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Rings of functions whose closure of discontinuity set is in an ideal of closed sets

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Abstract. Let $\mathcal{P}$ be an ideal of closed subsets of a T_1 topological space X. Suppose $C(X)_{\mathcal{P}}$ is the ring of all real-valued functions on X whose closure of discontinuity set is a member of $\mathcal{P}$. We investigate a number of properties associated with the ring structure of $C(X)_{\mathcal{P}}$ for different choices of $\mathcal{P}$. Mention may be made of the $\aleph_0$-self injectivity, Artinian-ness and/or Noetherian-ness of $C(X)_{\mathcal{P}}$. We further examine when is $C(X)_{\mathcal{P}}$ closed under uniform limit, for various choices of $\mathcal{P}$. We find out several conditions equivalent to the Von-Neumann regularity of $C(X)_{\mathcal{P}}$. We also determine via a number of conditions when does $C(X)_{\mathcal{P}}$ become a Bezout ring. The concept of $\mathcal{F}P$-space was introduced by Gharabaghi et. al. in [8]. In this paper, they established a result which essentially tells that if $C(X)$ is a Von-Neumann regular ring then X is an $\mathcal{F}P$-space. We show that this result might fail if X is not Tychonoff and we provide a suitable counterexample to prove this.

1. Introduction

We assume all our topological spaces (X, τ) to be T_1. Suppose, $\mathcal{P}$ is an ideal of closed subsets of (X, τ), this means that $\mathcal{P}$ is a collection of closed subsets of X satisfying the following conditions:

1. $A, B \in \mathcal{P} \implies A \cup B \in \mathcal{P}$, and
2. If $A \in \mathcal{P}$ and $B \subseteq A$ with B closed in X, then $B \in \mathcal{P}$.

For example, the set of all closed subsets of X is an ideal of closed subsets of X. Other examples include the set of all finite subsets of X, denoted by $\mathcal{P}_f$; the set of all closed compact subsets of X, denoted by $\mathcal{K}$, the set of all closed Lindelöf subsets of X, denoted by $\mathcal{L}$ and even the ideal $\mathcal{P}_{nd}$ of all closed nowhere dense subsets of X. Another trivial but important example of an ideal of closed sets is $\{\emptyset\}$. We introduce the triplet $(X, \tau, \mathcal{P})$ and call it a $\tau\mathcal{P}$-space. For a subset S of X, $\mathcal{P}_S = \{P \cap S : P \in \mathcal{P}\}$ is an ideal of closed subsets of S and we say that $(S, \tau_S, \mathcal{P}_S)$ is a $\tau\mathcal{P}$-subspace of X, here τ_S is the subspace topology on S induced by the topology τ on X.

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We define the set $C(X)_{\mathcal{P}} = \{f \in \mathbb{R}^X : \overline{D_f} \in \mathcal{P}\}$, where D_f denotes the set of all points of discontinuity of f. It can be easily seen that $C(X)_{\mathcal{P}}$ forms a commutative ring with identity, with respect to pointwise addition and multiplication. Further, on defining $(f \wedge g)(x) = \min\{f(x), g(x)\}, \forall x \in X$ and $(f \vee g)(x) = \max\{f(x), g(x)\}, \forall x \in X$, for any $f, g \in C(X)_{\mathcal{P}}$, $(C(X)_{\mathcal{P}}, +, \cdot, \vee, \wedge)$ forms a commutative lattice ordered ring. The set $C^*(X)_{\mathcal{P}}$ of all bounded functions in $C(X)_{\mathcal{P}}$ is a lattice ordered subring of $C(X)_{\mathcal{P}}$. Throughout this article, we say that X is a $\tau\mathcal{P}$-space, instead of writing $(X, \tau, \mathcal{P})$ and $C(X)_{\mathcal{P}}$ instead of $(C(X)_{\mathcal{P}}, +, \cdot, \vee, \wedge)$. It is clear that for $\mathcal{P} = \{\emptyset\}$, $C(X)_{\mathcal{P}} = C(X)$, the ring of real-valued continuous functions on X, which has been studied thoroughly in [9]. When $\mathcal{P}$ is the ideal of all closed subsets of X, then $C(X)_{\mathcal{P}} = \mathbb{R}^X$. Further, when $\mathcal{P}$ is the set of all finite subsets of X, then $C(X)_{\mathcal{P}} = C(X)_F$, the ring of functions on X having finitely many discontinuities [8] and when $\mathcal{P}$ is the set of all closed nowhere dense subsets of X, then $C(X)_{\mathcal{P}} = T'(X)$ (The ring of those real-valued continuous functions on X for which there exists a dense open subset D of X such that $f|_D \in C(D)$ [1]). We denote the ring $C(X)_{\mathcal{P}}$ as $C(X)_K$, when $\mathcal{P} = \mathcal{K}$.

The principal aim of this article is to investigate several relevant properties associated with the ring $C(X)_{\mathcal{P}}$.

In this article, all rings are assumed to be commutative with identity. An ideal of a ring is considered to be a proper subset of the ring throughout this article. An ideal of a ring R is said to be essential if it intersects every non-zero ideal of R non-trivially. The sum of all minimal ideals of R is called its socle. $rad(R)$ denotes the Jacobson radical of R, that is $rad(R)$ is the intersection of all maximal ideals of R. A ring S containing a reduced ring R is called a ring of quotients of R if for each $s \in S \setminus \{0\}$, there exists $r \in R$ such that $sr \in R \setminus \{0\}$. A ring R is said to be almost regular if for every non-unit element $a \in R$, there exists a non-identity element $b \in R$ such that $ab = a$. A ring S is called reduced if '0' is the only nilpotent member of S. We use the notation χ_A for the characteristic function of A on X, defined by

$$\chi_A(x) = \begin{cases} 1 \text{ when } x \in A, \\ 0 \text{ otherwise} \end{cases}.$$

For an ideal I of R, $Ann(I) = \{r \in R : ri = 0 \text{ for all } i \in I\}$ is called the annihilator ideal of I. A ring R is said to be an IN-ring if for any two ideals I and J, $Ann(I \cap J) = Ann(I) \cap Ann(J)$. R is said to be an SA-ring if for any two ideals I and J, there exists an ideal K of R such that $Ann(I) \cap Ann(J) = Ann(K)$. Clearly, an SA-ring is always an IN-ring. R is said to be a Baer ring if the annihilator of every non-zero ideal is generated by an idempotent. The next lemma, proved in [16], offers a characterisation for IN-rings amongst reduced rings.

Lemma 1.1. *Let R be a reduced ring. Then the following statements are equivalent.*

1. *For any two orthogonal ideals I and J of R, $Ann(I) + Ann(J) = R$. (Two ideals I and J of R are said to be orthogonal if $fg = 0$ for each $f \in I$ and each $g \in J$.)*
2. *For any two ideals I and J of R, $Ann(I) + Ann(J) = Ann(I \cap J)$.*

Furthermore, Birkenmeier, Ghirati and Taherifar established the following set of equivalence of the following conditions in [7].

Lemma 1.2. *Let R be a reduced ring. Then the following statements are equivalent:*

1. *R is a Baer ring.*
2. *R is a SA-ring.*
3. *The space of prime ideals of R equipped with the Zariski topology is extremally disconnected.*
4. *R is an IN-ring.*

In Section 2, we establish under the hypothesis that every open subspace of X is C-embedded, that there can be defined a lattice homomorphism on the family $\mathcal{I}(X)$ of all ideals of closed subsets of X into the collection $\mathcal{J}(X)$ of all subrings of $\mathbb{R}^X$ which contain $C(X)$ (Theorem 2.6). It is easy to give an example of a non-discrete T_1-space which satisfies the hypothesis of this theorem. Indeed if X is an irreeducible T_1-space (say for example X is an uncountable set with co-countable topology), then every open subspace of X is C-embedded in X because each real-valued continuous function on such a space is constant. (A space X is called irreducible if every non-empty open subspace of X is dense in X). We further take into our consideration that an irreducible space is far from being Tychonoff. However if in the hypothesis of Theorem 2.6, we assume that X is a Tychonoff space instead of just T_1 only, in which every open subspace is C-embedded then it becomes an extremally disconnected space [9, Exercise 1H] and also a P-space [9, Exercise 4J]. But if X is an extremally disconnected P-space such that the cardinality of X is non-measurable, then X becomes a discrete space [9, Exercise 12H]. The cardinality of X is called non-measurable if there does not exist any measure $\mu\colon \mathscr{P}(X) \longrightarrow \{0,1\}$ with the condition that $\mu(\{x\}) = 0$ for all $x \in X$ and $\mu(X) = 1$. Otherwise X is said to have measurable cardinal. It is still an open problem in axiomatic set theory whether there exists at all any measurable cardinal. [See the comments at the end of Section 12.1, Chapter 12 in [9].] Nevertheless if we allow the existence of a measurable cardinal, X then the Hewitt real compactification υX of such a set X, essentially with discrete topology furnishes us with an example of a non-discrete space which is an extremally disconnected P-space. Also, each open set in υX is C-embedded in it. These assertions are proved in Theorem 2.5 of the present article. We define zero sets of the form $Z_{\mathcal{P}}(f) = \{x \in X\colon f(x) = 0\}$, for $f \in C(X)_{\mathcal{P}}$. The cozero set of $f \in C(X)_{\mathcal{P}}$ is the complement of $Z_{\mathcal{P}}(f)$ in X and is denoted by $coz(f)$. We designate the collection of all zero sets of functions in $C(X)_{\mathcal{P}}$ by $Z_{\mathcal{P}}[X]$ and the set of all cozero sets of functions in $C(X)_{\mathcal{P}}$ by $coz[X]$. Also, for a subset $S \subseteq C(X)_{\mathcal{P}}$, we write $Z_{\mathcal{P}}[S] = \{Z_{\mathcal{P}}(f)\colon f \in S\}$ and $coz[S] = \{coz(f)\colon f \in S\}$. We also set for any subfamily $\mathcal{F}$ of $Z_{\mathcal{P}}[X]$, $Z_{\mathcal{P}}^{-1}(\mathcal{F}) = \{f \in C(X)_{\mathcal{P}}\colon Z_{\mathcal{P}}(f) \in \mathcal{F}\}$. Incidentally, we introduce the notion of $\mathcal{P}$-completely separated subsets of X and achieve their characterisation via zero sets of functions in $C(X)_{\mathcal{P}}$. Next, we define $z_{\mathcal{P}}$-filters on X and $z_{\mathcal{P}}$-ideals in $C(X)_{\mathcal{P}}$, and examine the duality existing between them. As expected it is realized that there exists a one-to-one correspondence between the set of all maximal ideals $Max(C(X)_{\mathcal{P}})$ of $C(X)_{\mathcal{P}}$ and the set $\mathcal{U}(X)_{\mathcal{P}}$ of all maximal $z_{\mathcal{P}}$-filters on X. We exploit this duality to show that $M(C(X)_{\mathcal{P}})$ equipped with the hull-kernel topology is homeomorphic to $\mathcal{U}(X)_{\mathcal{P}}$ equipped with the Stone topology [Theorem 2.20].

In Section 3, we examine when does $C(X)_{\mathcal{P}}$ become closed under uniform limit. If $C(X)_{\mathcal{P}}$ is closed under uniform limit, we then say X is a $\tau\mathcal{P}\mathcal{U}$-space. It is clear that with $\mathcal{P} = \{\emptyset\}$, $C(X)_{\mathcal{P}} = C(X)$, which is known to be closed under uniform limit for any topological space X. We also check that for any choice of $\mathcal{P}$, if the set of all non-isolated points of X is a member of $\mathcal{P}$, then $(X, \tau, \mathcal{P})$ is a $\tau\mathcal{P}\mathcal{U}$-space. For some special choice of $\mathcal{P}$, the converse of this result is seen to be valid [Theorem 3.3 and 3.6]. It is further noted that when $C(X)_{\mathcal{P}}$ is isomorphic to $C(Y)$, for some topological space Y, then X is a $\tau\mathcal{P}\mathcal{U}$-space. Using the above results, we have given an alternative proof of Theorem 3.4 in [8]. At the end of this section, we establish a result for a $\tau\mathcal{P}\mathcal{U}$-space, analogous to the Urysohn's extension Theorem for $C(X)$, stated in [9].

In Section 4, we address a few special problems related to the ring $C(X)_{\mathcal{P}}$ but this time with the additional hypothesis that each singleton set is a member of $\mathcal{P}$. It follows that $\chi_{\{x\}} \in C(X)_{\mathcal{P}}$, for all $x \in X$. Under this hypothesis, we see that $C(X)_{\mathcal{P}}$ is an almost regular ring and any $f \in C(X)_{\mathcal{P}}$ is either a zero divisor or an unit. We further show that $C(X)_{\mathcal{P}} = C(X)$ if and only if X is discrete if and only if $C(X)_{\mathcal{P}}$ is a ring of quotients of $C(X)$. We find out necessary and sufficient conditions under which an ideal of $C(X)_{\mathcal{P}}$ is a minimal ideal and establish that the socle of $C(X)_{\mathcal{P}}$ consists of all functions that vanish everywhere except on a finite set and is itself an essential ideal that is also free. We further note that $Soc(C(X)_{\mathcal{P}}) = C(X)_{\mathcal{P}} \iff X$ is finite. Exploiting these results, we establish that $C(X)_{\mathcal{P}}$ is an Artinian (Noetherian) ring if and only if X is finite. We complete this section by providing a set of conditions equivalent to $C(X)_{\mathcal{P}}$ being an IN-ring, SA-ring and Baer ring. We also provide counterexamples to show that these results may fail when the restriction "$\mathcal{P}$ contains all singleton subsets of X" is lifted.

In Section 5, we examine the Von-Neumann regularity, or simply the regularity of $C(X)_{\mathcal{P}}$. Here, we define a $\tau\mathcal{P}$-space, $(X, \tau, \mathcal{P})$ to be a $\mathcal{P}P$-space if $C(X)_{\mathcal{P}}$ is regular. We show that if a Tychonoff X is such that $C(X)$ is regular then $C(X)_{\mathcal{P}}$ is also regular. (Loosely speaking we may say that a P-space is a $\mathcal{P}P$-space.) However, with the help of a counterexample, we show that this statement may not be valid without the Tychonoff-ness hypothesis on X. We also construct a different counterexample to show that Proposition 6.1 in [8] fails when X is not Tychonoff. We provide several characterisations of a $\mathcal{P}P$-space using ideals of the ring $C(X)_{\mathcal{P}}$ as well as members of the ideal $\mathcal{P}$ of closed sets in X (Theorem 5.7). Using one such characterisation we realize that if X is a $\mathcal{P}P$-space, then the ring $C(X)_{\mathcal{P}}$ is a Bezout ring. Incidentally we state a few properties of $C(X)_{\mathcal{P}}$, when it is a Bezout ring and these properties themselves are shown to be pairwise equivalent. We conclude this section by providing a number of characterisations of $C(X)_{\mathcal{P}}$ to become a Bezout ring, under the additional hypothesis that $C(X)_{\mathcal{P}}$ is closed under uniform limit, that is, X is a $\tau\mathcal{PU}$-space (Theorem 5.10).

Finally, in the sixth section, we use the concept of ϕ-algebra and an algebra of measurable functions to establish a condition involving a $\tau\mathcal{PU}$-space, under which $C(X)_{\mathcal{P}}$ is $\aleph_0$-self injective. We also provide an example that supports that the condition X is a $\tau\mathcal{PU}$-space is not superfluous.

2. Definitions and preliminaries

Notation 2.1. Let $\mathcal{P}'$ be the ideal of all closed subsets of the set of isolated points of X.

Theorem 2.2. *$C(X)_{\mathcal{P}} = C(X)$ if and only if $\mathcal{P} \subseteq \mathcal{P}'$.*

Proof. Let $\mathcal{P} \nsubseteq \mathcal{P}'$. Then there exists $A \in \mathcal{P}$ such that there exists $x_0 \in A$ which is a non-isolated point in X. Let $f = \chi_{\{x_0\}}$. Then $f \in C(X)_{\mathcal{P}}$. However, since x_0 is a non-isolated point of X, $f \notin C(X)$. The converse is obvious. □

We have stated in the introduction that for $\mathcal{P} = \mathcal{P}_{nd}$, $C(X)_{\mathcal{P}} = T'(X)$, where $T'(X)$ is the ring of those real-valued continuous functions on X for which there exists a dense open subset D of X such that $f|_D \in C(D)$ [4]. We give a proof supporting this statement.

Theorem 2.3. $C(X)_{\mathcal{P}_{nd}} = T'(X)$

Proof. Let $f \in C(X)_{\mathcal{P}_{nd}}$, then $\overline{D_f}$ is nowhere dense. Therefore $X \setminus \overline{D_f}$ is dense in X. Also, $\overline{D_f}$ is closed in X $\implies$ $X \setminus \overline{D_f}$ is open in X. Finally, f is continuous on $X \setminus D_f \supseteq X \setminus \overline{D_f}$. Thus $f \in T'(X)$.

Conversely, let $g \in T'(X)$. Then there exists an open dense subset D of X such that $f|_D \in C(D)$ $\implies$ $D_f \subseteq X \setminus D$. So $\overline{D_f} \subseteq \overline{X \setminus D} = X \setminus D$. Since the complement of an open dense set is a closed nowhere dense set, $X \setminus D \in \mathcal{P}_{nd}$ and so $\overline{D_f} \in \mathcal{P}_{nd}$. Therefore $f \in C(X)_{\mathcal{P}_{nd}}$. □

We $\mathcal{I}(X)$ stand for the family of all ideals of closed subsets of X and $\mathcal{J}(X)$ to be the family of all subrings of $\mathbb{R}^X$ containing $C(X)$. Both of these families form a lattice with respect to the set inclusion.

It can be easily observed that for two rings $S_1, S_2 \in \mathcal{J}(X)$,

$S_1 \vee S_2 = \{\sum_{i=1}^{m} f_i g_i \colon f_i \in S_1, g_i \in S_2, i = 1, 2, ..., m.\}$ is the smallest subring of $\mathbb{R}^X$ containing $S_1 \cup S_2$ and for two ideals of closed sets $\mathcal{P}$ and $\mathcal{Q}$ in $\mathcal{I}(X)$, $\mathcal{P} \vee \mathcal{Q} = \{A \cup B \colon A \in \mathcal{P} \text{ and } B \in \mathcal{Q}\}$ is the smallest ideal of closed subsets of X containing $\mathcal{P}$ and $\mathcal{Q}$. It is obvious that for two rings $S_1, S_2 \in \mathcal{J}(X)$, $S_1 \wedge S_2 = S_1 \cap S_2$ and for two ideals of closed sets $\mathcal{P}$ and $\mathcal{Q}$ in $\mathcal{I}(X)$, $\mathcal{P} \wedge \mathcal{Q} = \mathcal{P} \cap \mathcal{Q}$. So $C(X)_{\mathcal{P}\wedge\mathcal{Q}} = C(X)_{\mathcal{P}} \wedge C(X)_{\mathcal{Q}}$. However, $C(X)_{\mathcal{P}\vee\mathcal{Q}} = C(X)_{\mathcal{P}} \vee C(X)_{\mathcal{Q}}$ may not hold for all T_1-spaces but is true for a topological space X if every open subspace of X is C-embedded (for example, when X is an irreducible space).

Theorem 2.4. *Let X be a topological space where every open subspace of X is C-embedded. Then for any $\mathcal{P}$ and $\mathcal{Q}$ in $\mathcal{I}(X)$, $C(X)_{\mathcal{P}\vee\mathcal{Q}} = C(X)_{\mathcal{P}} \vee C(X)_{\mathcal{Q}}$.*

Proof. Let $\alpha = \sum_{i=1}^{m} f_i g_i \in C(X)_{\mathcal{P}} \vee C(X)_{\mathcal{Q}}$. Then

$D_\alpha \subseteq \bigcup_{i=1}^{m} D_{f_i g_i} \subseteq \bigcup_{i=1}^{m} (D_{f_i} \cup D_{g_i})$ where $\overline{D_{f_i}} \in \mathcal{P}$ and $\overline{D_{g_i}} \in \mathcal{Q}$ for all $i = 1, 2, ..., m$. So $\overline{D_{f_i} \cup D_{g_i}} \in \mathcal{P} \vee \mathcal{Q}$ for all $i = 1, 2, ..., m$. Since $\mathcal{P} \vee \mathcal{Q}$ is an ideal of closed sets, $\overline{D_\alpha} \in \mathcal{P} \vee \mathcal{Q}$ and hence $\alpha \in C(X)_{\mathcal{P}\vee\mathcal{Q}}$. Conversely, let $f \in C(X)_{\mathcal{P}\vee\mathcal{Q}}$. Then $f|_{X\setminus\overline{D_f}}$ is continuous on the open subspace $X \setminus \overline{D_f}$ of X. By our hypothesis there exists $\widehat{f} \in C(X)$ such that $\widehat{f}|_{X\setminus\overline{D_f}} = f|_{X\setminus\overline{D_f}}$. Also $\overline{D_f} \in \mathcal{P} \vee \mathcal{Q}$. So $\overline{D_f} = A \cup B$ where $A \in \mathcal{P}$ and $B \in \mathcal{Q}$. We define

$$g\colon X \longrightarrow \mathbb{R} \text{ by } g(x) = \begin{cases} \widehat{f}(x) \text{ when } x \in X \setminus A \\ f(x) \text{ when } x \in A \setminus B \\ \frac{1}{2} f(x) \text{ when } x \in A \cap B \end{cases} \quad \text{and } h\colon X \longrightarrow \mathbb{R} \text{ by } h(x) = \begin{cases} 0 \text{ when } x \in X \setminus B \\ f(x) - \widehat{f}(x) \text{ when } x \in B \setminus A \\ \frac{1}{2} f(x) \text{ when } x \in A \cap B \end{cases}.$$

Then $\overline{D_g} \subseteq A$ and $\overline{D_h} \subseteq B$. Therefore $g \in C(X)_{\mathcal{P}}$ and $h \in C(X)_{Q}$ and $f = g + h \in C(X)_{\mathcal{P}} \vee C(X)_{Q}$. □

The next theorem decides a class of non-discrete Tychonoff spaces which satisfies the hypothesis (that every open subspace is C-embedded) of the above theorem.

Theorem 2.5. *Suppose X is a discrete topological space with measurable cardinal. Then υX is a non-discrete P-space which is also extremally disconnected. Furthermore every open subspace of υX is C-embedded in υX.*

Proof. Since the discrete space X is trivially a P-space it follows that $C(X)$ is a Von-Neumann regular ring. But the ring $C(X)$ is isomorphic to the ring $C(\upsilon X)$ under the map $f \longrightarrow f^*|_{\upsilon X}$, where $f^*\colon \beta X \longrightarrow \mathbb{R} \cup \{\infty\}$ is the Stone extension of f. Hence $C(\upsilon X)$ is also a Von-Neumann regular ring. In other words υX is a P-space. Since X is with measurable cardinal and a discrete space Y is realcompact if and only if its cardinal is non-measurable [9, Theorem 12.2], it follows that X is not a realcompact space. Hence $X \subsetneq \upsilon X$. This proves that υX is non-discrete. Now X is extremally disconnected implies βX is extremally disconnected [9, Exercise 6M]. Since every dense subset of an extremally disconnected space is extremally disconnected [9, Exercise 1H], it follows that υX is extremally disconnected. Finally since X is C-embedded in υX and is also dense in υX it follows that every open subspace of υX is C-embedded in υX. □

We summarize the facts in Theorem 2.4 in the following theorem.

Theorem 2.6. *Let X be a topological space in which every open subspace of X is C-embedded. Then $\phi\colon \mathcal{I}(X) \longrightarrow \mathcal{J}(X)$ defined by $\phi(\mathcal{P}) = C(X)_{\mathcal{P}}$ is a lattice homomorphism.*

We recall that for $f \in C(X)_{\mathcal{P}}$, $Z_{\mathcal{P}}(f) = \{x \in X\colon f(x) = 0\}$ is called the zero set of f. Also, a subset A of X is called a zero set if $A = Z_{\mathcal{P}}(f)$, for some $f \in C(X)_{\mathcal{P}}$. Let $Z_{\mathcal{P}}[X]$ be the set of all zero sets of X.

It is easy to verify that:

1. For $f \in C(X)_{\mathcal{P}}$, $Z_{\mathcal{P}}(f) = Z_{\mathcal{P}}(|f|) = Z_{\mathcal{P}}(f \wedge \mathbf{1}) = Z_{\mathcal{P}}(f^n)$, for all $n \in \mathbb{N}$.
2. $Z_{\mathcal{P}}(\mathbf{0}) = X$ and $Z_{\mathcal{P}}(\mathbf{1}) = \emptyset$.
3. For $f, g \in C(X)_{\mathcal{P}}$, $Z_{\mathcal{P}}(fg) = Z_{\mathcal{P}}(f) \cup Z_{\mathcal{P}}(g)$ and $Z_{\mathcal{P}}(|f| + |g|) = Z_{\mathcal{P}}(f^2 + g^2) = Z_{\mathcal{P}}(f) \cap Z_{\mathcal{P}}(g)$.
4. For $f \in C(X)_{\mathcal{P}}$, $r, s \in \mathbb{R}$, sets of the form $\{x \in X\colon f(x) \geq r\}$ and $\{x \in X\colon f(x) \leq s\}$ are zero sets as we have :
   (a) $\{x \in X\colon f(x) \geq r\} = Z_{\mathcal{P}}((f - \boldsymbol{r}) \wedge 0)$, and
   (b) $\{x \in X\colon f(x) \geq s\} = Z_{\mathcal{P}}((f - \boldsymbol{s}) \vee 0)$.

From the above observations, it follows that $Z_{\mathcal{P}}[X]$ is closed under finite union and finite intersection. We know that the zero sets in $Z[X]$ (zero sets of real-valued continuous functions on X) are G_δ-sets. The following result generalises this fact for functions in $C(X)_{\mathcal{P}}$.

Theorem 2.7. *Every zero set in X can be expressed as a disjoint union of a G_δ-set and a set A such that $\overline{A} \in \mathcal{P}$.*

Proof. Let $f \in C(X)_{\mathcal{P}}$. Then $Z_{\mathcal{P}}(f) = G \cup A$, where $G = Z_{\mathcal{P}}(f) \cap (X \setminus D_f)$ and $A = Z_{\mathcal{P}}(f) \cap D_f$. Then G is a zero set of the continuous map $f|_{X \setminus D_f}$ and is therefore a G_δ-set in $X \setminus D_f$. Since D_f is an F_σ-set in X, $X \setminus D_f$ is a G_δ-set in X. Thus G is a G_δ-set in X. On the other hand, $\overline{A} \subseteq \overline{D_f} \in \mathcal{P} \implies \overline{A} \in \mathcal{P}$. □

We know that all zero sets in $Z(X)$ are necessarily closed sets in X. In contrast however, each zero sets in $Z_{\mathcal{P}}[X]$ need not be closed. In fact, we establish a necessary and sufficient condition for this to happen.

Theorem 2.8. *For a $\tau\mathcal{P}$-space X, all zero sets in $Z_{\mathcal{P}}[X]$ are closed if and only if $C(X)_{\mathcal{P}} = C(X)$.*

Proof. Let $C(X)_{\mathcal{P}} \neq C(X)$. Then tracing the steps in Theorem 2.2, there exists a non-isolated point $x_0 \in X$ such that $\chi_{\{x_0\}} \in C(X)_{\mathcal{P}} \setminus C(X)$ and $Z_{\mathcal{P}}(\chi_{\{x_0\}}) = X \setminus \{x_0\}$, which is not closed in X. The converse is clear. □

The following observations are immediate.

Proposition 2.9. *Let X be a $\tau\mathcal{P}$-space. Then the following statements are true.*

1. *$C(X)_{\mathcal{P}}$ is reduced. (A commutative ring R is called reduced if it does not contain any non-zero nilpotent elements.)*
2. *f is a unit in $C(X)_{\mathcal{P}}$ if and only if $Z_{\mathcal{P}}(f) = \emptyset$.*

Definition 2.10. Two subsets A and B of X are said to be $\mathcal{P}$-completely separated if there exists $f \in C(X)_{\mathcal{P}}$ such that $f(X) \subseteq [0, 1]$, $f(A) = \{0\}$ and $f(B) = \{1\}$.

Thus $\mathcal{P}$-completely separated sets reduce to completely separated sets in [9], when $\mathcal{P} = \{\emptyset\}$.

Let τ_u be the usual topology on the set $\mathbb{R}$ of all real numbers and $\mathcal{P} = \{\emptyset, \{0\}\}$. Define $f \colon \mathbb{R} \to \mathbb{R}$ by: $f(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$ Then $f \in C(X)_{\mathcal{P}}$. Thus $(-\infty, 0]$ and $(0, \infty)$ are $\mathcal{P}$-completely separated in $\mathbb{R}$. However $(-\infty, 0]$ and $(0, \infty)$ are not completely separated in $\mathbb{R}$.

Proposition 2.11. *Two disjoint subsets A and B of a $\tau\mathcal{P}$-space X are $\mathcal{P}$-completely separated in X if and only if they are contained in disjoint zero sets in X.*

Proof. The first part of the theorem can be proved by closely following the proof of Theorem 1.2 in [9]. To prove the converse, let A and B be two disjoint subsets of a $\tau\mathcal{P}$-space X that are contained in disjoint zero sets, $Z_{\mathcal{P}}(f)$ and $Z_{\mathcal{P}}(g)$ respectively in X. Define

$$h(x) = \frac{|f|(x)}{(|f| + |g|)(x)}, \quad \forall x \in X.$$

Then $D_h \subseteq D_f \cup D_g$ and this implies that $\overline{D_h} \subseteq \overline{D_f} \cup \overline{D_g}$ Consequently $\overline{D_h} \in \mathcal{P}$ and hence $h \in C(X)_{\mathcal{P}}$. Also, $h(A) = \{0\}$ and $h(B) = \{1\}$. Hence A and B are $\mathcal{P}$-completely separated in X. □

Theorem 2.12. *Two disjoint subsets A and B of a $\tau\mathcal{P}$-space X are $\mathcal{P}$-completely separated in X if and only if there exists a $P \in \mathcal{P}$ such that $A \setminus P$ and $B \setminus P$ are completely separated in $X \setminus P$.*

Proof. The proof of this theorem is analogous to that of Proposition 2.3 of [8]. □

We now introduce the concept of $z_{\mathcal{P}}$-filters on X and $z_{\mathcal{P}}$-ideals of $C(X)_{\mathcal{P}}$.

Definition 2.13.

1. A non-empty subcollection $\mathcal{F}$ of $Z_{\mathcal{P}}[X]$ is called a $z_{\mathcal{P}}$-filter on X if $\mathcal{F}$ satisfies the following conditions:
   (a) $\emptyset \notin \mathcal{F}$,
   (b) $\mathcal{F}$ is closed under finite intersections, and
   (c) If $Z_1 \in \mathcal{F}$ and $Z_2 \in Z_{\mathcal{P}}[X]$ with $Z_1 \subseteq Z_2$, then $Z_2 \in \mathcal{F}$.
2. A $z_{\mathcal{P}}$-filter on X is said to be a $z_{\mathcal{P}}$-ultrafilter on X if it is not properly contained in any other $z_{\mathcal{P}}$-filter on X.
3. An ideal I of $C(X)_{\mathcal{P}}$ is called a $z_{\mathcal{P}}$-ideal if $Z_{\mathcal{P}}^{-1}Z_{\mathcal{P}}[I] = I$.

We realize that if I is a $z_{\mathcal{P}}$-ideal of the ring $C(X)_{\mathcal{P}}$, then whenever $f^n \in I$ for some $n \in \mathbb{N}$, we have $f \in I$. This ensures that I coincides with the collection $\{f \in C(X)_{\mathcal{P}} \colon f^n \in I \text{ for some } n \in \mathbb{N}\}$ which is nothing but the intersection of all prime ideals containing I [9, Theorem 0.18].

Theorem 2.14. *If I is a $z_{\mathcal{P}}$-ideal of $C(X)_{\mathcal{P}}$, then $I = \{f \in C(X)_{\mathcal{P}} \colon f^n \in I \text{ for some } n \in \mathbb{N}\}$ and it coincides with the intersection of all prime ideals containing I.*

Arguing analogously as in Theorems 2.3, 2.5, 2.9 and 2.11 of [9], we can prove the following theorems:

Theorem 2.15. 1. *For any ideal I of $C(X)_{\mathcal{P}}$, $Z_{\mathcal{P}}[I]$ is a $z_{\mathcal{P}}$-filter on X.*

2. *For a $z_{\mathcal{P}}$-filter $\mathcal{F}$, $Z_{\mathcal{P}}^{-1}(\mathcal{F}) = \{f \in C(X)_{\mathcal{P}} \colon Z_{\mathcal{P}}(f) \in \mathcal{F}\}$ is an ideal of $C(X)_{\mathcal{P}}$.*
3. *If M is a maximal ideal in $C(X)_{\mathcal{P}}$, then $Z_{\mathcal{P}}[M]$ is a $z_{\mathcal{P}}$-ultrafilter on X.*
4. *For a $z_{\mathcal{P}}$-ultrafilter $\mathcal{U}$ on X, $Z_{\mathcal{P}}^{-1}(\mathcal{U})$ is a maximal ideal of $C(X)_{\mathcal{P}}$.*
5. *The assignment $Z_{\mathcal{P}} \colon M \longrightarrow Z_{\mathcal{M}}$ renders a bijective correspondence between the maximal ideals in $C(X)_{\mathcal{P}}$ and the $z_{\mathcal{P}}$-ultrafilters on X.*

Theorem 2.16. *For any $z_{\mathcal{P}}$-ideal I of $C(X)_{\mathcal{P}}$, the following statements are equivalent:*

1. *I is a prime ideal in $C(X)_{\mathcal{P}}$.*
2. *I contains a prime ideal in $C(X)_{\mathcal{P}}$.*
3. *For all $g, h \in C(X)_{\mathcal{P}}$, if $gh = \mathbf{0}$, then $g \in I$ or $h \in I$.*
4. *For every $f \in C(X)_{\mathcal{P}}$, there is a zero-set in $Z[I]$ on which f does not change its sign.*

Theorem 2.17. *Every prime ideal in $C(X)_{\mathcal{P}}$ is contained in a unique maximal ideal.*

Corollary 2.18. *$C(X)_{\mathcal{P}}$ is a pm-ring (A commutative ring R with unity is called a pm-ring if every prime ideal of R is contained in a unique maximal ideal of R.).*

Now let us take up the study on the maximal ideal space of $C(X)_{\mathcal{P}}$. Suppose $Max(C(X)_{\mathcal{P}})$ is the collection of all maximal ideals of $C(X)_{\mathcal{P}}$. For $f \in C(X)_{\mathcal{P}}$, set $\mathcal{M}_f = \{M \in Max(C(X)_{\mathcal{P}}) \colon f \in M\}$. It is easy to see that $\mathcal{B} = \{\mathcal{M}_f \colon f \in C(X)_{\mathcal{P}}\}$ is a base for closed sets for some topology on $Max(C(X)_{\mathcal{P}})$, which is known as the hull-kernel topology.

Suppose $\beta_{\mathcal{P}}X$ is an index set for the family of all $z_{\mathcal{P}}$-ultrafilters on X. For $p \in \beta_{\mathcal{P}}X$ let the corresponding $z_{\mathcal{P}}$-ultrafilter on X be denoted by A^p with the agreement that if $p \in X$, $A^p = A_p = \{Z \in Z_{\mathcal{P}}[X] \colon p \in Z\}$. For $Z \in Z_{\mathcal{P}}[X]$, we set $\overline{Z} = \{p \in \beta_{\mathcal{P}}X \colon Z \in A^p\}$. Then $\mathcal{B}' = \{\overline{Z} \colon Z \in Z_{\mathcal{P}}[X]\}$ forms a base for the closed sets for some topology on $\beta_{\mathcal{P}}X$, called the Stone topology on $\beta_{\mathcal{P}}X$.

The following observations for $Z \in Z_{\mathcal{P}}[X]$ are immediate :

Theorem 2.19. 1. *$\overline{Z} \cap X = Z$.*

2. *$\overline{Z} = cl_{\beta_{\mathcal{P}}X} Z$.*

Theorem 2.20. *$Max(C(X)_{\mathcal{P}})$ is homeomorphic to $\beta_{\mathcal{P}}X$*

Proof. Define $\phi \colon \beta_{\mathcal{P}}X \longrightarrow Max(C(X)_{\mathcal{P}})$ by $\phi(p) = Z_{\mathcal{P}}^{-1}[A^p] = M^p$(say). Then M^p is a maximal ideal of $C(X)_{\mathcal{P}}$. Also, ϕ is a bijective map, by Theorem 2.15. Let $Z = Z_{\mathcal{P}}(f) \in Z_{\mathcal{P}}[X]$. Then $\phi(\overline{Z}) = \mathcal{M}_f \in \mathcal{B}$ and $\phi^{-1}(\mathcal{M}_f) = \overline{Z_{\mathcal{P}}(f)} \in \mathcal{B}'$. Therefore ϕ exchanges basic closed sets of $Max(C(X))_{\mathcal{P}}$ and $\beta_{\mathcal{P}}X$. Hence ϕ is a homeomorphism. □

Since $C(X)_{\mathcal{P}}$ contains unity, $Max(C(X)_{\mathcal{P}})$ is compact and hence $\beta_{\mathcal{P}}X$ is compact. Since $C(X)_{\mathcal{P}}$ is a $pm-$ring (by Theorem 2.17), by Theorem 1.2 of [15], $\beta_{\mathcal{P}}X$ is Hausdorff.

It can be easily seen that when $\mathcal{P} = \{\emptyset\}$, we have $\beta_{\mathcal{P}}X = \beta X$.

We further note that if $X = (0, 1) \cup \{2\}$ equipped with the subspace topology inherited from the usual topology of $\mathbb{R}$ and $\mathcal{P}$ the ideal of all closed subsets of $(0, 1)$, we have, $C(X)_{\mathcal{P}} = \mathbb{R}^X = C(X_d)$, where $X_d = X$ equipped with the discrete topology on X. Since for any Tychonoff space Y, each isolated point in Y is isolated in βY [9, 6.9(d)] and the maximal ideal space of $C(Y)$ is βY [9, Exercise 7N]. It follows that $\beta_{\mathcal{P}}X$ has uncountably many isolated points, but βX has only one isolated point. Thus $\beta_{\mathcal{P}}X$ is not homeomorphic to βX.

3. When is $C(X)_{\mathcal{P}}$ closed under uniform limit?

Definition 3.1. A sequence of functions $\{f_n\}$ in a subring S of $\mathbb{R}^X$ is said to converge uniformly to a function f on X if for a given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$ and for all $x \in X$.

A subring S of $\mathbb{R}^X$ is said to be closed under uniform limit if whenever $\{f_n\} \subseteq S$ converges uniformly to a function $f \in \mathbb{R}^X$, then $f \in S$.

We call a $\tau\mathcal{P}$-space $(X, \tau, \mathcal{P})$ a $\tau\mathcal{P}\mathcal{U}$-space if $C(X)_{\mathcal{P}}$ is closed under uniform limit.

It is well known and can also be independently observed that if $\mathcal{P} = \{\emptyset\}$, then X is a $\tau\mathcal{P}\mathcal{U}$-space. Another trivial example of a ring that is closed under uniform limit is $\mathbb{R}^X$. This means that if $C(X)_{\mathcal{P}} = \mathbb{R}^X$, then X is a $\tau\mathcal{P}\mathcal{U}$-space. Also, every function in $\mathbb{R}^X$ is continuous on all isolated points of X. In the light of these simple but useful facts, we have the following theorem.

Theorem 3.2. *Let $\mathcal{P}$ be an ideal of closed subsets of X such that the set of all non-isolated points in X is a member of $\mathcal{P}$. Then $C(X)_{\mathcal{P}} = \mathbb{R}^X$, and hence $C(X)_{\mathcal{P}}$ is closed under uniform limit, that is X is a $\tau\mathcal{P}\mathcal{U}$-space.*

The converse of the above theorem holds when $\mathcal{P} = \mathcal{P}_f$, as seen in Theorem 2.9 [14].

We now show that the converse of the above theorem also holds for a metrizable space X with $\mathcal{P} = \mathcal{K}$.

Theorem 3.3. *Let X be a metrizable space and $\mathcal{P} = \mathcal{K}$. If X is a $\tau\mathcal{P}\mathcal{U}$-space, then the set of all non-isolated points in X is a member of $\mathcal{K}$.*

Proof. Let T be the set of all non-isolated points of X. Assume that T is non-compact. Then T is not sequentially compact. So, $\exists$ a sequence $\{a_n\} \in T$ which has no convergent subsequence. Set $A = \{a_n : n \in \mathbb{N}\}$. Then A is a closed non-compact subset of X.

For each $m \in \mathbb{N}$, define f_m on X as follows :

$$f_m(x) = \begin{cases} \frac{1}{n} & x = a_n, n < m \\ 0 & otherwise. \end{cases}$$

Then, $f_m \in C(X)_K$, for each $m \in \mathbb{N}$ and $\{f_m\}$ is uniformly convergent to a function $f : X \longrightarrow \mathbb{R}$ where

$$f(x) = \begin{cases} \frac{1}{n} & x = a_n \\ 0 & otherwise. \end{cases}$$

Clearly, $\overline{D_f} = \overline{A} = A \notin \mathcal{K}$. Thus $f \notin C(X)_K$. Hence $C(X)_K$ is not closed under uniform limit. This completes the proof. □

It is well known that $C(Y)$ is closed under uniform limit for any topological space Y. It is natural to ask if $C(X)_{\mathcal{P}}$ is isomorphic to $C(Y)$, then is it the case that $C(X)_{\mathcal{P}}$ is also closed under uniform limit? An affirmative answer is given in the next theorem.

Theorem 3.4. *Let $\mathcal{P}$ be an ideal of closed subsets of a space X. If $C(X)_{\mathcal{P}}$ is isomorphic to $C(Y)$ for some topological space Y, then X is a $\tau\mathcal{P}\mathcal{U}$-space.*

Proof. Let $\phi \colon C(X)_{\mathcal{P}} \longrightarrow C(Y)$ be an isomorphism. First, we show that ϕ is an order preserving mapping. For that, let $g \in C(X)_{\mathcal{P}}$ be such that $g \geq 0$. Then $g = l^2$ for some $l \in C(X)_{\mathcal{P}}$. Thus $\phi(g) = \phi(l^2) = (\phi(l))^2 \geq 0$. So, ϕ is order preserving. For $f \in C(X)_{\mathcal{P}}$, $\phi(|f|) \geq 0$. Also, $(\phi(|f|))^2 = \phi(|f|^2) = \phi(f^2) = (\phi(f))^2$ which implies $\phi(|f|) = |\phi(f)|$, for all $f \in C(X)_{\mathcal{P}}$............(1). Since ϕ is an isomorphism, for any rational number r, $\phi(\boldsymbol{r}) = \boldsymbol{r}$................(2). On using the above arguments, we can show that (1) and (2) also hold for ϕ^{-1} as well. Let $\{f_n\}$ be a sequence in $C(X)_{\mathcal{P}}$ converging uniformly to a function $f \in \mathbb{R}^X$. We show that $f \in C(X)_{\mathcal{P}}$.

Let $\epsilon > 0$ be an arbitrary rational. Then there exists $k \in \mathbb{N}$ such that $|f_n - f_m| < \boldsymbol{\epsilon}$ for all $n, m \geq k$. Since ϕ is order preserving, and using (1) and (2), we have $|\phi(f_n) - \phi(f_m)| < \phi(\boldsymbol{\epsilon}) = \boldsymbol{\epsilon}$, for all $n, m \geq k$. Since $C(Y)$

is closed under uniform limit it follows that there exists $h \in C(Y)$ such that $\{\phi(f_n)\}$ converges uniformly to $h \in C(Y)$. By hypothesis, ϕ is onto. Therefore there exists $g \in C(X)_{\mathcal{P}}$ such that $\phi(g) = h$.

Now, for a given rational $\epsilon > 0$, as $\{\phi(f_n)\}$ converges uniformly to $h \in C(Y)$, there exists $k \in \mathbb{N}$ such that

$$|\phi(f_n) - h| < \epsilon \quad \forall n \geq k$$
$$\implies |\phi(f_n) - \phi(g)| < \epsilon \quad \forall n \geq k$$
$$\implies |f_n - g| = |\phi^{-1}(\phi(f_n)) - \phi^{-1}\phi(g)| < \phi^{-1}(\epsilon) = \epsilon \quad \forall n \geq k.$$

Thus $\{f_n\}$ converges uniformly to the function g. But $\{f_n\}$ converges uniformly already to f. Hence $f = g \in C(X)_{\mathcal{P}}$. Thus $C(X)_{\mathcal{P}}$ is closed under uniform limit. □

Corollary 3.5. *In a metric space X the following statements are equivalent :*

1. *$C(X)_K$ is closed under uniform limit.*
2. *The set of all non-isolated points in X is compact.*
3. *$C(X)_K$ is isomorphic to $C(Y)$ for some topological space Y.*

Proof. (1) $\implies$ (2) follows from Theorem 3.3.

From Theorem 3.2, we get (2) $\implies$ $C(X)_K = \mathbb{R}^X = C(X_d)$, where X_d denotes the space X with discrete topology. This proves (3).

(3) $\implies$ (1) follows from Theorem 3.4 for $\mathcal{P} = \mathcal{K}$. □

If we consider the ideal $\mathcal{P} = \mathcal{P}_f$ on a topological space X, then we have the following theorem where we provide an alternative proof for Theorem 3.4 of [8].

Theorem 3.6. *For a topological space X, the following statements are equivalent:*

1. *$C(X)_F$ is closed under uniform limit.*
2. *X has only finitely many non-isolated points.*
3. *$C(X)_F$ is isomorphic to $C(Y)$, for some topological space Y.*

Proof. (1) $\iff$ (2) follows from Theorem 2.9 in [14].

From Theorem 3.2, we get (2) $\implies$ $C(X)_F = \mathbb{R}^X = C(X_d)$, where X_d denotes the set X with discrete topology. This proves (3).

(3) $\implies$ (1) follows from Theorem 3.4 for $\mathcal{P} = \mathcal{P}_f$. □

The above examples of rings that are closed under uniform limit are all essentially isomorphic to $C(Y)$ for some topological space Y. We provide some more examples of $\tau\mathcal{PU}$-spaces.

Examples 3.7. Let $X = \mathbb{R}$ be endowed with usual topology.

- Let $X = \mathbb{R}$ and $\mathcal{P} = \{\emptyset, \{c\}\}$ for some $c \in \mathbb{R}$. Then $(X, \tau, \mathcal{P})$ is a $\tau\mathcal{PU}$-space.

  More generally, if A is a finite subset of X, then $(X, \tau, \mathcal{P})$ is a $\tau\mathcal{PU}$-space for $\mathcal{P}$ =set of all subsets of A.

- If we take X to be a P-space and A a closed proper subset of X, then $(X, \tau, \mathcal{P})$ is a $\tau\mathcal{PU}$-space for $\mathcal{P}$ = the set of all closed subsets of A.

- Consider X to be a P-space and $\mathcal{P}$ = the set of all closed countable sets in X. Then $(X, \tau, \mathcal{P})$ is a $\tau\mathcal{PU}$-space.

In fact, we have the following result that unifies the above examples.

Theorem 3.8. *Let X be an arbitrary topological space and $\mathcal{P}$ is an ideal of closed subsets of X with the property that it is closed under countable union, then $(X, \tau, \mathcal{P})$ is a $\tau\mathcal{PU}$-space.*

The fact that $C(X)$ is closed under uniform limit is used crucially to prove the Urysohn's Extension Theorem (Theorem 1.17 in [9]). Our aim is to achieve an analog of that result. We need the following definitions for that purpose.

Definition 3.9. A subspace S of X is said to be $C_{\mathcal{P}}$-embedded if every $f \in C(S)_{\mathcal{P}_S}$ can be extended to a function in $C(X)_{\mathcal{P}}$.

A subspace S of X is said to be $C^*_{\mathcal{P}}$-embedded if every $f \in C^*(S)_{\mathcal{P}_S}$ can be extended to a function in $C^*(X)_{\mathcal{P}}$.

The following theorem places the Urysohn's Extension Theorem (Theorem 1.17 in [9]) in sense of continuous function on a wider setting, and can be proved by closely following the proof of Theorem 1.17 in [9].

Theorem 3.10. *Let X be a $\tau\mathcal{P}\mathcal{U}$-space. Then a subspace S of X is $C^*_{\mathcal{P}}$-embedded in X if and only if any two $\mathcal{P}_S$-completely separated sets in S are $\mathcal{P}$-completely separated sets in X.*

Further, as is seen in case of $C(X)$, a $C^*_{\mathcal{P}}$-embedded subspace of X may not be $C_{\mathcal{P}}$-embedded.

Theorem 3.11. *A $C^*_{\mathcal{P}}$-embedded subspace of X is $C_{\mathcal{P}}$-embedded if and only if it is $\mathcal{P}$-completely separated from every zero set disjoint from it.*

This can be proved by closely following the proof of Theorem 1.18 in [9].

4. When $\mathcal{P}$ contains all singleton subsets of X

In this section, we study properties of $C(X)_{\mathcal{P}}$ under the restriction that $\mathcal{P}$ contain all singleton subsets of X. Since X is T_1 and all singleton subsets of X are in $\mathcal{P}$, we have $\chi_{\{x\}} \in C(X)_{\mathcal{P}}$ for every $x \in X$. Two of these type of rings, viz, $T'(X)$ and $C(X)_F$ have been studied in [8], [2], [14] and [1].

Throughout this section, we assume that any ideal $\mathcal{P}$ of closed subsets of a $\tau\mathcal{P}$-space X contains all singleton subsets of X (unless otherwise specified).

For any non-unit element $f \in C(X)_{\mathcal{P}}$, there exists $x_0 \in X$ such that $f(x_0) = 0$. This gives $f\chi_{X\setminus\{x_0\}} = f$. This shows that $C(X)_{\mathcal{P}}$ is almost regular which is summarised in the following theorem.

Theorem 4.1. *For a $\tau\mathcal{P}$-space X, $C(X)_{\mathcal{P}}$ is an almost regular ring.*

Also, for a non-unit element $f \in C(X)_{\mathcal{P}}$, there exists $y \in Z_{\mathcal{P}}(f)$ such that $f\chi_{\{y\}} = \mathbf{0}$. So f is a zero divisor. Thus we have the following result.

Proposition 4.2. *For a $\tau\mathcal{P}$-space X, $f \in C(X)_{\mathcal{P}}$ is either a zero divisor or an unit.*

The above result might fail if we remove the condition that $\mathcal{P}$ contains all singleton subsets of X.

Counterexample 4.3. Let us consider $X = \mathbb{R}$ with usual topology and $\mathcal{P}$ = the collection of all closed subsets of $(0, \infty)$. Then $f(x) = |x|$ is such that $f \in C(X)_{\mathcal{P}}$. Since $Z_{\mathcal{P}}(f) \neq \emptyset$, f is a non-unit element. Now, let $g \in C(X)_{\mathcal{P}}$ be such that $fg = \mathbf{0}$. Then as $f(x) \neq 0$ for all $x \neq 0$, we must have, $g(x) = 0$ for all $x \neq 0$.

If $g(0) \neq 0$, then $\overline{D_g} = \{0\} \notin \mathcal{P}$, which contradicts that $g \in C(X)_{\mathcal{P}}$. So, $g(0) = 0$ and thus $g = \mathbf{0}$. Therefore f is not a zero divisor.

Our next aim is to generalise Proposition 3.1 of [8] in the following way.

Proposition 4.4. *The following statements are equivalent for a $\tau\mathcal{P}$-space, $(X, \tau, \mathcal{P})$.*

1. $C(X) = C(X)_{\mathcal{P}}$.
2. *X is discrete.*
3. *$C(X)_{\mathcal{P}}$ is a ring of quotients of $C(X)$.*

Proof. If X is discrete, then all functions in $\mathbb{R}^X$ are continuous. So, $\mathbb{R}^X = C(X) = C(X)_{\mathcal{P}}$. Further, when $C(X) = C(X)_{\mathcal{P}}$, then for every $x \in X$, $\chi_{\{x\}} \in C(X)_{\mathcal{P}} = C(X)$ which implies that $\{x\}$ is open. Thus X is discrete. This shows that 1 and 2 are equivalent. To show that 3 and 2 are equivalent, it is already seen that if X is discrete, then $C(X) = C(X)_{\mathcal{P}}$. So, for every $f \in C(X)_{\mathcal{P}}$, we have $f \cdot \mathbf{1} = f \in C(X)_{\mathcal{P}} = C(X)$. This proves 3. Finally, let 3 hold. Then for each $x \in X$, there exists $f \in C(X)$ such that $f\chi_{\{x\}} \in C(X) \setminus \{\mathbf{0}\}$. This implies that $f(x) \neq 0$. But $\chi_{\{x\}} = \frac{1}{f(x)} f\chi_{\{x\}} \in C(X)$. This shows that X is discrete. □

Similarly, following the proof of Theorem 3.2 of [8] we have:

Theorem 4.5. *The following are equivalent for a $\tau\mathcal{P}$-space, X:*

1. *X is finite.*
2. *Each proper ideal of $C(X)_{\mathcal{P}}$ is fixed.*
3. *Each maximal ideal of $C(X)_{\mathcal{P}}$ is fixed.*
4. *Each proper ideal of $C^*(X)_{\mathcal{P}}$ is fixed.*
5. *Each maximal ideal of $C^*(X)_{\mathcal{P}}$ is fixed.*

However, this result may fail even if $\mathcal{P}$ fails to contain all singleton subsets of X.

Counterexample 4.6. Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ be the subspace of real line and $\mathcal{P} = \{\emptyset, \{1\}\}$. Then as 1 is an isolated point,
$C(X)_{\mathcal{P}} = C(X)$. Also X is compact, which implies that every maximal ideal of $C(X) = C(X)_{\mathcal{P}}$ is fixed; even though X is an infinite set.

We next move on to discuss certain rings properties of $C(X)_{\mathcal{P}}$. We start by discussing the structure of the minimal ideals and the socle of the ring.

Theorem 4.7. *The following assertions are true for a $\tau\mathcal{P}$-space X.*

1. *A non zero ideal I of $C(X)_{\mathcal{P}}$ is minimal if and only if there exists an $\alpha \in X$ such that $I =< \chi_{\{\alpha\}} >$ if and only if $|Z_{\mathcal{P}}[I]| = 2$.*
2. *The socle of $C(X)_{\mathcal{P}}$ consists of all functions that vanish everywhere except on a finite set.*
3. *The socle of $C(X)_{\mathcal{P}}$ is essential and free.*

Proof. 1. Let I be a non zero minimal ideal of $C(X)_{\mathcal{P}}$. For $f \in I \setminus \{\mathbf{0}\}$, there exists $\alpha \in X$ such that $f(\alpha) \neq 0$. Therefore $\chi_{\{\alpha\}} = \frac{1}{f(\alpha)}\chi_{\{\alpha\}} f \in I$. Since I is a minimal ideal of $C(X)_{\mathcal{P}}$, it follows that $I =< \chi_{\{\alpha\}} >$. This shows that $Z_{\mathcal{P}}[I] = \{Z_{\mathcal{P}}(\mathbf{0}), Z_{\mathcal{P}}(\chi_{\{\alpha\}})\} = \{X, X \setminus \{\alpha\}\}$. Thus, $|Z_{\mathcal{P}}[I]| = 2$.

Next we show that $< \chi_{\{\alpha\}} >$ is a minimal ideal of $C(X)_{\mathcal{P}}$. Let I be an ideal of $C(X)_{\mathcal{P}}$, $\{\mathbf{0}\} \subsetneq I \subseteq< \chi_{\{\alpha\}} >$. Then there exists $f \in I \setminus \{\mathbf{0}\} \subseteq< \chi_{\{\alpha\}} >$. So $f = g\chi_{\{\alpha\}}$, for some $g \in C(X)_{\mathcal{P}}$. But $f = g\chi_{\{\alpha\}} = g(\alpha)\chi_{\{\alpha\}} \implies \chi_{\{\alpha\}} = \frac{1}{g(\alpha)} f \in I$. Therefore $I =< \chi_{\{\alpha\}} >$.

Finally we assume that $|Z_{\mathcal{P}}[I]| = 2$ and show that I is a minimal ideal of $C(X)_{\mathcal{P}}$. There exists $f \in I$ such that $f(\alpha) \neq 0$ for some $\alpha \in X$. So $\chi_{\{\alpha\}} = \frac{1}{f(\alpha)}\chi_{\{\alpha\}} f \in I$. By our assumption, for any non zero function $g \in I$, $Z_{\mathcal{P}}(g) = Z_{\mathcal{P}}(\chi_{\{\alpha\}}) = X \setminus \{\alpha\}$. So every non zero $g \in I$ is of the form $g = g(\alpha)\chi_{\{\alpha\}}$. Therefore $I = \{c\chi_{\{\alpha\}} : c \in \mathbb{R}\} =< \chi_{\{\alpha\}} >$, which is a minimal ideal, as seen above.

2. By 1, the socle of $C(X)_{\mathcal{P}}$,

$$Soc(C(X)_{\mathcal{P}}) = \sum_{\alpha \in X} < \chi_{\{\alpha\}} >=< \left\{\chi_{\{\alpha\}} : \alpha \in X\right\} > .$$

Thus every function in $Soc(C(X)_{\mathcal{P}})$ vanishes everywhere except for a finitely many points of X. Conversely, let $f \in C(X)_{\mathcal{P}}$ be such that it vanishes everywhere except for a finitely many points, that is $Z_{\mathcal{P}}(f) = X \setminus \{\alpha_i : \alpha_i \in X, i = 1, ...n\}$ where $n \in \mathbb{N}$. Then

$$f = \sum_{i=1}^{n} f(\alpha_i)\chi_{\{\alpha_i\}} \in Soc(C(X)_{\mathcal{P}}).$$

3. Let I be a non zero ideal of $C(X)_{\mathcal{P}}$. Then there exists $f \in I$ such that $f(\alpha) \neq 0$ for some $\alpha \in X$. From 1, we have $\chi_{\{\alpha\}} \in Soc(C(X)_{\mathcal{P}})$ and $\chi_{\{\alpha\}} = \frac{1}{f(\alpha)}\chi_{\{\alpha\}}f \in I$. This ensures that $Soc(C(X)_{\mathcal{P}}) \cap I \neq \emptyset$. Thus $Soc(C(X)_{\mathcal{P}})$ is an essential ideal. Also, for an arbitrary $\alpha \in X$, $\chi_{\{\alpha\}} \in Soc(C(X)_{\mathcal{P}})$ and $\chi_{\{\alpha\}}(\alpha) = 1$. So $\alpha \notin Z_{\mathcal{P}}[Soc(C(X)_{\mathcal{P}})]$. This ensures that $Soc(C(X)_{\mathcal{P}})$ is a free ideal. □

We shall note here that the condition that $\mathcal{P}$ contains all singleton subsets of X is not a necessary condition for 2 in the above theorem. This can be seen by taking a Tychonoff space X and $\mathcal{P} = \{\emptyset\}$. Here $C(X)_{\mathcal{P}} = C(X)$. The rest follows from Proposition 2.2 in [4].

Using the above results, we establish a condition under which $C(X)_{\mathcal{P}}$ is an Artinian Ring. We need the following result to do this.

Proposition 4.8. *$Soc(C(X)_{\mathcal{P}}) = C(X)_{\mathcal{P}}$ if and only if X is finite.*

Proof. Let $X = \{x_1, x_2, ..., x_n\}$. Then $\mathbf{1} = \sum_{i=1}^{n} \chi_{\{x_i\}} \in Soc(C(X)_{\mathcal{P}})$. Thus $C(X)_{\mathcal{P}} = Soc(C(X)_{\mathcal{P}})$. Conversely, let $Soc(C(X)_{\mathcal{P}}) = C(X)_{\mathcal{P}}$. Then there exists $f_i \in C(X)_{\mathcal{P}}$ for $i = 1, 2, ..., n$ such that $\mathbf{1} = \sum_{i=1}^{n} f_i\chi_{\{x_i\}} = \sum_{i=1}^{n} f_i(x_i)\chi_{\{x_i\}}$. So, for each $x \in X$,

$$1 = \sum_{i=1}^{n} f_i(x_i)\chi_{\{x_i\}}(x) = f_i(x_i)\chi_{\{x_i\}}(x) \text{ for some } i \in \{1, 2, ..., n\}$$

which implies $\chi_{\{x_i\}}(x) = 1 \implies x = x_i$. Thus X is finite. □

[6] tells us that a commutative ring R with unity is semisimple if and only if $rad(R) = \{0\}$. Further R is Artinian semisimple if and only if R equals the sum of its minimal ideals.

In the ring $C(x)_{\mathcal{P}}$, it is easy to see that $\bigcap_{p\in X} M_p = \{\mathbf{0}\}$. So $rad(C(X)_{\mathcal{P}}) = \{\mathbf{0}\}$. Thus $C(X)_{\mathcal{P}}$ is semisimple.

Under the assumption that $\mathcal{P}$ contains all singleton subsets of X and using the above discussions, we have the following corollary.

Corollary 4.9. *$C(X)_{\mathcal{P}}$ is an Artinian ring if and only if X is finite.*

An obvious question arises here : When is $C(X)_{\mathcal{P}}$ Noetherian? We are able to answer this question when $\mathcal{P}$ contains all singleton subsets of X.

Theorem 4.10. *$C(X)_{\mathcal{P}}$ is a Noetherian ring if and only if X is finite.*

Proof. By 4.9 and the fact that an Artinian commutative ring is Noetherian, X is finite implies that $C(X)_{\mathcal{P}}$ is Noetherian. Conversely let X be an infinite set. Then X contains a countably infinite set $\{x_n : n \in \mathbb{N}\}$. Then $< \chi_{\{x_1\}} >\subsetneq< \chi_{\{x_1,x_2\}} >\subsetneq< \chi_{\{x_1,x_2,x_3\}} >\subsetneq ...$ gives an unbounded ascending chain of ideals of $C(X)_{\mathcal{P}}$. Thus $C(X)_{\mathcal{P}}$ is not Noetherian. □

It is important to note that the condition $\mathcal{P}$ contains all singleton subsets of X is not superfluous in 4.7(1), 4.9 and 4.10. We see that in the following example.

Counterexample 4.11. Let $X = \mathbb{R}$ with cofinite topology and $\mathcal{P} = \{\emptyset\}$. Then $C(X)_{\mathcal{P}} = C(X)$ which consists of only the constant functions on $\mathbb{R}$. Thus $C(X)_{\mathcal{P}}$ is isomorphic to $\mathbb{R}$ and the only ideals of $C(X)_{\mathcal{P}}$ are $\{\mathbf{0}\}$ and itself. So $\{\mathbf{0}\}$ is the only minimal ideal of $C(X)_{\mathcal{P}}$ and is not generated by $\chi_{\{x\}}$ for any $x \in X$. Also $|Z_{\mathcal{P}}[\{\mathbf{0}\}]| = 1$. Further $C(X)_{\mathcal{P}}$ is both Artinian and Noetherian, even though X is an infinite set.

We continue the study of ring properties of $C(X)_{\mathcal{P}}$ and establish a set of equivalent conditions to determine when is $C(X)_{\mathcal{P}}$ an *IN*-ring, *SA*-ring and/or a Baer ring.

Theorem 4.12. *The following statements are equivalent for a $\tau\mathcal{P}$-space $(X, \tau, \mathcal{P})$.*

1. *Any two disjoint subsets of X are $\mathcal{P}$-completely separated.*
2. *$C(X)_{\mathcal{P}}$ is an IN-ring.*

3. *$C(X)_{\mathcal{P}}$ is an SA-ring.*
4. *$C(X)_{\mathcal{P}}$ is an Baer ring.*
5. *The space of all prime ideals of $C(X)_{\mathcal{P}}$ is extremally disconnected.*
6. *Any subset of X is of the form $coz(e)$ for some idempotent $e \in C(X)_{\mathcal{P}}$.*
7. *For any subset A of X, there exists $P \in \mathcal{P}$ such that $A \setminus P$ is a clopen subset of $X \setminus P$.*

To prove this result, we need the following two lemmas.

Lemma 4.13. *For any subset A of X, there exists a subset S of $C(X)_{\mathcal{P}}$ such that*

$$A = \bigcup coz[S] = \bigcup \{coz(f) : f \in S\}.$$

This follows directly since

$$A = \bigcup coz[S] \text{ where } S = \{\chi_{\{x\}} : x \in A\} \text{ and } \chi_{\{x\}} \in C(X)_{\mathcal{P}} \text{ for all } x \in X.$$

Lemma 4.14.

1. *Let I and J be ideals of $C(X)_{\mathcal{P}}$. Then $Ann(I) \subseteq Ann(J)$ if and only if $\bigcap Z_{\mathcal{P}}[I] \subseteq \bigcap Z_{\mathcal{P}}[J]$ if and only if $\bigcap coz[J] \subseteq \bigcap coz[I]$.*
2. *For any subset S of $C(X)_{\mathcal{P}}$, $Ann(S) = \{f \in C(X)_{\mathcal{P}} : \bigcup coz[S] \subseteq Z_{\mathcal{P}}(f)\}$.*

Proof. 1. Let $Ann(I) \subseteq Ann(J)$ and $x \in \bigcap Z_{\mathcal{P}}[I]$. Then $f(x) = 0$ for all $f \in I \implies \chi_{\{x\}} f = \mathbf{0}$ for all $f \in I$. Therefore $\chi_{\{x\}} \in Ann(I) \subseteq Ann(J) \implies \chi_{\{x\}} g = \mathbf{0}$ for all $g \in J$. So $g(x) = 0$ for all $g \in J \implies x \in \bigcap Z_{\mathcal{P}}[J]$. Conversely, let $\bigcap Z_{\mathcal{P}}[I] \subseteq \bigcap Z_{\mathcal{P}}[J]$ and $f \in Ann(I)$. Then $fh = \mathbf{0}$ for all $h \in I$. So $coz(f) \subseteq \bigcap Z_{\mathcal{P}}[I] \subseteq \bigcap Z_{\mathcal{P}}[J]$. Thus $fh_1 = \mathbf{0}$ for all $h_1 \in J$ and hence $f \in Ann(J)$.

2. Let $f \in Ann(S)$. Then $fg = \mathbf{0}$ for all $g \in S$. Therefore for $x \in \bigcup coz[S]$, $f(x) = 0$. Conversely, let $f \in C(X)_{\mathcal{P}}$ be such that $\bigcup coz[S] \subseteq Z_{\mathcal{P}}(f)$ and $g \in S$. Then $coz(g) \subseteq \bigcup coz[S] \subseteq Z_{\mathcal{P}}(f)$. Therefore $fg = \mathbf{0}$. Hence $f \in Ann(S)$. □

We now prove Theorem 4.12.

Proof. Since $C(X)_{\mathcal{P}}$ is a reduced commutative ring, it follows from Lemma 1.2 that the statements from (2) to (5) are equivalent. We use Lemma 1.1 to prove (1) is equivalent to (2). Let (1) hold and let I and J be orthogonal ideals of $C(X)_{\mathcal{P}}$. Then $\bigcup coz[I]$ and $\bigcup coz[J]$ are disjoint subsets of X. By (1) there exists disjoint zero sets in $C(X)_{\mathcal{P}}$, $Z_{\mathcal{P}}(f_1)$ and $Z_{\mathcal{P}}(f_2)$ such that $\bigcup coz[I] \subseteq Z_{\mathcal{P}}(f_1)$ and $\bigcup coz[J] \subseteq Z_{\mathcal{P}}(f_2)$. This implies that $f_1 \in Ann(I)$ and $f_2 \in Ann(J)$. So ${f_1}^2 + {f_2}^2$ is a unit in $Ann(I) + Ann(J)$. This proves (2). Next let (2) be true and also let A and B be disjoint subsets of X. By Lemma 4.13, there exists subsets $S_A, S_A \subseteq C(X)_{\mathcal{P}}$ such that $A = \bigcup coz[S_A]$ and $B = \bigcup coz[S_B]$. Let $I =< S_A >$ and $J =< S_B >$. Then $\bigcup coz[I]$ and $\bigcup coz[J]$ are disjoint sets (as A and B are disjoint). Therefore I and J are orthogonal ideals of $C(X)_{\mathcal{P}}$. By (2) and Lemma 1.1, $Ann(I) + Ann(J) = C(X)_{\mathcal{P}}$. So there exists $h_1 \in Ann(I)$ and $h_2 \in Ann(J)$ such that $h_1 + h_2 = \mathbf{1}$, which is a unit. Therefore $Z_{\mathcal{P}}(h_1)$ and $Z_{\mathcal{P}}(h_1)$ are disjoint. Further $A = \bigcup coz[S_A] \subseteq \bigcup coz[I] \subseteq Z_{\mathcal{P}}(h_1)$ (since $h_1 \in Ann(I)$). Similarly $B \subseteq Z_{\mathcal{P}}(h_2)$. This proves (1).

We next show that (4) is equivalent to (6). Let $A \subseteq X$. Then there exists $S \subseteq C(X)_{\mathcal{P}}$ (by Lemma 4.13) such that $A = \bigcup coz[S]$. Define I to be the ideal generated by S. By (4) there exists an idempotent $e' \in C(X)_{\mathcal{P}}$ such that $Ann(I) =< e' >= Ann(< e >)$, where $e = \mathbf{1} - e'$ is also an idempotent. By Lemma 4.14, we have $\bigcup coz[I] = \bigcup coz[< e >]$. It can be easily seen that $\bigcup coz[S] = \bigcup coz[I]$. Thus $A = \bigcup coz[S] = \bigcup coz[< e >] = X \setminus Z_{\mathcal{P}}(e)$. This proves (6). Let (6) be true and I be an ideal of $C(X)_{\mathcal{P}}$. By (6) there exists an idempotent $e \in C(X)_{\mathcal{P}}$ such that $\bigcup coz[I] = coz(e)$. By Lemma 4.14, $Ann(I) = \{f \in C(X)_{\mathcal{P}} : \bigcup coz[I] \subseteq Z_{\mathcal{P}}(f)\} = \{f \in C(X)_{\mathcal{P}} : coz(e) \subseteq Z_{\mathcal{P}}(f)\} = Ann(e) =< (\mathbf{1} - e) >$. This shows that $C(X)_{\mathcal{P}}$ is a Baer ring.

Finally we show that (6) and (7) are equivalent Let $A \subseteq X$. By (6), $A = coz(e)$ for some idempotent $e \in C(X)_{\mathcal{P}}$. Let $P = \overline{D_e} \in \mathcal{P}$. It is easy to see that $coz(e) = Z_{\mathcal{P}}(\mathbf{1} - e)$. Thus $A \setminus P = X \setminus Z(e|_{X \setminus P}) = Z((\mathbf{1} - e)|_{X \setminus P})$ is clopen in $X \setminus P$. Let (7) hold and $A \subseteq X$. Then by (7), there exists $P \in \mathcal{P}$ such that $A \setminus P$ is clopen in $X \setminus P$. Define $e = \chi_A$. Then $e|_{A \setminus P}$ is continuous on $X \setminus P$. Therefore $\overline{D_e} \subseteq P \in \mathcal{P}$. So $e \in C(X)_{\mathcal{P}}$ and $A = coz(e)$. □

5. $C(X)_{\mathcal{P}}$ as a Von-Neumann regular ring and/or a Bezout ring

Definition 5.1. A commutative ring R with unity is said to be a regular ring (in the sense of Von-Neumann) if for every element $a \in R$, there exists an $x \in R$ such that $a = a^2x$.

R is said to be an Bezout ring if every finitely generated ideal of R is a principal ideal.

A Tychonoff space X is called a P-space if the ring $C(X)$ is Von-Neumann regular. This holds if and only if each G_δ-set in X is open. We would like to mention that this equivqlence may fail if the hypothesis that X is Tychonoff is omitted.

Counterexample 5.2. *Let $X = \mathbb{R}$ equipped with co-finite topology. Then $C(X)$ consists of all real-valued constant functions on $\mathbb{R}$. Therefore, $C(X)$ is isomorphic to the field $\mathbb{R}$ which is regular. Thus $C(X)$ is regular. We define $G_r = X \setminus \{r\}$ for each $r \in \mathbb{Q}$. Then G_r is open in X for all $r \in \mathbb{Q}$, and $G = \bigcap_{r\in\mathbb{Q}} G_r = \mathbb{R} \setminus \mathbb{Q}$ is a G_δ-set which is not open in X.*

Definition 5.3. A $\tau\mathcal{P}$-space $(X, \tau, \mathcal{P})$ is called a $\mathcal{P}P$-space if $C(X)_{\mathcal{P}}$ is a regular ring.

It may be mentioned in this context that a $\mathcal{P}P$-space may well be just T_1 without being Tychonoff.

We also note that the conclusion of Proposition 6.1 in [8] may fail when X is not a Tychonoff space. To see this, we consider the next counterexample.

Counterexample 5.4. Let $X = \mathbb{Q}^* = \mathbb{Q} \cup \{\infty\}$, the one-point compactification of $\mathbb{Q}$. Then every function in $C(\mathbb{Q}^*)$ is constant. Thus $C(X)$ is isomorphic to $\mathbb{R}$, which is regular. However, $C(\mathbb{Q})$ is not regular, even though $\mathbb{Q}$ is a subspace of $\mathbb{Q}^*$. Next, we show that $C(X)_F$ is not regular.

We define $f(x) = \begin{cases} \sin(\pi x) & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x = \infty \end{cases}$. Then $f \in C(X)_F$. If possible, let there exists $g \in C(X)_F$ such that $f = f^2g$, then $g(x) = \frac{1}{\sin(\pi x)}$ for all $x \in X \setminus \mathbb{Z}$. For any $n \in \mathbb{Z}$, g is unbounded in any neighbourhood of n in $\mathbb{Q}^*$. Therefore, $D_g \supseteq \mathbb{Z}$, which contradicts $g \in C(X)_F$. This shows that the regularity of $C(X)$ might not imply the regularity of $C(X)_F$, that is, a P-space may not be an $\mathcal{F}P$-space.

However, if we assume X to be Tychonoff, then the following is true.

Example 5.5. If X is a P-space, then it is $\mathcal{P}P$-space.

Proof. Let X be a P-space and $f \in C(X)_{\mathcal{P}}$. Then $f \in C(X \setminus \overline{D_f})$, where $X \setminus \overline{D_f}$ is a P-space, as it is a subspace of a P-space (by 4K in [9]). So, $C(X \setminus \overline{D_f})$ is regular. Therefore, $\exists\, g \in C(X \setminus \overline{D_f})$ such that $f|_{X\setminus\overline{D_f}} = (f|_{X\setminus\overline{D_f}})^2 g$.

Define g^* on X by $g^*(x) = \begin{cases} g(x) & \text{when } x \in X \setminus \overline{D_f} \\ \frac{1}{f(x)} & \text{when } x \in \overline{D_f} \setminus Z_{\mathcal{P}}(f) \\ 0 & \text{when } x \in \overline{D_f} \cap Z_{\mathcal{P}}(f) \end{cases}$. Then $D_{g^*} \subseteq \overline{D_f}$. Therefore, $g^* \in C(X)_{\mathcal{P}}$ and $f = f^2g^*$.

Thus $C(X)_{\mathcal{P}}$ is regular, and so X is a $\mathcal{P}P$-space. □

The following result gives a generalisation of Theorem 6.2 (1) $\Longleftrightarrow$ (2) [8].

Theorem 5.6. *X is a $\mathcal{P}P$-space if and only if for any zero set $Z \in Z_{\mathcal{P}}[X]$, there exists a set $P \in \mathcal{P}$ such that $Z \setminus P$ is a clopen set in $X \setminus P$.*

Proof. Let X be a $\mathcal{P}P$-space and $f \in C(X)_{\mathcal{P}}$. Then there exists $g \in C(X)_{\mathcal{P}}$ such that $f = f^2g$. Let $P = \overline{D_f} \cup \overline{D_g} \in \mathcal{P}$. Now $f|_{X\setminus P}, (\mathbf{1} - fg)|_{X\setminus P}$ are continuous. Also, $Z_{\mathcal{P}}(f) \setminus P = Z(f|_{X\setminus P}) = (X \setminus P) \setminus Z((\mathbf{1} - fg)|_{X\setminus P})$ is a clopen subset of $X \setminus P$. Conversely, let the given condition hold and let $f \in C(X)_{\mathcal{P}}$. Then $Z_{\mathcal{P}}(f) \setminus P$ is a clopen subset of $X \setminus P$ for some $P \in \mathcal{P}$. Define $g \colon X \longrightarrow \mathbb{R}$ by $g(x) = \begin{cases} \frac{1}{f(x)}, & x \notin Z_{\mathcal{P}}(f) \\ 0, & x \in Z_{\mathcal{P}}(f) \end{cases}$. Then $\overline{D_g} \subseteq \overline{P} \cup \overline{D_f} \in \mathcal{P} \implies \overline{D_g} \in \mathcal{P}$.

So, $g \in C(X)_{\mathcal{P}}$ and $f = f^2g$. Hence X is a $\mathcal{P}P$-space. □

We aim to provide some more characterisations for $C(X)_{\mathcal{P}}$ to be a regular ring.

Theorem 5.7. *The following statements are equivalent for a $\tau\mathcal{P}$-space $(X,\tau,\mathcal{P})$.*

1. *$(X,\tau,\mathcal{P})$ is a $\mathcal{P}P$-space.*
2. *Every prime ideal of $C(X)_{\mathcal{P}}$ is maximal.*
3. *For $f \in C(X)_{\mathcal{P}}$, there exists $g \in C(X)_{\mathcal{P}}$ with $Z_{\mathcal{P}}(f) = X \setminus Z_{\mathcal{P}}(g)$.*
4. *Every ideal of $C(X)_{\mathcal{P}}$ is a $z_{\mathcal{P}}$-ideal of $C(X)_{\mathcal{P}}$.*
5. *Every ideal of $C(X)_{\mathcal{P}}$ is an intersection of prime ideals.*
6. *Every ideal of $C(X)_{\mathcal{P}}$ is an intersection of maximal ideals.*
7. *For every $f,g \in C(X)_{\mathcal{P}}$, $< f,g >=< f^2+g^2 >$.*
8. *Every set of the form $X \setminus Z_{\mathcal{P}}(f)$ for some $f \in C(X)_{\mathcal{P}}$ is $C_{\mathcal{P}}$-embedded.*
9. *Every principal ideal is generated by an idempotent.*
10. *For $f \in C(X)_{\mathcal{P}}$ $Z_{\mathcal{P}}(f)$, there exists a set $P \in \mathcal{P}$ such that $Z_{\mathcal{P}}(f) \setminus P$ is a clopen set in $X \setminus P$.*

Proof. It is well known that a reduced ring R is Von-Neumann regular if and only if each prime ideal of R is maximal [10, Theorem 1.16]. This proves (1) is equivalent to (2).

To establish the equivalence of (1) and (3) let us first assume that $(X,\tau,\mathcal{P})$ is a $\mathcal{P}P$-space. Then for each $f \in C(X)_{\mathcal{P}}$, there exists $g \in C(X)_{\mathcal{P}}$ such that $f = f^2 g$. This gives us $Z_{\mathcal{P}}(f) = X \setminus Z_{\mathcal{P}}(\mathbf{1} - fg)$. Conversely let us assume the condition (3) and $f \in C(X)_{\mathcal{P}}$. Then there exists $g \in C(X)_{\mathcal{P}}$ such that $Z_{\mathcal{P}}(f) = X \setminus Z_{\mathcal{P}}(g)$. Choose $P = \overline{D_f} \cup \overline{D_g} \in \mathcal{P}$. Then it is immediate that $Z_{\mathcal{P}}(f) \setminus P = Z(f|_{X\setminus P})$ is clopen in $X \setminus P$. Define $h\colon X \longrightarrow \mathbb{R}$ as follows:

$$h(x) = \begin{cases} \frac{1}{f(x)} \text{ if } x \notin Z_{\mathcal{P}}(f) \setminus P \\ 0 \text{ if } x \in Z_{\mathcal{P}}(f) \end{cases} .$$

It follows that $h|_{X\setminus P}$ is continuous on $X \setminus P$. Since P is a closed set in X, h is continuous at each point in $X \setminus P$. Hence $D_h \subseteq P$ and so $h \in C(X)_{\mathcal{P}}$ and clearly $f = f^2 h$. Thus (1) and (3) are equivalent statements.

Next let $C(X)_{\mathcal{P}}$ be a Von-Neumann regular ring and I be an ideal in $C(X)_{\mathcal{P}}$. Also let $Z_{\mathcal{P}}(f) = Z_{\mathcal{P}}(g)$ with $g \in I$ and $f \in C(X)_{\mathcal{P}}$. Then there exists $h \in C(X)_{\mathcal{P}}$ such that $g = g^2 h$ and so $Z_{\mathcal{P}}(g) = X \setminus Z_{\mathcal{P}}(\mathbf{1} - gh)$. Hence we have $Z_{\mathcal{P}}(f) = X \setminus Z_{\mathcal{P}}(\mathbf{1} - gh)$ which implies that $f(\mathbf{1} - gh) = \mathbf{0}$ and hence $f = fgh \in I$ as $g \in I$. Thus I is a $z_{\mathcal{P}}$-ideal. Conversely let each ideal in $C(X)_{\mathcal{P}}$ be a $z_{\mathcal{P}}$-ideal and let $f \in C(X)$. Then by our hypothesis, $I =< f^2 >$ is a $z_{\mathcal{P}}$-ideal. Since $Z_{\mathcal{P}}(f) = Z_{\mathcal{P}}(f^2)$, it follows that $f \in I =< f^2 >$ and so there exists $g \in C(X)_{\mathcal{P}}$ such that $f = f^2 g$. Thus $C(X)_{\mathcal{P}}$ is a Von-Neumann regular ring. This proves the equivalence between (1) and (4).

The fact that (4) implies (5) follows from Theorem 2.14. To prove (5) implies (1), assume that the condition (5) is true and let $f \in C(X)_{\mathcal{P}}$. Then by our hypothesis $I =< f^2 >$ coincides with the intersection of all prime ideals containing I. Thus we have $I = \{f \in C(X)_{\mathcal{P}} \colon f^n \in I \text{ for some } n \in \mathbb{N}\}$ [9, Theorem 0.18]. Since $f^2 \in I$, we thus have $f \in I =< f^2 >$. This ensures that $f = f^2 g$ for some $g \in C(X)_{\mathcal{P}}$. Thus, we have that the conditions (1) through (5) are equivalent.

Equivalence between the statements (1) and (6) follows from the equivalence among the conidtions (1), (2) and (5).

(7) implies (1) is immediate by taking $g = \mathbf{0}$ in the statement (7). We show that (4) implies (7). Let $f,g \in C(X)_{\mathcal{P}}$. Then $< f^2+g^2 >\subseteq< f,g >$. To prove the reverse implication, see that $Z_{\mathcal{P}}(f^2+g^2) \subseteq Z_{\mathcal{P}}(f)$. It follows from our hypothesis that $< f^2+g^2 >$ is a $z_{\mathcal{P}}$-ideal in $C(X)_{\mathcal{P}}$. Thus we have $f \in< f^2+g^2 >$. Analogously we get $g \in< f^2+g^2 >$. This shows that $< f^2+g^2 >=< f,g >$.

We now prove the implication (1) $\Longrightarrow$ (8). Let $f \in C(X)_{\mathcal{P}}$ and $h \in C(S)_{\mathcal{P}_S}$ where $S = X \setminus Z_{\mathcal{P}}(f)$. Since (1) $\Longrightarrow$ (3), there exists $g \in C(X)_{\mathcal{P}}$ such that $Z_{\mathcal{P}}(f) = X \setminus Z_{\mathcal{P}}(g)$. See that $\overline{D_f}$ and $\overline{D_g}$ are members of $\mathcal{P}$ and $cl_S(D_h) \in \mathcal{P}_S$. Therefore there exists a $P_1 \in \mathcal{P}$ such that $cl_S(D_h) = P_1 \cap S$ where $P_1 \in \mathcal{P}$. See that $D_h \subseteq cl_S(D_h)$ and so $D_h \subseteq P_1$. This implies that $\overline{D_h} \subseteq P_1 \in \mathcal{P}$, and we have $\overline{D_h} \in \mathcal{P}$. Define $P = \overline{D_f} \cup \overline{D_g} \cup \overline{D_h} \in \mathcal{P}$. See that the functions $f|_{X\setminus P}$, $g|_{X\setminus P}$, $h|_{X\setminus(Z_{\mathcal{P}}(f)\cup P)}$ are continuous functions and $Z(f|_{X\setminus P}) = (X \setminus P) \setminus Z(g|_{X\setminus P})$ is clopen in $X \setminus P$. Define $\widehat{h}\colon X \longrightarrow \mathbb{R}$ as follows:

$$\widehat{h}(x) = \begin{cases} h(x) \text{ if } x \notin Z_{\mathcal{P}}(f) \\ 0 \text{ if } x \in Z_{\mathcal{P}} \end{cases} .$$

It is immediate that $\widehat{h}|_{X\setminus P}$ is continuous and since P is closed in X, $\widehat{h}$ is continuous at each point in $X \setminus P$. This ensures that $D_{\widehat{h}} \subseteq P$ and thus $\widehat{h} \in C(X)_{\mathcal{P}}$ is an extension of h. Conversely let us assume the statement (8) holds and $f \in C(X)_{\mathcal{P}}$. Define $g(x) = \frac{1}{f(x)}$ for all $x \in X \setminus Z_{\mathcal{P}}$. Then $D_g \subseteq D_f \cap S$ which implies that $cl_S(D_g) \subseteq cl_S(D_f \cap S) \subseteq \overline{D_f}$ and hence $cl_S(D_g) \subseteq \overline{D_f} \cap S$ where $\overline{D_f} \in \mathcal{P}$. This ensures that $cl_S(D_g) \in \mathcal{P}_S$ and thus $g \in C(S)_{\mathcal{P}_S}$ where $S = X \setminus Z_{\mathcal{P}}(f)$. By our hypothesis, there exists $\widehat{g} \in C(X)_{\mathcal{P}}$ such that $\widehat{g}|_{X\setminus Z_{\mathcal{P}}(f)} = g$. Thus we have $f = f^2\widehat{g}$.

To prove the equivalence between (1) and (9), we first assume that $C(X)_{\mathcal{P}}$ is Von-Neumann regular and $f \in C(X)_{\mathcal{P}}$. Then there exists $g \in C(X)_{\mathcal{P}}$ such that $f = f^2 g$. Define $e = fg \in C(X)_{\mathcal{P}}$. Then e is an idempotent and it can be easily checked that $< f >=< e >$. Conversely let each principal ideal in $C(X)_{\mathcal{P}}$ be generated by an idempotent. Let $f \in C(X)_{\mathcal{P}}$. By our hypothesis, there exists an idempotent $e \in C(X)_{\mathcal{P}}$ such that $< f >=< e >$. This implies that there exists $g, h \in C(X)_{\mathcal{P}}$ such that $e = fg$ and $f = eh$. It follows directly that $f = f^2\widehat{f}$ where $\widehat{f} = g^2 h \in C(X)_{\mathcal{P}}$.

Finally the equivalence between (1) and (10) has already been established in Theorem 5.6. This completes the proof. □

See that if $C(X)_{\mathcal{P}}$ is a Von-Neumann regular ring, then it follows from the above theorem that $< f, g >=< f^2 + g^2 >$ for any $f, g \in C(X)_{\mathcal{P}}$. Thus every finitely generated ideal of $C(X)_{\mathcal{P}}$ is principal. Hence $C(X)_{\mathcal{P}}$ is a Bezout ring.

It follows directly that the ring $T'(X)$ ([1]) is a Bezout ring.

To see when is $C(X)_{\mathcal{P}}$ a Bezout ring, we first establish a few properties of Bezout rings.

Theorem 5.8. *If $C(X)_{\mathcal{P}}$ is a Bezout ring, then the following conditions hold.*

1. *For each $f \in C(X)_{\mathcal{P}}$ there exists $k \in C(X)_{\mathcal{P}}$ such that $f = k|f|$.*
2. *For each $f \in C(X)_{\mathcal{P}}$, the sets pos $f (= \{x \in X : f(x) > 0\})$ and neg $f (= \{x \in X : f(x) < 0\})$ are $\mathcal{P}$-completely separated in X.*
3. *For each For each $f \in C(X)_{\mathcal{P}}$, there exists $P \in \mathcal{P}$ such that the sets pos $f \setminus P$ and neg $f \setminus P$ are completely separated in $X \setminus P$.*

Before proving this theorem we establish that the properties in Theorem 5.8 are pairwise equivalent for any $\tau\mathcal{P}$-space $C(X)_{\mathcal{P}}$.

Proposition 5.9. *The following statements are equivalent for any $\tau\mathcal{P}$-space $C(X)_{\mathcal{P}}$.*

1. *For each $f \in C(X)_{\mathcal{P}}$ there exists $k \in C(X)_{\mathcal{P}}$ such that $f = k|f|$.*
2. *For each $f \in C(X)_{\mathcal{P}}$, the sets pos $f (= \{x \in X : f(x) > 0\})$ and neg $f (= \{x \in X : f(x) < 0\})$ are $\mathcal{P}$-completely separated in X.*
3. *For each For each $f \in C(X)_{\mathcal{P}}$, there exists $P \in \mathcal{P}$ such that the sets pos $f \setminus P$ and neg $f \setminus P$ are completely separated in $X \setminus P$.*

Proof. To prove (1) implies (2), let $f \in C(X)_{\mathcal{P}}$. By (1), there exists a $k \in C(X)_{\mathcal{P}}$ such that $f = k|f|$. Now define $\widehat{k} = k \wedge \mathbf{1}$. Then $\widehat{k} \in C(X)_{\mathcal{P}}$ and $f = \widehat{k}|f|$ with $|\widehat{k}| \leq 1$. Thus $k(pos\ f) = \{1\}$ and $k(neg\ f) = \{-1\}$.

Next we assume the statement (2) to be true and prove (3). Let $f \in C(X)_{\mathcal{P}}$. Then there exists $k \in C(X)_{\mathcal{P}}$ such that $|k| \leq 1$, $k(pos\ f) = \{1\}$ and $k(neg\ f) = \{-1\}$. Then $\overline{D_f}, \overline{D_k} \in \mathcal{P}$. Choose $P = \overline{D_f} \cup \overline{D_k} \in \mathcal{P}$. Then $k|_{X\setminus P}$ is continuous on $X \setminus P$. This ensures that $pos\ f \setminus P$ and $neg\ f \setminus P$ are completely separated in $X \setminus P$ by $k|_{X\setminus P}$.

Finally we need to show that (3) $\Longrightarrow$ (1). Let $f \in C(X)_{\mathcal{P}}$. Then by our hypothesis, there exists $P \in \mathcal{P}$ such that $pos\ f \setminus P$ and $neg\ f \setminus P$ are completely separated in $X \setminus P$. This means that there exists $k \in C(X \setminus P)$ such that $|k| \leq 1$, $k(pos\ f \setminus P) = \{1\}$ and $k(neg\ f \setminus P) = \{-1\}$. Define $\widehat{k} : X \longrightarrow \mathbb{R}$ as follows:

$$\widehat{k}(x) = \begin{cases} k(x) & \text{when } x \in X \setminus P \\ 1 & \text{when } x \in P \cap pos\ f \\ -1 & \text{when } x \in P \cap neg\ f \\ 0 & \text{when } x \in P \cap Z_{\mathcal{P}}(f) \end{cases}.$$

We note that $\widehat{k}|_{X\setminus P} = k$ which is continuous and P is closed in X. Hence $\widehat{k}$ is continuous at each point in $X \setminus P$ and hence $D_{\widehat{k}} \subseteq P \in \mathcal{P}$. Therefore $\widehat{k} \in C(X)_{\mathcal{P}}$ with $f = \widehat{k}|f|$. $\square$

We are now ready to prove Theorem 5.8.

Proof. [Proof of Theorem 5.8] We will just show that the statement (2) is true. The rest will follow from Proposition 5.9. Let $f \in C(X)_{\mathcal{P}}$. Then by our hypothesis $I =< f, |f| >$ is generated by a single element, $d \in C(X)_{\mathcal{P}}$. So there exists $g, h, s, t \in C(X)_{\mathcal{P}}$ such that $f = gd$, $|f| = hd$, and $d = sf + |f|t$. It follows that $sg + th = \mathbf{1}$ on $X \setminus Z_{\mathcal{P}}(f)$, $g = h$ on *pos* f and $g = -h$ on *neg* f. This ensures that

$$pos\ f \subseteq Z_{\mathcal{P}}(g - h) \cap Z_{\mathcal{P}}(sg + th - \mathbf{1}) = Z_1(\text{say}) \text{ and}$$

$$neg\ f \subseteq Z_{\mathcal{P}}(g + h) \cap Z_{\mathcal{P}}(sg + th - \mathbf{1}) = Z_2(\text{say}),$$

where $Z_1 \cap Z_2 = \emptyset$. Thus *pos* f and *neg* f are $\mathcal{P}$-completely separated in X. $\square$

It follows from the Proposition 5.9 that the ring $C(\mathbb{R})_F$ is not a Bezout ring. Indeed define $f : X \longrightarrow \mathbb{R}$ as $f(x) = \sin x$ for all $x \in X$. Then $f \in C(\mathbb{R})_F$. But there does not exist any $k \in C(\mathbb{R})_F$ such that $f = k|f|$.

We would like to mention in this context that the ring $C(\mathbb{R})_F$ is not closed under uniform limit. This can be easily seen on using Theorem 3.6 taking care of the fact that $\mathbb{R}$ has infinitely many non-isolated points.

This suggests us to characterise Bezout rings in the entire family of rings of the form $C(X)_{\mathcal{P}}$ which are closed under uniform limit.

Theorem 5.10. *The following statements are equivalent for a $\tau\mathcal{PU}$-space $(X, \tau, \mathcal{P})$.*

1. *For each $f \in C(X)_{\mathcal{P}}$ there exists $k \in C(X)_{\mathcal{P}}$ such that $f = k|f|$.*
2. *For each $f \in C(X)_{\mathcal{P}}$, the sets pos $f(= \{x \in X : f(x) > 0\})$ and neg $f(= \{x \in X : f(x) < 0\})$ are $\mathcal{P}$-completely separated in X.*
3. *For each $f \in C(X)_{\mathcal{P}}$, there exists $P \in \mathcal{P}$ such that the sets pos $f \setminus P$ and neg $f \setminus P$ are completely separated in $X \setminus P$.*
4. *Every set of the form $X \setminus Z_{\mathcal{P}}(f)$, for some $f \in C(X)_{\mathcal{P}}$, is $C^*_{\mathcal{P}}$-embedded in X.*
5. *Every ideal in $C(X)_{\mathcal{P}}$ is convex.*
6. *For each $f, g \in C(X)_{\mathcal{P}}$, $< f, g >=< |f| + |g| >$.*
7. *Every finitely generated ideal in $C(X)_{\mathcal{P}}$ is principal, i.e., $C(X)_{\mathcal{P}}$ is a Bezout ring.*

Proof. It follows from the proof of Proposition 5.9 that the statements (1) to (3) are equivalent.

To prove that (2) implies (4), we use Theorem 3.10 which is permissible since X is a $\tau\mathcal{PU}$-space. Let $f \in C(X)_{\mathcal{P}}$ and $S = X \setminus Z_{\mathcal{P}}(f)$. Assume that A and B are $\mathcal{P}_S$-completely separated in S. It is enough to show that A and B are $\mathcal{P}$-completely separated in X. It follows that there exists $k \in C^*(S)_{\mathcal{P}_S}$ such that $|k| \leq 1$, $k(A) = \{1\}$ and $k(B) = \{-1\}$. Define $g : X \longrightarrow \mathbb{R}$ as follows:

$$g(x) = \begin{cases} 0 & \text{when } x \in Z_{\mathcal{P}}(f) \\ k(x)|f(x)| & \text{when } x \notin Z_{\mathcal{P}}(f) \end{cases}.$$

Choose $P = \overline{D_k} \cup \overline{D_f}$. Then $P \in \mathcal{P}$ (see the arguments made in the proof of Theorem 5.7) and $g|_{X\setminus P}$ is continuous on $X \setminus P$ since k is bounded. Since P is a closed subset of X, g is continuous at each point in $X \setminus P$ ensuring that $g \in C(X)_{\mathcal{P}}$. Also note that $A \subseteq pos\ g$ and $B \subseteq neg\ g$. By our hypothesis *pos* f and *neg* f (and hence A and B) are $\mathcal{P}$-completely separated in X.

Next we show that (4) implies (5). Let I be an ideal in $C(X)_{\mathcal{P}}$ and $\mathbf{0} \leq f \leq g$ with $f \in C(X)_{\mathcal{P}}$ and $g \in I$. Define $s = \frac{f}{g}$ on $X \setminus Z_{\mathcal{P}}(g)$. Then $s \in C^*(S)_{\mathcal{P}_S}$. By our hypothesis there exists $\widehat{s} \in C(X)_{\mathcal{P}}$ such that $\widehat{s}|_S = s$. It is then immediate that $f = \widehat{s}g \in I$, as $g \in I$. This ensures that I is a convex ideal.

Let us now assume the condition (5) to be true and let $f, g \in C(X)_{\mathcal{P}}$. Since $-(|f| + |g|) \leq f \leq (|f| + |g|)$, by convexity of the ideal $< |f| + |g| >$, $f \in< |f| + |g| >$. Analogously we get $g \in< |f| + |g| >$. Thus

$< f, g > \subseteq < |f| + |g| >$. Again see that $-|f| \leq f \leq |f|$. By convexity of $< |f| >$, we have $f \in < |f| >$ which implies that there exists $h \in C(X)_{\mathcal{P}}$ such that $f = |f|h \implies |f| = fh \in < f, g >$. Similarly we can show that $|g| \in < f, g >$. This ensures that $< |f| + |g| > \subseteq < f, g >$.

(6) implies (7) is trivial and it follows from Theorem 5.8 that (7) implies (1). □

6. $\aleph_0$-Self injectiveness of $C(X)_{\mathcal{P}}$

In this section, we establish conditions under which $C(X)_{\mathcal{P}}$ is $\aleph_0$-self injective. In order to achieve this, we need the following definitions and theory.

Definition 6.1. ([3]) A ring R is said to be $\aleph_0$-self injective if every module homomorphism $\phi\colon I \longrightarrow R$ can be extended to a module homomorphism $\widehat{\phi}\colon R \longrightarrow R$ where I is a countably generated ideal of R.

A lattice-ordered vector space or vector lattice is a partially ordered vector space where the order structure forms a lattice.

Definition 6.2. ([13]) An element x of a vector lattice X is called a weak order unit in X if $x \geq 0$ and also for all $y \in X$, $\inf\{x, |y|\} = 0$ implies $y = 0$.

Definitions 6.3. ([13]) By a lattice-ordered ring $(A, +, ., \vee, \wedge)$, we mean a lattice-ordered group that is a ring in which the product of positive elements is positive. If, in addition, A is a (real) vector lattice, then A is said to be a lattice-ordered algebra.

A lattice-ordered ring A is said to be Archimedean if, for each non-zero element $a \in A$, the set $\{na\colon n \in \mathbb{Z} \setminus \{0\}\}$ is unbounded.

By a ϕ-algebra A, we mean an Archimedean, lattice-ordered algebra over the real field $\mathbb{R}$ which has identity element 1 that is a weak order unit in A.

A ϕ-algebra A of real-valued functions is said to be uniformly closed if it is closed under uniform convergence.

Definitions 6.4. ([12]) Let A be a ϕ-algebra. We denote $\mathcal{M}(A)$ as the compact space of maximal absolutely convex ring ideals of A carrying Stone topology. Further, we denote $\mathcal{R}(A)$ to be the set of all real ideals of A.

Definitions 6.5. ([3]) For a subset Q of a ring R, $Ann(Q) = \{r \in R\colon qr = 0 \text{ for all } q \in Q\}$. A subset, P of a ring R is said to be orthogonal if the product of any two distinct elements of P is zero. Suppose P and Q are disjoint subsets of R whose union is an orthogonal subset of R. Then, an element $a \in R$ is said to separate P from Q if

1. $p^2a = p$ for all $p \in P$, and
2. $a \in Ann(Q)$.

We shall use Theorem 2.3 in [12] to show that $C(X)_{\mathcal{P}}$ is isomorphic to an algebra of measurable functions.

We assume X to be a $\mathcal{P}P$-space and a $\tau\mathcal{PU}$-space, so that $C(X)_{\mathcal{P}}$ is regular and closed under uniform convergence. Further, for any $f \in C(X)_{\mathcal{P}} \setminus \{\mathbf{0}\}$, there exists $x \in X$ such that $f(x) \neq 0$. So, the set $\{\boldsymbol{n}f\colon n \in \mathbb{Z} \setminus \{\mathbf{0}\}\}$ is unbounded. We have already seen that $C(X)_{\mathcal{P}}$ is a lattice ordered group. It is also easy to see that it forms a real vector space and for any two positive elements $f, g \in C(X)_{\mathcal{P}}$, fg is also positive. Also, $C(X)_{\mathcal{P}}$ has the identity element $\mathbf{1}$ which is clearly a weak order unit. Thus, $C(X)_{\mathcal{P}}$ is a ϕ-algebra which is closed under uniform convergence. That is, $C(X)_{\mathcal{P}}$ forms a uniformly closed ϕ-algebra.

We have also seen that all maximal ideals of $C(X)_{\mathcal{P}}$ are $z_{\mathcal{P}}$-ideals which are in turn absolutely convex. Therefore, all maximal ideals of $C(X)_{\mathcal{P}}$ are in $Max(C(X)_{\mathcal{P}})$.

Define for each $p \in X$, $M_p = \{f \in C(X)_{\mathcal{P}}\colon f(p) = 0\}$. Then, $C(X)_{\mathcal{P}}/M_p$ is isomorphic to $\mathbb{R}$, for each $p \in X$. Thus, M_p is a real maximal ideal, for each $p \in X$ and is thus a member of $\mathcal{R}(C(X)_{\mathcal{P}})$.

Theorem 6.6. ([12, Theorem 2.3]) *The following conditions on the ϕ-algebra A are equivalent.*

(a) A is uniformly closed, regular, and $\bigcap\{M\colon M \in \mathcal{R}(A)\} = \{0\}$.

(b) A is isomorphic to an algebra of measurable functions.

We have

$$\bigcap_{p\in X} M_p = \{\mathbf{0}\} \implies \bigcap\{M\colon M \in \mathcal{R}(C(X)_{\mathcal{P}})\} = \{\mathbf{0}\}$$

From the above theorem, we have $C(X)_{\mathcal{P}}$ is isomorphic to an algebra of measurable functions.
Next we show that $\aleph_0$-self injectiveness of a ring is invariant under ring isomorphism.

Theorem 6.7. *If a ring R is isomorphic to an $\aleph_0$-self injective reduced ring S, then R is also $\aleph_0$-self injective.*

Proof. We shall use Theorem 2.2 of [11] to prove the result.
Let $\psi\colon S \longrightarrow R$ be the given isomorphism. It is easy to see that, since S is reduced, so is $\psi(S) = R$.
Further, by Theorem 2.2 of [11], S is regular. Therefore, $R = \psi(S)$ is also a regular ring.
Let us now consider two disjoint subsets of R, P and Q whose union is a countable orthogonal subset of R. Then, $\psi^{-1}(P)$ and $\psi^{-1}(Q)$ are disjoint and their union is countable. Also, for any $s, s' \in \psi^{-1}(P) \cup \psi^{-1}(Q)$ with $s \neq s'$, $\psi(s), \psi(s') \in P \cup Q$ with $\psi(s) \neq \psi(s')$ (since ψ is injective). As $P \cup Q$ is orthogonal,

$$\psi(s)\psi(s') = 0 \implies \psi(ss') = 0 \implies ss' = 0, \text{ since } \psi \text{ is injective.}$$

Thus, $\psi^{-1}(P) \cup \psi^{-1}(Q)$ is orthogonal. As S is $\aleph_0$-self injective, by Theorem 2.2 of [11], there exists $a \in S$ that separates $\psi^{-1}(P)$ from $\psi^{-1}(Q)$.
We now show that $\psi(a)$ separates P from Q.

1. Let $p \in P$, then $\psi^{-1}(p^2\psi(a)) = (\psi^{-1}(p))^2 a = \psi^{-1}(p)$, as $a \in S$ that separates $\psi^{-1}(P)$ from $\psi^{-1}(Q)$. It follows from the injectivity of ψ^{-1} that $p^2\psi(a) = p$. Thus, $p^2\psi(a) = p$ for all $p \in P$.
2. Let $q \in Q$, then $\psi^{-1}(q) \in \psi^{-1}(Q)$. As $a \in S$ that separates $\psi^{-1}(P)$ from $\psi^{-1}(Q)$, $\psi^{-1}(q)a = 0$ which shows that $q\psi(a) = 0$. Thus, $\psi(a) \in Ann(Q)$.

Thus, we get an element in R ($\psi(a)$) that separates P from Q. It follows from Theorem 2.2 of [11] that R is $\aleph_0$-self injective. □

Finally we use the above theory to establish the following theorem.

Theorem 6.8. *For a $\tau\mathcal{PU}$-space, the following conditions are equivalent.*
(a) X is a $\mathcal{P}P$-space.
(b) $C(X)_{\mathcal{P}}$ is isomorphic to an algebra of measurable functions.
(c) $C(X)_{\mathcal{P}}$ is $\aleph_0$-self injective.

Proof. (a) $\implies$ (b) follows from the above discussions. (b) $\implies$ (c) can be seen from Theorem 7 of [3] and the above theorem. Finally, (c) $\implies$ (a) follows directly from Theorem 2.2 of [11]. □

We cannot omit the condition that X is a $\tau\mathcal{PU}$-space. This can be seen from the following example.

Counterexample 6.9. Let us consider

$$X = \mathbb{N} \cup \bigcup_{k\in\mathbb{N}}\{\frac{1}{n} + k\colon n \in \mathbb{N}\}$$

endowed with the subspace topology inherited from $\mathbb{R}_u$. Also let $\mathcal{P} = \mathcal{P}_f$, that is, the ideal of all finite subsets of X. Then, $C(X)_{\mathcal{P}} = C(X)_F$, which is not uniformly closed (by Theorem 2.9 in [14]). Now, we consider the following subsets of X:

$$A = \bigcup_{k\in\mathbb{N}}\{\frac{1}{2n} + k\colon n \in \mathbb{N}\} \text{ and } B = \bigcup_{k\in\mathbb{N}}\{\frac{1}{2n-1} + k\colon n \in \mathbb{N}\}.$$

Define $P = \{\chi_{\{x\}}\colon x \in A\}$ and $Q = \{\chi_{\{x\}}\colon x \in B\}$ Then, P and Q are disjoint and $P \cup Q$ is countable and orthogonal. If there exists an $f \in \mathbb{R}^X$ that separates P from Q, then $f(A) = \{1\}$ and $f(B) = \{0\}$. Thus, every point in $\mathbb{N}$ is a point of discontinuity of f. Therefore $f \notin C(X)_F$. This ensures from Theorem 2.2 of [11] that $C(X)_F$ is not $\aleph_0$-self injective.

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