Filomat 38:27 (2024), 9557–9566
https://doi.org/10.2298/FIL2427557P

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On (strongly) regular relations on soft topological hypergroups

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Abstract. In the first step, we determine some conditions for having a topological complete hypergroup by defining the geometric space obtained from each strongly regular relation. Then applying set theory, we introduce the concepts of "soft geometric spaces," "soft topological hypergroups," and "(strongly) regular relations" on soft hypergroups. Finally, to find some conditions for having a soft topological (complete) hypergroup, using the results of the first step, we determine the soft geometric space obtained from a strongly regular relation on soft hypergroups.

1. Introduction and preliminaries

The concept of hypergroup as a generalization of algebraic structures was first introduced in 1934 by Marty [14]. Afterwards, This concept was investigated in some publications such as [2, 3, 7, 15, 16, 24]. Then, Vougiouklis [26] presented the notation of H_v-structure as a generalization of algebraic hyperstructures in 1990. Davvaz, Spartalis, Dramalidis, Leoreanu-Fotea, S. Hoskova, and others have tried to develop this concept in different directions. All the results until 2018 were collected by Davvaz and Vougiouklis [6].

Recall from [4, 6] that a *hypergroupoid* $(H, \circ)$ is a nonempty set H with a map $\circ : H \times H \to \mathcal{P}^*(H)$ called *(binary) hyperoperation*, where $\mathcal{P}^*(H)$ is the set of all nonempty subsets of H. The image of pair (x, y) is denoted by $x \circ y$. For every $A, B \in \mathcal{P}^*(H)$ and $x \in H$, consider

$$A \circ B = \bigcup_{\substack{a\in A\\ b\in B}} a \circ b, \quad A \circ x = A \circ \{x\}, \quad \text{and} \quad x \circ B = \{x\} \circ B.$$

Let $(H, \circ)$ be a hypergroupoid.

- It is called *commutative* if $x \circ y = y \circ x$ for all $x, y \in H$.
- It is called a *semihypergroup* if $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in H$.
- It is called an H_v*-semigroup* if $x \circ (y \circ z) \bigcap (x \circ y) \circ z \neq \emptyset$ for all $x, y, z \in H$.
- It is called a *quasihypergroup* if it has the reproduction axiom (i.e., $a \circ H = H \circ a = H$ for every $a \in H$).

2020 *Mathematics Subject Classification*. Primary 20N20; Secondary 54A40
Keywords. Soft hypergroups, soft geometric spaces, strongly regular relations, topological hypergroups
Received: 25 September 2023; Revised: 22 August 2024; Accepted: 30 August 2024
Communicated by Ljubiša D. R. Kočinac
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Recall that a semihypergroup $(H, \circ)$ is a *hypergroup* if it is a quasihypergroup. An H_v-semigroup $(H, \circ)$ is an H_v-*group* if it has the reproduction axiom.

Let $(H, \circ)$ be a semihypergroup with an equivalence relation ρ. If A and B are nonempty subsets of H, then

$$\begin{aligned} A\overline{\rho}B \text{ means that } & \text{ for all } a \in A, \text{ there exists } b \in B \text{ such that } a\rho b \\ \text{and } & \text{ for all } b' \in B, \text{ there exists } a' \in A \text{ such that } a'\rho b'; \\ A\overline{\overline{\rho}}B \text{ means that } & \text{ for all } a \in A \text{ and for all } b \in B, \text{ we have } a\rho b. \end{aligned}$$

The equivalence relation ρ is called

- *regular on the right* (resp., *on the left*) if for all $x \in H$, it follows from $a\rho b$ that $(a \circ x)\overline{\rho}(b \circ x)$ (resp., $(x \circ a)\overline{\rho}(x \circ b)$);
- *strongly regular on the right* (resp., *on the left*) if for all $x \in H$, it follows from $a\rho b$ that $(a \circ x)\overline{\overline{\rho}}(b \circ x)$ (resp., $(x \circ a)\overline{\overline{\rho}}(x \circ b)$);
- *regular* (resp., *strongly regular*) if it is regular (resp., strongly regular) on the right and on the left.

Let $(H, \circ)$ be a hypergroup with an equivalence relation ρ. Then ρ is regular (resp., strongly regular) if and only if $\left(\frac{H}{\rho}, \circledcirc\right)$ is a hypergroup (resp., group), where $\rho(x) \circledcirc \rho(y) = \rho(z)$ for all $x, y \in H$ and $z \in x \circ y$. Moreover, $\rho(a)$ is the ρ-class of $a \in H$.

The *fundamental equivalence relation* β^* on a hypergroup (H_v-group) $(H, \circ)$ is the smallest equivalence relation on H such that the quotient $\frac{H}{\beta^*}$ is a group.

The notion of topological hypergroup was introduced by Heidari, Davvaz, and Modarres [9]. Then Singha, Das, and Davvaz [23] defined the concept of topological complete hypergroups and investigated some of their properties. This concept was also studied in some other papers [1, 10–12, 22]. Also, we [25] extended the concept of topological hypergroup to the topological H_v-group by the concept of geometric spaces, which was defined by Freni [8]. By defining the geometric space obtained from the fundamental relation β^* on the H_v-group, we determined some conditions for having a topological complete H_v-group. In this paper, we determine some conditions for having a topological complete hypergroup by defining the geometric space obtained from each strongly regular relation.

A powerful tool for modeling uncertain problems is "soft set theory", which was initiated in 1999 by Molodtsov [17]. Many researchers investigated soft set theory and combined it with other objects in mathematics. One of these objects is an algebraic hyperstructure. For example, Yamak, Kazanci, and Davvaz [27] introduced the notion of "soft hypergroupoids". Selvachandran and Salleh [21] defined "soft hypergroups" and "soft hypergroup homomorphism". Furthermore, they discussed some basic properties and structural characteristics of these notions. Recently, Oguz and Davvaz [18] combined it with topological spaces and initiated the concept of "soft topological hypergroupoids". Ostadhadi-Dehkordi and Shum [19] defined "(strongly) regular relations" on soft hyperrings (see also [5]). In this paper, we introduce the concepts of "soft geometric spaces" and "soft topological hypergroups" and state some conditions of soft geometric spaces. Then, we define the concept of "(strongly) regular relations" on soft hypergroups. To find some conditions for having a soft topological (complete) hypergroup, we determine the soft geometric space obtained from a strongly regular relation on a soft hypergroup.

Recall from [8] that a *geometric space* is a pair $(S, \mathcal{B})$ such that S is a nonempty set in which its elements are called *points* and $\mathcal{B}$ is a nonempty family of subsets of S, which its elements are called *blocks*. If C is a subset of S, then it is called a $\mathcal{B}$-*part* of S if $B \cap C \neq \emptyset$ implies $B \subseteq C$ for every $B \in \mathcal{B}$. For a subset $X \subseteq S$, the intersection of all $\mathcal{B}$-parts of S containing X is denoted by $\Gamma(X)$. For any subsets X and Y of a geometric space $(S, \mathcal{B})$, the following properties are true:

(P1) $X \subseteq \Gamma(X)$,

(P2) $X \subseteq Y \Rightarrow \Gamma(X) \subseteq \Gamma(Y)$,

(P3) $\Gamma\Big(\Gamma(X)\Big) = \Gamma(X)$,

(P4) $\Gamma(X) = \bigcup_{x\in X} \Gamma(x)$, where $\Gamma(x) = \Gamma(\{x\})$.

Corollary 1.1 ([8]). *Let $(S, \mathcal{B})$ be a geometric space and let B be an element of $\mathcal{B}$. Then,*

1. $\Gamma(x) = \Gamma(y)$ *for each* $x, y \in B$.
2. $\Gamma(B) = \Gamma(x)$ *for all* $x \in B$.

The n-tuple $(B_1, B_2, \ldots, B_n)$ of blocks of a geometric space $(S, \mathcal{B})$ is called a *polygonal* if $B_i \cap B_{i+1} \neq \emptyset$ for each $1 \leq i < n$. Freni [8] defined the relation $\approx$ as follows:

$$x \approx y \text{ if and only if } x = y \text{ or there exists a polygonal } (B_1, B_2, \ldots, B_n) \text{ such that } x \in B_1 \text{ and } y \in B_n.$$

The relation $\approx$ is an equivalence and coincides with the transitive closure of the following relation:

$$x \sim y \text{ if and only if } x = y \text{ or there exists } B \in \mathcal{B} \text{ such that } \{x, y\} \subseteq B.$$

Hence $\approx$ is equal to $\bigcup_{n\geq 1} \sim^n$, where $\sim^n = \underbrace{\sim \circ \sim \circ \cdots \circ \sim}_{n \text{ times}}$. Freni [8] proved that $y \sim^n x$ if and only if $y \in \Gamma_n(x)$ and the $\approx$-class of x in S coincides with $\Gamma(x)$.

We [25] showed that for the family $\mathcal{F}_{\mathcal{B}}(S)$ of all $\mathcal{B}$-parts of S, there are two topologies on S; see the following proposition.

Proposition 1.2 ([25]). *Let $(S, \mathcal{B})$ be a geometric space; then the following properties hold:*

1. *The family $\mathcal{F}_{\mathcal{B}}(S)$ is the family of open subsets of topology $\mathcal{T}^o_{\mathcal{B}}(S)$ (which is called the* open topology corresponding to $\mathcal{B}$*).*
2. *The family $\mathcal{F}_{\mathcal{B}}(S)$ is the family of closed subsets of topology $\mathcal{T}^c_{\mathcal{B}}(S)$ on S (which is called the* closed topology corresponding to $\mathcal{B}$*).*

Open sets of topology $\mathcal{T}^o_{\mathcal{B}}(S)$ are closed sets of topology $\mathcal{T}^c_{\mathcal{B}}(S)$, and vice versa. Throughout the paper, whenever the type of these topologies is not important or their properties are common, we call it *corresponding topology* and denote it by $\mathcal{T}_{\mathcal{B}}(S)$.

Recall from [25] that a block $B \in \mathcal{B}$ of a geometric space $(S, \mathcal{B})$ is *complete* if B is a $\mathcal{B}$-part. In other words, a block B is complete if and only if $\Gamma(B) = B$. A geometric space is *complete* if each of its block is complete. A geometric space $(S, \mathcal{B})$ with a topology τ on S is called *τ-complete* if every open set of S is a $\mathcal{B}$-part. In other words, a geometric space $(S, \mathcal{B})$ with a topology τ on S is τ-complete if $\Gamma(U) = U$ for every $U \in \tau$. A geometric space $(S, \mathcal{B})$ is called *τ-open* (resp., *τ-closed*) if every block $B \in \mathcal{B}$ is an open (resp., closed) subset of (S, τ).

Proposition 1.3 ([25]). *If $(S, \mathcal{B})$ is a complete geometric space such that $S = \bigcup_{B\in\mathcal{B}} B$, then $\mathcal{B}$ is an open basis of topology $\mathcal{T}^o_{\mathcal{B}}(S)$ and a closed subbasis of topology $\mathcal{T}^c_{\mathcal{B}}(S)$. Moreover, $(S, \mathcal{B})$ is transitive.*

Proposition 1.4 ([25]). *If a geometric space $(S, \mathcal{B})$ is τ-open and τ-complete for a topology τ on S, then it is complete.*

Proposition 1.5 ([25]). *Let $(S_i, \mathcal{B}_i)$ be geometric spaces with topology τ_i on S_i for $i = 1, 2, \ldots, n$. Then, $(\prod_{i=1}^n S_i, \prod_{i=1}^n \mathcal{B}_i)$ is a geometric space and the following properties hold:*

1. *If $(S_i, \mathcal{B}_i)$ is (strongly) transitive for $i = 1, 2, \ldots, n$, then $(\prod_{i=1}^n S_i, \prod_{i=1}^n \mathcal{B}_i)$ is (strongly) transitive.*
2. *If $(S_i, \mathcal{B}_i)$ is τ_i-open (resp., τ_i-closed) for $i = 1, 2, \ldots, n$, then $(\prod_{i=1}^n S_i, \prod_{i=1}^n \mathcal{B}_i)$ is $(\prod_{i=1}^n \tau_i)$-open (resp., $(\prod_{i=1}^n \tau_i)$-closed).*
3. *If $(S_i, \mathcal{B}_i)$ is τ_i-complete (resp., complete) for $i = 1, 2, \ldots, n$, then $(\prod_{i=1}^n S_i, \prod_{i=1}^n \mathcal{B}_i)$ is $(\prod_{i=1}^n \tau_i)$-complete (resp., complete).*

Let (S, τ) be a topological space. Then, the family $\mathcal{U}$ consisting of all sets

$$S_U = \{V \in \mathcal{P}^*(S) \mid V \subseteq U\}, \text{ where } U \in \tau,$$

is a base for a topology on $\mathcal{P}^*(S)$; see [11]. This topology is denoted by τ^*.

Recall from [25] that for the geometric space $(S, \mathcal{B})$, the *induced geometric space* on $\mathcal{P}^*(S)$ is $\left(\mathcal{P}^*(S), \mathcal{B}^*\right)$ if

$$\mathcal{B}^* = \left\{B^* \mid B \in \mathcal{B}\right\}, \text{ where } B^* = \left\{V \in \mathcal{P}^*(S) \mid V \subseteq B\right\}.$$

Proposition 1.6 ([25]). *Let $(S, \mathcal{B})$ be a geometric space with topology τ on S and let $\left(\mathcal{P}^*(S), \mathcal{B}^*\right)$ be the induced geometric space with topology τ^* on $\mathcal{P}^*(S)$. Then,*

1. *$(S, \mathcal{B})$ is complete if and only if $\left(\mathcal{P}^*(S), \mathcal{B}^*\right)$ is complete;*
2. *$(S, \mathcal{B})$ is τ-open implies that $\left(\mathcal{P}^*(S), \mathcal{B}^*\right)$ is τ^*-open;*
3. *$(S, \mathcal{B})$ is τ-complete implies that $\left(\mathcal{P}^*(S), \mathcal{B}^*\right)$ is τ^*-complete.*

A map $f : (S_1, \mathcal{B}_1) \to (S_2, \mathcal{B}_2)$ between the geometric spaces, is called a *good morphism* if $x \sim y$ implies $f(x) \sim f(y)$ for all $x, y \in S_1$ (see [25]).

Proposition 1.7 ([25]). *Let $f : (S_1, \mathcal{B}_1) \to (S_2, \mathcal{B}_2)$ be a map between the geometric spaces. If f is a good morphism, then*

1. *$x \approx y$ implies $f(x) \approx f(y)$ for all $x, y \in S_1$.*
2. *$f : (S_1, \mathcal{T}_{\mathcal{B}_1}(S_1)) \to (S_2, \mathcal{T}_{\mathcal{B}_2}(S_2))$ is continuous.*

Proposition 1.8 ([25]). *Let $(S_i, \mathcal{B}_i)$ be a geometric space with topology τ_i on S_i for $i = 1, 2$ and let $f : (S_1, \mathcal{B}_1) \to (S_2, \mathcal{B}_2)$ be a good morphism. If $(S_1, \mathcal{B}_1)$ is τ_1-open such that $S_1 = \bigcup_{B \in \mathcal{B}_1} B$ and $(S_2, \mathcal{B}_2)$ is τ_2-complete, then $f : (S_1, \tau_1) \to (S_2, \tau_2)$ is continuous.*

2. Topological hypergroups

n this section, we introduce the induced geometric space from a hypergroup with respect to an arbitrary strongly regular relation ρ such that the relation "$\sim$" in geometric spaces coincides with the relation ρ in the hypergroup. Then, we state some conditions for having a topological (complete) hypergroup. First, we remind some concepts of hypergroups in algebras and some relative properties of them; for more information, see [4, 6, 20].

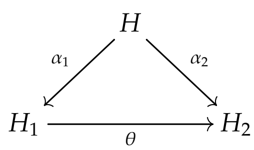
Let $(H_1, \circ)$ and $(H_2, \bullet)$ be two hypergroups (resp., H_v-groups). A map $f : H_1 \to H_2$ is called a *homomorphism* or a *good (or strong) homomorphism* if

$$f(x \circ y) = f(x) \bullet f(y) \quad \text{for all } x, y \in H_1.$$

Proposition 2.1 ([6]). *If ρ is a regular equivalence relation on a (semi)hypergroup $(H, \circ)$, then the canonical projection $f : (H, \circ) \to \left(\frac{H}{\rho}, \circledcirc\right)$ is an onto good homomorphism.*

By Proposition 2.1, for a regular equivalence ρ on a (semi)hypergroup $(H, \circ)$, we have $\rho(x \circ y) = \rho(x) \circledcirc \rho(y)$ for every $x, y \in H$.

Proposition 2.2 ([4]). *Let α_i be a good homomorphism from semihypergroup $(H, \bullet)$ onto semihypergroup $(H_i, \circ_i)$ for $i = 1, 2$, such that $\alpha_1^{-1} \circ \alpha_1 \subseteq \alpha_2^{-1} \circ \alpha_2$. Then, there exists a unique good homomorphism $\theta : (H_1, \circ_1) \to (H_2, \circ_2)$ such that $\theta \circ \alpha_1 = \alpha_2$; that is, the following diagram commutes:*



Proposition 2.3 ([4])**.** *If ρ_1 and ρ_2 are strongly regular relations on a semihypergroup H such that $\rho_1 \subseteq \rho_2$, then there exists a good homomorphism from $\frac{H}{\rho_1}$ onto $\frac{H}{\rho_2}$.*

Proposition 2.4. *Let $f : (H_1, \circ) \to (H_2, \bullet)$ be a good homomorphism between hypergroups and let ρ be an equivalence relation on H_1. Let ${}^{f}\rho$ be a relation on $f(H_1)$ such that $f(x)\,{}^{f}\rho\, f(y)$ if and only if $f(x) = f(y)$ or $x\rho y$. Then, ${}^{f}\rho$ is a regular (resp., strongly regular) relation on $f(H_1)$ if ρ is a regular (resp., strongly regular) relation on H_1.*

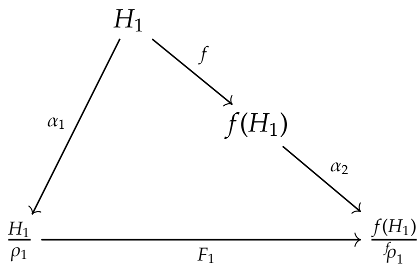
Proof. Let $a, b, x \in H_1$ such that $f(a)\,{}^{f}\rho\, f(b)$. There are two cases: Case 1: If $f(a) = f(b)$, then $f(a) \bullet f(x) = f(b) \bullet f(x)$. Case 2: If $f(a) \neq f(b)$, then $a\rho b$. So $(a \circ x)\overline{\rho}(b \circ x)$ (resp.,$(a \circ x)\overline{\overline{\rho}}(b \circ x)$). Since ρ is regular (resp., strongly regular), $f(a \circ x)\overline{{}^{f}\rho} f(b \circ x)$ (resp., $f(a \circ x)\overline{\overline{{}^{f}\rho}} f(b \circ x)$). Since f is a good homomorphism, $\big(f(a) \bullet f(x)\big)\overline{{}^{f}\rho}\big(f(b) \bullet f(x)\big)$ (resp., $\big(f(a) \bullet f(x)\big)\overline{\overline{{}^{f}\rho}}\big(f(b) \bullet f(x)\big)$). Therefore, ${}^{f}\rho$ is regular (resp., strongly regular) on the right. By similar argument, ${}^{f}\rho$ is regular (resp., strongly regular) on the left. Hence ${}^{f}\rho$ is regular (resp., strongly regular). □

Theorem 2.5. *Let $f : (H_1, \circ) \to (H_2, \bullet)$ be a good homomorphism between hypergroups and let ρ_i be an equivalence relation on H_i for $i = 1, 2$, such that ${}^{f}\rho_1 \subseteq \rho_2$.*

1. *f preserves the relation ρ_1 (i.e., $x\rho_1 y$ yields $f(x)\rho_2 f(y)$ for all $x, y \in H_1$).*
2. *If ρ_1 is a strongly regular relation, then the induced map $F : \left(\frac{H_1}{\rho_1}, \circledcirc\right) \to \left(\frac{H_2}{\rho_2}, \odot\right)$ is a group homomorphism.*

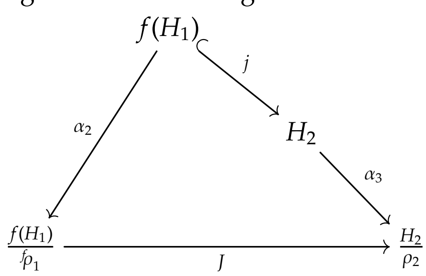
Proof.

1. Clearly, $f\big(\rho_1(x)\big) \subseteq {}^{f}\rho_1\big(f(x)\big)$ for all $x \in H_1$. Since ${}^{f}\rho_1 \subseteq \rho_2$, then $f\big(\rho_1(x)\big) \subseteq \rho_2\big(f(x)\big)$ for all $x \in H_1$. Thus f preserves the relation ρ_1.
2. Let $\alpha_1 : H_1 \to \frac{H_1}{\rho_1}$ and $\alpha_2 : f(H_1) \to \frac{f(H_1)}{{}^{f}\rho_1}$ be the canonical maps. For every $x \in H_1$, we have $\alpha_1^{-1} \circ \alpha_1(x) = \rho_1(x) \subseteq {}^{f}\rho_1\big(f(x)\big) = (\alpha_2 \circ f)^{-1} \circ (\alpha_2 \circ f)(x)$. Therefore, by Proposition 2.2, there exists a unique good homomorphism F_1 such that $F_1 \circ \alpha_1 = \alpha_2 \circ f$. Consider the following commutative diagram:



By Proposition 2.4, the relation ${}^{f}\rho_1$ is strongly regular on $f(H_1)$. Hence $\frac{f(H_1)}{{}^{f}\rho_1}$ is a group. Since ρ_1 is a strongly regular relation, $\frac{H_1}{\rho_1}$ is a group. Therefore, F_1 is a group homomorphism since it is a good homomorphism and $\frac{H_1}{\rho_1}$ and $\frac{f(H_1)}{{}^{f}\rho_1}$ are groups.

Let $\alpha_3 : H_2 \to \frac{H_2}{\rho_2}$ be the canonical map. Since $\alpha_2^{-1} \circ \alpha_2(f(x)) = {}^{f}\rho_1\big(f(x)\big) \subseteq \rho_2\big(f(x)\big) = (\alpha_3 \circ j)^{-1} \circ (\alpha_3 \circ j)(f(x))$ for all $x \in H_1$, by Proposition 2.2, there exists a unique good homomorphism J such that $J \circ \alpha_2 = \alpha_3 \circ j$. Consider the following commutative diagram:



Since $\frac{f(H_1)}{{}^{f}\rho_1}$ and $\frac{H_2}{\rho_2}$ are groups and J is a good homomorphism, J is a group homomorphism. Now, consider the following commutative diagram:

$$\begin{array}{ccccc} H_1 & \xrightarrow{f} & f(H_1) & \xhookrightarrow{j} & H_2 \\ \alpha_1 \downarrow & & \alpha_2 \downarrow & & \downarrow \alpha_3 \\ \frac{H_1}{\rho_1} & \xrightarrow[F_1]{} & \frac{f(H_1)}{{}^f\rho_1} & \xrightarrow[J]{} & \frac{H_2}{\rho_2} \end{array} \tag{1}$$

Obviously, $F = J \circ F_1$. Thus F is a group homomorphism.
□

Remark 2.6. *Let $(H, \circ)$ be a hypergroup and let ρ be a strongly regular relation on H. Then $\big(H, \rho(H)\big)$ is a geometric space, where $\rho(H) = \{\rho(x) | x \in H\}$. Clearly, the relation "$\sim$" coincides with the relation ρ.*

Theorem 2.7. *Let $f : (H_1, \circ) \to (H_2, \bullet)$ be a good homomorphism between hypergroups and let ρ_i be an equivalence relation on H_i for $i = 1, 2$, such that ${}^f\rho_1 \subseteq \rho_2$. Then the induced map $f : \big(H_1, \rho_1(H_1)\big) \to \big(H_2, \rho_2(H_2)\big)$ is a good morphism between geometric spaces. Thus $f : \big(H_1, \mathcal{T}_{\rho_1(H_1)}(H_1)\big) \to \big(H_2, \mathcal{T}_{\rho_2(H_2)}(H_2)\big)$ is continuous. If f is an isomorphism on hypergroups, then f is a homeomorphism on the corresponding topological spaces.*

Proof. Since ${}^f\rho_1 \subseteq \rho_2$, by Theorem 2.5, f preserves the relation ρ_1. By Remark 2.6, the relation $\sim$ in geometric space $\big(H_1, \rho_1(H_1)\big)$ coincides with ρ_1. Therefore, f preserves the relation $\sim$ in geometric spaces. So f is a good morphism between geometric spaces. Thus by Theorem 1.7, f is continuous. If f is isomorphism, then there is a good homomorphism $g : H_2 \to H_1$, such that $g \circ f$ and $f \circ g$ are identities on H_1 and H_2, respectively, which completes the proof. □

Proposition 2.8. *Let $(H, \circ)$ be a hypergroup and let ρ be a strongly regular relation on H. Then the canonical map $q : H \to \frac{H}{\rho}$ is a quotient map with respect to corresponding topology.*

Proof. By Proposition 2.1, q is an onto good homomorphism. So by Theorem 2.7, q is continuous. It is remained to show that it is an open (resp., closed) map. Let U be an open (resp., closed) subset of H in open (resp., closed) corresponding topology. Then, by Proposition 1.2, $U = \Gamma(U)$. Therefore, $q\big(\Gamma(U)\big) = \Gamma\big(q(U)\big)$, and hence q is open (resp., closed). □

Recall from [9] that for a hypergroup $(H, \circ)$ with topology τ on H, the system $(H, \circ, \tau)$ is called a *topological hypergroup* if the following conditions hold:

1. The mapping $(x, y) \mapsto x \circ y$ from $H \times H$ to $\mathcal{P}^*(H)$ is continuous.
2. The mapping $(x, y) \mapsto \frac{x}{y}$ from $H \times H$ to $\mathcal{P}^*(H)$ is continuous, where $\frac{x}{y} = \{z \in H \mid x \in z \circ y\}$.

Here, $H \times H$ is equipped with the product topology and $\mathcal{P}^*(H)$ is equipped with the topology τ^*. If the first condition holds, then $(H, \circ, \tau)$ is called a *paratopological hypergroup*.

Lemma 2.9. *Let $(H, \circ)$ be a hypergroup and let ρ be a strongly regular relation on H. Then, the hyperoperation $\circ : H \times H \to \mathcal{P}^*(H)$ that maps (x, y) to $x \circ y$, is continuous on the topological space $\big(H, \mathcal{T}_{\rho(H)}(H)\big)$. Hence $\big(H, \circ, \mathcal{T}_{\rho(H)}(H)\big)$ is a paratopological hypergroup.*

Proof. If we show that $\circ : \big(H, \rho(H)\big) \times \big(H, \rho(H)\big) \to \big(\mathcal{P}^*(H), \big(\rho(H)\big)^*\big)$ preserves the relation $\sim$, then by Theorem 2.7, it is continuous in the topological space $\big(H, \mathcal{T}_{\rho(H)}(H)\big)$. Let $(x_1, y_1) \sim (x_2, y_2)$, that is, $x_1 \sim x_2$ and $y_1 \sim y_2$. Therefore, $\Gamma(x_1) = \Gamma(x_2)$ and $\Gamma(y_1) = \Gamma(y_2)$. Note that ρ coincides with the relation $\sim$ and that $\frac{H}{\rho}$ is a group by the operation $\rho(x_1) \odot \rho(y_1) = \rho(z_1)$, where $z_1 \in x_1 \circ y_1$. Hence $\Gamma(x_1) \odot \Gamma(y_1) = \Gamma(z_1)$, so

$$\Gamma(z_1) = \Gamma(x_1) \odot \Gamma(y_1) = \Gamma(x_2) \odot \Gamma(y_2) = \Gamma(z_2),$$

where $z_2 \in x_2 \circ y_2$. However, it follows from (P1) that $\Gamma(z_i) \subseteq \Gamma(x_i \circ y_i)$, for $i = 1, 2$. Thus $\Gamma(x_1 \circ y_1) \cap \Gamma(x_2 \circ y_2) \neq \emptyset$. Therefore, $(x_1 \circ y_1) \approx (x_2 \circ y_2)$ in the geometric space $\left(\mathcal{P}^*(H), \left(\rho(H)\right)^*\right)$. Since $\rho(H)$ is a cover for H, $\left(\rho(H)\right)^*$ is a cover for $\mathcal{P}^*(H)$, so by Proposition 1.3, $\left(\mathcal{P}^*(H), \left(\rho(H)\right)^*\right)$ is a transitive geometric space. Therefore, $(x_1 \circ y_1) \sim (x_2 \circ y_2)$, which implies that $\circ$ is a good morphism. □

In the following lemma, we investigate the continuity of inversion of hypergroups.

Lemma 2.10. *Let $(H, \circ)$ be a hypergroup and let ρ be a strongly regular relation on H. Then, the map $f : H \times H \to \mathcal{P}^*(H)$ that maps (x, y) to $\frac{x}{y} = \{z \in H \mid x \in z \circ y\}$ is continuous in the topological space $\left(H, \mathcal{T}_{\rho(H)}(H)\right)$.*

Proof. By Theorem 2.7, it is enough to show that f preserves the relation $\sim$. Let $(x_1, y_1) \sim (x_2, y_2)$. Then $x_1 \sim x_2$ and $y_1 \sim y_2$. Indeed, for each $z_1 \in \frac{x_1}{y_1}$ and $z_2 \in \frac{x_2}{y_2}$, we have $x_1 \in z_1 \circ y_1$ and $x_2 \in z_2 \circ y_2$. Since $\left(\frac{H}{\rho}, \odot\right)$ is a group and ρ coincides with the relation $\sim$, we have $\rho(z_1) \odot \rho(y_1) = \rho(x_1)$, where $x_1 \in z_1 \circ y_1$ yields $\Gamma(z_1) \odot \Gamma(y_1) = \Gamma(x_1)$. Indeed $\Gamma(x_1) = \Gamma(x_2)$ and $\Gamma(y_1) = \Gamma(y_2)$, so

$$\Gamma(z_1) \odot \Gamma(y_2) = \Gamma(z_1) \odot \Gamma(y_1) = \Gamma(x_1) = \Gamma(x_2) = \Gamma(z_2) \odot \Gamma(y_2).$$

By the properties of the group $\left(\frac{H}{\rho}, \odot\right)$,

$$\Gamma(z_1) \odot \Gamma(y_2) = \Gamma(z_2) \odot \Gamma(y_2) \Rightarrow \Gamma(z_1) = \Gamma(z_2).$$

Similar the proof of Lemma 2.9, we have $\Gamma(z_1) = \Gamma(z_2)$ in the transitive geometric space $\left(\mathcal{P}^*(H), \left(\rho(H)\right)^*\right)$, so $z_1 \sim z_2$. □

The two above lemmas yield the following theorem.

Theorem 2.11. *Let $(H, \circ)$ be a hypergroup and ρ be a strongly regular relation on H. Then, $\left(H, \circ, \mathcal{T}_{\rho(H)}(H)\right)$ is a topological hypergroup.*

Theorem 2.12. *Let $(H, \circ)$ be a hypergroup with topology τ on H. If $\left(H, \rho(H)\right)$ is τ-complete and τ-open for some strongly regular relation ρ on H, then $(H, \circ, \tau)$ is a topological complete hypergroup.*

Proof. Similar the proof of Lemma 2.9, we have $\circ : H \times H \to \mathcal{P}^*(H)$ is a good morphism. Therefore, by Propositions 1.5, 1.6, and 1.8, it is continuous. Similar to the proof of Lemma 2.10, the map $f : H \times H \to \mathcal{P}^*(H)$ that maps (x, y) to $\frac{x}{y} = \{z \in H \mid x \in z \circ y\}$ is a good morphism. So by Propositions 1.5, 1.6, and 1.8, f is continuous. Proposition 1.4 completes the proof. □

3. Soft topological hypergroups

In this section, we introduce the concept of soft geometric space over a geometric space. Then we define the induced soft geometric space from a soft hypergroup with respect to an arbitrary strongly regular relation ρ such that the relation "$\sim$" in geometric spaces coincides with the relation ρ in the hypergroup. Then, we introduce the concept of soft topological hypergroup and state some conditions for having a soft topological (complete) hypergroup. First, we recall concepts of soft sets, soft hypergroups, soft topological hypergroupoids, and some relative properties of them; for more information, see [13, 17, 18, 21, 27].

Recall form [17] that a pair (F, A) is called a *soft set* over an initial universe X, where $F : A \to \mathcal{P}(X)$ is a mapping, E is a set of parameters, and $A \subseteq E$. In other words, a soft set over X is a parametrization of $\mathcal{P}(X)$. Recall from [13] that the *support* of (F, A) is defined as follows:

$$\mathrm{Supp}(F, A) = \left\{a \in A \,\middle|\, F(a) \neq \emptyset\right\}.$$

If (F, A) is defined by $F(a) = \emptyset$, for each $a \in A$, then (F, A) is called a *null soft set* over X, and it is denoted by $\tilde{\phi}_A$.

Definition 3.1. Let $(S, \mathcal{B})$ be a geometric space with $\emptyset \neq K \subseteq S$; then the geometric space (K, C) is called *geometric subspace* if $\emptyset \neq C \subseteq \mathcal{B}$. The *induced geometric subspace* of K, denoted by $K \leqslant S$, is a geometric subspace $\big(K, \mathcal{B}(K)\big)$, where $\mathcal{B}(K) = \{B \in \mathcal{B} \mid B \subseteq K\}$. If there is no ambiguity, we delete the expression "geometric". The subspace $(\emptyset, \emptyset)$ is the induced subspace of $\emptyset$, and we call it *trivial subspace*.

Obviously, if $\big(H, \rho(H)\big)$ and $\big(K, \rho(K)\big)$ are the corresponding geometric spaces of a hypergroup $(H, \circ)$ and subhypergroup $K \leqslant H$, with respect to a strongly regular relation ρ, respectively, then $\big(K, \rho(K)\big)$ is an induced geometric subspace of $\big(H, \rho(H)\big)$.

Definition 3.2. A soft set (F, A) over a geometric space $(S, \mathcal{B})$ is called a soft geometric space if $F(a)$ is an induced geometric subspace of S, for all $a \in A$.

Proposition 3.3. *Let $(S, \mathcal{B})$ be a geometric space. Then $\big(\Gamma(x), \mathcal{B}[x]\big)$ is an induced geometric subspace for each $x \in S$, where $\mathcal{B}[x] = \Big\{B \in \mathcal{B} \;\Big|\; B \subseteq \Gamma(x)\Big\}$.*

Proof. Let $x \in S$; then $\Gamma(x) \subseteq S$. If $B \in \mathcal{B}$ such that $B \subseteq \Gamma(x)$, then by the construction of $\Gamma(x)$, we have $B \in \mathcal{B}[x]$. Hence $(\Gamma(x), \mathcal{B}[x])$ is an induced geometric space of $(S, \mathcal{B})$. □

Immediately, we have the following corollary.

Corollary 3.4. *Let $(S, \mathcal{B})$ be a geometric space. Then (Γ, S) is a soft geometric space over $(S, \mathcal{B})$.*

Definition 3.5. Let $(F, A) \neq \tilde{\phi}_A$ be a soft geometric space over a geometric space $(S, \mathcal{B})$ with topology τ on S. Then, the soft geometric space (F, A) is called *τ-open (resp., τ-closed, τ-complete, or complete)* if the geometric space $\Big(F(a), \mathcal{B}\big(F(a)\big)\Big)$ is τ_a-open (resp., τ_a-closed, τ_a-complete, or complete), for all $a \in \mathrm{Supp}(F, A)$, where τ_a is the subspace topology on $F(a)$.

Proposition 3.6. *Let $(F, A) \neq \tilde{\phi}_A$ be a soft geometric space over a geometric space $(S, \mathcal{B})$ with topology τ on S. If the geometric space $(S, \mathcal{B})$ is τ-open (resp., τ-closed, τ-complete, or complete), then the soft geometric space (F, A) is τ-open (resp., τ-closed, τ-complete, or complete).*

Proof. Let $a \in \mathrm{Supp}(F, A)$ and let $B \in \mathcal{B}\big(F(a)\big) \subseteq \mathcal{B}$. Then B is an open (resp., closed) subset of topological space (S, τ), since $(S, \mathcal{B})$ is a τ-open (resp., τ-closed) geometric space. Indeed $B \subseteq F(a)$, so B is an open (resp., closed) subset of subspace topology τ_a. Therefore, $\Big(F(a), \mathcal{B}\big(F(a)\big)\Big)$ is a τ_a-open (resp., τ_a-closed) geometric space.

Assume that $U \in \tau_a$. There exists an open subset $V \in \tau$ such that $U = V \cap F(a)$. Since $(S, \mathcal{B})$ is τ-complete, ${}^{S}\Gamma(V) = V$ (where ${}^{S}\Gamma(V)$ is $\Gamma(V)$ in the geometric space $(S, \mathcal{B})$). Let $B \in \mathcal{B}\big(F(a)\big)$ such that $B \subseteq {}^{F(a)}\Gamma(U)$. By (P2), we have $B \subseteq {}^{S}\Gamma(U) \subseteq {}^{S}\Gamma(V) = V$, so $B \subseteq V \cap F(a) = U$. Thus ${}^{F(a)}\Gamma(U) \subseteq U$. Hence by (P1), ${}^{F(a)}\Gamma(U) = U$, which completes the proof. Let $B \in \mathcal{B}\big(F(a)\big)$. Since $(S, \mathcal{B})$ is complete, then ${}^{S}\Gamma(B) = B$, so ${}^{F(a)}\Gamma(B) = B$. Therefore, $\Big(F(a), \mathcal{B}\big(F(a)\big)\Big)$ is complete. □

Clearly, the inverse of above proposition is not necessarily true, unless $F(a) = S$ for some $a \in \mathrm{Supp}(F, A)$.

Recall from [27] that a nonnull soft set (F, A) over a hypergroupoid $(H, \circ)$ is called a *soft hypergroupoid* over H if $F(a)$ is a subhypergroupoid of H for all $a \in \mathrm{Supp}(F, A)$. If $(H, \circ)$ is a hypergroup and $F(a)$ is a subhypergroup of H for all $a \in \mathrm{Supp}(F, A)$, then (F, A) is called *soft hypergroup*(see [21]).

Let $(H, \circ)$ be a hypergroupoid with topology τ on H. Assume that $\mathcal{P}_\circ(H)$ is the set of all subhypergroupoids of H and A that is the set of parameters. Recall from [18] that if $F : A \to \mathcal{P}_\circ(H)$ is a map, then the pair (F, A) is called a *soft topological hypergroupoid* over H with the topology τ whenever the following conditions are satisfied:

1. $F(a)$ is a subhypergroupoid of H for all $a \in \text{Supp}(F, A)$,
2. The hyperoperation $\circ : F(a) \times F(a) \to \mathcal{P}^*\big(F(a)\big)$ is continuous, for all $a \in \text{Supp}(F, A)$,

Here $F(a) \times F(a)$ is equipped with the product topology, and $\mathcal{P}^*\big(F(a)\big)$ is equipped with the topology τ^*.

In other words, (F, A) is a soft topological hypergroupoid over H with the topology τ if (F, A) is a soft hypergroupoid and the hyperoperation $\circ : F(a) \times F(a) \to \mathcal{P}^*\big(F(a)\big)$ is continuous, for all $a \in \text{Supp}(F, A)$, with respect to the product topology on $F(a) \times F(a)$ and the topology τ^* on $\mathcal{P}^*\big(F(a)\big)$.

As above, we can define the concept of soft topological hypergroup.

Definition 3.7. Let $(H, \circ)$ be a hypergroup with topology τ on H, and let $(F, A) \neq \tilde{\phi}_A$ be a soft hypergroup over H. Then, the triple (F, A, τ) is called a *soft topological hypergroup* over H if the following conditions hold:

1. The mapping $(x, y) \mapsto x \circ y$ from $F(a) \times F(a)$ to $\mathcal{P}^*\big(F(a)\big)$ is continuous, for all $a \in \text{Supp}(F, A)$.
2. The mapping $(x, y) \mapsto \frac{x}{y}$, from $F(a) \times F(a)$ to $\mathcal{P}^*\big(F(a)\big)$ is continuous, for all $a \in \text{Supp}(F, A)$, where $\frac{x}{y} = \{z \in H \mid x \in z \circ y\}$.

Moreover, the product topology on $F(a) \times F(a)$ and the topology τ^* on $\mathcal{P}^*\big(F(a)\big)$ are considered.

In other words, (F, A, τ) is a soft topological hypergroup if $\big(F(a), \circ, \tau_a\big)$ is a topological hypergroup, for all $a \in \text{Supp}(F, A)$, where τ_a is the subspace topology on $F(a)$.

Clearly, if $(H, \circ, \tau)$ is a topological hypergroup and (F, A) is a soft hypergroup over H, then the triple (F, A, τ) is a soft topological hypergroup over H. In other words, the soft topological hypergroup (F, A, τ) is a parametrization of subhypergroups of topological hypergroup $(H, \circ, \tau)$.

To find the conditions that a soft hypergroup is a soft topological hypergroup, we need some concepts and propositions, which are discussed below.

Definition 3.8. Let $(F, A) \neq \tilde{\phi}_A$ be a soft hypergroup over a hypergroup $(H, \circ)$. A relation ρ on H is called *equivalence relation* on (F, A) if ρ is an equivalence relation on $F(a)$, for all $a \in \text{Supp}(F, A)$. An equivalence relation ρ on (F, A) is called *(strongly) regular* on soft hypergroup (F, A) if it is (strongly) regular on the subhypergroup $F(a)$, for all $a \in \text{Supp}(F, A)$.

Proposition 3.9. *Let $(F, A) \neq \tilde{\phi}_A$ be a soft hypergroup over a hypergroup $(H, \circ)$ and let ρ be a relation on H. If ρ is an equivalence relation (resp., (strongly) regular relation) on the hypergroup $(H, \circ)$. Then ρ is an equivalence relation (resp., (strongly) regular relation) on the soft hypergroup (F, A).*

Proof. It is straightforward. □

Clearly, the inverse of above proposition is not necessarily true, unless $F(a) = H$ for some $a \in A$.

Theorem 3.10. *Let $(F, A) \neq \tilde{\phi}_A$ be a soft hypergroup over a hypergroup $(H, \circ)$ and let ρ be an equivalence relation on (F, A). Then $\left(\frac{F}{\rho}, A\right)$ defined by $\frac{F}{\rho}(a) = \frac{F(a)}{\rho}$, for all $a \in \text{Supp}(F, A)$ is a soft hypergroup (resp., soft group) over $\left(\frac{H}{\rho}, \odot\right)$ if and only if ρ is a regular (resp., strongly regular) relation on H.*

Proof. Since ρ is regular (resp., strongly regular) if and only if $\left(\frac{H}{\rho}, \odot\right)$ is a hypergroup (resp., group), so Proposition 3.9 completes the proof. □

Example 3.11. Let $(F, A) \neq \tilde{\phi}_A$ be a soft hypergroup over a hypergroup $(H, \circ)$ and let ρ be a strongly regular relation on H. Then (F, A) is a soft geometric space over the geometric space $\big(H, \rho(H)\big)$. Clearly, the relation "$\sim$" coincides with the relation ρ.

Theorem 3.12. *Let* $(F, A) \neq \tilde{\phi}_A$ *be a soft hypergroup over a hypergroup* $(H, \circ)$ *with topology* τ *on* H*. If the soft geometric space* (F, A) *over* $\big(H, \rho(H)\big)$ *is* τ*-complete and* τ*-open for some strongly regular relation* ρ *on* H*, then* (F, A, τ) *is a soft topological complete hypergroup.*

Proof. Let $a \in \text{Supp}(F, A)$. Since the soft geometric space (F, A) over $\big(H, \rho(H)\big)$ is τ-complete and τ-open, the geometric space $\big(F(a), \rho\big(F(a)\big)\big)$ is τ_a-open and τ_a-complete, respectively. Hence by Theorem 2.12, $\big(F(a), \circ, \tau_a\big)$ is a topological complete hypergroup, which completes the proof. □

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