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The category of $\top$-filter spaces

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Abstract. $\top$-filters serve as an important tool to define mathematical structures and deserve more and more attention. This paper aims to investigate categorical properties of $\top$-filter spaces. Firstly, it is shown that the category $\top$-**Fil** of $\top$-fiter spaces is Cartesian closed, extensional and productive for quotient mappings. Secondly, the concepts of $\top$-semi-Cauchy spaces and complete $\top$-filter spaces are proposed. It is proved that the categories of $\top$-semi-Cauchy and $\top$-Cauchy spaces, as bireflective subcategories of $\top$-**Fil**, are Cartesian closed, and the category of complete $\top$-filter spaces, as a bicoreflective subcategory of $\top$-**Fil**, is strongly Cartesian closed and is isomorphic to that of symmetric Kent $\top$-convergence spaces.

1. Introduction

Filters play an important role in topology. Cartan [5] first used filters to investigate convergence. Later, Choquet [7] and Kowalsky [29] presented their theories which involve an axiomatization of the concept of convergence via filters. In this approach, Fischer [16] and Kent [27] further considered convergence structures. From the categorical aspect, Edgar [9] proved the category of convergence spaces is Cartesian closed. Combined with uniform structures, Weil [43] introduced the concept of uniform convergence structures. Afterwards, Cook and Fischer [6] redefined uniform convergence structures by modifying the axioms in the sense of Weil. Then Lechicki and Ziemińska [30] studied a general notion of a uniform convergence structure. In order to establish the relationship between convergence structures and uniform convergence structures, Bently [3] et al. formalized filter structures, which can be considered as a characterization of filter merotopic structures in the sense of Katetov [26]. Since then many scholars studied these structures [4, 28, 33, 41].

The above-mentioned mathematical structures are all defined via filters. These filter-based structures not only can be used to describe topology, but also have nice categorical properties, including Cartesian-closedness [2, 34], extensionality [8, 32] and productivity of quotient mappings [35, 39]. This topic has became an interesting research area known as Convenient Topology [40].

With the development of lattice-valued theory, filters have been generalized to the lattice-valued case, which leads to a representative type of lattice-valued filters, called stratified L-filters. Many scholars used stratified L-filters to define different types of lattice-valued mathematical structures. Jäger defined

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stratified L-generalized convergence structures [21] and L-uniform convergence structures [24], and studied their Cartesian-closedness as well as their relationships with stratified L-topology. Considering fuzzy inclusion orders between L-subsets, Fang [10] and Li [31] proposed L-ordered convergence structures and investigated their relationships with L-convergence structures. Fang [11] introduced stratified L-semiuniform convergence structures and L-ordered semiuniform convergence structures, and studied their Cartesian-closedness. Fang also proposed L-ordered quasiuniform limit structures [12] and stratified L-preuniform convergence structures [13], and presented their categorical properties. Yang and Li [44] proposed (L, M)-filter tower structures and studied their completion. Pang et al. introduced stratified L-filter structures [36], stratified L-ordered filter structures [37, 47] and stratified L-convergence tower structures [38], and investigated their categorical properties. Zhang et al. [49] used stratified L-filters to define (L, M)-semiuniform convergence tower structures and discussed its categorical relationships with (L, M)-filter tower structures. Up to now, lattice-valued mathematical structures via stratified L-filters have been extensively discussed.

Since $\top$-filters have some advantages compared with stratified L-filters, especially on the generalizations of lattice background, $\top$-filters are receiving increasing attention. Yu and Fang [45] first used $\top$-filters to define $\top$-convergence structures and studied the Cartesian-closedness of the resulting category. Afterwards, Fang and Yue discussed $\top$-diagonal conditions and continuous extension theorem in $\top$-convergence spaces [14] and constructed a $\top$-filter monad to study its applications in $\top$-convergence spaces [46]. Reid and Richardson [42] introduced $\top$-Cauchy structures and $\top$-uniform limit structures and investigated their completions. Recently, Jäger and Yue [25] studied $\top$-uniform structures in more detail. Zhang and Pang [48] proposed the concept of $\top$-convergence groups via combining a $\top$-convergence structure and a group, and investigated its characterization theorems. Motivated by lattice-valued structures via $\top$-filters, we will focus on lattice-valued filter structures via $\top$-filters, called $\top$-filter structures in this paper. Actually, it can be considered as generalizations of $\top$-Cauchy structures [42] and $\top$-quasi Cauchy structures [23].

As the first aim of our paper, we will explore the categorical properties of $\top$-filter spaces, including Cartesian-closedness, extensionality and productivity of quotient mappings. As the second aim, we will include $\top$-semi-Cauchy spaces, $\top$-Cauchy spaces and complete $\top$-filter spaces into the framework of $\top$-filter spaces from a categorical aspect, and also investigate their categorical properties.

2. Preliminaries

In this section, we recall some basic notations and concepts that will be needed in the sequel.

Definition 2.1. ([19]) A complete residuated lattice is a triple $(L, \leq, *)$, where $(L, \leq)$ is a complete lattice with the top element $\top$ and the bottom element $\bot$, and $*$ is a commutative, associative binary operation such that
(1) $\top$ is the unit element for $*$;
(2) $*$ is distributive over arbitrary joins, i.e., $(\bigvee_{i\in I} \alpha_i) * \beta = \bigvee_{i\in I}(\alpha_i * \beta)$.

For a given complete residuated lattice L, the binary operation $\rightarrow$ on L can be computed by

$$\alpha \rightarrow \beta = \bigvee \{\gamma \in L \,|\, \alpha * \gamma \leq \beta\}.$$

The binary operation $\rightarrow$ is called the implication operation on L with respect to $*$. Further, $*$ and $\rightarrow$ form an adjoint pair in the sense of $\alpha * \gamma \leq \beta \Longleftrightarrow \gamma \leq \alpha \rightarrow \beta$ for all $\alpha, \beta, \gamma \in L$. In this paper, we will often use a complete residuated lattice that satisfies the following distributive law

$$\text{(MID)} \quad \alpha \wedge \bigvee_{i\in I} \beta_i = \bigvee_{i\in I} (\alpha \wedge \beta_i) \quad \forall \alpha \in L, \{\beta_i\}_{i\in I} \subseteq L.$$

An L-subset of X is a mapping from X to L, and the family of all L-subsets on X will be denoted by L^X, called the L-power set of X. $\top_X$ represents the constant L-subset with the value $\top$ and $\bot_X$ represents the constant L-subset with the value $\bot$. For a universal set X, the set of all subsets of X is denoted by $\mathcal{P}(X)$.

All algebraic operations on L can be extended to the L-power set L^X in a pointwise way. For each A, $B \in L^X$, $\alpha \in L$ and $x \in X$,

(1) $(A \vee B)(x) = A(x) \vee B(x)$;
(2) $(A \wedge B)(x) = A(x) \wedge B(x)$;
(3) $(A * B)(x) = A(x) * B(x)$ and $(\alpha * A)(x) = \alpha * A(x)$;
(4) $(A \to B)(x) = A(x) \to B(x)$ and $(\alpha \to B)(x) = \alpha \to B(x)$.

Let $\varphi : X \longrightarrow Y$ be a mapping. Define $\varphi^{\to} : L^X \longrightarrow L^Y$ and $\varphi^{\leftarrow} : L^Y \longrightarrow L^X$ by $\varphi^{\to}(A)(y) = \bigvee_{\varphi(x)=y} A(x)$ for all $A \in L^X$ and $y \in Y$, and $\varphi^{\leftarrow}(B)(x) = B(\varphi(x))$ for all $B \in L^Y$ and $x \in X$.

For a given set X, there is a binary mapping $\mathcal{S}_X(-,-) : L^X \times L^X \longrightarrow L$, defined by

$$\mathcal{S}_X(A, B) = \bigwedge_{x \in X} (A(x) \to B(x))$$

for any pair $(A, B) \in L^X \times L^X$. $\mathcal{S}_X(A, B)$ can be interpreted as the degree of A being a subset of B. $\mathcal{S}_X(-,-)$ is also called the fuzzy inclusion order between L-subsets.

Lemma 2.2. ([1],[25]) For each $A, B, C, D \in L^X$, it holds that

(1) $A \leq B \Longleftrightarrow \mathcal{S}_X(A, B) = \top$;
(2) $\mathcal{S}_X(A, B) * \mathcal{S}_X(B, C) \leq \mathcal{S}_X(A, C)$;
(3) $\mathcal{S}_X(A, B) * \mathcal{S}_X(C, D) \leq \mathcal{S}_X(A * C, B * D)$;
(4) $\mathcal{S}_X(A, B) * \mathcal{S}_X(C, D) \leq \mathcal{S}_X(A \wedge C, B \wedge D)$;
(5) $\mathcal{S}_X(A, B) \wedge \mathcal{S}_X(C, D) \leq \mathcal{S}_X(A \wedge C, B \wedge D)$;
(6) $\mathcal{S}_X(A, B) \wedge \mathcal{S}_X(C, D) \leq \mathcal{S}_X(A \vee C, B \vee D)$;
(7) $A \leq B$ implies $\mathcal{S}_X(C, A) \leq \mathcal{S}_X(C, B)$ and $\mathcal{S}_X(B, D) \leq \mathcal{S}_X(A, D)$.

Lemma 2.3. ([1]) *Let $\varphi : X \longrightarrow Y$ be a mapping. For each $A, B \in L^X$ and $C, D \in L^Y$, it holds that*

(1) $\mathcal{S}_X(A, B) \leq \mathcal{S}_Y(\varphi^{\to}(A), \varphi^{\to}(B))$;
(2) $\mathcal{S}_Y(C, D) \leq \mathcal{S}_X(\varphi^{\leftarrow}(C), \varphi^{\leftarrow}(D))$;
(3) $\mathcal{S}_Y(\varphi^{\to}(A), C) = \mathcal{S}_X(A, \varphi^{\leftarrow}(C))$.

The notion of a $\top$-filter and that of a $\top$-filter base are due to Höhle [20]. A particular version which follows here is due to Fang and Yue [14].

Definition 2.4. ([14, 20]) A $\top$-filter on X is a nonempty subset $\mathbb{F} \subseteq L^X$ with the following properties:

($\mathbb{F}$1) if $A \in L^X$ with $\bigvee_{C \in \mathbb{F}} \mathcal{S}_X(C, A) = \top$, then $A \in \mathbb{F}$;
($\mathbb{F}$2) $A_1 \wedge A_2 \in \mathbb{F}$ for all $A_1, A_2 \in \mathbb{F}$;
($\mathbb{F}$3) $\bigvee_{x \in X} A(x) = \top$ for all $A \in \mathbb{F}$.

The family of all $\top$-filters on X is denoted by $\mathcal{F}_L^{\top}(X)$. Given a point $x \in X$, then $[x] = \{A \in L^X \,|\, A(x) = \top\}$ is a $\top$-filter, and called the point $\top$-filter of x.

Definition 2.5. ([14, 20]) A nonempty subset $\mathbb{B} \subseteq L^X$ is called a $\top$-filter base on X if it satisfies:

($\mathbb{B}$1) $\bigvee_{B \in \mathbb{B}} \mathcal{S}_X(B, C \wedge D) = \top$ for all $C, D \in \mathbb{B}$;
($\mathbb{B}$2) $\bigvee_{x \in X} C(x) = \top$ for all $C \in \mathbb{B}$.

It is obvious that each $\top$-filter is a $\top$-filter base. For a $\top$-filter base $\mathbb{B}$, a $\top$-filter can be generated in the following way:

$$\mathbb{F}_{\mathbb{B}} = \left\{ A \in L^X \,\Big|\, \bigvee_{B \in \mathbb{B}} \mathcal{S}_X(B, A) = \top \right\}.$$

Then $\mathbb{B}$ is called a base of $\mathbb{F}_{\mathbb{B}}$.

Proposition 2.6. ([48]) *Let* $\mathbb{F}, \mathbb{G} \in \mathcal{F}_L^{\top}(X)$ *and* $\mathbb{B}_{\mathbb{F}}, \mathbb{B}_{\mathbb{G}}$ *be a* $\top$*-filter base of* $\mathbb{F}, \mathbb{G}$*. Then* $\{A \vee B \in L^X \,|\, A \in \mathbb{B}_{\mathbb{F}}, B \in \mathbb{B}_{\mathbb{G}}\}$ *and* $\{A \vee B \in L^X \,|\, A \in \mathbb{F}, B \in \mathbb{G}\}$ *are both* $\top$*-filter bases of* $\mathbb{F} \cap \mathbb{G}$*.*

Take any $A \in L^X$ and $B \in L^Y$. Then $A \times B \in L^{X\times Y}$ is defined by $(A \times B)(x, y) = A(x) \wedge B(y)$.

Definition 2.7. ([45]) Let $\mathbb{F} \in \mathcal{F}_L^{\top}(X)$ and $\mathbb{G} \in \mathcal{F}_L^{\top}(Y)$. Then

$$\mathbb{F} \times \mathbb{G} = \left\{ D \in L^{X\times Y} \,\Big|\, \bigvee_{A\in\mathbb{F}, B\in\mathbb{G}} \mathcal{S}_{X\times Y}(A \times B, D) = \top \right\}$$

is a $\top$-filter on $X \times Y$, which is called the product of $\mathbb{F}$ and $\mathbb{G}$.

Definition 2.8. ([14]) Let $\varphi : X \longrightarrow Y$ be a mapping, $\mathbb{F} \in \mathcal{F}_L^{\top}(X)$ and $\mathbb{G} \in \mathcal{F}_L^{\top}(Y)$.

(1) The set $\{\varphi^{\rightarrow}(A) \in L^Y \,|\, A \in \mathbb{F}\}$ is a $\top$-filter base on Y and its generated $\top$-filter is denoted by $\varphi^{\Rightarrow}(\mathbb{F})$. That is

$$\varphi^{\Rightarrow}(\mathbb{F}) = \left\{ B \in L^Y \,\Big|\, \bigvee_{A\in\mathbb{F}} \mathcal{S}_Y(\varphi^{\rightarrow}(A), B) = \top \right\}.$$

Then $\varphi^{\Rightarrow}(\mathbb{F})$ is called the image of $\mathbb{F}$ under φ. Obviously, $B \in \varphi^{\Rightarrow}(\mathbb{F})$ iff $\varphi^{\leftarrow}(B) \in \mathbb{F}$.

(2) The set $\{\varphi^{\leftarrow}(B) \in L^X \,|\, B \in \mathbb{G}\}$ is a $\top$-filter base on X when $\bigvee_{y\in\varphi(X)} B(y) = \top$ holds for all $B \in \mathbb{G}$. If

$$\varphi^{\Leftarrow}(\mathbb{G}) = \left\{ A \in L^X \,\Big|\, \bigvee_{B\in\mathbb{G}} \mathcal{S}_X(\varphi^{\leftarrow}(B), A) = \top \right\}$$

is a $\top$-filter on X, then $\varphi^{\Leftarrow}(\mathbb{G})$ is called the inverse image of $\mathbb{G}$ under φ.

Proposition 2.9. ([45]) *Let* $\varphi : X \longrightarrow Y$ *be a mapping and* $\mathbb{F}, \mathbb{G} \in \mathcal{F}_L^{\top}(X)$*,* $\mathbb{H} \in \mathcal{F}_L^{\top}(Y)$*. Then*

(1) $\varphi^{\Rightarrow}(\mathbb{F} \cap \mathbb{G}) = \varphi^{\Rightarrow}(\mathbb{F}) \cap \varphi^{\Rightarrow}(\mathbb{G})$;

(2) $\varphi^{\Leftarrow}(\varphi^{\Rightarrow}(\mathbb{F})) \subseteq \mathbb{F}$*, if* φ *is injective, then* $\varphi^{\Leftarrow}(\varphi^{\Rightarrow}(\mathbb{F})) = \mathbb{F}$;

(3) $\mathbb{H} \subseteq \varphi^{\Rightarrow}(\varphi^{\Leftarrow}(\mathbb{H}))$ *when* $\varphi^{\Leftarrow}(\mathbb{H})$ *exists, if* φ *is surjective, then* $\mathbb{H} = \varphi^{\Rightarrow}(\varphi^{\Leftarrow}(\mathbb{H}))$.

Proposition 2.10. ([45],[48]) *Let* $\varphi : X \longrightarrow U$ *and* $\psi : Y \longrightarrow V$ *be mappings,* $pr_X : X\times Y \longrightarrow X$*,* $pr_Y : X\times Y \longrightarrow Y$ *be projection mappings and* $\mathbb{F} \in \mathcal{F}_L^{\top}(X)$*,* $\mathbb{G} \in \mathcal{F}_L^{\top}(Y)$*,* $\mathbb{K} \in \mathcal{F}_L^{\top}(X \times Y)$*. Then*

(1) $\varphi^{\Rightarrow}(\mathbb{F}) \times \psi^{\Rightarrow}(\mathbb{G}) \subseteq (\varphi \times \psi)^{\Rightarrow}(\mathbb{F} \times \mathbb{G})$*, if* L *satisfies* (MID)*, then* $\varphi^{\Rightarrow}(\mathbb{F}) \times \psi^{\Rightarrow}(\mathbb{G}) = (\varphi \times \psi)^{\Rightarrow}(\mathbb{F} \times \mathbb{G})$;

(2) $pr_X^{\Rightarrow}(\mathbb{F} \times \mathbb{G}) = \mathbb{F}$, $pr_Y^{\Rightarrow}(\mathbb{F} \times \mathbb{G}) = \mathbb{G}$;

(3) $pr_X^{\Rightarrow}(\mathbb{K}) \times pr_Y^{\Rightarrow}(\mathbb{K}) \subseteq \mathbb{K}$.

For other notions on residuated lattices we refer to Bělohlávek [1]; for other notions on $\top$-filters we refer to Höhle [19] and Yu and Fang [45]; for category theory we refer to Preuss [40].

3. $\top$-filter spaces

In this section, we will introduce the concept of $\top$-filter spaces and present its product space, subspace and quotient space from the aspect of the resulting category.

Definition 3.1. A nonempty subset γ of $\mathcal{F}_L^{\top}(X)$ is called a $\top$-filter structure on X provided that

(TF1) $\forall x \in X$, $[x] \in \gamma$;

(TF2) $\forall \mathbb{F}, \mathbb{G} \in \mathcal{F}_L^{\top}(X)$, $\mathbb{F} \in \gamma$ and $\mathbb{F} \subseteq \mathbb{G}$ imply $\mathbb{G} \in \gamma$.

For a $\top$-filter structure γ on X, the pair (X, γ) is called a $\top$-filter space.

A mapping $\varphi : (X, \gamma_X) \longrightarrow (Y, \gamma_Y)$ between $\top$-filter spaces is called Cauchy continuous provided that $\mathbb{F} \in \gamma_X$ implies $\varphi^{\Rightarrow}(\mathbb{F}) \in \gamma_Y$ for all $\mathbb{F} \in \mathcal{F}_L^{\top}(X)$.

It is easy to check that all $\top$-filter spaces and Cauchy continuous mappings form a category, denoted by $\top$-**Fil**.

Theorem 3.2. $\top$-**Fil** *is a topological category over* **Set**.

Proof. Given a source $\left\{\varphi_j : X \longrightarrow (X_j, \gamma_{X_j})\right\}_{j\in J}$ in $\top$-**Fil**, define $\gamma_X \subseteq \mathcal{F}_L^{\top}(X)$ by

$$\gamma_X = \left\{\mathbb{F} \in \mathcal{F}_L^{\top}(X) \,|\, \forall j \in J, \varphi_j^{\Rightarrow}(\mathbb{F}) \in \gamma_{X_j}\right\}.$$

It is straightforward to verify that γ_X is the initial structure with respect to the source $\{\varphi_j : X \longrightarrow (X_j, \gamma_{X_j})\}_{j\in J}$. Further, it is easy to show the fiber-smallness and terminal separator property. □

By choosing special sources in $\top$-**Fil**, the product space and the subspace of $\top$-filter spaces in $\top$-**Fil** can be defined in a natural way.

Definition 3.3. Let $\{(X_\lambda, \gamma_{X_\lambda})\}_{\lambda\in\Lambda}$ be a family of $\top$-filter spaces and $\{pr_\lambda : \prod_{\mu\in\Lambda} X_\mu \longrightarrow X_\lambda\}_{\lambda\in\Lambda}$ be the family of the projection mappings. Then the initial structure with respect to the source $\{pr_\lambda : \prod_{\mu\in\Lambda} X_\mu \longrightarrow (X_\lambda, \gamma_\lambda)\}_{\lambda\in\Lambda}$ is called the product $\top$-filter structure, denoted by $\prod_{\lambda\in\Lambda} \gamma_{X_\lambda}$. The pair $\left(\prod_{\lambda\in\Lambda} X_\lambda, \prod_{\lambda\in\Lambda} \gamma_{X_\lambda}\right)$ is called the product space of $\{(X_\lambda, \gamma_{X_\lambda})\}_{\lambda\in\Lambda}$. Explicitly,

$$\prod_{\lambda\in\Lambda} \gamma_{X_\lambda} = \left\{\mathbb{H} \in \mathcal{F}_L^{\top}\left(\prod_{\lambda\in\Lambda} X_\lambda\right) \middle|\, \forall \lambda \in \Lambda, pr_\lambda^{\Rightarrow}(\mathbb{H}) \in \gamma_{X_\lambda}\right\}.$$

Definition 3.4. Let (X, γ) be a $\top$-filter space, $Y \subseteq X$ and $i_Y : Y \longrightarrow X$ be the inclusion mapping. Then the initial structure with respect to the source $i_Y : Y \longrightarrow (X, \gamma)$ is called the sub-$\top$-filter structure, denoted by $\gamma|_Y$. The pair $(Y, \gamma|_Y)$ is called the subspace of (X, γ). Explicitly,

$$\gamma|_Y = \{\mathbb{F} \in \mathcal{F}_L^{\top}(Y) \,|\, i_Y^{\Rightarrow}(\mathbb{F}) \in \gamma_X\}.$$

Since $\top$-**Fil** is a topological category over **Set**, there exists a final structure with respect to any sink $\{\varphi_j : (X_j, \gamma_{X_j}) \longrightarrow X\}_{j\in J}$. Now let us explore the concrete form of the final structure.

Proposition 3.5. *Let* $\{(X_j, \gamma_{X_j})\}_{j\in J}$ *be a family of* $\top$*-filter spaces and* $\{\varphi_j : X_j \longrightarrow X\}_{j\in J}$ *be a family of mappings. Then* $\gamma_X \subseteq \mathcal{F}_L^{\top}(X)$ *defined by*

$$\gamma_X = \left\{\mathbb{H} \in \mathcal{F}_L^{\top}(X) \,|\, \exists j \in J \textit{ and } \exists \mathbb{F}_j \in \gamma_{X_j} \textit{ such that } \varphi_j^{\Rightarrow}(\mathbb{F}_j) \subseteq \mathbb{H}\right\} \cup \left\{[x] \in \mathcal{F}_L^{\top}(X) \,|\, x \in X\right\}$$

is the final structure with respect to the sink $\left\{\varphi_j : (X_j, \gamma_{X_j}) \longrightarrow X\right\}_{j\in J}$*. In addition, if the sink* $\{\varphi_j : (X_j, \gamma_{X_j}) \longrightarrow X\}_{j\in J}$ *is surjective* (i.e., $X = \bigcup_{j\in J} \varphi_j(X_j)$)*, then it holds that*

$$\gamma_X = \left\{\mathbb{H} \in \mathcal{F}_L^{\top}(X) \,|\, \exists j \in J \textit{ and } \exists \mathbb{F}_j \in \gamma_{X_j} \textit{ such that } \varphi_j^{\Rightarrow}(\mathbb{F}_j) \subseteq \mathbb{H}\right\}.$$

Proof. First, we show that γ_X satisfies (TF1) and (TF2). (TF1) is straightforward.

(TF2) Let $\mathbb{F} \in \gamma_X$ and $\mathbb{F} \subseteq \mathbb{G}$. If $\mathbb{F} = [x]$ for some $x \in X$, then $[x] = \mathbb{G}$ since $[x]$ is maximal. This implies that $\mathbb{G} \in \gamma_X$. If $\mathbb{F} \neq [x]$ for all $x \in X$, then there exists some $j \in J$ and some $\mathbb{F}_j \in \gamma_{X_j}$ such that $\varphi_j^{\Rightarrow}(\mathbb{F}_j) \subseteq \mathbb{F}$. This implies that $\varphi_j^{\Rightarrow}(\mathbb{F}_j) \subseteq \mathbb{G}$. By the definition of γ_X, we obtain $\mathbb{G} \in \gamma_X$.

Next, it suffices to verify that γ_X is the final structure on X such that for each (Y, γ_Y) in $\top$-**Fil** and for each mapping $\varphi : X \longrightarrow Y$, the mapping $\varphi : (X, \gamma_X) \longrightarrow (Y, \gamma_Y)$ is Cauchy continuous if and only if the mapping $\varphi \circ \varphi_j : (X_j, \gamma_{X_j}) \longrightarrow (Y, \gamma_Y)$ is Cauchy continuous for each $j \in J$. The necessity is obvious. For the sufficiency, take any $\mathbb{F} \in \gamma_X$. If $\mathbb{F} = [x]$ for some $x \in X$, then $\varphi^{\Rightarrow}(\mathbb{F}) = [\varphi(x)] \in \gamma_Y$. If $\mathbb{F} \neq [x]$ for any $x \in X$, then there exists some $j \in J$ and some $\mathbb{F}_j \in \gamma_{X_j}$ such that $\varphi_j^{\Rightarrow}(\mathbb{F}_j) \subseteq \mathbb{F}$. By the Cauchy continuity of $\varphi \circ \varphi_j$, we have $\varphi^{\Rightarrow} \circ \varphi_j^{\Rightarrow}(\mathbb{F}_j) \in \gamma_Y$. Since $\varphi^{\Rightarrow} \circ \varphi_j^{\Rightarrow}(\mathbb{F}_j) \subseteq \varphi^{\Rightarrow}(\mathbb{F})$, we get $\varphi^{\Rightarrow}(\mathbb{F}) \in \gamma_Y$.

If the sink $\{\varphi_j : (X_j, \gamma_{X_j}) \longrightarrow X\}_{j\in J}$ is surjective, i.e., $X = \bigcup_{j\in J} \varphi_j(X_j)$, then there exists some $j \in J$ and some $x_j \in X_j$ such that $\varphi_j(x_j) = x$ for any $x \in X$. Thus, there exists $j \in J$ and $[x_j] \in \gamma_{X_j}$ such that $\varphi_j^{\Rightarrow}([x_j]) = [x]$. Then it follows that

$$\left\{[x] \in \mathcal{F}_L^{\top}(X) \,|\, x \in X\right\} \subseteq \left\{\mathbb{H} \in \mathcal{F}_L^{\top}(X) \,|\, \exists j \in J \text{ and } \exists \mathbb{F}_j \in \gamma_{X_j} \text{ such that } \varphi_j^{\Rightarrow}(\mathbb{F}_j) \subseteq \mathbb{H}\right\}.$$

This implies that

$$\gamma_X = \left\{\mathbb{H} \in \mathcal{F}_L^{\top}(X) \,|\, \exists j \in J \text{ and } \exists \mathbb{F}_j \in \gamma_{X_j} \text{ such that } \varphi_j^{\Rightarrow}(\mathbb{F}_j) \subseteq \mathbb{H}\right\}.$$

□

As a special final structure in $\top$-**Fil**, a quotient structure of a $\top$-filter space is defined as follows.

Definition 3.6. Let (X, γ_X) be $\top$-filter space, Y be a nonempty set and $\varphi : X \longrightarrow Y$ be a surjective mapping. The final structure on Y with respect to the sink $\varphi : (X, \gamma_X) \longrightarrow Y$ is called a quotient structure on Y, denoted by γ_Y. Explicitly,

$$\gamma_Y = \{\mathbb{G} \in \mathcal{F}_L^{\top}(Y) \,|\, \exists \mathbb{F} \in \gamma_X \text{ such that } \varphi^{\Rightarrow}(\mathbb{F}) \subseteq \mathbb{G}\}.$$

The pair (Y, γ_Y) called a quotient space of (X, γ_X). In this sense, φ is called a quotient mapping.

4. Convenient properties of $\top$-Fil

Preuss [40] proposed some convenient properties for a topological category $\mathscr{C}$, namely
(CP1) $\mathscr{C}$ is Cartesian closed.
(CP2) $\mathscr{C}$ is extensional.
(CP3) The product of quotient mappings in $\mathscr{C}$ is a quotient mapping.

According to the terminology of [40], a topological category $\mathscr{C}$ is called
(1) strongly Cartesian closed provided that $\mathscr{C}$ fulfills (CP1) and (CP3);
(2) a topological universe provided that $\mathscr{C}$ fulfills (CP1) and (CP2);
(3) a strong topological universe provided that $\mathscr{C}$ fulfills (CP1)-(CP3).

In this section, we will show that $\top$-**Fil** is a strong topological universe.

4.1. Cartesian-closedness of $\top$-**Fil**

Recall that a category $\mathscr{C}$ is called Cartesian closed provided that the following conditions are satisfied:

(1) For each pair (Y, Z) of $\mathscr{C}$-objects, there exists a product $Y \times Z$ in $\mathscr{C}$.

(2) For each pair (Y, Z) of $\mathscr{C}$-objects, there exists a $\mathscr{C}$-object Z^Y (called power object) and a $\mathscr{C}$-morphism $ev : Z^Y \times Y \longrightarrow Z$ (called evaluation morphism) such that for each $\mathscr{C}$-object X and each $\mathscr{C}$-morphism $\varphi : X \times Y \longrightarrow Z$, there exists a unique $\mathscr{C}$-morphism $\varphi^* : X \longrightarrow Z^Y$ such that $ev \circ (\varphi^* \times id_Y) = \varphi$.

Since $\top$-**Fil** is a topological category, it remains to show that $\top$-**Fil** satisfies (2). For each $\top$-filter space, we denote the set of Cauchy continuous mappings from (X, γ_X) to (Y, γ_Y) by $[X, Y]$, i.e.,

$$[X, Y] = \{\varphi : (X, \gamma_X) \longrightarrow (Y, \gamma_Y) \,|\, \varphi \text{ is Cauchy continuous}\}.$$

Define $\top_\varphi \in L^{[X,Y]}$ by $\top_\varphi(\phi) = \top$ when $\phi = \varphi$ and $\top_\varphi(\phi) = \bot$ otherwise.

Proposition 4.1. *Let (X, γ_X) and (Y, γ_Y) be $\top$-filter spaces. Define $\gamma_{[X,Y]} \subseteq \mathcal{F}_L^{\top}([X, Y])$ by*

$$\gamma_{[X,Y]} = \left\{\mathbb{H} \in \mathcal{F}_L^{\top}([X, Y]) \,|\, \forall \mathbb{F} \in \mathcal{F}_L^{\top}(X), \mathbb{F} \in \gamma_X \text{ implies } ev^{\Rightarrow}(\mathbb{H} \times \mathbb{F}) \in \gamma_Y\right\}.$$

Then $\gamma_{[X,Y]}$ is a $\top$-filter structure on $[X, Y]$.

Proof. It suffices to verify that $\gamma_{[X,Y]}$ satisfies (TF1) and (TF2). (TF2) is straightforward.

(TF1) Take any $\varphi \in [X, Y]$ and $\mathbb{F} \in \gamma_X$. Then $\varphi^{\Rightarrow}(\mathbb{F}) \in \gamma_Y$. For each $B \in \varphi^{\Rightarrow}(\mathbb{F})$, $\phi \in [X, Y]$ and $x \in X$, it follows that

$$(\top_\varphi \times \varphi^{\leftarrow}(B))(\phi, x) = \top_\varphi(\phi) \wedge \varphi^{\leftarrow}(B)(x) \leqslant B(\phi(x)) = ev^{\leftarrow}(B)(\phi, x),$$

which means that $\top_\varphi \times \varphi^{\leftarrow}(B) \leqslant ev^{\leftarrow}(B)$. Since $\top_\varphi \times \varphi^{\leftarrow}(B) \in [\varphi] \times \mathbb{F}$, we know $ev^{\leftarrow}(B) \in [\varphi] \times \mathbb{F}$, i.e., $B \in ev^{\Rightarrow}([\varphi] \times \mathbb{F})$. By the arbitrariness of B, we obtain $\varphi^{\Rightarrow}(\mathbb{F}) \subseteq ev^{\Rightarrow}([\varphi] \times \mathbb{F})$. Then it follows from (TF2) that $ev^{\Rightarrow}([\varphi] \times \mathbb{F}) \in \gamma_Y$. This shows $[\varphi] \in \gamma_{[X,Y]}$. □

Proposition 4.2. *Let (X, γ_X) and (Y, γ_Y) be $\top$-filter spaces. Then the evaluation mapping $ev : ([X, Y], \gamma_{[X,Y]}) \times (X, \gamma_X) \longrightarrow (Y, \gamma_Y)$ is Cauchy continuous.*

Proof. Take any $\mathbb{K} \in \gamma_{[X,Y]} \times \gamma_X$. Then it follows from Definition 3.3 that $pr^{\Rightarrow}_{[X,Y]}(\mathbb{K}) \in \gamma_{[X,Y]}$ and $pr^{\Rightarrow}_X(\mathbb{K}) \in \gamma_X$. By Proposition 4.1, we have $\mathbb{F} \in \gamma_X$ implies $ev^{\Rightarrow}(pr^{\Rightarrow}_{[X,Y]}(\mathbb{K}) \times \mathbb{F}) \in \gamma_Y$ for all $\mathbb{F} \in \mathcal{F}^{\top}_L(X)$. Then we get $ev^{\Rightarrow}(pr^{\Rightarrow}_{[X,Y]}(\mathbb{K}) \times pr^{\Rightarrow}_X(\mathbb{K})) \in \gamma_Y$. By Proposition 2.10, it follows that $pr^{\Rightarrow}_{[X,Y]}(\mathbb{K}) \times pr^{\Rightarrow}_X(\mathbb{K}) \subseteq \mathbb{K}$. Thus, we obtain $ev^{\Rightarrow}(\mathbb{K}) \in \gamma_Y$. □

Let $\varphi : X_1 \times X_2 \longrightarrow X_3$ be a mapping. For each $x_1 \in X_1$, define a mapping $\varphi_{x_1} : X_2 \longrightarrow X_3$ by $\varphi_{x_1}(x_2) = \varphi(x_1, x_2)$ for all $x_2 \in X_2$.

Proposition 4.3. *Let (X_1, γ_{X_1}), (X_2, γ_{X_2}) and (X_3, γ_{X_3}) be $\top$-filter spaces. If $\varphi : (X_1, \gamma_{X_1}) \times (X_2, \gamma_{X_2}) \longrightarrow (X_3, \gamma_{X_3})$ is Cauchy continuous, then $\varphi_{x_1} : (X_2, \gamma_{X_2}) \longrightarrow (X_3, \gamma_{X_3})$ is Cauchy continuous for all $x_1 \in X_1$.*

Proof. It suffices to show that $\mathbb{F} \in \gamma_{X_2}$ implies $\varphi^{\Rightarrow}_{x_1}(\mathbb{F}) \in \gamma_{X_3}$. By the Cauchy continuity of φ, we know $\varphi^{\Rightarrow}([x_1] \times \mathbb{F}) \in \gamma_{X_3}$ since $[x_1] \times \mathbb{F} \in \gamma_{X_1} \times \gamma_{X_2}$. Take any $C \in \varphi^{\Rightarrow}([x_1] \times \mathbb{F})$, i.e., $\varphi^{\leftarrow}(C) \in [x_1] \times \mathbb{F}$. Then it follows that

$$\bigvee_{A\in[x_1],B\in\mathbb{F}} \mathcal{S}_{X_3}(\varphi^{\rightarrow}(A \times B), C) = \bigvee_{A\in[x_1],B\in\mathbb{F}} \mathcal{S}_{X_1\times X_2}(A \times B, \varphi^{\leftarrow}(C)) = \top.$$

For each $x_3 \in X_3$, $A \in [x_1]$ and $B \in \mathbb{F}$, we have

$$\varphi^{\rightarrow}_{x_1}(B)(x_3) = \bigvee_{\varphi_{x_1}(x_2)=x_3} B(x_2) = \bigvee_{\varphi(x_1,x_2)=x_3} A(x_1) \wedge B(x_2) \leqslant \bigvee_{\varphi(u,v)=x_3} A(u) \wedge B(v) = \varphi^{\rightarrow}(A \times B)(x_3).$$

This implies that $\varphi^{\rightarrow}_{x_1}(B) \leqslant \varphi^{\rightarrow}(A \times B)$. Then it follows that

$$\begin{aligned}\top &= \bigvee_{A\in[x_1],B\in\mathbb{F}} \mathcal{S}_{X_3}(\varphi^{\rightarrow}(A \times B), C)\\ &\leqslant \bigvee_{A\in[x_1],B\in\mathbb{F}} \mathcal{S}_{X_3}(\varphi^{\rightarrow}_{x_1}(B), C)\\ &= \bigvee_{B\in\mathbb{F}} \mathcal{S}_{X_3}(\varphi^{\rightarrow}_{x_1}(B), C),\end{aligned}$$

which implies that $C \in \varphi^{\Rightarrow}_{x_1}(\mathbb{F})$. By the arbitrariness of C, we have $\varphi^{\Rightarrow}([x_1] \times \mathbb{F}) \subseteq \varphi^{\Rightarrow}_{x_1}(\mathbb{F})$. Then it follows from (TF2) that $\varphi^{\Rightarrow}_{x_1}(\mathbb{F}) \in \gamma_{X_3}$. □

By Proposition 4.3, we can define a mapping $\varphi^* : X_1 \longrightarrow [X_2, X_3]$ by $\varphi^*(x_1) = \varphi_{x_1}$ for all $x_1 \in X_1$.

Proposition 4.4. *Suppose that L satisfies* (MID). *Let (X_1, γ_{X_1}), (X_2, γ_{X_2}) and (X_3, γ_{X_3}) be $\top$-filter spaces. If $\varphi : (X_1, \gamma_{X_1}) \times (X_2, \gamma_{X_2}) \longrightarrow (X_3, \gamma_{X_3})$ is Cauchy continuous, then $\varphi^* : (X_1, \gamma_{X_1}) \longrightarrow ([X_2, X_3], \gamma_{[X_2,X_3]})$ is Cauchy continuous.*

Proof. Take any $\mathbb{F} \in \gamma_{X_1}$. For each $\mathbb{G} \in \gamma_{X_2}$, we have $\varphi^{\Rightarrow}(\mathbb{F} \times \mathbb{G}) \in \gamma_{X_3}$. Since

$$ev \circ (\varphi^* \times id_{X_2})(x_1, x_2) = ev(\varphi_{x_1}, x_2) = \varphi_{x_1}(x_2) = \varphi(x_1, x_2),$$

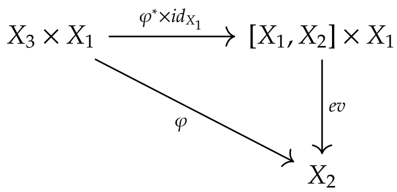
we get $(ev \circ (\varphi^* \times id_{X_2}))^{\Rightarrow}(\mathbb{F} \times \mathbb{G}) = \varphi^{\Rightarrow}(\mathbb{F} \times \mathbb{G})$. By Proposition 2.10, it follows that

$$\begin{aligned} ev^{\Rightarrow}((\varphi^*)^{\Rightarrow}(\mathbb{F}) \times \mathbb{G}) &= ev^{\Rightarrow}((\varphi^* \times id_{X_2})^{\Rightarrow}(\mathbb{F} \times \mathbb{G})) \\ &= (ev \circ (\varphi^* \times id_{X_2}))^{\Rightarrow}(\mathbb{F} \times \mathbb{G}) \\ &= \varphi^{\Rightarrow}(\mathbb{F} \times \mathbb{G}) \in \gamma_{X_3}. \end{aligned}$$

By the definition of $\gamma_{[X_2,X_3]}$, we get $(\varphi^*)^{\Rightarrow}(\mathbb{F}) \in \gamma_{[X_2,X_3]}$. □

Theorem 4.5. *Suppose that L satisfies* (MID). *Then the category* ⊤-**Fil** *is Cartesian closed.*

Proof. Let (X_1, γ_{X_1}) and (X_2, γ_{X_2}) be ⊤-filter spaces. By Propositions 4.1 and 4.2, there exists a ⊤-filter space $([X_1, X_2], \gamma_{[X_1,X_2]})$ and a Cauchy continuous evaluation mapping $ev : ([X_1, X_2], \gamma_{[X_1,X_2]}) \times (X_1, \gamma_{X_1}) \longrightarrow (X_2, \gamma_{X_2})$. Further, for each ⊤-filter space (X_3, γ_{X_3}) and Cauchy continuous mapping $\varphi : (X_3 \times X_1, \gamma_{X_3} \times \gamma_{X_1}) \longrightarrow (X_2, \gamma_{X_2})$, by Proposition 4.4, there exists a unique Cauchy continuous mapping $\varphi^* : (X_3, \gamma_{X_3}) \longrightarrow ([X_1, X_2], \gamma_{[X_1,X_2]})$ satisfying $ev \circ (\varphi^* \times id_{X_1}) = \varphi$, i.e., the triangle

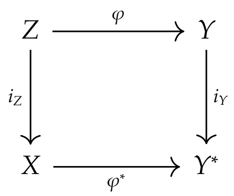


commutes. This shows the Cartesian-closedness of ⊤-**Fil**. □

4.2. Extensionality of ⊤-**Fil**

For convenience, suppose that X is a nonempty set and $\infty_X \notin X$. Put $X^* = X \cup \{\infty_X\}$ and $i_X : X \longrightarrow X^*$ be the embedding mapping. Define $\top_{\infty_X} : X^* \longrightarrow L$ by $\top_{\infty_X}(x^*) = \top$ whenever $x^* = \infty_X$, and $\top_{\infty_X}(x^*) = \bot$ otherwise.

Recall that in a topological category $\mathscr{C}$, a partial morphism from X to Y is a $\mathscr{C}$-morphism $\varphi : Z \longrightarrow Y$ whose domain is a subobject of X. A topological category $\mathscr{C}$ is called extensional provided that every $\mathscr{C}$-object Y has a *one-point extension* Y^*, in the sense that every $\mathscr{C}$-object Y can be embedded via the addition of a single point ∞_Y into a $\mathscr{C}$-object Y^* such that for every partial morphism $\varphi : Z \longrightarrow Y$, the mapping $\varphi^* : X \longrightarrow Y^*$ defined by $\varphi^*(x) = \varphi(x)$ whenever $x \in Z$, and $\varphi^*(x) = \infty_Y$ whenever $x \notin Z$, is a $\mathscr{C}$-morphsim and the following diagram



commutes.

Proposition 4.6. ([15]) *Let* $\mathbb{F} \in \mathcal{F}_L^{\top}(X)$ *and* $\mathbb{F}^* = i_X^{\Rightarrow}(\mathbb{F}) \cap [\infty_X]$. *Then* $i_X^{\Leftarrow}(\mathbb{F}^*) = \mathbb{F}$.

Proposition 4.7. *Let* (X, γ_X) *be a* ⊤*-filter space. Define* $\gamma_{X^*} \subseteq \mathcal{F}_L^{\top}(X^*)$ *by*

$$\gamma_{X^*} = \left\{\mathbb{F} \in \mathcal{F}_L^{\top}(X^*) \,|\, i_X^{\Leftarrow}(\mathbb{F}) \text{ exists and } i_X^{\Leftarrow}(\mathbb{F}) \in \gamma_X\right\} \cup \left\{\mathbb{F} \in \mathcal{F}_L^{\top}(X^*) \,|\, i_X^{\Leftarrow}(\mathbb{F}) \text{ does not exist}\right\}.$$

Then (X^*, γ_{X^*}) *is a* ⊤*-filter space.*

Proof. It suffices to verify that γ_{X^*} satisfies (TF1) and (TF2).

(TF1) For each $x \in X^*$, if $x \in X$, then $i_X^{\Leftarrow}([x])$ exists and $i_X^{\Leftarrow}([x]) = [x] \in \gamma_X$. If $x = \infty_X$, then $i_X^{\Leftarrow}([\infty_X])$ does not exist, i.e., $[\infty_X] \in \gamma_{X^*}$. This implies that $[x] \in \gamma_{X^*}$ for all $x \in X^*$.

(TF2) Let $\mathbb{F} \in \gamma_{X^*}$ and $\mathbb{F} \subseteq \mathbb{G}$. If $i_X^{\Leftarrow}(\mathbb{G})$ does not exist, then $\mathbb{G} \in \gamma_{X^*}$. If $i_X^{\Leftarrow}(\mathbb{G})$ exists, then $i_X^{\Leftarrow}(\mathbb{F})$ exists. This implies that $i_X^{\Leftarrow}(\mathbb{F}) \in \gamma_X$. Since $i_X^{\Leftarrow}(\mathbb{F}) \subseteq i_X^{\Leftarrow}(\mathbb{G})$, we obtain $i_X^{\Leftarrow}(\mathbb{G}) \in \gamma_X$. Hence $\mathbb{G} \in \gamma_{X^*}$. □

Theorem 4.8. $\top$-**Fil** *is extensional.*

Proof. Let (X, γ_X) be a $\top$-filter space. By Proposition 4.7, we obtain a $\top$-filter structure γ_{X^*} on X^*. First, we show that (X, γ_X) is a subspace of (X^*, γ_{X^*}), i.e., $\gamma_{X^*}|_X = \gamma_X$, where $\gamma_{X^*}|_X = \left\{\mathbb{F} \in \mathcal{F}_L^{\top}(X) \,|\, i_X^{\Rightarrow}(\mathbb{F}) \in \gamma_{X^*}\right\}$. For each $\mathbb{F} \in \gamma_{X^*}|_X$, we obtain $i_X^{\Rightarrow}(\mathbb{F}) \in \gamma_{X^*}$. Take any $A \in i_X^{\Rightarrow}(\mathbb{F})$. Then it follows from $i_X^{\leftarrow}(A) \in \mathbb{F}$ that

$$\top = \bigvee_{x \in X} i_X^{\leftarrow}(A)(x) = \bigvee_{x \in X} A(i_X(x)) = \bigvee_{x \in X} A(x).$$

Then $i_X^{\Leftarrow}(i_X^{\Rightarrow}(\mathbb{F}))$ exists. This implies $i_X^{\Leftarrow}(i_X^{\Rightarrow}(\mathbb{F})) \in \gamma_X$. Since $\mathbb{F} = i_X^{\Leftarrow}(i_X^{\Rightarrow}(\mathbb{F}))$, we obtain $\mathbb{F} \in \gamma_X$. Thus $\gamma_{X^*}|_X \subseteq \gamma_X$. Conversely, for each $\mathbb{F} \in \gamma_X$, $i_X^{\Leftarrow}(i_X^{\Rightarrow}(\mathbb{F}))$ exists and $i_X^{\Leftarrow}(i_X^{\Rightarrow}(\mathbb{F})) = \mathbb{F}$ imply $i_X^{\Rightarrow}(\mathbb{F}) \in \gamma_{X^*}$. Hence $\mathbb{F} \in \gamma_{X^*}|_X$. This shows $\gamma_X \subseteq \gamma_{X^*}|_X$.

Next, we show that (X^*, γ_{X^*}) is the one-point extension of (X, γ_X). Let (Y, γ_Y) be a $\top$-filter space, (Z, γ_Z) be a subspace of (Y, γ_Y) and $\varphi : (Z, \gamma_Z) \longrightarrow (X, \gamma_X)$ be a Cauchy continuous mapping. Define $\varphi^* : Y \longrightarrow X^*$ by $\varphi^*(y) = \varphi(y)$ whenever $y \in Z$, and $\varphi^*(y) = \infty_X$ otherwise. There is a commutative diagram as follows:

$$\begin{array}{ccc} (Z, \gamma_Z) & \xrightarrow{\ \varphi\ } & (X, \gamma_X) \\ \Big\downarrow{\scriptstyle i_Z} & & \Big\downarrow{\scriptstyle i_X} \\ (Y, \gamma_Y) & \xrightarrow[\ \varphi^*\ ]{} & (X^*, \gamma_{X^*}) \end{array}$$

In order to show the Cauchy continuity of $\varphi^* : (Y, \gamma_Y) \longrightarrow (X^*, \gamma_{X^*})$, it suffices to verify that $\mathbb{G} \in \gamma_Y$ implies $(\varphi^*)^{\Rightarrow}(\mathbb{G}) \in \gamma_{X^*}$ for all $\mathbb{G} \in \mathcal{F}_L^{\top}(Y)$.

Case 1: $i_Z^{\Leftarrow}(\mathbb{G})$ does not exist. Then there exists $B \in \mathbb{G}$ such that $\bigvee_{z \in Z} B(z) < \top$. Let $\alpha = \bigvee_{z \in Z} B(z)$. Define $\alpha_{X^*} : X^* \longrightarrow L$ by $\alpha_{X^*}(x) = \alpha$ for all $x \in X^*$. Let $\beta = \alpha_{X^*} \vee \top_{\infty_X}$. Then

$$(\varphi^*)^{\leftarrow}(\beta)(y) = \beta(\varphi^*(y)) = \begin{cases} \alpha, & y \in Z, \\ \top, & y \notin Z. \end{cases}$$

This means $B \leqslant (\varphi^*)^{\leftarrow}(\beta)$. Thus, we get $(\varphi^*)^{\leftarrow}(\beta) \in \mathbb{G}$, i.e., $\beta \in (\varphi^*)^{\Rightarrow}(\mathbb{G})$. Since

$$\bigvee_{x \in i_X^{\rightarrow}(X)} \beta(x) = \bigvee_{x \in X} \beta(x) = \bigvee_{x \in X} (\alpha_{X^*} \vee \top_{\infty_X})(x) = \alpha < \top,$$

we know $i_X^{\Leftarrow}((\varphi^*)^{\Rightarrow}(\mathbb{G}))$ does not exist. By the definition of γ_{X^*}, it follows that $(\varphi^*)^{\Rightarrow}(\mathbb{G}) \in \gamma_{X^*}$.

Case 2: $i_Z^{\Leftarrow}(\mathbb{G})$ exists. Since $\mathbb{G} \subseteq i_Z^{\Rightarrow}(i_Z^{\Leftarrow}(\mathbb{G}))$, $\mathbb{G} \in \gamma_Y$ and (Z, γ_Z) is a subspace of (Y, γ_Y), we know $i_Z^{\Leftarrow}(\mathbb{G}) \in \gamma_Z$. By the Cauchy continuity of φ, we obtain $\varphi^{\Rightarrow}(i_Z^{\Leftarrow}(\mathbb{G})) \in \gamma_X$. Let $\mathbb{H} = \varphi^{\Rightarrow}(i_Z^{\Leftarrow}(\mathbb{G}))$. By Proposition 4.6, we get $i_X^{\Leftarrow}(\mathbb{H}^*) = \mathbb{H}$, where $\mathbb{H}^* = i_X^{\Rightarrow}(\varphi^{\Rightarrow}(i_Z^{\Leftarrow}(\mathbb{G}))) \cap [\infty_X]$. Then it follows from the definition of γ_{X^*} that $\mathbb{H}^* \in \gamma_{X^*}$. Next, we will prove $\mathbb{H}^* \subseteq (\varphi^*)^{\Rightarrow}(\mathbb{G})$ by the following two steps.

Step 1: $(\varphi^*)^{\Rightarrow}(\mathbb{G})$ has the $\top$-filter base $\mathbb{B}_1 = \{(\varphi^*)^{\rightarrow}(B) \,|\, B \in \mathbb{G}\}$. By Proposition 2.6, $\mathbb{H}^*$ has the $\top$-filter base $\mathbb{B}_2 = \left\{i_X^{\rightarrow}(\varphi^{\rightarrow}(i_Z^{\leftarrow}(B))) \vee \top_{\infty_X} \,|\, B \in \mathbb{G}\right\}$. Since

$$i_X^{\rightarrow}(\varphi^{\rightarrow}(i_Z^{\leftarrow}(B)))(x^*) = \bigvee_{i_X(x) = x^*} \varphi^{\rightarrow}(i_Z^{\leftarrow}(B))(x) = \begin{cases} \bigvee_{\varphi(z) = x^*, z \in Z} B(z), & x^* \in X, \\ \bot, & x^* = \infty_X, \end{cases}$$

and

$$(\varphi^*)^{\rightarrow}(B)(x^*) = \bigvee_{\varphi^*(y)=x^*} B(y) = \begin{cases} \bigvee_{\varphi(z)=x^*, z\in Z} B(z), & x^* \in X, \\ \bigvee_{z\in Y/Z} B(z), & x^* = \infty_X, \end{cases}$$

it follows that $i_X^{\rightarrow}(\varphi^{\rightarrow}(i_Z^{\leftarrow}(B))) = (\varphi^*)^{\rightarrow}(B) \wedge \top_X$. This implies that $\mathbb{B}_2 = \{((\varphi^*)^{\rightarrow}(B) \wedge \top_X) \vee \top_{\infty_X} \mid B \in \mathbb{G}\}$.

Step 2: Let $A \in \mathbb{H}^*$. Then

$$\begin{aligned} \top &= \bigvee_{C\in\mathbb{B}_2} \mathcal{S}_{X^*}(C, A) \\ &= \bigvee_{B\in\mathbb{G}} \mathcal{S}_{X^*}\Big(((\varphi^*)^{\rightarrow}(B) \wedge \top_X) \vee \top_{\infty_X}, A\Big) \\ &\leqslant \bigvee_{B\in\mathbb{G}} \mathcal{S}_{X^*}\Big((\varphi^*)^{\rightarrow}(B), A\Big) \\ &= \bigvee_{D\in\mathbb{B}_1} \mathcal{S}_{X^*}(D, A). \end{aligned}$$

Hence $A \in (\varphi^*)^{\Rightarrow}(\mathbb{G})$.

By **Step 1** and **Step 2**, we obtain $\mathbb{H}^* \subseteq (\varphi^*)^{\Rightarrow}(\mathbb{G})$. Then it follows from (TF2) that $(\varphi^*)^{\Rightarrow}(\mathbb{G}) \in \gamma_{X^*}$. Thus, $\varphi^* : (Y, \gamma_Y) \longrightarrow (X^*, \gamma_{X^*})$ is Cauchy continuous. □

4.3. *Productivity of quotient mappings in* $\top$-**Fil**

In this subsection, we will define the product of an arbitrary family of $\top$-filters, which can include the product of two $\top$-filters as a special case. To this end, we first give the following propositions.

Proposition 4.9. *Let $\{\mathbb{F}_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{F}_L^{\top}(X)$. Then the following statements are equivalent.*

(1) *There exists $\mathbb{H} \in \mathcal{F}_L^{\top}(X)$ such that $\mathbb{F}_\lambda \subseteq \mathbb{H}$ for all $\lambda \in \Lambda$.*

(2) *For each $n \in \mathbb{N}$, $\{\lambda_i\}_{i=1}^n \subseteq \Lambda$, $\bigvee_{x\in X} \bigwedge_{i=1}^n A_i(x) = \top$ where $A_i \in \mathbb{F}_{\lambda_i}$ for all $i = 1, \cdots, n$.*

Proof. (1) $\Longrightarrow$ (2) It is straightforward.

(2) $\Longrightarrow$ (1) Let

$$\mathbb{H} = \Big\{A \in L^X \mid \bigvee_{n\in\mathbb{N}} \bigvee_{\{\lambda_i\}_{i=1}^n\subseteq\Lambda} \bigvee_{\forall i=1,\cdots,n, A_i\in\mathbb{F}_{\lambda_i}} \mathcal{S}_X\Big(\bigwedge_{i=1}^n A_i, A\Big) = \top\Big\}.$$

Then we will show $\mathbb{H}$ satisfies ($\mathbb{F}$1)–($\mathbb{F}$3).

($\mathbb{F}$1) If $\bigvee_{B\in\mathbb{H}} \mathcal{S}_X(B, A) = \top$, then

$$\top = \bigvee_{B\in\mathbb{H}} \Big(\mathcal{S}_X(B, A) * \bigvee_{n\in\mathbb{N}} \bigvee_{\{\lambda_i\}_{i=1}^n\subseteq\Lambda} \bigvee_{\forall i=1,\cdots,n, A_i\in\mathbb{F}_{\lambda_i}} \mathcal{S}_X\Big(\bigwedge_{i=1}^n A_i, B\Big)\Big) \leqslant \bigvee_{n\in\mathbb{N}} \bigvee_{\{\lambda_i\}_{i=1}^n\subseteq\Lambda} \bigvee_{\forall i=1,\cdots,n, A_i\in\mathbb{F}_{\lambda_i}} \mathcal{S}_X\Big(\bigwedge_{i=1}^n A_i, A\Big).$$

This shows $A \in \mathbb{H}$.

($\mathbb{F}$2) Take any $C, D \in \mathbb{H}$. Then

$$\begin{aligned} \top &= \bigvee_{m\in\mathbb{N}} \bigvee_{\{\lambda_i\}_{i=1}^m\subseteq\Lambda} \bigvee_{\forall i=1,\cdots,m, A_i\in\mathbb{F}_{\lambda_i}} \mathcal{S}_X\Big(\bigwedge_{i=1}^m A_i, C\Big) * \bigvee_{n\in\mathbb{N}} \bigvee_{\{\mu_j\}_{j=1}^n\subseteq\Lambda} \bigvee_{\forall j=1,\cdots,n, B_j\in\mathbb{F}_{\mu_j}} \mathcal{S}_X\Big(\bigwedge_{j=1}^n B_j, D\Big) \\ &\leqslant \bigvee_{m\in\mathbb{N}} \bigvee_{\{\lambda_i\}_{i=1}^m\subseteq\Lambda} \bigvee_{n\in\mathbb{N}} \bigvee_{\{\mu_j\}_{j=1}^n\subseteq\Lambda} \bigvee_{\forall i=1,\cdots,m, A_i\in\mathbb{F}_{\lambda_i}} \bigvee_{\forall j=1,\cdots,n, B_j\in\mathbb{F}_{\mu_j}} \mathcal{S}_X\Big(\bigwedge_{i=1}^m A_i \wedge \bigwedge_{j=1}^n B_j, C \wedge D\Big) \\ &\leqslant \bigvee_{m+n\in\mathbb{N}} \bigvee_{\{\beta_q\}_{q=1}^{m+n}\subseteq\Lambda} \bigvee_{\forall q=1,\cdots,m,m+1,\cdots,m+n, E_q\in\mathbb{F}_{\beta_q}} \mathcal{S}_X\Big(\bigwedge_{q=1}^{m+n} E_q, C \wedge D\Big) \end{aligned}$$

where $\{\beta_q\}_{q=1}^{m+n} = \{\lambda_1, \cdots, \lambda_m, \mu_1, \cdots, \mu_n\}$. Hence $C \wedge D \in \mathbb{H}$.

($\mathbb{F}$3) Take any $A \in \mathbb{H}$. Then

$$\begin{aligned}\top &= \bigvee_{n\in\mathbb{N}} \bigvee_{\{\lambda_i\}_{i=1}^n \subseteq \Lambda} \bigvee_{\forall i=1,\cdots,n, A_i \in \mathbb{F}_{\lambda_i}} \mathcal{S}_X\Big(\bigwedge_{i=1}^n A_i, A\Big) * \Big(\bigvee_{x\in X} \bigwedge_{i=1}^n A_i(x)\Big) \\ &= \bigvee_{x\in X} \bigvee_{n\in\mathbb{N}} \bigvee_{\{\lambda_i\}_{i=1}^n \subseteq \Lambda} \bigvee_{\forall i=1,\cdots,n, A_i \in \mathbb{F}_{\lambda_i}} \mathcal{S}_X\Big(\bigwedge_{i=1}^n A_i, A\Big) * \bigwedge_{i=1}^n A_i(x) \leqslant \bigvee_{x\in X} A(x).\end{aligned}$$

This implies that $\bigvee_{x\in X} A(x) = \top$ for all $A \in \mathbb{H}$. $\square$

Proposition 4.9 implies that the supremum of an arbitrary family of $\top$-filters exists when it satisfies (2). As a corollary of Proposition 4.9, we present the concrete form of the supremum when it exists.

Corollary 4.10. *Let $\{\mathbb{F}_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{F}_L^\top(X)$. If for each $n \in \mathbb{N}$, $\{\lambda_i\}_{i=1}^n \subseteq \Lambda$, $\bigvee_{x\in X} \bigwedge_{i=1}^n A_i(x) = \top$ where $A_i \in \mathbb{F}_{\lambda_i}$ for each $i = 1, \cdots, n$, then*

$$\bigvee_{\lambda\in\Lambda} \mathbb{F}_\lambda = \Big\{A \in L^X \,\Big|\, \bigvee_{n\in\mathbb{N}} \bigvee_{\{\lambda_i\}_{i=1}^n \subseteq \Lambda} \bigvee_{\forall i=1,\cdots,n, A_i \in \mathbb{F}_{\lambda_i}} \mathcal{S}_X\Big(\bigwedge_{i=1}^n A_i, A\Big) = \top\Big\}.$$

In particular, for $\mathbb{F}_1, \mathbb{F}_2 \in \mathcal{F}_L^\top(X)$, by Proposition 4.9, we know that $\mathbb{F}_1 \vee \mathbb{F}_2$ exists when $\bigvee_{x\in X} A(x) \wedge B(x) = \top$ for all $A \in \mathbb{F}_1$ and $B \in \mathbb{F}_2$. Then

$$\mathbb{F}_1 \vee \mathbb{F}_2 = \Big\{C \in L^X \,\Big|\, \bigvee_{A\in\mathbb{F}_1, B\in\mathbb{F}_2} \mathcal{S}_X(A \wedge B, C) = \top\Big\}.$$

This is coincident with that in [18].

Proposition 4.11. *Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of nonempty sets and $\{\mathbb{F}_\lambda\}_{\lambda\in\Lambda}$ be a family of $\top$-filters, where for each $\lambda \in \Lambda$, $\mathbb{F}_\lambda \in \mathcal{F}_L^\top(X_\lambda)$. For each $\lambda \in \Lambda$, $pr_\lambda : \prod_{\mu\in\Lambda} X_\mu \longrightarrow X_\lambda$ is the projection mapping. Then $\bigvee_{\lambda\in\Lambda} pr_\lambda^{\Leftarrow}(\mathbb{F}_\lambda)$ exists.*

Proof. For convenience, let $X = \prod_{\mu\in\Lambda} X_\mu$. By Proposition 4.9, it is enough to show that for each $n \in \mathbb{N}$, $\{\lambda_i\}_{i=1}^n \subseteq \Lambda$ and $A_i \in pr_{\lambda_i}^{\Leftarrow}(\mathbb{F}_{\lambda_i})$ for all $i = 1, \cdots, n$, $\bigvee_{x\in X} \bigwedge_{i=1}^n A_i(x) = \top$ holds. By Definition 2.8, we know $\bigvee_{B_i\in\mathbb{F}_{\lambda_i}} \mathcal{S}_X(p_{\lambda_i}^{\leftarrow}(B_i), A_i) = \top$ for each $i = 1, \cdots, n$. This implies

$$\bigvee_{B_1\in\mathbb{F}_{\lambda_1}} \mathcal{S}_X(p_{\lambda_1}^{\leftarrow}(B_1), A_1) * \cdots * \bigvee_{B_i\in\mathbb{F}_{\lambda_i}} \mathcal{S}_X(p_{\lambda_i}^{\leftarrow}(B_i), A_i) * \cdots * \bigvee_{B_n\in\mathbb{F}_{\lambda_n}} \mathcal{S}_X(p_{\lambda_n}^{\leftarrow}(B_n), A_n) = \top.$$

For each $i = 1, \cdots, n$, take $B_i \in \mathbb{F}_{\lambda_i}$. Then

$$\begin{aligned}\bigvee_{x\in X} \Big(\bigwedge_{i=1}^n pr_{\lambda_i}^{\leftarrow}(B_i)\Big)(x) &= \bigvee_{x=(x_\lambda)\in X} \bigwedge_{i=1}^n B_i(pr_{\lambda_i}(x)) \\ &= \bigvee_{\forall i=1,\cdots,n, x_{\lambda_i}\in X_{\lambda_i}} B_1(x_{\lambda_1}) \wedge \cdots \wedge B_n(x_{\lambda_n}) \\ &\geqslant \bigvee_{\forall i=1,\cdots,n, x_{\lambda_i}\in X_{\lambda_i}} B_1(x_{\lambda_1}) * \cdots * B_n(x_{\lambda_n}) \\ &= \bigvee_{x_{\lambda_1}\in X_{\lambda_1}} B_1(x_{\lambda_1}) * \cdots * \bigvee_{x_{\lambda_n}\in X_{\lambda_1}} B_n(x_{\lambda_n}) \\ &= \top.\end{aligned}$$

Further, it follows that

$$\begin{aligned}
\top &= \bigvee_{B_1\in\mathbb{F}_{\lambda_1}} \mathcal{S}_X(p_{\lambda_1}^{\leftarrow}(B_1),A_1)*\cdots*\bigvee_{B_i\in\mathbb{F}_{\lambda_i}} \mathcal{S}_X(p_{\lambda_i}^{\leftarrow}(B_i),A_i)*\cdots*\bigvee_{B_n\in\mathbb{F}_{\lambda_n}} \mathcal{S}_X(p_{\lambda_n}^{\leftarrow}(B_n),A_n)\\
&= \bigvee_{\forall i=1,\cdots,n,B_i\in\mathbb{F}_{\lambda_i}} \mathcal{S}_X(p_{\lambda_1}^{\leftarrow}(B_1),A_1)*\cdots*\mathcal{S}_X(pr_{\lambda_i}^{\leftarrow}(B_i),A_i)*\cdots*\mathcal{S}_X(p_{\lambda_n}^{\leftarrow}(B_n),A_n)\\
&\leqslant \bigvee_{\forall i=1,\cdots,n,B_i\in\mathbb{F}_{\lambda_i}} \mathcal{S}_X\Big(\bigwedge_{i=1}^{n} pr_{\lambda_i}^{\leftarrow}(B_i),\bigwedge_{i=1}^{n}A_i\Big)\\
&= \bigvee_{\forall i=1,\cdots,n,B_i\in\mathbb{F}_{\lambda_i}} \bigg(\mathcal{S}_X\Big(\bigwedge_{i=1}^{n} pr_{\lambda_i}^{\leftarrow}(B_i),\bigwedge_{i=1}^{n}A_i\Big)*\bigvee_{x\in X}\Big(\bigwedge_{i=1}^{n} pr_{\lambda_i}^{\leftarrow}(B_i)\Big)(x)\bigg)\\
&= \bigvee_{x\in X}\bigg(\bigvee_{i=1,\cdots,n,B_i\in\mathbb{F}_{\lambda_i}} \mathcal{S}_X\Big(\bigwedge_{i=1}^{n} pr_{\lambda_i}^{\leftarrow}(B_i),\bigwedge_{i=1}^{n}A_i\Big)*\Big(\bigwedge_{i=1}^{n} pr_{\lambda_i}^{\leftarrow}(B_i)\Big)(x)\bigg)\\
&\leqslant \bigvee_{x\in X}\bigwedge_{i=1}^{n}A_i(x),
\end{aligned}$$

as desired. □

By Propositions 4.9 and 4.11, the product $\prod_{\lambda\in\Lambda}\mathbb{F}_\lambda$ of a family of $\top$-filters $\{\mathbb{F}_\lambda\}_{\lambda\in\Lambda}$ can be defined via the supremum of $\{pr_\lambda^{\Leftarrow}(\mathbb{F}_\lambda)\}_{\lambda\in\Lambda}$.

Definition 4.12. Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of nonempty sets and $\{\mathbb{F}_\lambda\}_{\lambda\in\Lambda}$ be a family of $\top$-filters with $\mathbb{F}_\lambda\in\mathcal{F}_L^{\top}(X_\lambda)$ for each $\lambda\in\Lambda$. Then

$$\prod_{\lambda\in\Lambda}\mathbb{F}_\lambda=\bigvee_{\lambda\in\Lambda}pr_\lambda^{\Leftarrow}(\mathbb{F}_\lambda)=\bigg\{A\in L^X \,\Big|\, \bigvee_{n\in\mathbb{N}}\bigvee_{\{\lambda_i\}_{i=1}^n\subseteq\Lambda}\bigvee_{\forall i=1,\cdots,n,A_i\in pr_{\lambda_i}^{\Leftarrow}(\mathbb{F}_{\lambda_i})}\mathcal{S}_X\Big(\bigwedge_{i=1}^{n}A_i,A\Big)=\top\bigg\}$$

is called the product of $\{\mathbb{F}_\lambda\}_{\lambda\in\Lambda}$.

Proposition 4.13. *Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of nonempty sets, $pr_\lambda:\prod_{\mu\in\Lambda}X_\mu\longrightarrow X_\lambda$ be the projection mapping, $\mathbb{F}_\lambda\in\mathcal{F}_L^{\top}(X_\lambda)$ for each $\lambda\in\Lambda$ and $\mathbb{F}\in\mathcal{F}_L^{\top}(\prod_{\lambda\in\Lambda}X_\lambda)$. Then the following statements hold:*
(1) *$\mathbb{F}_\lambda\subseteq pr_\lambda^{\Rightarrow}(\prod_{\mu\in\Lambda}\mathbb{F}_\mu)$ for all $\lambda\in\Lambda$;*
(2) *$\prod_{\lambda\in\Lambda}pr_\lambda^{\Rightarrow}(\mathbb{F})\subseteq\mathbb{F}$.*

Proof. (1) For each $\lambda\in\Lambda$, it follows that

$$\mathbb{F}_\lambda\subseteq pr_\lambda^{\Rightarrow}(pr_\lambda^{\Leftarrow}(\mathbb{F}_\lambda))\subseteq pr_\lambda^{\Rightarrow}\bigg(\bigvee_{\mu\in\Lambda}pr_\mu^{\Leftarrow}(\mathbb{F}_\mu)\bigg)=pr_\lambda^{\Rightarrow}\bigg(\prod_{\mu\in\Lambda}\mathbb{F}_\mu\bigg).$$

(2) Take any $B\in\prod_{\lambda\in\Lambda}pr_\lambda^{\Rightarrow}(\mathbb{F})$. Then

$$\bigvee_{n\in\mathbb{N}}\bigvee_{\{\lambda_i\}_{i=1}^n\subseteq\Lambda}\bigvee_{\forall i=1,\cdots,n,A_i\in pr_{\lambda_i}^{\Leftarrow}(pr_{\lambda_i}^{\Rightarrow}(\mathbb{F}))}\mathcal{S}_X\Big(\bigwedge_{i=1}^{n}A_i,B\Big)=\top.$$

Since $pr_\lambda^{\Leftarrow}(pr_\lambda^{\Rightarrow}(\mathbb{F}))\subseteq\mathbb{F}$ for all $\lambda\in\Lambda$, we get

$$\top=\bigvee_{n\in\mathbb{N}}\bigvee_{\{\lambda_i\}_{i=1}^n\subseteq\Lambda}\bigvee_{\forall i=1,\cdots,n,A_i\in\mathbb{F}}\mathcal{S}_X\Big(\bigwedge_{i=1}^{n}A_i,B\Big)\leqslant\bigvee_{n\in\mathbb{N}}\bigvee_{\bigwedge_{i=1}^n A_i\in\mathbb{F}}\mathcal{S}_X\Big(\bigwedge_{i=1}^{n}A_i,B\Big)\leqslant\bigvee_{A\in\mathbb{F}}\mathcal{S}_X(A,B),$$

which implies that $B\in\mathbb{F}$. By the arbitrariness of B, we obtain $\prod_{\lambda\in\Lambda}pr_\lambda^{\Rightarrow}(\mathbb{F})\subseteq\mathbb{F}$, as desired. □

Proposition 4.14. *Let $\{\mathbb{F}_\lambda\}_{\lambda\in\Lambda}$ be a family of $\top$-filters with $\mathbb{F}_\lambda \in \mathcal{F}_L^\top(X_\lambda)$. Then*

$$\prod_{\lambda\in\Lambda}\mathbb{F}_\lambda = \Big\{A \in L^{\prod_\lambda X_\lambda} \mid \bigvee_{n\in\mathbb{N}}\bigvee_{\{\lambda_i\}_{i=1}^n\subseteq\Lambda}\bigvee_{\forall i=1,\cdots,n,B_{\lambda_i}\in\mathbb{F}_{\lambda_i}} \mathcal{S}_{\prod_{\lambda\in\Lambda}X_\lambda}\Big(\bigwedge_{i=1}^n pr_{\lambda_i}^{\leftarrow}(B_{\lambda_i}), A\Big) = \top\Big\}.$$

Proof. By Definition 4.12, we have $\prod_{\lambda\in\Lambda}\mathbb{F}_\lambda = \bigvee_{\lambda\in\Lambda} pr_\lambda^{\Leftarrow}(\mathbb{F}_\lambda)$. Then

$$A \in \prod_{\lambda\in\Lambda}\mathbb{F}_\lambda \Longleftrightarrow A \in \bigvee_{\lambda\in\Lambda} pr_\lambda^{\Leftarrow}(\mathbb{F}_\lambda) \Longleftrightarrow \bigvee_{n\in\mathbb{N}}\bigvee_{\{\lambda_i\}_{i=1}^n\subseteq\Lambda}\bigvee_{\forall i=1,\cdots,n,A_{\lambda_i}\in pr_{\lambda_i}^{\Leftarrow}(\mathbb{F}_{\lambda_i})} \mathcal{S}_{\prod_{\lambda\in\Lambda}X_\lambda}\Big(\bigwedge_{i=1}^n A_{\lambda_i}, A\Big) = \top.$$

Since

$$\begin{aligned}
&\bigvee_{n\in\mathbb{N}}\bigvee_{\{\lambda_i\}_{i=1}^n\subseteq\Lambda}\bigvee_{\forall i=1,\cdots,n,A_{\lambda_i}\in pr_{\lambda_i}^{\Leftarrow}(\mathbb{F}_{\lambda_i})} \mathcal{S}_{\prod_{\lambda\in\Lambda}X_\lambda}\Big(\bigwedge_{i=1}^n A_{\lambda_i}, A\Big)\\
&= \bigvee_{n\in\mathbb{N}}\bigvee_{\{\lambda_i\}_{i=1}^n\subseteq\Lambda}\bigvee_{\forall i=1,\cdots,n,A_{\lambda_i}\in pr_{\lambda_i}^{\Leftarrow}(\mathbb{F}_{\lambda_i})} \mathcal{S}_{\prod_{\lambda\in\Lambda}X_\lambda}\Big(\bigwedge_{i=1}^n A_{\lambda_i}, A\Big) *\\
&\qquad\Big(\bigvee_{B_{\lambda_1}\in\mathbb{F}_{\lambda_1}} \mathcal{S}_{\prod_{\lambda\in\Lambda}X_\lambda}(pr_{\lambda_1}^{\leftarrow}(B_{\lambda_1}), A_{\lambda_1}) * \cdots * \bigvee_{B_{\lambda_n}\in\mathbb{F}_{\lambda_n}} \mathcal{S}_{\prod_{\lambda\in\Lambda}X_\lambda}(pr_{\lambda_n}^{\leftarrow}(B_{\lambda_n}), A_{\lambda_n})\Big)\\
&= \bigvee_{n\in\mathbb{N}}\bigvee_{\{\lambda_i\}_{i=1}^n\subseteq\Lambda}\bigvee_{\forall i=1,\cdots,n,A_{\lambda_i}\in pr_{\lambda_i}^{\Leftarrow}(\mathbb{F}_{\lambda_i})}\bigvee_{\forall i=1,\cdots,n,B_{\lambda_i}\in\mathbb{F}_{\lambda_i}} \mathcal{S}_{\prod_{\lambda\in\Lambda}X_\lambda}\Big(\bigwedge_{i=1}^n A_{\lambda_i}, A\Big) *\\
&\qquad \mathcal{S}_{\prod_{\lambda\in\Lambda}X_\lambda}(pr_{\lambda_1}^{\leftarrow}(B_{\lambda_1}), A_{\lambda_1}) * \cdots * \mathcal{S}_{\prod_{\lambda\in\Lambda}X_\lambda}(pr_{\lambda_n}^{\leftarrow}(B_{\lambda_n}), A_{\lambda_n})\\
&\leqslant \bigvee_{n\in\mathbb{N}}\bigvee_{\{\lambda_i\}_{i=1}^n\subseteq\Lambda}\bigvee_{\forall i=1,\cdots,n,A_{\lambda_i}\in pr_{\lambda_i}^{\Leftarrow}(\mathbb{F}_{\lambda_i})}\bigvee_{\forall i=1,\cdots,n,B_{\lambda_i}\in\mathbb{F}_{\lambda_i}} \mathcal{S}_{\prod_{\lambda\in\Lambda}X_\lambda}\Big(\bigwedge_{i=1}^n A_{\lambda_i}, A\Big) * \mathcal{S}_{\prod_{\lambda\in\Lambda}X_\lambda}\Big(\bigwedge_{i=1}^n pr_{\lambda_i}^{\leftarrow}(B_{\lambda_i}), \bigwedge_{i=1}^n A_{\lambda_i}\Big)\\
&\leqslant \bigvee_{n\in\mathbb{N}}\bigvee_{\{\lambda_i\}_{i=1}^n\subseteq\Lambda}\bigvee_{\forall i=1,\cdots,n,B_{\lambda_i}\in\mathbb{F}_{\lambda_i}} \mathcal{S}_{\prod_{\lambda\in\Lambda}X_\lambda}\Big(\bigwedge_{i=1}^n pr_{\lambda_i}^{\leftarrow}(B_{\lambda_i}), A\Big)\\
&\leqslant \bigvee_{n\in\mathbb{N}}\bigvee_{\{\lambda_i\}_{i=1}^n\subseteq\Lambda}\bigvee_{\forall i=1,\cdots,n,pr_{\lambda_i}^{\leftarrow}(B_{\lambda_i})\in pr_{\lambda_i}^{\Leftarrow}(\mathbb{F}_{\lambda_i})} \mathcal{S}_{\prod_{\lambda\in\Lambda}X_\lambda}\Big(\bigwedge_{i=1}^n pr_{\lambda_i}^{\leftarrow}(B_{\lambda_i}), A\Big)\\
&\leqslant \bigvee_{n\in\mathbb{N}}\bigvee_{\{\lambda_i\}_{i=1}^n\subseteq\Lambda}\bigvee_{\forall i=1,\cdots,n,A_{\lambda_i}\in pr_{\lambda_i}^{\Leftarrow}(\mathbb{F}_{\lambda_i})} \mathcal{S}_{\prod_{\lambda\in\Lambda}X_\lambda}\Big(\bigwedge_{i=1}^n A_{\lambda_i}, A\Big),
\end{aligned}$$

it follows that

$$\bigvee_{n\in\mathbb{N}}\bigvee_{\{\lambda_i\}_{i=1}^n\subseteq\Lambda}\bigvee_{\forall i=1,\cdots,n,B_{\lambda_i}\in\mathbb{F}_{\lambda_i}} \mathcal{S}_{\prod_{\lambda\in\Lambda}X_\lambda}\Big(\bigwedge_{i=1}^n pr_{\lambda_i}^{\leftarrow}(B_{\lambda_i}), A\Big) = \bigvee_{n\in\mathbb{N}}\bigvee_{\{\lambda_i\}_{i=1}^n\subseteq\Lambda}\bigvee_{\forall i=1,\cdots,n,A_{\lambda_i}\in pr_{\lambda_i}^{\Leftarrow}(\mathbb{F}_{\lambda_i})} \mathcal{S}_{\prod_{\lambda\in\Lambda}X_\lambda}\Big(\bigwedge_{i=1}^n A_{\lambda_i}, A\Big).$$

Hence we obtain

$$\begin{aligned}
A \in \prod_{\lambda\in\Lambda}\mathbb{F}_\lambda &\Longleftrightarrow \bigvee_{n\in\mathbb{N}}\bigvee_{\{\lambda_i\}_{i=1}^n\subseteq\Lambda}\bigvee_{\forall i=1,\cdots,n,A_{\lambda_i}\in pr_{\lambda_i}^{\Leftarrow}(\mathbb{F}_{\lambda_i})} \mathcal{S}_{\prod_{\lambda\in\Lambda}X_\lambda}\Big(\bigwedge_{i=1}^n A_{\lambda_i}, A\Big) = \top\\
&\Longleftrightarrow \bigvee_{n\in\mathbb{N}}\bigvee_{\{\lambda_i\}_{i=1}^n\subseteq\Lambda}\bigvee_{\forall i=1,\cdots,n,B_{\lambda_i}\in\mathbb{F}_{\lambda_i}} \mathcal{S}_{\prod_{\lambda\in\Lambda}X_\lambda}\Big(\bigwedge_{i=1}^n pr_{\lambda_i}^{\leftarrow}(B_{\lambda_i}), A\Big) = \top.
\end{aligned}$$

□

Corollary 4.15. *Let* $\mathbb{F}_1 \in \mathcal{F}_L^{\top}(X_1)$ *and* $\mathbb{F}_2 \in \mathcal{F}_L^{\top}(X_2)$. *Then*

$$\mathbb{F}_1 \times \mathbb{F}_2 = \left\{ A \in L^{X_1 \times X_2} \,|\, \bigvee_{B_1 \in \mathbb{F}_1, B_2 \in \mathbb{F}_2} \mathcal{S}_{X_1 \times X_2}(B_1 \times B_2, A) = \top \right\}.$$

Note that the product of two $\top$-filters in Corollary 4.15 coincides with that in [45] and it is obvious that $\{B_1 \times B_2 \in L^{X_1 \times X_2} \,|\, B_1 \in \mathbb{F}_1, B_2 \in \mathbb{F}_2\}$ is a $\top$-filter base of $\mathbb{F}_1 \times \mathbb{F}_2$. This demonstrates that the product of an arbitrary family of $\top$-filters defined herein can be considered a reasonable generalization of product of filters.

Lemma 4.16. *Suppose that* L *satisfies* (MID). *Let* $\{\varphi_\lambda : X_\lambda \longrightarrow Y_\lambda\}_{\lambda \in \Lambda}$ *be a family of surjective mappings and* $\{\mathbb{F}_\lambda\}_{\lambda \in \Lambda}$ *be a family of* $\top$*-filters with* $\mathbb{F}_\lambda \in \mathcal{F}_L^{\top}(X_\lambda)$. *Then*

$$\left(\prod_{\lambda \in \Lambda} \varphi_\lambda\right)^{\Rightarrow} \left(\prod_{\lambda \in \Lambda} \mathbb{F}_\lambda\right) = \prod_{\lambda \in \Lambda} \varphi_\lambda^{\Rightarrow}(\mathbb{F}_\lambda).$$

Proof. Let

$$\begin{array}{ccc} \prod_{\lambda \in \Lambda} X_\lambda & \xrightarrow{\prod_{\lambda \in \Lambda} \varphi_\lambda} & \prod_{\lambda \in \Lambda} Y_\lambda \\ {\scriptstyle pr_\lambda}\downarrow & & \downarrow{\scriptstyle qr_\lambda} \\ X_\lambda & \xrightarrow[\varphi_\lambda]{} & Y_\lambda \end{array}$$

be the product commutation diagram. First, we verify

$$\left(\prod_{\lambda \in \Lambda} \varphi_\lambda\right)^{\Rightarrow} \left(\prod_{\lambda \in \Lambda} \mathbb{F}_\lambda\right) \subseteq \prod_{\lambda \in \Lambda} \varphi_\lambda^{\Rightarrow}(\mathbb{F}_\lambda)$$

by the following three steps:

Step 1: Take any $A \in (\prod_{\lambda \in \Lambda} \varphi_\lambda)^{\Rightarrow}(\prod_{\lambda \in \Lambda} \mathbb{F}_\lambda)$. Then $(\prod_{\lambda \in \Lambda} \varphi_\lambda)^{\leftarrow}(A) \in \prod_{\lambda \in \Lambda} \mathbb{F}_\lambda$. By Proposition 4.14, we have

$$\begin{aligned} \top &= \bigvee_{n \in \mathbb{N}} \bigvee_{\{\lambda_i\}_{i=1}^n \subseteq \Lambda} \bigvee_{\forall i=1,\cdots,n, B_{\lambda_i} \in \mathbb{F}_{\lambda_i}} \mathcal{S}_{\prod_{\lambda \in \Lambda} X_\lambda}\left(\bigwedge_{i=1}^n pr_{\lambda_i}^{\leftarrow}(B_{\lambda_i}), \left(\prod_{\lambda \in \Lambda} \varphi_\lambda\right)^{\leftarrow}(A) \right) \\ &= \bigvee_{n \in \mathbb{N}} \bigvee_{\{\lambda_i\}_{i=1}^n \subseteq \Lambda} \bigvee_{\forall i=1,\cdots,n, B_{\lambda_i} \in \mathbb{F}_{\lambda_i}} \mathcal{S}_{\prod_{\lambda \in \Lambda} X_\lambda}\left(\prod_{\lambda \in \Lambda} B_\lambda, \left(\prod_{\lambda \in \Lambda} \varphi_\lambda\right)^{\leftarrow}(A) \right) \\ &\qquad (\text{where } B_\lambda = \top_{X_\lambda} \text{ when } \lambda \notin \{\lambda_1, \cdots, \lambda_n\}). \end{aligned}$$

For each $n \in \mathbb{N}$, $\{\lambda_i\}_{i=1}^n \subseteq \Lambda$, let $B_{\lambda_i} \in \mathbb{F}_{\lambda_i}$ for all $i = 1, \cdots, n$ and let $B_\lambda = \top_{X_\lambda}$ when $\lambda \notin \{\lambda_1, \cdots, \lambda_n\}$. Then let $E_\lambda = \varphi_\lambda^{\rightarrow}(B_\lambda)$ for any $\lambda \in \Lambda$. Since φ_λ is a surjective mapping, we obtain $E_\lambda = \top_{Y_\lambda}$ when $\lambda \notin \{\lambda_1, \cdots, \lambda_n\}$. Since $B_{\lambda_i} \in \mathbb{F}_{\lambda_i}$ and $B_{\lambda_i} \leqslant \varphi_{\lambda_i}^{\leftarrow}(\varphi_{\lambda_i}^{\rightarrow}(B_{\lambda_i}))$ for all $i = 1, \cdots, n$, we have $\varphi_{\lambda_i}^{\leftarrow}(\varphi_{\lambda_i}^{\rightarrow}(B_{\lambda_i})) \in \mathbb{F}_{\lambda_i}$, i.e., $\varphi_{\lambda_i}^{\leftarrow}(E_{\lambda_i}) \in \mathbb{F}_{\lambda_i}$.

Step 2: For each $y \in \prod_{\lambda\in\Lambda} Y_\lambda$, we get

$$\begin{aligned}
\Big(\prod_{\lambda\in\Lambda} E_\lambda\Big)(y) = \Big(\prod_{\lambda\in\Lambda} \varphi_\lambda^{\rightarrow}(B_\lambda)\Big)(y) &= \bigwedge_{\lambda\in\Lambda} \varphi_\lambda^{\rightarrow}(B_\lambda)(y_\lambda) \\
&= \bigwedge_{\lambda\in\Lambda} \bigvee_{\varphi_\lambda(x_\lambda)=y_\lambda} B_\lambda(x_\lambda) \\
&= \bigwedge_{i=1}^{n} \bigvee_{\varphi_{\lambda_i}(x_{\lambda_i})=y_{\lambda_i}} B_{\lambda_i}(x_{\lambda_i}) \\
&= \bigvee_{\varphi_{\lambda_1}(x_{\lambda_1})=y_{\lambda_1}} B_{\lambda_1}(x_{\lambda_1}) \wedge \bigvee_{\varphi_{\lambda_2}(x_{\lambda_2})=y_{\lambda_2}} B_{\lambda_2}(x_{\lambda_2}) \wedge \cdots \bigvee_{\varphi_{\lambda_n}(x_{\lambda_n})=y_{\lambda_n}} B_{\lambda_n}(x_{\lambda_n}) \\
&= \bigvee_{\varphi_{\lambda_1}(x_{\lambda_1})=y_{\lambda_1}} \cdots \bigvee_{\varphi_{\lambda_n}(x_{\lambda_n})=y_{\lambda_n}} \bigwedge_{i=1}^{n} B_{\lambda_i}(x_{\lambda_i}) \qquad \text{(by MID)} \\
&= \bigvee_{\forall\lambda\in\Lambda, \varphi_\lambda(x_\lambda)=y_\lambda} \bigwedge_{\lambda\in\Lambda} B_\lambda(x_\lambda) \\
&= \bigvee_{(\prod_{\lambda\in\Lambda}\varphi_\lambda)(x)=y} \Big(\prod_{\lambda\in\Lambda} B_\lambda\Big)(x) \\
&= \Big(\prod_{\lambda\in\Lambda}\varphi_\lambda\Big)^{\rightarrow}\Big(\prod_{\lambda\in\Lambda} B_\lambda\Big)(y).
\end{aligned}$$

By the arbitrariness of y, we obtain $\prod_{\lambda\in\Lambda} E_\lambda = \Big(\prod_{\lambda\in\Lambda}\varphi_\lambda\Big)^{\rightarrow}\Big(\prod_{\lambda\in\Lambda} B_\lambda\Big)$.

Step 3: Since

$$\begin{aligned}
\top &= \bigvee_{n\in\mathbb{N}} \bigvee_{\{\lambda_i\}_{i=1}^n\subseteq\Lambda} \bigvee_{\forall i=1,\cdots,n, B_{\lambda_i}\in\mathbb{F}_{\lambda_i}} \mathcal{S}_{\prod_{\lambda\in\Lambda} X_\lambda}\Bigg(\prod_{\lambda\in\Lambda} B_\lambda, \Big(\prod_{\lambda\in\Lambda}\varphi_\lambda\Big)^{\leftarrow}(A)\Bigg) \\
&= \bigvee_{n\in\mathbb{N}} \bigvee_{\{\lambda_i\}_{i=1}^n\subseteq\Lambda} \bigvee_{\forall i=1,\cdots,n, B_{\lambda_i}\in\mathbb{F}_{\lambda_i}} \mathcal{S}_{\prod_{\lambda\in\Lambda} Y_\lambda}\Bigg(\Big(\prod_{\lambda\in\Lambda}\varphi_\lambda\Big)^{\rightarrow}\Big(\prod_{\lambda\in\Lambda} B_\lambda\Big), A\Bigg) \\
&\leqslant \bigvee_{n\in\mathbb{N}} \bigvee_{\{\lambda_i\}_{i=1}^n\subseteq\Lambda} \bigvee_{\forall i=1,\cdots,n, \varphi_{\lambda_i}^{\leftarrow}(E_{\lambda_i})\in\mathbb{F}_{\lambda_i}} \mathcal{S}_{\prod_{\lambda\in\Lambda} Y_\lambda}\Big(\prod_{\lambda\in\Lambda} E_\lambda, A\Big) \quad \text{(by } \textbf{Step 1} \text{ and } \textbf{Step 2}\text{)} \\
&= \bigvee_{n\in\mathbb{N}} \bigvee_{\{\lambda_i\}_{i=1}^n\subseteq\Lambda} \bigvee_{\forall i=1,\cdots,n, E_{\lambda_i}\in\varphi_{\lambda_i}^{\Rightarrow}(\mathbb{F}_{\lambda_i})} \mathcal{S}_{\prod_{\lambda\in\Lambda} Y_\lambda}\Big(\prod_{\lambda\in\Lambda} E_\lambda, A\Big) \\
&= \bigvee_{n\in\mathbb{N}} \bigvee_{\{\lambda_i\}_{i=1}^n\subseteq\Lambda} \bigvee_{\forall i=1,\cdots,n, E_{\lambda_i}\in\varphi_{\lambda_i}^{\Rightarrow}(\mathbb{F}_{\lambda_i})} \mathcal{S}_{\prod_{\lambda\in\Lambda} Y_\lambda}\Big(\bigwedge_{i=1}^{n} qr_{\lambda_i}^{\leftarrow}(E_{\lambda_i}), A\Big),
\end{aligned}$$

it follows from Proposition 4.14 that $A \in \prod_{\lambda\in\Lambda}\varphi_\lambda^{\Rightarrow}(\mathbb{F}_\lambda)$. By the arbitrariness of A, we have $(\prod_{\lambda\in\Lambda}\varphi_\lambda)^{\Rightarrow}(\prod_{\lambda\in\Lambda}\mathbb{F}_\lambda) \subseteq \prod_{\lambda\in\Lambda}\varphi_\lambda^{\Rightarrow}(\mathbb{F}_\lambda)$.

Conversely, by Proposition 4.13, we have

$$\begin{aligned}\prod_{\lambda\in\Lambda}\varphi_\lambda^{\Rightarrow}(\mathbb{F}_\lambda) &\subseteq \prod_{\lambda\in\Lambda}\varphi_\lambda^{\Rightarrow}\Big(pr_\lambda^{\Rightarrow}\Big(\prod_{\mu\in\Lambda}\mathbb{F}_\mu\Big)\Big)\\ &= \prod_{\lambda\in\Lambda}(\varphi_\lambda\circ pr_\lambda)^{\Rightarrow}\Big(\prod_{\mu\in\Lambda}\mathbb{F}_\mu\Big)\\ &= \prod_{\lambda\in\Lambda}\Big(qr_\lambda\circ\prod_{\mu\in\Lambda}\varphi_\mu\Big)^{\Rightarrow}\Big(\prod_{\mu\in\Lambda}\mathbb{F}_\mu\Big)\\ &= \prod_{\lambda\in\Lambda}qr_\lambda^{\Rightarrow}\bigg(\Big(\prod_{\mu\in\Lambda}\varphi_\mu\Big)^{\Rightarrow}\Big(\prod_{\mu\in\Lambda}\mathbb{F}_\mu\Big)\bigg)\\ &\subseteq \Big(\prod_{\lambda\in\Lambda}\varphi_\lambda\Big)^{\Rightarrow}\Big(\prod_{\lambda\in\Lambda}\mathbb{F}_\lambda\Big),\end{aligned}$$

where the second equality holds since $\varphi_\lambda\circ pr_\lambda = qr_\lambda\circ\prod_{\mu\in\Lambda}\varphi_\mu$. This proves that

$$\Big(\prod_{\lambda\in\Lambda}\varphi_\lambda\Big)^{\Rightarrow}\Big(\prod_{\lambda\in\Lambda}\mathbb{F}_\lambda\Big) = \prod_{\lambda\in\Lambda}\varphi_\lambda^{\Rightarrow}(\mathbb{F}_\lambda).$$

□

Theorem 4.17. *Suppose that L satisfies* (MID). *Let $\{\varphi_\lambda : (X_\gamma, \gamma_{X_\lambda}) \longrightarrow (Y_\lambda, \gamma_{Y_\lambda})\}_{\lambda\in\Lambda}$ be a family of quotient mappings in $\top$-**Fil**. Then the product mapping*

$$\prod_{\lambda\in\Lambda}\varphi_\lambda : \bigg(\prod_{\lambda\in\Lambda}X_\lambda, \prod_{\lambda\in\Lambda}\gamma_{X_\lambda}\bigg)\longrightarrow\bigg(\prod_{\lambda\in\Lambda}Y_\lambda, \prod_{\lambda\in\Lambda}\gamma_{Y_\lambda}\bigg)$$

is a quotient mapping.

Proof. Define

$$(X,\gamma_X) = \bigg(\prod_{\lambda\in\Lambda}X_\lambda, \prod_{\lambda\in\Lambda}\gamma_{X_\lambda}\bigg)\quad\text{and}\quad (Y,\gamma_Y) = \bigg(\prod_{\lambda\in\Lambda}Y_\lambda, \prod_{\lambda\in\Lambda}\gamma_{Y_\lambda}\bigg).$$

By Proposition 3.3, we have

$$\gamma_Y = \Big\{\mathbb{H}\in\mathcal{F}_L^{\top}(Y)\,|\,\forall\lambda\in\Lambda, qr_\lambda^{\Rightarrow}(\mathbb{H})\in\gamma_{X_\lambda}\Big\}.$$

By Definition 3.6, we know

$$\gamma'_Y = \bigg\{\mathbb{K}\in\mathcal{F}_L^{\top}(Y)\,\Big|\,\exists\mathbb{G}\in\gamma_X, \text{s.t.}, \Big(\prod_{\lambda\in\Lambda}\varphi_\lambda\Big)^{\Rightarrow}(\mathbb{G})\subseteq\mathbb{K}\bigg\}.$$

In order to show that $\prod_{\lambda\in\Lambda}\varphi_\lambda$ is a quotient mapping, it suffices to verify that $\gamma_Y = \gamma'_Y$. For each $\mathbb{K}\in\gamma'_Y$, there exists $\mathbb{G}\in\gamma_X$ such that $(\prod_{\lambda\in\Lambda}\varphi_\lambda)^{\Rightarrow}(\mathbb{G})\subseteq\mathbb{K}$. By the definition of γ_X, we know $pr_\lambda^{\Rightarrow}(\mathbb{G})\in\gamma_{X_\lambda}$ for all $\lambda\in\Lambda$. Since φ_γ is a quotient mapping, it follows that

$$qr_\lambda^{\Rightarrow}\bigg(\Big(\prod_{\lambda\in\Lambda}\varphi_\lambda\Big)\bigg)^{\Rightarrow}(\mathbb{G}) = \Big(qr_\lambda\circ\prod_{\lambda\in\Lambda}\varphi_\lambda\Big)^{\Rightarrow}(\mathbb{G}) = (\varphi_\lambda\circ pr_\lambda)^{\Rightarrow}(\mathbb{G}) = \varphi_\lambda^{\Rightarrow}(pr_\lambda^{\Rightarrow}(\mathbb{G}))\in\gamma_{Y_\lambda}.$$

By $qr_\lambda^{\Rightarrow}\circ\Big(\prod_{\lambda\in\Lambda}\varphi_\lambda\Big)^{\Rightarrow}(\mathbb{G})\subseteq qr_\lambda^{\Rightarrow}(\mathbb{K})$, we have $qr_\lambda^{\Rightarrow}(\mathbb{K})\in\gamma_{Y_\lambda}$ for each $\lambda\in\Lambda$, which implies $\mathbb{K}\in\gamma_Y$. This shows $\gamma'_Y\subseteq\gamma_Y$.

Conversely, let $\mathbb{H} \in \gamma_Y$. By the definition of γ_Y, we have $qr_\lambda^{\Rightarrow}(\mathbb{H}) \in \gamma_{Y_\lambda}$ for each $\lambda \in \Lambda$. Then for each $\lambda \in \Lambda$, there exists $\mathbb{F}_\lambda \in \gamma_{X_\lambda}$ such that $\varphi_\lambda^{\Rightarrow}(\mathbb{F}_\lambda) \subseteq qr_\lambda^{\Rightarrow}(\mathbb{H})$ since φ_λ is a quotient mapping. Let

$$\mathcal{F}_\lambda = \left\{\mathbb{F}_\lambda \in \mathcal{F}_L^\top(X_\lambda) \,|\, \mathbb{F}_\lambda \in \gamma_{X_\lambda} \text{ and } \varphi_\lambda^{\Rightarrow}(\mathcal{F}_\lambda) \subseteq qr_\lambda^{\Rightarrow}(\mathbb{H})\right\}$$

for each $\lambda \in \Lambda$ and let

$$\prod_{\lambda\in\Lambda} \mathcal{F}_\lambda = \left\{f : \Lambda \longrightarrow \coprod_{\lambda\in\Lambda} \mathcal{F}_\lambda \,\Big|\, \forall \lambda \in \Lambda, f(\lambda) \in \mathcal{F}_\lambda\right\}$$

be the set of choice functions, i.e.,

$$\forall \lambda \in \Lambda, \exists \mathbb{F}_\lambda \in \gamma_{X_\lambda}, \text{s.t.}, \varphi_\lambda^{\Rightarrow}(\mathbb{F}_\lambda) \subseteq qr_\lambda^{\Rightarrow}(\mathbb{H}) \Longleftrightarrow \exists f \in \prod_{\lambda\in\Lambda} \mathcal{F}_\lambda, \text{s.t.}, \forall \lambda \in \Lambda, \varphi_\lambda^{\Rightarrow}(f(\lambda)) \subseteq qr_\lambda^{\Rightarrow}(\mathbb{H}).$$

Then there exists $f \in \prod_{\lambda\in\Lambda} \mathcal{F}_\lambda$ such that $\varphi_\lambda^{\Rightarrow}(f(\lambda)) \subseteq qr_\lambda^{\Rightarrow}(\mathbb{H})$ for each $\lambda \in \Lambda$. It follows from Proposition 2.9 that $qr_\lambda^{\Leftarrow} \circ \varphi_\lambda^{\Rightarrow}(f(\lambda)) \subseteq \mathbb{H}$ for each $\lambda \in \Lambda$. This implies that $\bigvee_{\lambda\in\Lambda} qr_\lambda^{\Leftarrow} \circ \varphi_\lambda^{\Rightarrow}(f(\lambda)) \subseteq \mathbb{H}$, i.e., $\prod_{\lambda\in\Lambda} \varphi_\lambda^{\Rightarrow}(f(\lambda)) \subseteq \mathbb{H}$. By Lemma 4.16, we obtain there exists $\prod_{\lambda\in\Lambda} f(\lambda) \in \gamma_X$ such that $\left(\prod_{\lambda\in\Lambda} \varphi_\lambda\right)^{\Rightarrow}\left(\prod_{\lambda\in\Lambda} f(\lambda)\right) = \prod_{\lambda\in\Lambda} \varphi_\lambda^{\Rightarrow}(f(\lambda)) \subseteq \mathbb{H}$. Then it follows from the definition of γ'_Y that $\mathbb{H} \in \gamma'_Y$. By the arbitrariness of $\mathbb{H}$, we obtain that $\gamma_Y \subseteq \gamma'_Y$. □

By Theorems 4.5, 4.8 and 4.17, we obtain the following theorem.

Theorem 4.18. *Suppose that L satisfies* (MID). *Then* $\top$-**Fil** *is a strong topological universe.*

5. Subcategories of ⊤-Fil

In this section, we will propose $\top$-semi-Cauchy structures, $\top$-Cauchy structures and complete $\top$-filter structures, which can be considered as generalizations of semi-Cauchy structures, Cauchy structures and complete filter structures respectively. Then we will establish their categorical relationships with $\top$-filter structures as well as their categorical properties.

5.1. ⊤-**SChy**

Definition 5.1. A $\top$-filter structure γ on X is called $\top$-semi-Cauchy provided that

(TSChy) If there exist $\mathbb{F}_1, \cdots, \mathbb{F}_n \in \gamma$ such that $\bigcap_{i=1}^{n} \mathbb{F}_i \times \mathbb{F}_i \subseteq \mathbb{F} \times \mathbb{F}$, then $\mathbb{F} \in \gamma$.

For a $\top$-semi-Cauchy structure γ on X, the pair (X, γ) is called a $\top$-semi-Cauchy space.

The category of $\top$-semi-Cauchy spaces, as a full subcategory of $\top$-**Fil**, is denoted by $\top$-**SChy**. For convenience, we use $I : \top$-**SChy** $\longrightarrow \top$-**Fil** to denote the inclusion functor.

Proposition 5.2. *Let* (X, γ) *be a* $\top$*-filter space. Define* $\gamma^\diamond \subseteq \mathcal{F}_L^\top(X)$ *by*

$$\gamma^\diamond = \left\{\mathbb{F} \in \mathcal{F}_L^\top(X) \,\Big|\, \exists \mathbb{F}_1, \cdots, \mathbb{F}_n \in \gamma, \text{s.t.}, \bigcap_{i=1}^{n} \mathbb{F}_i \times \mathbb{F}_i \subseteq \mathbb{F} \times \mathbb{F}\right\}.$$

Then $(X, \gamma^\diamond)$ *is a* $\top$*-semi-Cauchy space.*

Proof. (TF1) and (TF2) are obvious. It remains to verify (TSChy). Suppose that $\mathbb{G}_1, \cdots, \mathbb{G}_n \in \gamma^\diamond$ and $\mathbb{G} \in \mathcal{F}_L^\top(X)$ such that $\bigcap_{i=1}^n \mathbb{G}_i \times \mathbb{G}_i \subseteq \mathbb{G} \times \mathbb{G}$. For each $\mathbb{G}_i$, by the definition of $\gamma^\diamond$, there exist $\mathbb{F}_{i1}, \cdots, \mathbb{F}_{im_i}$ such that $\bigcap_{j=1}^{m_i} \mathbb{F}_{ij} \times \mathbb{F}_{ij} \subseteq \mathbb{G}_i \times \mathbb{G}_i$. This implies that there exist $\mathbb{F}_{11}, \cdots, \mathbb{F}_{1m_1}, \cdots, \mathbb{F}_{i1}, \cdots, \mathbb{F}_{im_i}, \cdots, \mathbb{F}_{n1}, \cdots, \mathbb{F}_{nm_n}$ such that

$$\bigcap_{q=1}^{m_1+\cdots+m_n} \mathbb{F}_q \times \mathbb{F}_q \subseteq \bigcap_{i=1}^{n} \mathbb{G}_i \times \mathbb{G}_i \subseteq \mathbb{G} \times \mathbb{G}.$$

This shows $\mathbb{G} \in \gamma^\diamond$, as desired. □

Proposition 5.3. *Suppose that L satisfies* (MID). *If $\varphi : (X, \gamma_X) \longrightarrow (Y, \gamma_Y)$ is a Cauchy continuous mapping between ⊤-filter spaces, then $\varphi : (X, \gamma_X^\diamond) \longrightarrow (Y, \gamma_Y^\diamond)$ is a Cauchy continuous mapping between ⊤-semi-Cauchy spaces.*

Proof. Take any $\mathbb{F} \in \gamma_X^\diamond$. Then there exist $\mathbb{F}_1, \cdots, \mathbb{F}_n \in \gamma_X$ such that $\bigcap_{i=1}^n \mathbb{F}_i \times \mathbb{F}_i \subseteq \mathbb{F} \times \mathbb{F}$. Since $\varphi : (X, \gamma_X) \longrightarrow (Y, \gamma_Y)$ is Cauchy continuous, there exist $\varphi^\Rightarrow(\mathbb{F}_1), \cdots, \varphi^\Rightarrow(\mathbb{F}_n) \in \gamma_Y$ such that

$$\begin{aligned}
\bigcap_{i=1}^{n} \varphi^\Rightarrow(\mathbb{F}_i) \times \varphi^\Rightarrow(\mathbb{F}_i) &= \bigcap_{i=1}^{n} (\varphi \times \varphi)^\Rightarrow(\mathbb{F}_i \times \mathbb{F}_i) \\
&= (\varphi \times \varphi)^\Rightarrow\Big(\bigcap_{i=1}^{n} \mathbb{F}_i \times \mathbb{F}_i\Big) \\
&\subseteq (\varphi \times \varphi)^\Rightarrow(\mathbb{F} \times \mathbb{F}) \\
&= \varphi^\Rightarrow(\mathbb{F}) \times \varphi^\Rightarrow(\mathbb{F}),
\end{aligned}$$

where that the first and the last equalities follow from Proposition 2.10. By Proposition 5.2, we obtain $\varphi^\Rightarrow(\mathbb{F}) \in \gamma_Y^\diamond$. This shows $\varphi : (X, \gamma_X^\diamond) \longrightarrow (Y, \gamma_Y^\diamond)$ is a Cauchy continuous mapping. □

By Propositions 5.2 and 5.3, we get a functor.

$$F : \begin{cases} \top\text{-}\mathbf{Fil} & \longrightarrow & \top\text{-}\mathbf{SChy} \\ (X, \gamma) & \longmapsto & (X, \gamma^\diamond) \\ \varphi & \longmapsto & \varphi \end{cases}$$

Proposition 5.4. *Suppose that L satisfies* (MID). *Then F is a left adjoint to I.*

Proof. It is easy to verify that $F \circ I = id_{\top\text{-}\mathbf{SChy}}$ and $I \circ F(X, \gamma) = (X, \gamma^\diamond) \supseteq (X, \gamma)$ for each ⊤-semi-Cauchy space (X, γ). Thus, F is a left adjoint to I. □

By Proposition 5.4 and Theorem 2.2.12 in [40], we get

Corollary 5.5. *Suppose that L satisfies* (MID). *Then* ⊤-**SChy** *is a bireflective subcategory of* ⊤-**Fil**.

Corollary 5.6. *Suppose that L satisfies* (MID). *Then* ⊤-**SChy** *is a topological category.*

Lemma 5.7 ([25]). *If L is distributive, then for each $\mathbb{F}, \mathbb{G} \in \mathcal{F}_L^\top(X)$ and $\mathbb{H} \in \mathcal{F}_L^\top(Y)$,*

$$(\mathbb{F} \cap \mathbb{G}) \times \mathbb{H} = (\mathbb{F} \times \mathbb{H}) \cap (\mathbb{G} \times \mathbb{H}).$$

Lemma 5.8. *Suppose that L satisfies* (MID). *Let $\mathbb{H}_1, \mathbb{H}_2, \mathbb{H} \in \mathcal{F}_L^\top(X)$ and $\mathbb{F} \in \mathcal{F}_L^\top(Y)$. If $(\mathbb{H}_1 \times \mathbb{H}_1) \cap (\mathbb{H}_2 \times \mathbb{H}_2) \subseteq \mathbb{H} \times \mathbb{H}$, then*

$$\Big((\mathbb{H}_1 \times \mathbb{F}) \times (\mathbb{H}_1 \times \mathbb{F})\Big) \cap \Big((\mathbb{H}_2 \times \mathbb{F}) \times (\mathbb{H}_2 \times \mathbb{F})\Big) \subseteq (\mathbb{H} \times \mathbb{F}) \times (\mathbb{H} \times \mathbb{F}).$$

Proof. Define a mapping $\varphi : (X \times X) \times (Y \times Y) \longrightarrow (X \times Y) \times (X \times Y)$ by

$$\varphi\Big((x_1, x_2), (y_1, y_2)\Big) = \Big((x_1, y_1), (x_2, y_2)\Big).$$

Then φ is bijective. By Corollary 4.15, we know $\mathbb{B}_{\mathbb{H}\times\mathbb{H}} = \{A \times B \,|\, A, B \in \mathbb{H}\}$ is a $\top$-filter base of $\mathbb{H} \times \mathbb{H}$ and $\mathbb{B}_{\mathbb{F}\times\mathbb{F}} = \{C \times D \,|\, C, D \in \mathbb{F}\}$ is a $\top$-filter base of $\mathbb{F} \times \mathbb{F}$. This implies that

$$\mathbb{B}_1 = \{\varphi^{\rightarrow}\Big((A \times B) \times (C \times D) \,|\, A \times B \in \mathbb{B}_{\mathbb{H}\times\mathbb{H}}, C \times D \in \mathbb{B}_{\mathbb{F}\times\mathbb{F}}\Big)\}$$

is a $\top$-filter base of $\varphi^{\Rightarrow}\Big((\mathbb{H} \times \mathbb{H}) \times (\mathbb{F} \times \mathbb{F})\Big)$ and

$$\mathbb{B}_2 = \{(A \times C) \times (B \times D) \,|\, A, B \in \mathbb{H}, C, D \in \mathbb{F}\}$$

is a $\top$-filter base of $(\mathbb{H} \times \mathbb{H}) \times (\mathbb{F} \times \mathbb{F})$. Since φ is bijective, it is easy to verify that $\mathbb{B}_1 = \mathbb{B}_2$. This implies that

$$\varphi^{\Rightarrow}\Big((\mathbb{H} \times \mathbb{H}) \times (\mathbb{F} \times \mathbb{F})\Big) = (\mathbb{H} \times \mathbb{F}) \times (\mathbb{H} \times \mathbb{F}).$$

Since $(\mathbb{H}_1 \times \mathbb{H}_1) \cap (\mathbb{H}_2 \times \mathbb{H}_2) \subseteq \mathbb{H} \times \mathbb{H}$, it follows from Lemma 5.7 that

$$\Big((\mathbb{H}_1 \times \mathbb{H}_1) \times (\mathbb{F} \times \mathbb{F})\Big) \cap \Big((\mathbb{H}_2 \times \mathbb{H}_2) \times (\mathbb{F} \times \mathbb{F})\Big) \subseteq (\mathbb{H} \times \mathbb{H}) \times (\mathbb{F} \times \mathbb{F}).$$

This implies that

$$\begin{aligned}
&\Big((\mathbb{H}_1 \times \mathbb{F}) \times (\mathbb{H}_1 \times \mathbb{F})\Big) \cap \Big((\mathbb{H}_2 \times \mathbb{F}) \times (\mathbb{H}_2 \times \mathbb{F})\Big) \\
&= \varphi^{\Rightarrow}\Big((\mathbb{H}_1 \times \mathbb{H}_1) \times (\mathbb{F} \times \mathbb{F})\Big) \cap \varphi^{\Rightarrow}\Big((\mathbb{H}_2 \times \mathbb{H}_2) \times (\mathbb{F} \times \mathbb{F})\Big) \\
&= \varphi^{\Rightarrow}\Big(\Big((\mathbb{H}_1 \times \mathbb{H}_1) \times (\mathbb{F} \times \mathbb{F})\Big) \cap \Big((\mathbb{H}_2 \times \mathbb{H}_2) \times (\mathbb{F} \times \mathbb{F})\Big)\Big) \\
&\subseteq \varphi^{\Rightarrow}\Big((\mathbb{H} \times \mathbb{H}) \times (\mathbb{F} \times \mathbb{F})\Big) \\
&= (\mathbb{H} \times \mathbb{F}) \times (\mathbb{H} \times \mathbb{F}),
\end{aligned}$$

as desired. □

Theorem 5.9. *Suppose that L satisfies* (MID). *Then* $\top$-**SChy** *is Cartesian closed.*

Proof. By Corollaries 5.5 and 5.6, we only need to verify that $\top$-**SChy** is closed under the formation of power objects in $\top$-**Fil**. Let (X, γ_X) be a $\top$-filter space and (Y, γ_Y) be a $\top$-semi-Cauchy space. By Proposition 4.1, the power object in $\top$-**Fil** has the following form

$$\gamma_{[X,Y]} = \Big\{\mathbb{H} \in \mathcal{F}_L^{\top}([X, Y]) \,|\, \forall \mathbb{F} \in \mathcal{F}_L^{\top}(X), \mathbb{F} \in \gamma_X \text{ implies } ev^{\Rightarrow}(\mathbb{H} \times \mathbb{F}) \in \gamma_Y\Big\}.$$

It remains to show that $\gamma_{[X,Y]}$ satisfies (TSChy). If there exist $\mathbb{H}_1, \cdots, \mathbb{H}_n \in \gamma_{[X,Y]}$ such that $\bigcap_{i=1}^n \mathbb{H}_i \times \mathbb{H}_i \subseteq \mathbb{H} \times \mathbb{H}$, then it follows from Lemma 5.8 that

$$\bigcap_{i=1}^{n} (\mathbb{H}_i \times \mathbb{F}) \times (\mathbb{H}_i \times \mathbb{F}) \subseteq (\mathbb{H} \times \mathbb{F}) \times (\mathbb{H} \times \mathbb{F})$$

for each $\mathbb{F} \in \gamma_X$. Since $\mathbb{H}_i \in \gamma_{[X,Y]}$ for any $i = 1, \cdots, n$, it follows that $ev^{\Rightarrow}(\mathbb{H}_i \times \mathbb{F}) \in \gamma_Y$. This shows that there exist $ev^{\Rightarrow}(\mathbb{H}_1 \times \mathbb{F}), \cdots, ev^{\Rightarrow}(\mathbb{H}_n \times \mathbb{F}) \in \gamma_Y$ such that

$$\bigcap_{i=1}^{n} ev^{\Rightarrow}(\mathbb{H}_i \times \mathbb{F}) \times ev^{\Rightarrow}(\mathbb{H}_i \times \mathbb{F}) = (ev \times ev)^{\Rightarrow}\left(\bigcap_{i=1}^{n} (\mathbb{H}_i \times \mathbb{F}) \times (\mathbb{H}_i \times \mathbb{F})\right) \subseteq ev^{\Rightarrow}(\mathbb{H} \times \mathbb{F}) \times ev^{\Rightarrow}(\mathbb{H} \times \mathbb{F}).$$

Since (Y, γ_Y) is a $\top$-semi-Cauchy space, we obtain $ev^{\Rightarrow}(\mathbb{H} \times \mathbb{F}) \in \gamma_Y$. By the definition of $\gamma_{[X,Y]}$, we have $\mathbb{H} \in \gamma_{[X,Y]}$. □

5.2. $\top$-**Chy**

Definition 5.10. ([42]) A $\top$-filter structure γ on X is called $\top$-Cauchy provided that

(TChy) $\mathbb{F} \cap \mathbb{G} \in \gamma$ whenever $\mathbb{F}, \mathbb{G} \in \gamma$ and $\mathbb{F} \vee \mathbb{G}$ exists.

For a $\top$-Cauchy sturcture γ on X, the pair (X, γ) is called a $\top$-Cauchy space.

The category of $\top$-Cauchy spaces, as a full subcategory of $\top$-**Fil**, is denoted by $\top$-**Chy**. For convenience, we use $I : \top\text{-}\mathbf{Chy} \longrightarrow \top\text{-}\mathbf{Fil}$ to denote the inclusion functor.

Let $\gamma(X) = \{\overline{\gamma} \mid (X, \overline{\gamma}) \text{ is a } \top\text{-Cauchy space}\}$.

Proposition 5.11. *Let (X, γ) be a $\top$-filter space. Define $\gamma^\star \subseteq \mathcal{F}_L^\top(X)$ by*

$$\gamma^\star = \bigcap \left\{\overline{\gamma} \subseteq \mathcal{F}_L^\top(X) \,|\, \overline{\gamma} \in \gamma(X) \text{ and } \gamma \subseteq \overline{\gamma}\right\}.$$

Then $(X, \gamma^\star)$ is a $\top$-Cauchy space.

Proof. It is easy and is omitted. □

Proposition 5.12. *Let (Y, γ_Y) be a $\top$-Cauchy space and $\varphi : X \longrightarrow Y$ be a mapping. Then $\gamma^* = \left\{\mathbb{F} \in \mathcal{F}_L^\top(X) \,|\, \varphi^\Rightarrow(\mathbb{F}) \in \gamma_Y\right\}$ is a $\top$-Cauchy structure on X.*

Proof. It is straightforward to verify that γ^* satisfies (TF1) and (TF2).

(TChy) Let $\mathbb{F}, \mathbb{G} \in \gamma^*$ such that $\mathbb{F} \vee \mathbb{G}$ exists. Then $\varphi^\Rightarrow(\mathbb{F}) \in \gamma_Y$ and $\varphi^\Rightarrow(\mathbb{G}) \in \gamma_Y$. For each $A \in \varphi^\Rightarrow(\mathbb{F})$ and $B \in \varphi^\Rightarrow(\mathbb{G})$, it follows that $\varphi^\leftarrow(A) \in \mathbb{F}$ and $\varphi^\leftarrow(B) \in \mathbb{G}$. Since $\mathbb{F} \vee \mathbb{G}$ exists, we have

$$\bigvee_{y \in Y}(A \wedge B)(y) \geq \bigvee_{y \in \varphi(X)} (A \wedge B)(y) = \bigvee_{x \in X}(A \wedge B)(\varphi(x)) = \bigvee_{x \in X}(\varphi^\leftarrow(A) \wedge \varphi^\leftarrow(B))(x) = \top.$$

This implies that $\varphi^\Rightarrow(\mathbb{F}) \vee \varphi^\Rightarrow(\mathbb{G})$ exists. By (TChy), we obtain $\varphi^\Rightarrow(\mathbb{F} \cap \mathbb{G})$=$\varphi^\Rightarrow(\mathbb{F}) \cap \varphi^\Rightarrow(\mathbb{G}) \in \gamma_Y$. Thus, $\mathbb{F} \cap \mathbb{G} \in \gamma^*$. □

Proposition 5.13. *If $\varphi : (X, \gamma_X) \longrightarrow (Y, \gamma_Y)$ is a Cauchy continuous mapping between $\top$-filter spaces, then $\varphi : (X, \gamma_X^\star) \longrightarrow (Y, \gamma_Y^\star)$ is a Cauchy continuous mapping between $\top$-Cauchy spaces.*

Proof. By Proposition 5.12, we know $\gamma_X^* = \{\mathbb{F} \in \mathcal{F}_L^\top(X) \,|\, \varphi^\Rightarrow(\mathbb{F}) \in \gamma_Y^\star\}$ is a $\top$-Cauchy structure on X. By the Cauchy continuity of $\varphi : (X, \gamma_X) \longrightarrow (Y, \gamma_Y)$ and $\gamma_Y \subseteq \gamma_Y^\star$, we get $\gamma_X \subseteq \gamma_X^*$. This shows that γ_X^* is a $\top$-Cauchy structure satisfying $\gamma_X \subseteq \gamma_X^*$. Then it follows that $\gamma_X^\star \subseteq \gamma_X^*$. Take any $\mathbb{F} \in \gamma_X^\star$. Then $\mathbb{F} \in \gamma_X^*$. By the definition of γ_X^*, we have $\varphi^\Rightarrow(\mathbb{F}) \in \gamma_Y^\star$. □

By Propositions 5.11 and 5.13, we construct a functor.

$$G : \begin{cases} \top\text{-}\mathbf{Fil} & \longrightarrow & \top\text{-}\mathbf{Chy} \\ (X, \gamma) & \longmapsto & (X, \gamma^\star) \\ \varphi & \longmapsto & \varphi \end{cases}$$

Proposition 5.14. *G is a left adjoint to I.*

Proof. It follows immediately from the facts that $G \circ I(X, \gamma) = (X, \gamma)$ for each $\top$-Cauchy space (X, γ) and $I \circ G(X, \gamma) = (X, \gamma^\star) \supseteq (X, \gamma)$ for each $\top$-filter space (X, γ). □

By Proposition 5.14 and Theorem 2.2.12 in [40], we obtain

Corollary 5.15. $\top$-**Chy** *is a bireflective subcategory of* $\top$-**Fil**.

Corollary 5.16. $\top$-**Chy** *is a topological category over* **Set**.

Proposition 5.17. *Suppose that L satisfies* (MID). *Then* $\top$-**Chy** *is Cartesian closed.*

Proof. By Corollaries 5.15 and 5.16, it suffices to show that $\top$-**Chy** is closed under formation of power objects in $\top$-**Fil**. Let (X, γ_X) be a $\top$-filter space and (Y, γ_Y) be a $\top$-Cauchy space. Then

$$\gamma_{[X,Y]} = \left\{\mathbb{H} \in \mathcal{F}_L^\top([X, Y]) \,|\, \forall \mathbb{F} \in \mathcal{F}_L^\top(X), \mathbb{F} \in \gamma_X \text{ implies } ev^\Rightarrow(\mathbb{H} \times \mathbb{F}) \in \gamma_Y\right\}.$$

Next, we will verify that $\gamma_{[X,Y]}$ satisfies (TChy). Take any $\mathbb{H}_1, \mathbb{H}_2 \in \gamma_{[X,Y]}$ such that $\mathbb{H}_1 \vee \mathbb{H}_2$ exists. In order to show $\mathbb{H}_1 \cap \mathbb{H}_2 \in \gamma_{[X,Y]}$, we divide into three steps.

Step 1: Take any $\Phi_1 \in \mathbb{H}_1$, $\Phi_2 \in \mathbb{H}_2$ and $A_1, A_2 \in \mathbb{F}$. Then

$$\begin{aligned}
&\bigvee_{(\varphi,x)\in[X,Y]\times X} (\Phi_1 \times A_1)(\varphi, x) \wedge (\Phi_2 \times A_2)(\varphi, x) \\
&= \bigvee_{(\varphi,x)\in[X,Y]\times X} (\Phi_1 \wedge \Phi_2)(\varphi) \wedge (A_1 \wedge A_2)(x) \\
&\geqslant \bigvee_{\varphi\in[X,Y]} (\Phi_1 \wedge \Phi_2)(\varphi) * \bigvee_{x\in X} (A_1 \wedge A_2)(x) \\
&= \top.
\end{aligned}$$

Then for each $\Psi_1 \in \mathbb{H}_1 \times \mathbb{F}$ and $\Psi_2 \in \mathbb{H}_2 \times \mathbb{F}$, it follows that

$$\begin{aligned}
\top &= \bigvee_{\Phi_1\in\mathbb{H}_1, A_1\in\mathbb{F}} \mathcal{S}_{[X,Y]\times X}(\Phi_1 \times A_1, \Psi_1) * \bigvee_{\Phi_2\in\mathbb{H}_2, A_2\in\mathbb{F}} \mathcal{S}_{[X,Y]\times X}(\Phi_2 \times A_2, \Psi_2) \\
&\leqslant \bigvee_{\Phi_1\in\mathbb{H}_1, A_1\in\mathbb{F}} \bigvee_{\Phi_2\in\mathbb{H}_2, A_2\in\mathbb{F}} \mathcal{S}_{[X,Y]\times X}((\Phi_1 \times A_1) \wedge (\Phi_2 \times A_2), \Psi_1 \wedge \Psi_2) \\
&\qquad * \left(\bigvee_{(\varphi,x)\in[X,Y]\times X} (\Phi_1 \times A_1) \wedge (\Phi_2 \times A_2)(\varphi, x)\right) \\
&= \bigvee_{\Phi_1\in\mathbb{H}_1, A_1\in\mathbb{F}} \bigvee_{\Phi_2\in\mathbb{H}_2, A_2\in\mathbb{F}} \bigvee_{(\varphi,x)\in[X,Y]\times X} \Big(\mathcal{S}_{[X,Y]\times X}((\Phi_1 \times A_1) \wedge (\Phi_2 \times A_2), \Psi_1 \wedge \Psi_2)\Big) \\
&\qquad * \Big((\Phi_1 \times A_1) \wedge (\Phi_2 \times A_2)\Big)(\varphi, x) \\
&\leqslant \bigvee_{(\varphi,x)\in[X,Y]\times X} (\Psi_1 \wedge \Psi_2)(\varphi, x).
\end{aligned}$$

By Corollary 4.10, we know $(\mathbb{H}_1 \times \mathbb{F}) \vee (\mathbb{H}_2 \times \mathbb{F})$ exists.

Step 2: Take any $\mathbb{G}_1 \in ev^\Rightarrow(\mathbb{H}_1 \times \mathbb{F})$ and $\mathbb{G}_2 \in ev^\Rightarrow(\mathbb{H}_2 \times \mathbb{F})$. Then

$$\begin{aligned}
\top &= \bigvee_{(\varphi,x)\in[X,Y]\times X} (ev^\leftarrow(\mathbb{G}_1) \wedge ev^\leftarrow(\mathbb{G}_2))(\varphi, x) \quad \text{(by } \textbf{Step 1}\text{)} \\
&= \bigvee_{(\varphi,x)\in[X,Y]\times X} \mathbb{G}_1(ev(\varphi, x)) \wedge \mathbb{G}_2(ev(\varphi, x)) \\
&= \bigvee_{(\varphi,x)\in[X,Y]\times X} \mathbb{G}_1(\varphi(x)) \wedge \mathbb{G}_2(\varphi(x)) \\
&= \bigvee_{y\in\varphi(X)} (\mathbb{G}_1 \wedge \mathbb{G}_2)(y) \\
&\leqslant \bigvee_{y\in Y} (\mathbb{G}_1 \wedge \mathbb{G}_2)(y).
\end{aligned}$$

Hence $ev^{\Rightarrow}(\mathbb{H}_1 \times \mathbb{F}) \vee ev^{\Rightarrow}(\mathbb{H}_2 \times \mathbb{F})$ exists.

Step 3: Take any $\mathbb{F} \in \gamma_X$. Then $ev^{\Rightarrow}(\mathbb{H}_1 \times \mathbb{F}) \in \gamma_Y$ and $ev^{\Rightarrow}(\mathbb{H}_2 \times \mathbb{F}) \in \gamma_Y$. By **Step 2**, we know $ev^{\Rightarrow}(\mathbb{H}_1 \times \mathbb{F}) \vee ev^{\Rightarrow}(\mathbb{H}_2 \times \mathbb{F})$ exists. Since γ_Y satisfies (TChy), we obtain $ev^{\Rightarrow}(\mathbb{H}_1 \times \mathbb{F}) \cap ev^{\Rightarrow}(\mathbb{H}_2 \times \mathbb{F}) \in \gamma_Y$. By Proposition 2.10 and Lemma 5.7, it follows that $ev^{\Rightarrow}((\mathbb{H}_1 \cap \mathbb{H}_2) \times \mathbb{F}) \in \gamma_Y$. This shows $\mathbb{H}_1 \cap \mathbb{H}_2 \in \gamma_{[X,Y]}$. Thus, $\gamma_{[X,Y]}$ satisfies (TChy). □

5.3. $\top$-**CFil**

Definition 5.18. A $\top$-filter structure γ on X is called complete provided that

(TC) For any $\mathbb{F} \in \gamma$, there exists $x \in X$ such that $\mathbb{F} \cap [x] \in \gamma$.

For a complete $\top$-filter structure γ on X, the pair (X, γ) is called a complete $\top$-filter space.

The category of complete $\top$-filter spaces, as a full subcategory of $\top$-**Fil**, is denoted by $\top$-**CFil**. For convenience, we use $I : \top\text{-}\mathbf{CFil} \longrightarrow \top\text{-}\mathbf{Fil}$ to denote the inclusion functor.

Proposition 5.19. *Let (X, γ) be a $\top$-filter space. Define $\gamma^c \subseteq \mathcal{F}_L^{\top}(X)$ by*

$$\gamma^c = \left\{ \mathbb{F} \in \mathcal{F}_L^{\top}(X) \mid \exists x \in X, \text{s.t.}, \mathbb{F} \cap [x] \in \gamma \right\}.$$

Then (X, γ^c) is a complete $\top$-filter space and $\gamma^c \subseteq \gamma$.

Proof. It is easy to check γ^c satisfies (TF1), (TF2) and (TC). Take any $\mathbb{F} \in \gamma^c$. Then there exists $x \in X$ such that $\mathbb{F} \cap [x] \in \gamma$. By (TF2), we obtain $\mathbb{F} \in \gamma$. Thus, $\gamma^c \subseteq \gamma$. □

Proposition 5.20. *If $\varphi : (X, \gamma_X) \longrightarrow (Y, \gamma_Y)$ between $\top$-filter spaces is Cauchy continuous, then $\varphi : (X, \gamma_X^c) \longrightarrow (Y, \gamma_Y^c)$ between complete $\top$-filter spaces is Cauchy continuous.*

Proof. Take any $\mathbb{F} \in \gamma_X^c$. Then there exists $x \in X$ such that $\mathbb{F} \cap [x] \in \gamma_X$. Since $\varphi : (X, \gamma_X) \longrightarrow (Y, \gamma_Y)$ is Cauchy continuous, it follows that there exists $\varphi(x) \in Y$ such that

$$\varphi^{\Rightarrow}(\mathbb{F}) \cap [\varphi(x)] = \varphi^{\Rightarrow}(\mathbb{F} \cap [x]) \in \gamma_Y.$$

By the definition of γ_Y^c, we obtain $\varphi^{\Rightarrow}(\mathbb{F}) \in \gamma_Y^c$. □

Thus, we get a functor.

$$H : \begin{cases} \top\text{-}\mathbf{Fil} & \longrightarrow & \top\text{-}\mathbf{CFil} \\ (X, \gamma) & \longmapsto & (X, \gamma^c) \\ \varphi & \longmapsto & \varphi \end{cases}$$

Proposition 5.21. *H is a right adjoint to I.*

Proof. For each $\top$-filter space (X, γ), we get $I \circ H(X, \gamma) = (X, \gamma^c) \subseteq (X, \gamma)$. Then $H \circ I = id_{\top\text{-}\mathbf{CFil}}$ and $I \circ H \subseteq id_{\top\text{-}\mathbf{Fil}}$. This implies that H is a right adjoint to I. □

Further, we can get the following conclusions.

Corollary 5.22. $\top$-**CFil** *is a bicoreflective subcategory of* $\top$-**Fil**.

Corollary 5.23. $\top$-**CFil** *is a topological category.*

Theorem 5.24. *Suppose that L satisfies* (MID). *Then* $\top$-**CFil** *is strongly Cartesian closed.*

Proof. It suffices to show that $\top$-**CFil** satisfies (CP1) and (CP3). By Corollaries 5.22 and 5.23, it is enough to check $\top$-**CFil** is closed under formation of products in $\top$-**Fil** [40] (see Corollary 3.1.7 and Proposition 3.2). Let (X, γ_X) and (Y, γ_Y) be two complete $\top$-filter spaces. Then their product in $\top$-**Fil** is

$$\gamma_X \times \gamma_Y = \left\{\mathbb{H} \in \mathcal{F}_L^\top(X \times Y) \,|\, pr_X^{\Rightarrow}(\mathbb{H}) \in \gamma_X, pr_Y^{\Rightarrow}(\mathbb{H}) \in \gamma_Y\right\}.$$

Now it remains to prove that $\gamma_X \times \gamma_Y$ satisfies (TC). Take any $\mathbb{H} \in \gamma_X \times \gamma_Y$. Then $pr_X^{\Rightarrow}(\mathbb{H}) \in \gamma_X$ and $pr_Y^{\Rightarrow}(\mathbb{H}) \in \gamma_Y$. Since (X, γ_X) and (Y, γ_Y) satisfy (TC), it follows that there exists $x \in X$ and $y \in Y$ such that $pr_X^{\Rightarrow}(\mathbb{H}) \cap [x] \in \gamma_X$ and $pr_Y^{\Rightarrow}(\mathbb{H}) \cap [y] \in \gamma_Y$. Then $pr_X^{\Rightarrow}(\mathbb{H} \cap [(x, y)]) \in \gamma_X$ and $pr_Y^{\Rightarrow}(\mathbb{H} \cap [(x, y)]) \in \gamma_Y$, which implies $\mathbb{H} \cap [(x, y)] \in \gamma_X \times \gamma_Y$. Hence, by the definition of $\gamma_X \times \gamma_Y$, we obtain $\gamma_X \times \gamma_Y$ satisfies (TC). □

In the classical case, there exist close relationships between complete filter spaces and symmetric Kent convergence spaces. Next, we will introduce the concept of symmetric Kent $\top$-convergence spaces and study its relationships with complete $\top$-filter spaces.

Definition 5.25. ([17]) A mapping $\lim : \mathcal{F}_L^\top(X) \longrightarrow \mathcal{P}(X)$ satisfying the following conditions:
(TC1) $x \in \lim[x]$;
(TC2) $\mathbb{F} \subseteq \mathbb{G}$ implies $\lim \mathbb{F} \subseteq \lim \mathbb{G}$;
(TCK) $x \in \lim \mathbb{F} \Rightarrow x \in \lim(\mathbb{F} \cap [x])$;
is called a Kent $\top$-convergence structure on X. The pair $(X, \lim)$ is called a Kent $\top$-convergence space.

The category of Kent $\top$-convergence spaces is denoted by $\top$-**KConv**.

Definition 5.26. A Kent $\top$-convergence structure lim on X is called symmetric provided that for each $\mathbb{F}, \mathbb{G} \in \mathcal{F}_L^\top(X)$ and $x, y \in X$,

(TCSK) $y \in \lim \mathbb{G}$ and $\mathbb{G} \subseteq \mathbb{F} \cap [x]$ imply $x \in \lim \mathbb{F}$.

The pair $(X, \lim)$ is called a symmetric Kent $\top$-convergence space.

The category of symmetric Kent $\top$-convergence spaces, as a full subcategory of $\top$-**KConv**, is denoted by $\top$-**SKConv**.

Proposition 5.27. *Let $(X, \lim)$ be a* Kent *$\top$-convergence space. The following statements are equivalent.*
(TCSK) $y \in \lim \mathbb{G}$ *and* $\mathbb{G} \subseteq \mathbb{F} \cap [x]$ *imply* $x \in \lim \mathbb{F}$.
(TCSK′) $y \in \lim(\mathbb{F} \cap [x])$ *implies* $x \in \lim \mathbb{F}$.
(TCSK″) $y \in \lim \mathbb{F}$ *and* $\bigwedge_{A \in \mathbb{F}} A(x) = \top$ *imply* $x \in \lim \mathbb{F}$.

Proof. (TCSK) $\Longrightarrow$ (TCSK′) It is straightforward.

(TCSK′) $\Longrightarrow$ (TCSK″) Suppose that $y \in \lim \mathbb{F}$ and $\bigwedge_{A \in \mathbb{F}} A(x) = \top$. Then $\mathbb{F} \subseteq [x]$. This implies that $y \in \lim \mathbb{F} = \lim(\mathbb{F} \cap [x])$. Hence $x \in \lim \mathbb{F}$.

(TCSK″) $\Longrightarrow$ (TCSK) Suppose that $y \in \lim \mathbb{G}$ and $\mathbb{G} \subseteq \mathbb{F} \cap [x]$. Then $\bigwedge_{A \in \mathbb{G}} A(x) = \top$. By (TCSK″), we get $x \in \lim \mathbb{G}$. Thus, $x \in \lim \mathbb{F}$. □

Proposition 5.28. *Let (X, γ) be a $\top$-filter space. Define $\lim_\gamma : \mathcal{F}_L^\top(X) \longrightarrow \mathcal{P}(X)$ by*

$$\lim_\gamma \mathbb{F} = \{x \in X \,|\, \mathbb{F} \cap [x] \in \gamma\}.$$

Then $(X, \lim_\gamma)$ is a symmetric Kent *$\top$-convergence space.*

Proof. (TF1) and (TF2) are straightforward.

(TCK) For each $\mathbb{F} \in \mathcal{F}_L^\top(X)$ and $x \in X$, we have

$$x \in \lim_\gamma \mathbb{F} \Longleftrightarrow \mathbb{F} \cap [x] \in \gamma \Longleftrightarrow \mathbb{F} \cap [x] \cap [x] \in \gamma \Longleftrightarrow x \in \lim_\gamma(\mathbb{F} \cap [x]).$$

(TCSK′) Let $y \in \lim_\gamma(\mathbb{F} \cap [x])$. Then $\mathbb{F} \cap [x] \cap [y] \in \gamma$. Hence, we obtain $x \in \lim_\gamma(\mathbb{F} \cap [y]) \subseteq \lim_\gamma \mathbb{F}$. □

Proposition 5.29. *Let* $(X, \lim)$ *be a* Kent $\top$*-convergence space. Define* $\gamma_{\lim} \subseteq \mathcal{F}_L^{\top}(X)$ *by*

$$\gamma_{\lim} = \left\{\mathbb{F} \in \mathcal{F}_L^{\top}(X) \,|\, \exists x \in X, \text{s.t.}, x \in \lim \mathbb{F}\right\}.$$

Then $(X, \gamma_{\lim})$ *is a complete* $\top$*-filter space.*

Proof. (TF1) and (TF2) are obvious. It is enough to show that $\gamma_{\lim}$ satisfies (TC).

(TC) Let $\mathbb{F} \in \gamma_{\lim}$. Then there exists $x \in X$ such that $x \in \lim \mathbb{F}$. Since $(X, \lim)$ is a Kent $\top$-convergence space, we obtain $x \in \lim(\mathbb{F} \cap [x])$. This shows $\mathbb{F} \cap [x] \in \gamma_{\lim}$. □

Proposition 5.30. (1) *If* $\varphi : (X, \gamma_X) \longrightarrow (Y, \gamma_Y)$ *between* $\top$*-filter spaces is Cauchy continuous, then* $\varphi : (X, \lim_{\gamma_X}) \longrightarrow (Y, \lim_{\gamma_Y})$ *between symmetric* Kent $\top$*-convergence spaces is continuous.*

(2) *If* $\varphi : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ *between* Kent $\top$*-convergence spaces is continuous, then* $\varphi : (X, \gamma_{\lim_X}) \longrightarrow (Y, \gamma_{\lim_Y})$ *between complete* $\top$*-filter spaces is Cauchy continuous.*

Proof. (1) Take each $\mathbb{F} \in \mathcal{F}_L^{\top}(X)$ and $x \in X$ such that $x \in \lim_{\gamma_X} \mathbb{F}$. Then $\mathbb{F} \cap [x] \in \gamma_X$. Since $\varphi : (X, \gamma_X) \longrightarrow (Y, \gamma_Y)$ is Cauchy continuous, it follows that $\varphi^{\Rightarrow}(\mathbb{F}) \cap [\varphi(x)] \in \gamma_Y$. By Proposition 5.28, we obtain $\varphi(x) \in \lim_{\gamma_Y} \varphi^{\Rightarrow}(\mathbb{F})$.

(2) Take each $\mathbb{F} \in \gamma_{\lim_X}$. Then there exists $x \in X$ such that $x \in \lim_X \mathbb{F}$. By the continuity of $\varphi : (X, \lim_X) \longrightarrow (Y, \lim_Y)$, we know $\varphi(x) \in \lim_Y \varphi^{\Rightarrow}(\mathbb{F})$. By Proposition 5.29, we obtain $\varphi^{\Rightarrow}(\mathbb{F}) \in \gamma_{\lim_Y}$. □

Theorem 5.31. $\top$-**CFil** *is isomorphic to* $\top$-**SKConv**.

Proof. It suffices to show that $\gamma_{\lim_\gamma} = \gamma$ and $\lim = \lim_{\gamma_{\lim}}$ for each complete $\top$-filter space (X, γ) and each symmetric Kent $\top$-convergence space $(X, \lim)$.

First, we prove $\gamma_{\lim_\gamma} = \gamma$. Take any $\mathbb{F} \in \mathcal{F}_L^{\top}(X)$. Then

$$\mathbb{F} \in \gamma_{\lim_\gamma} \Longleftrightarrow \exists x \in X, \text{s.t.}, x \in \lim_\gamma \mathbb{F} \Longleftrightarrow \exists x \in X, \text{s.t.}, \mathbb{F} \cap [x] \in \gamma \Longrightarrow \mathbb{F} \in \gamma.$$

Since γ satisfies (TC), $\mathbb{F} \in \gamma$ implies that there exists $x \in X$ such that $\mathbb{F} \cap [x] \in \gamma$. Thus, $\gamma_{\lim_\gamma} = \gamma$.

Next, we show $\lim = \lim_{\gamma_{\lim}}$. Take each $x \in X$ and $\mathbb{F} \in \mathcal{F}_L^{\top}(X)$. Then

$$x \in \lim_{\gamma_{\lim}} \mathbb{F} \Longleftrightarrow \mathbb{F} \cap [x] \in \gamma_{\lim} \Longleftrightarrow \exists y \in X, \text{s.t.}, y \in \lim(\mathbb{F} \cap [x]).$$

Since lim satisfies (TCSK'), we obtain $x \in \lim \mathbb{F}$. If $x \in \lim \mathbb{F}$, by (TCK), we obtain $x \in \lim(\mathbb{F} \cap [x])$. Hence $x \in \lim_{\gamma_{\lim}} \mathbb{F}$. This shows $\lim_{\gamma_{\lim}} = \lim$, as desired. □

6. Conclusions

In this paper, we introduced the notion of $\top$-filter spaces and its product space, subspace and quotient space. We investigated some convenient properties of $\top$-**Fil** and proved $\top$-**Fil** is a strong topological universe. Additionally, the concrete form of the product of an arbitrary family of $\top$-filters was presented. Further, we got $\top$-**SChy** and $\top$-**Chy** are bireflective subcategories of $\top$-**Fil** and $\top$-**CFil** is a bicoreflective subcategory of $\top$-**Fil**. Moreover, we showed that $\top$-**SChy** and $\top$-**Chy** are Cartesian closed, and $\top$-**CFil** is strongly Cartesian closed.

Reid and Richardson [42] investigated several types of completions of $\top$-Cauchy spaces and Jäger [23] studied completions of $\top$-quasi-Cauchy spaces. This implies that the framework where completion is discussed can be extended. Yang and Li [44] studied completions of (L, M)-filter tower spaces. This motivates us to consider completions of $\top$-filter spaces and provide a unified approach to different completions of $\top$-Cauchy spaces.

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