 4.2, we have

$$\begin{aligned}(x^{\#})^n((aw)^{\#})^n &= ((aw)^{n+1}x)(x^{n+1}aw) \\ &= ((aw)^{n+1}x^{n+1})xaw \\ &= (awx^2)aw \\ &= xaw.\end{aligned}$$

(2) By Lemma 3.2 (3) and Corollary 4.4, we have

$$\begin{aligned}((aw)^{\#})^n(x^{\#})^n &= (x^{n+1}aw)((aw)^{n+1}x) \\ &= (x^{n+1}(aw)^{n+1})awx \\ &= x(aw)^2x \\ &= awx.\end{aligned}$$

□

By Lemma 3.1, Theorem 3.3 (1), and Corollaries 4.2, 4.4, 4.5, we get the following results.

Theorem 4.6. *Let $(aw)^{\textcircled{\#}} = x$ and $n, m \in \mathbb{Z}_+$. Then*
(1) $[(aw)^n]^{\textcircled{\#}}$ exists, and $[(aw)^n]^{\textcircled{\#}} = x^n$.
(2) $((x^n)^{\#})^m = (aw)^{nm+1}x$.
(3) $(((aw)^n)^{\#})^m = x^{nm+1}aw$.
(4) $((x^n)^{\#})^m(((aw)^n)^{\#})^m = xaw$.
(5) $(((aw)^n)^{\#})^m((x^n)^{\#})^m = awx$.

Proof. (1) It follows from $(aw)^{\textcircled{\#}} = x$, Theorem 3.3 (1) and Lemma 3.1.

(2) By Corollary 4.2 and Lemma 3.2 (1), we have

$$\begin{aligned}((x^n)^\#)^m &= ((aw)^n)^{m+1}x^n \\ &= (aw)^{nm+n}x^n \\ &= (aw)^{nm}((aw)^n x^n) \\ &= (aw)^{nm}awx \\ &= (aw)^{nm+1}x.\end{aligned}$$

(3) By Corollary 4.4 and Lemma 3.2 (3), we have

$$\begin{aligned}(((aw)^n)^\#)^m &= (x^n)^{m+1}(aw)^n \\ &= x^{nm+n}(aw)^n \\ &= x^{nm}(x^n(aw)^n) \\ &= x^{nm}xaw \\ &= x^{nm+1}aw.\end{aligned}$$

(4) According to (2), (3) and Lemma 3.2 (1), we have

$$\begin{aligned}((x^n)^\#)^m(((aw)^n)^\#)^m &= ((aw)^{nm+1}x)(x^{nm+1}aw) \\ &= (awx)(xaw) \\ &= (awx^2)aw \\ &= xaw.\end{aligned}$$

(5) From (2), (3) and Lemma 3.2 (3), we have

$$\begin{aligned}(((aw)^n)^\#)^m((x^n)^\#)^m &= (x^{nm+1}aw)((aw)^{nm+1}x) \\ &= (xaw)(awx) \\ &= x(aw)^2x \\ &= awx.\end{aligned}$$

□

Remark 4.7. *From Corollaries 4.2, 4.4 and Theorem 4.6, one can see that* $((x^\#)^n)^m = ((x^n)^\#)^m$ *and* $(((aw)^\#)^n)^m = (((aw)^n)^\#)^m$ *for each* $m, n \in \mathbb{Z}_+$.

If one is interested in other results, we think use the conclusions in Section 3, there should be many other interesting results. Next, we consider some concrete w-core inverses.

Theorem 4.8. *Let* $a \in R^\# \cap R^\dagger$. *Then* $a^{\textcircled{\#}}_{(a^\dagger)^*} = aa^\# a^* a^\# aa^\dagger$.

Proof. Note that

$$a(a^\dagger)^*(aa^\# a^* a^\# aa^\dagger) = a(a^\dagger)^* a^* a^\# aa^\dagger = a^2 a^\dagger a^\# aa^\dagger = aa^\# aa^\dagger = aa^+ = (aa^\dagger)^* = (a(a^\dagger)^* aa^\# a^* a^\# aa^\dagger)^*,$$

$$a(a^\dagger)^*(aa^\# a^* a^\# aa^\dagger)^2 = aa^\dagger(aa^\# a^* a^\# aa^\dagger) = aa^\# a^* a^\# aa^\dagger,$$

and

$$(aa^{\#}a^{*}a^{\#}aa^{\dagger})a(a^{\dagger})^{*}a = aa^{\#}a^{*}a^{\#}a(a^{\dagger})^{*}a = aa^{\#}a^{*}(a^{\dagger})^{*}a = aa^{\#}a^{\dagger}a^{2} = aa^{\#}a = a.$$

Hence, $a^{\textcircled{\#}}_{(a^{\dagger})^{*}} = aa^{\#}a^{*}a^{\#}aa^{\dagger}$. □

Corollary 4.9. *Let $a \in R^{\#} \cap R^{\dagger}$ and $n \in \mathbb{Z}_{+}$. Then*
(1) $((aa^{\#}a^{*}a^{\#}aa^{\dagger})^{\#})^{n} = (a(a^{\dagger})^{*})^{n}aa^{\dagger}$.
(2) $((a(a^{\dagger})^{*})^{\#})^{n} = (aa^{\#}a^{*}a^{\#}aa^{\dagger})^{n}aa^{\#}$.
(3) $((aa^{\#}a^{*}a^{\#}aa^{\dagger})^{\#})^{n}((a(a^{\dagger})^{*})^{\#})^{n} = aa^{\#}$.
(4) $((a(a^{\dagger})^{*})^{\#})^{n}((aa^{\#}a^{*}a^{\#}aa^{\dagger})^{\#})^{n} = aa^{\dagger}$.

Proof. (1) By Theorem 4.8 and Corollary 4.2, we have

$$\begin{aligned}
((aa^{\#}a^{*}a^{\#}aa^{\dagger})^{\#})^{n} &= (a(a^{\dagger})^{*})^{n+1}aa^{\#}a^{*}a^{\#}aa^{\dagger} \\
&= (a(a^{\dagger})^{*})^{n}a((a^{\dagger})^{*}aa^{\#})a^{*}a^{\#}aa^{\dagger} \\
&= (a(a^{\dagger})^{*})^{n}a((a^{\dagger})^{*}a^{*})a^{\#}aa^{\dagger} \\
&= (a(a^{\dagger})^{*})^{n}a(aa^{\dagger})^{*}a^{\#}aa^{\dagger} \\
&= (a(a^{\dagger})^{*})^{n}(a^{2}a^{\dagger}a^{\#})aa^{\dagger} \\
&= (a(a^{\dagger})^{*})^{n}aa^{\#}aa^{\dagger} \\
&= (a(a^{\dagger})^{*})^{n-1}a((a^{\dagger})^{*}aa^{\#})aa^{\dagger} \\
&= (a(a^{\dagger})^{*})^{n-1}a(a^{\dagger})^{*}aa^{\dagger} \\
&= (a(a^{\dagger})^{*})^{n}aa^{\dagger}.
\end{aligned}$$

(2) According to Theorem 4.8 and Corollary 4.4, we have

$$\begin{aligned}
((a(a^{\dagger})^{*})^{\#})^{n} &= (aa^{\#}a^{*}a^{\#}aa^{\dagger})^{n+1}a(a^{\dagger})^{*} \\
&= (aa^{\#}a^{*}a^{\#}aa^{\dagger})^{n}aa^{\#}a^{*}a^{\#}(aa^{\dagger}a)(a^{\dagger})^{*} \\
&= (aa^{\#}a^{*}a^{\#}aa^{\dagger})^{n}aa^{\#}a^{*}(a^{\#}a(a^{\dagger})^{*}) \\
&= (aa^{\#}a^{*}a^{\#}aa^{\dagger})^{n}aa^{\#}(a^{*}(a^{\dagger})^{*}) \\
&= (aa^{\#}a^{*}a^{\#}aa^{\dagger})^{n}aa^{\#}(a^{\dagger}a)^{*} \\
&= (aa^{\#}a^{*}a^{\#}aa^{\dagger})^{n}a(a^{\#}a^{\dagger}a) \\
&= (aa^{\#}a^{*}a^{\#}aa^{\dagger})^{n}aa^{\#}.
\end{aligned}$$

(3) From Theorem 4.8 and Corollary 4.5, we have

$$\begin{aligned}
((aa^{\#}a^{*}a^{\#}aa^{\dagger})^{\#})^{n}((a(a^{\dagger})^{*})^{\#})^{n} &= aa^{\#}a^{*}a^{\#}(aa^{\dagger}a)(a^{\dagger})^{*} \\
&= aa^{\#}a^{*}(a^{\#}a(a^{\dagger})^{*}) \\
&= aa^{\#}a^{*}(a^{\dagger})^{*} \\
&= aa^{\#}(a^{\dagger}a)^{*} \\
&= a(a^{\#}a^{\dagger}a) \\
&= aa^{\#}.
\end{aligned}$$

(4) It follows from Theorem 4.8 and Corollary 4.5 that

$$
\begin{aligned}
((a(a^{\dagger})^{*})^{\#})^{n}((aa^{\#}a^{*}a^{\#}aa^{\dagger})^{\#})^{n} &= a((a^{\dagger})^{*}aa^{\#})a^{*}a^{\#}aa^{\dagger} \\
&= a((a^{\dagger})^{*}a^{*})a^{\#}aa^{\dagger} \\
&= a(aa^{\dagger})^{*}a^{\#}aa^{\dagger} \\
&= (a^{2}a^{\dagger}a^{\#})aa^{\dagger} \\
&= (aa^{\#}a)a^{\dagger} \\
&= aa^{+}.
\end{aligned}
$$

□

Theorem 4.10. *Let $a \in R^{\#} \cap R^{\dagger}$ and $n \in \mathbb{Z}_{+}$. Then*
(1) $(a^{\dagger})^{\textcircled{\#}}_{a^{*}}$ exists, and $(a^{\dagger})^{\textcircled{\#}}_{a^{*}} = (a^{\#})^{*}a$.
(2) $(((a^{\#})^{*}a)^{\#})^{n} = (a^{\dagger}a^{*})^{n}a^{\dagger}a$.
(3) $((a^{\dagger}a^{*})^{\#})^{n} = ((a^{\#})^{*}a)^{n}(aa^{\#})^{*}$.

Proof. (1) It follows from a straightforward verification.
(2) From (1) and Corollary 4.2, we have

$$
\begin{aligned}
(((a^{\#})^{*}a)^{\#})^{n} &= (a^{\dagger}a^{*})^{n+1}(a^{\#})^{*}a \\
&= (a^{\dagger}a^{*})^{n}(a^{\dagger}a^{*}(a^{\#})^{*})a \\
&= (a^{\dagger}a^{*})^{n}a^{\dagger}a.
\end{aligned}
$$

(3) By (1) and Corollary 4.4, we have

$$
\begin{aligned}
(a^{\dagger}a^{*})^{n} &= ((a^{\#})^{*}a)^{n+1}a^{\dagger}a^{*} \\
&= ((a^{\#})^{*}a)^{n}((a^{\#})^{*}aa^{\dagger}a^{*}) \\
&= ((a^{\#})^{*}a)^{n}(a(aa^{\dagger})a^{\#})^{*} \\
&= ((a^{\#})^{*}a)^{n}(aa^{\#})^{*}.
\end{aligned}
$$

□

Theorem 4.11. *Let $a \in R^{\#} \cap R^{\dagger}$ and $n \in \mathbb{Z}_{+}$. Then*
(1) $a^{\textcircled{\#}}_{a^{*}a}$ exists, and $a^{\textcircled{\#}}_{a^{*}a} = a^{\#}(a^{\dagger})^{*}a^{\dagger}$.
(2) $((a^{\#}(a^{\dagger})^{*}a^{\dagger})^{\#})^{n} = (aa^{*}a)^{n}aa^{\dagger}$.
(3) $((aa^{*}a)^{\#})^{n} = (a^{\#}(a^{\dagger})^{*}a^{\dagger})^{n}a^{\#}a$.

Proof. (1) It follows from a straightforward verification.
(2) By (1) and Corollary 4.2, we have

$$
\begin{aligned}
((a^{\#}(a^{\dagger})^{*}a^{\dagger})^{\#})^{n} &= (aa^{*}a)^{n+1}a^{\#}(a^{\dagger})^{*}a^{\dagger} \\
&= (aa^{*}a)^{n}aa^{*}(aa^{\#}(a^{\dagger})^{*})a^{\dagger} \\
&= (aa^{*}a)^{n}a(a^{*}(a^{\dagger})^{*})a^{\dagger} \\
&= (aa^{*}a)^{n}(aa^{\dagger}a)a^{\dagger} \\
&= (aa^{*}a)^{n}aa^{\dagger}.
\end{aligned}
$$

(3) From (1) and Corollary 4.4, we have

$$
\begin{aligned}
((aa^*a)^\#)^n &= (a^\#(a^\dagger)^*a^\dagger)^{n+1}aa^*a \\
&= (a^\#(a^\dagger)^*a^\dagger)^n a^\#((a^\dagger)^*a^\dagger a)a^*a \\
&= (a^\#(a^\dagger)^*a^\dagger)^n a^\#(a^\dagger aa^\dagger)^*a^*a \\
&= (a^\#(a^\dagger)^*a^\dagger)^n a^\#((a^\dagger)^*a^*)a \\
&= (a^\#(a^\dagger)^*a^\dagger)^n a^\# aa^\dagger a \\
&= (a^\#(a^\dagger)^*a^\dagger)^n a^\# a.
\end{aligned}
$$

□

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Conflict of Interest

The authors declared that they have no conflict of interest.

References

[1] L. F. Cao, L. You, J. C. Wei, *EP elements of* $\mathbb{Z}[x]/(x^2+x)$, Filomat **37(22)** (2023) 7467-7478.
[2] D. Mosić, *Generalized inverses*, Faculty of Sciences and Mathematics, University of Niš, Niš (2018).
[3] D. Mosić, D. S. Djordjević, J. J. Koliha, *EP elements in rings*, Linear Algebra Appl. **431** (2009) 527-535.
[4] D. Mosić, D. S. Djordjević, *Moore-Penrose-invertible normal and Hermitian elements in rings*, Linear Algebra Appl. **431** (2009) 732-745.
[5] D. Mosić, D. S. Djordjević, *New characterizations of EP, generalized normal and generalized Hermitian elements in rings*, Appl. Math. Comput. **218** (2012) 6702-6710.
[6] D. Mosić, D. S. Djordjević, *Further results on partial isometries and EP elements in rings with involution*, Math. Comput. Model **54** (2011) 460-465.
[7] D. Mosić, D. S. Djordjević, *Partial isometries and EP elements in rings with involution*, Electron. J. Linear Algebra **18** (2009) 761-772.
[8] R. Penrose, *A generalized inverse for matrices*, Proc. Cambridge Philos. Soc. **51** (1955) 406-413.
[9] L. Y. Shi, J. C. Wei, *Some new characterizations of normal elements*, Filomat **33(13)** (2019) 4115-4120.
[10] L. Y. Wu, H. H. Zhu, *Weighted w-core inverses in rings*, Czecho. Math. J. **73** (2023) 581-602.
[11] Z. C. Xu, R. J. Chen, J. C. Wei, *Strongly EP elements in a ring with involution*, Filomat **34(6)** (2020) 2101-2107.
[12] D. D. Zhao, J. C. Wei, *Strongly EP elements in rings with involution*, J. Algebra Appl. (2022), 2250088, 10pages, DOI: 10.1142/S0219498822500888.
[13] D. D. Zhao, J. C. Wei, *Some new characterizations of partial isometries in rings with involution*, Intern. Eletron. J. Algebra **30** (2021) 304-311.
[14] R. J. Zhao, H. Yao, J. C. Wei, *Characterizations of partial isometries and two special kinds of EP elements*, Czecho. Math. J. **70(2)** (2020) 539-551.
[15] H. H. Zhu, L. Y. Wu, J. L. Chen, *A new class of generalized inverses in semigroups and rings with involution*, Comm. Algebra **51(5)** (2023) 2098-2113.
[16] H. H. Zhu, L. Y. Wu, D. Mosić, *One-sided w-core inverses in rings with an involution*, Linear Algebra Appl. **71(4)** (2023) 528-544.