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Structures induced on hypersurfaces of meta-Golden Riemannian manifolds

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Abstract. In this paper, our aim is to examine the hypersurfaces in almost meta-Golden Riemannian manifolds. First, properties of the induced structure on a hypersurface by meta-Golden Riemannian structures were investigated. After that a necessary and sufficient condition obtained for a hypersurface of a meta-Golden Riemannian manifold to be invariant. Then, totally geodesic, minimal and totally umbilical hypersurfaces were analyzed in the meta-Golden Riemann manifold, respectively. Invariant and non-invariant hypersurfaces of meta-Golden Riemann manifolds were also characterized. The relationships between the eigenvalues of the golden structure and the invariant and non-invariant hypersurfaces of the meta-Golden Riemann manifolds were investigated. Finally three examples of such hypersurfaces were given.

1. Introduction

In recent years, a new polynomial structure, called as Golden structure, and its generalization, metallic structure, have been studied by many authors, and the geometry of these structures has been investigated. Crasmareanu and Hretcanu initiated the theory of Golden manifolds by defining a polynomial structure on a differentiable manifold which is $Q(X) = X^2 - X - I$, [3]. Structure polynomials are useful tools for producing new geometric structures on differentiable manifolds from the class C^∞. Özkan and Peltek, similarly defined a silver structure and a bronze structure on a differentiable manifold with structure polynomials $Q(X) = X^2 - 2X - I$ and $Q(X) = X^2 - X - 2I$, respectively, based on a similar idea in [11, 12].

The ways in which the golden ratio appears in nature and applied sciences are well known. In recent years, the golden ratio has frequently emerged in modern physics research and holds a significant place in nuclear physics. A close connection between the transition from Newtonian physics to relativistic mechanics and the golden ratio has been revealed, and the golden rectangle has been used in the theory of special relativity to derive time dilation and Lorentz contraction. Moreover, thanks to the golden ratio, interesting and important results have been produced in Kantor spacetime, in conformal field theory, in the topology of 4-manifolds, in mathematical probability theory, in Kantor fractal theory, and in El Naschie's field theory, [10]. These cases reveal the research of numberless objects that satisfy the golden ratio necessity through

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the world. One of them was the view that a logarithmic spiral provides the Golden ratio. However, Barlett [1] has shown that this assertion is untrue and also proves that an important class of logarithmic spirals delivers the meta-Golden Chi ratio wonderfully. In [1], same fulfillment was built ground the meta-Golden ratio $\chi = \frac{1+\sqrt{4\text{₵}+5}}{2\text{₵}}$, where $\text{₵} = \frac{1+\sqrt{5}}{2}$.

In Riemannian (also semi-Riemannian) manifolds, different geometric structure allow important consequences to occur while investigating differential and geometric properties of submanifolds. Manifolds with such differential geometric structures have been studied by several authors in [2–7, 11, 12, 14–16].

In the light of above discussions, Şahin and Şahin [13] introduced a novel manifold called as meta-Golden Riemannian manifold. This manifold was erected by means of the Golden manifolds and the meta-Golden ratio.

This paper is divided into three parts. In section 2, meta-Golden structure, meta-Golden Chi ratio, Golden structure, structure induced on hypersurfaces of Golden Riemannian manifold are mentioned. In the third section, properties of induced structures on hypersurfaces in meta-Golden manifolds with a special view towards minimal, totally umbilic and totally geodesic hypersurfaces are investigate, respectively. As well as, three examples are given.

Note: After that for the sake of shortness, the terms AMGR manifold instead of almost meta-Golden Riemannian manifold and MGR manifold instead of meta-Golden Riemannian manifold will be used for the remainder of the article.

2. Preliminaries

In studies conducted until 2019, it was claimed that the logarithmic spiral satisfies the Golden ratio. However, in 2019, Bartlett ([1]) demonstrated that this argument is not true and proved that an important class of logarithmic spirals perfectly satisfies the meta-Golden-Chi ratio. The geometric interpretation of the meta-Golden-Chi ratio $\dot{\chi}$ is similar to the geometric interpretation of the Golden ratio ₵. The relationship between the meta-Golden-Chi ratio $\dot{\chi}$ and continued fractions was established by Huylebrouck, who also provided its geometric interpretation, [8]. From Figure 1 in [13], the authors obtain $\dot{\chi} = \frac{1}{\text{₵}} + \frac{1}{\dot{\chi}}$, therefore they get

$$\dot{\chi}^2 - \frac{1}{\text{₵}}\dot{\chi} - 1 = 0. \tag{1}$$

Thus, the roots of (1) find as

$$\frac{\frac{1}{\text{₵}} \mp \sqrt{4 + \frac{1}{\text{₵}^2}}}{2}.$$

The correlation between the meta-Golden Chi ratio $\dot{\chi}$ and continued fractions was found in [8]. We denote the positive roots by

$$\dot{\chi} = \frac{\frac{1}{\text{₵}} + \sqrt{4 + \frac{1}{\text{₵}^2}}}{2},$$

which is called the silver mean of inverse of golden mean and the negative roots by

$$\ddot{\chi} = \frac{\frac{1}{\text{₵}} - \sqrt{4 + \frac{1}{\text{₵}^2}}}{2}.$$

Also, by direct computation, it is easy to see that

$$\ddot{\chi} = \frac{1}{\text{₵}} - \dot{\chi}. \tag{2}$$

Then we have

$$\not{c}\dot{\chi}^2 = \not{c} + \dot{\chi} \tag{3}$$

and

$$\not{c}\ddot{\chi}^2 = \not{c} + \ddot{\chi}. \tag{4}$$

Hretcanu and Crasmareanu [3], introduced that $\bar{\beta}$ is an almost Golden structure which is an endomorphism on a manifold $\mathcal{M}^*$, if

$$\bar{\beta}^2\mathbb{X}_1 = \bar{\beta}\mathbb{X}_1 + \mathbb{X}_1 \tag{5}$$

is provided for $\mathbb{X}_1 \in \Gamma(T\mathcal{M}^*)$. Hence, let $\bar{g}$ be a Riemannian metric on $\bar{\mathcal{M}}^*$, then $(\bar{g}, \bar{\beta})$ is called an almost Golden Riemannian structure if

$$\bar{g}(\bar{\beta}\mathbb{X}_1, \mathbb{Y}_1) = \bar{g}(\mathbb{X}_1, \bar{\beta}\mathbb{Y}_1) \tag{6}$$

for $\mathbb{X}_1$, $\mathbb{Y}_1 \in \Gamma(T\mathcal{M}^*)$. From (6), we get

$$\bar{g}(\bar{\beta}\mathbb{X}_1, \bar{\beta}\mathbb{Y}_1) = \bar{g}(\mathbb{X}_1, \bar{\beta}\mathbb{Y}_1) + \bar{g}(\mathbb{X}_1, \mathbb{Y}_1). \tag{7}$$

Therefore $(\bar{\mathcal{M}}^*, \bar{\beta}, \bar{g})$ is called as almost Golden Riemannian manifold.

Definition 2.1. *Let $\bar{\mathfrak{I}}$ be an endomorphism on an almost Golden manifold $(\bar{\mathcal{M}}^*, \bar{\beta})$ which satisfied*

$$\bar{\beta}\bar{\mathfrak{I}}^2\mathbb{X}_1 = \bar{\beta}\mathbb{X}_1 + \bar{\mathfrak{I}}\mathbb{X}_1 \tag{8}$$

for every $\mathbb{X}_1 \in \Gamma(T\bar{\mathcal{M}}^*)$. Then $\bar{\mathfrak{I}}$ is called as an almost meta-Golden structure and $(\bar{\mathcal{M}}^*, \bar{\beta}, \bar{\mathfrak{I}})$ is called as an almost meta-Golden manifold, [13].

Theorem 2.2. *Let $\bar{\mathfrak{I}}$ be an endomorphism on an almost Golden manifold $(\bar{\mathcal{M}}^*, \bar{\beta})$. At that case, $\bar{\mathfrak{I}}$ is almost meta-Golden structure iff*

$$\bar{\mathfrak{I}}^2 = \bar{\beta}\bar{\mathfrak{I}} - \bar{\mathfrak{I}} + I \tag{9}$$

where I is the identity map, [13].

Definition 2.3. *Let $\bar{\mathfrak{I}}$ is almost meta-Golden structure on $(\bar{\mathcal{M}}^*, \bar{\beta}, \bar{g})$. If $\bar{\mathfrak{I}}$ is compatible with g on $\bar{\mathcal{M}}^*$, namely*

$$\bar{g}(\bar{\mathfrak{I}}\mathbb{X}_1, \mathbb{Y}_1) = \bar{g}(\mathbb{X}_1, \bar{\mathfrak{I}}\mathbb{Y}_1), \quad \forall \mathbb{X}_1, \mathbb{Y}_1 \in \Gamma(T\mathcal{M}^*) \tag{10}$$

or

$$\bar{g}(\bar{\mathfrak{I}}\mathbb{X}_1, \bar{\mathfrak{I}}\mathbb{Y}_1) = \bar{g}(\bar{\beta}\mathbb{X}_1, \bar{\mathfrak{I}}\mathbb{Y}_1) - \bar{g}(\mathbb{X}_1, \bar{\mathfrak{I}}\mathbb{Y}_1) + \bar{g}(\mathbb{X}_1, \mathbb{Y}_1) \tag{11}$$

then $(\bar{\mathcal{M}}^*, \bar{\beta}, \bar{\mathfrak{I}}, \bar{g})$ is called almost meta-Golden Riemannian manifold (AMGR) for $\mathbb{X}_1$, $\mathbb{Y}_1 \in \Gamma(T\bar{\mathcal{M}}^*)$, [13].

Proposition 2.4. *$\bar{\mathfrak{I}}$ is an isomorphism on $T_p\bar{\mathcal{M}}^*$, for every $p \in \bar{\mathcal{M}}^*$, [13].*

Proposition 2.5. *Let $(\bar{\mathcal{M}}^*, \bar{\beta}, \bar{\mathfrak{I}}, \bar{g})$ be an AMGR manifold. In that case;*

1 If $\not{c}$ is the eigenvalue of the Golden structure $\bar{\beta}$, then $\dot{\chi}$ and $\ddot{\chi}$ are the eigenvalues of the meta-Golden structure.

2 If $1-\not{c}$ is the eigenvalue of the Golden structure $\bar{\beta}$, then

$$G_m = \frac{\frac{1}{1-\not{c}} + \sqrt{4 + \frac{1}{(1-\not{c})^2}}}{2}, \bar{G}_{\overline{m}} = \frac{\frac{1}{1-\not{c}} - \sqrt{4 + \frac{1}{(1-\not{c})^2}}}{2}$$

are the eigenvalues of the meta-Golden structure, [13].

Proposition 2.6. *Let $(\bar{\mathcal{M}}^*, \bar{g})$ be an $m-$dimensional Riemannian manifold and $\mathcal{J}$ is almost product structure on $\bar{\mathcal{M}}^*$. Then $\mathcal{J}$ induces two meta-Golden structure on $(\bar{\mathcal{M}}^*, \bar{g})$, as follow;*

$$\bar{\mathfrak{I}} = \mathcal{A}_*\mathcal{J} + \mathcal{B}_*\mathcal{I} \tag{12}$$

where $\not{c}(\mathcal{A}_* + \mathcal{B}_*)^2 = \not{c} + (\mathcal{A}_* + \mathcal{B}_*)$, [13].

Theorem 2.7. *Let $(\bar{\mathcal{M}}^*, \bar{\beta}, \bar{\mathfrak{I}}, \bar{g})$ be an AMGR manifold, $\bar{\mathfrak{I}}$ is integrable if the Codazzi-like equation is ensured for any $\mathbb{X}_1, \mathbb{Y}_1 \in \Gamma(T\bar{\mathcal{M}}^*)$ given as follows;*

$$(\nabla_{\bar{\mathfrak{I}}\mathbb{X}_1}\bar{\mathfrak{I}})\mathbb{Y}_1 - \bar{\mathfrak{I}}(\nabla_{\mathbb{X}_1}\bar{\mathfrak{I}})\mathbb{Y}_1 = 0.$$

Also if $\nabla\bar{\mathfrak{I}} = 0$, therefore $(\bar{\mathcal{M}}^*, \bar{\beta}, \bar{\mathfrak{I}}, \bar{g})$ is a meta-Golden manifold and meta-Golden structure $\bar{\mathfrak{I}}$ is integrable. If $\nabla\bar{\mathfrak{I}} = 0$ then $\nabla\bar{\beta} = 0$, [13].

Now let's mention about structures induced on hypersurfaces of a Golden Riemannian manifold. We admit the covariant differential in $\bar{\mathcal{M}}^*$ by $\overline{\nabla}$ and in $\mathcal{M}^*$ by ∇. We admit by A the Weingarten operator on $T\mathcal{M}^*$ with respect to the local unit normal vector field N of $\mathcal{M}^*$ in $\bar{\mathcal{M}}^*$.

Proposition 2.8. *Let $\mathcal{M}^*$ be a hypersurface of $(\bar{\mathcal{M}}^*, \bar{\beta}, \overline{g})$ almost Golden Riemannian manifold. Then the induced structure $\Pi = (\beta, g, u, \xi_1, c)$ on $\mathcal{M}^*$ provides the following equalities:*

1 $\beta^2(\mathbb{X}_1) = \beta(\mathbb{X}_1) + \mathbb{X}_1 - u(\mathbb{X}_1)\xi_1$

2 $u(\beta(\mathbb{X}_1)) = (1-c)u(\mathbb{X}_1)$

3 $u(\xi_1) = 1 + c - c^2$

4 $\beta(\xi_1) = (1-c)\xi_1$

5 $u(\mathbb{X}_1) = g(\mathbb{X}_1, \xi_1)$

6 $g(\beta\mathbb{X}_1, \mathbb{Y}_1) = g(\mathbb{X}_1, \beta\mathbb{Y}_1)$

7 $g(\beta\mathbb{X}_1, \beta\mathbb{Y}_1) = g(\mathbb{X}_1, \beta\mathbb{Y}_1) + g(\mathbb{X}_1, \mathbb{Y}_1) + u(\mathbb{X}_1)u(\mathbb{Y}_1)$

for every $\mathbb{X}_1, \mathbb{Y}_1 \in \Gamma(T\mathcal{M}^*)$, $c \in C^\infty(\mathcal{M}^*)$, u is a $1-$form, [3, 9].
Moreover, we have

$$\begin{array}{lclcl} \bar{\beta}\mathbb{X}_1 & = & \beta\mathbb{X}_1 + u(\mathbb{X}_1)\xi & \text{and} & \beta: \ \Gamma(T\mathcal{M}^*) \to \Gamma(T\mathcal{M}^*) \\ \bar{\beta}N & = & \xi_1 + cN & & (\bar{\beta}\mathbb{X}_1)^\top = \beta\mathbb{X}_1 \end{array}$$

3. Hypersurfaces of Meta-Golden Riemannian Manifolds

In this section, we will investigate the equations of the structure reduced onto the hypersurface of a meta-Golden Riemannian manifold and the properties it satisfies. Furthermore, we will define invariant and anti-invariant hypersurfaces and examine the geometries of these hypersurfaces.

Let $(\bar{\mathcal{M}}^*, \bar{\beta}, \bar{\mathfrak{I}}, \bar{g})$ be a AMGR manifold and $\mathcal{M}^*$ is a hypersurface of $\bar{\mathcal{M}}^*$. We admit by A the Weingarten operator on $T\mathcal{M}^*$ with respect to the local unit normal vector field N of $\mathcal{M}^*$ in $\bar{\mathcal{M}}^*$.

Let $\mathfrak{I}$ be $(1,1)$ type tensor field on hypersurface $\mathcal{M}^*$, $\bar{\mathfrak{I}}$ be the meta-Golden structure, g be the Riemannian metric, v be a $1-$form on hypersurface. Then for each $p \in \mathcal{M}^*$, we can write

$$T_p\bar{\mathcal{M}}^* = T_p\mathcal{M}^* \oplus T_p\mathcal{M}^{*\perp}.$$

Via above equation for any $\mathbb{X}_1, \xi \in \Gamma(T\mathcal{M}^*)$, $N \in \Gamma(T\mathcal{M}^{*\perp})$, we get

$$\bar{\mathfrak{I}}\mathbb{X}_1 = \mathfrak{I}\mathbb{X}_1 + v(\mathbb{X}_1)N, \tag{13}$$

$$\bar{\mathfrak{I}}N = \xi + bN, \quad b \in C^\infty(\mathcal{M}^*) \tag{14}$$

and

$$\mathfrak{I} : \Gamma(T\mathcal{M}^*) \to \Gamma(T\mathcal{M}^*), \quad \mathfrak{I}\mathbb{X}_1 = (\bar{\mathfrak{I}}\mathbb{X}_1)^\top.$$

On the other hand, Gauss and Weingarten formulas are given as follows;

$$\overline{\nabla}_{\mathbb{X}_1}\mathbb{Y}_1 = \nabla_{\mathbb{X}_1}\mathbb{Y}_1 + h(\mathbb{X}_1, \mathbb{Y}_1)N, \tag{15}$$

$$\overline{\nabla}_{\mathbb{X}_1}N = -A_N\mathbb{X}_1 \tag{16}$$

where $h(\mathbb{X}_1, \mathbb{Y}_1) = g(A_N\mathbb{X}_1, \mathbb{Y}_1)$ is the second fundamental form in $T\mathcal{M}^{*\perp}$ and $\mathbb{X}_1, \mathbb{Y}_1 \in \Gamma(T\mathcal{M}^*)$. If we apply $\bar{\mathfrak{I}}$ to equation (13) and consider equations (9) and (10), we get

$$\bar{\beta}\bar{\mathfrak{I}}\mathbb{X}_1 - \bar{\mathfrak{I}}\mathbb{X}_1 + \mathbb{X}_1 = \bar{\mathfrak{I}}\mathfrak{I}\mathbb{X}_1 + v(\mathbb{X}_1)\bar{\mathfrak{I}}N. \tag{17}$$

Then by using (13) and (14) in (17), we obtain

$$\bar{\beta}\mathfrak{I}\mathbb{X}_1 + v(\mathbb{X}_1)\bar{\beta}N - \mathfrak{I}\mathbb{X}_1 - v(\mathbb{X}_1)N + \mathbb{X}_1 = \mathfrak{I}^2\mathbb{X}_1 + v(\mathfrak{I}\mathbb{X}_1)N + v(\mathbb{X}_1)\xi + bv(\mathbb{X}_1)N. \tag{18}$$

Here, let's assume that the structure induced from the Golden structure to the tangent bundle of hypersurface of a Golden Riemannian manifold is invariant. Therefore $\bar{\beta}(T\mathcal{M}^*) \subset T\mathcal{M}^*$ and $\bar{\beta}(T\mathcal{M}^{*\perp}) \subset T\mathcal{M}^{*\perp}$. Then let $\bar{\beta}N = cN$, where $c = \rlap{/}c$ or $c = 1 - \rlap{/}c$ and from (18), we have

$$\bar{\beta}\mathfrak{I}\mathbb{X}_1 + v(\mathbb{X}_1)cN - \mathfrak{I}\mathbb{X}_1 - v(\mathbb{X}_1)N + \mathbb{X}_1 = \mathfrak{I}^2\mathbb{X}_1 + v(\mathfrak{I}\mathbb{X}_1)N + v(\mathbb{X}_1)\xi + bv(\mathbb{X}_1)N. \tag{19}$$

By equating the tangent and normal components of equation (19), we respectively obtain the following equations;

$$\mathfrak{I}^2\mathbb{X}_1 = \bar{\beta}\mathfrak{I}\mathbb{X}_1 - \mathfrak{I}\mathbb{X}_1 + \mathbb{X}_1 - v(\mathbb{X}_1)\xi, \tag{20}$$

$$v(\mathfrak{I}\mathbb{X}_1) = (c - 1 - b)v(\mathbb{X}_1). \tag{21}$$

Also for a meta-Golden structure, if we apply $\bar{\mathfrak{I}}$ to the equation (14), we obtain

$$\bar{\mathfrak{I}}^2N = \bar{\mathfrak{I}}\xi + b\bar{\mathfrak{I}}N.$$

Then via equations (9) and (10), we get

$$\bar{\beta}\xi + b\bar{\beta}N - \xi + (1 - b)N = \mathfrak{I}\xi + v(\mathfrak{I}\xi)N + b\xi + b^2N. \tag{22}$$

By equating the tangent and normal components of equation (22), we have

$$\mathfrak{I}\xi = \bar{\beta}\xi - (1 + b)\xi,$$

$$b\bar{\beta}N + (1 - b)N = v(\xi)N + b^2N.$$

Then for $\bar{\beta}N = cN$, we obtain

$$v(\xi) = b(c-1) + 1 - b^2.$$

As well as, for $\mathbb{X}_1 \in \Gamma(T\mathcal{M}^*)$ and $N \in \Gamma(T\mathcal{M}^{*\perp})$, by using (10), (13), (14), we find

$$v(\mathbb{X}_1) = g(\mathbb{X}_1, \xi) \tag{23}$$

and from $g(\bar{\mathfrak{I}}\mathbb{X}_1, \mathbb{Y}_1) = g(\mathbb{X}_1, \bar{\mathfrak{I}}\mathbb{Y}_1)$, we have

$$g(\mathfrak{I}\mathbb{X}_1, \mathbb{Y}_1) = g(\mathbb{X}_1, \mathfrak{I}\mathbb{Y}_1). \tag{24}$$

Then if we apply (9) and (10) to equation (13) and consider that $\bar{\beta}$ is invariant, we get

$$\begin{aligned} g(\mathfrak{I}\mathbb{X}_1 + v(\mathbb{X}_1)N, \mathfrak{I}\mathbb{Y}_1 + v(\mathbb{Y}_1)N) &= g(\bar{\beta}\mathbb{X}_1, \mathfrak{I}\mathbb{Y}_1 + v(\mathbb{Y}_1)N) \\ &\quad - g(\mathbb{X}_1, \mathfrak{I}\mathbb{Y}_1 + v(\mathbb{Y}_1)N) + g(\mathbb{X}_1, \mathbb{Y}_1). \end{aligned}$$

Finally, we obtain

$$g(\mathfrak{I}\mathbb{X}_1, \mathfrak{I}\mathbb{Y}_1) + v(\mathbb{X}_1)v(\mathbb{Y}_1) = g(\bar{\beta}\mathbb{X}_1, \mathfrak{I}\mathbb{Y}_1) - g(\mathbb{X}_1, \mathfrak{I}\mathbb{Y}_1) + g(\mathbb{X}_1, \mathbb{Y}_1). \tag{25}$$

Thus, we can give the following proposition.

Proposition 3.1. *Let $(\bar{\mathcal{M}}^*, \bar{\beta}, \bar{\mathfrak{I}}, \bar{g})$ be a AMGR manifold and $\mathcal{M}^*$ be a hypersurface of $\bar{\mathcal{M}}^*$. If the structure induced from Golden structure to the tangent bundle of hypersurface of a MGR manifold is invariant, then the structure $\Pi = (\mathfrak{I}, \bar{\beta}, g, v, \xi, b)$ induced on $\mathcal{M}^*$ by the meta-Golden structure $\bar{\mathfrak{I}}$, satisfies the following equalities:*

1 $\mathfrak{I}^2\mathbb{X}_1 = \bar{\beta}\mathfrak{I}\mathbb{X}_1 - \mathfrak{I}\mathbb{X}_1 + \mathbb{X}_1 - v(\mathbb{X}_1)\xi$

2 $v(\mathfrak{I}\mathbb{X}_1) = (c - 1 - b)v(\mathbb{X}_1)$

3 $v(\xi) = b(c-1) + 1 - b^2$

4 $\mathfrak{I}\xi = \bar{\beta}\xi - (1+b)\xi$

5 $v(\mathbb{X}_1) = g(\mathbb{X}_1, \xi)$

6 $g(\mathfrak{I}\mathbb{X}_1, \mathbb{Y}_1) = g(\mathbb{X}_1, \mathfrak{I}\mathbb{Y}_1)$

7 $g(\mathfrak{I}\mathbb{X}_1, \mathfrak{I}\mathbb{Y}_1) = g(\bar{\beta}\mathbb{X}_1, \mathfrak{I}\mathbb{Y}_1) - g(\mathbb{X}_1, \mathfrak{I}\mathbb{Y}_1) + g(\mathbb{X}_1, \mathbb{Y}_1) - v(\mathbb{X}_1)v(\mathbb{Y}_1)$.

From here on, it will be assumed that the golden structure of the meta-Golden structure is invariant on the hypersurface of the MGR manifold.

Remark 3.2. *If $b = c - 1$ and b is a constant function on $\mathcal{M}^*$, in this case $v \circ \mathfrak{I} = 0$, $\mathfrak{I}v = \bar{\beta}\xi - c\xi$ and $\|v\|^2 = cb - b + 1 - b^2$. Namely, $\|v\|^2 = 1$ or more generally, if M is a non-invariant hypersurface according to the meta-Golden structure, we have $\mathrm{Im}(b) \in (-\frac{\not{c} \pm \sqrt{\not{c}^2+4}}{2}, -\frac{(1-\not{c}) \pm \sqrt{(1-\not{c})^2+4}}{2})$ and $\|v\| = \sqrt{cb - b + 1 - b^2}$.*

The following result characterizes the invariance of the real hypersurface of the meta-Golden Riemannian manifold.

Proposition 3.3. *$\mathcal{M}^*$ is an invariant and orientable hypersurface in $(\bar{\mathcal{M}}^*, \bar{\beta}, \bar{\mathfrak{I}}, \bar{g})$ AMGR manifold if and only if*

$$b = -\frac{\not{c} \pm \sqrt{\not{c}^2 + 4}}{2}$$

or

$$b = -\frac{(1 - \not{c}) \pm \sqrt{(1 - \not{c})^2 + 4}}{2}$$

in the induced structure from $(\bar{\beta}, \bar{\mathfrak{I}}, \bar{g})$ meta-Golden structure to $(\mathfrak{I}, \bar{\beta}, g, v, \xi, b)$.

Theorem 3.4. *Let $\mathcal{M}^*$ be hypersurface of a MGR manifold with structure $(\bar{\mathcal{M}}^*, \bar{\beta}, \bar{\mathfrak{I}}, \bar{g})$. If the meta-Golden structure $\bar{\mathfrak{I}}$ is parallel with respect to the Levi-Civita connection $\overline{\nabla}$ denoted on $\bar{g}$, then the structure $\Pi = (\mathfrak{I}, \bar{\beta}, g, v, \xi, b)$ induced on $\mathcal{M}^*$ by the structure $\bar{\mathfrak{I}}$ has following properties:*

1 $(\nabla_{\mathbb{X}_1}\mathfrak{I})\mathbb{Y}_1 = g(A_N\mathbb{X}_1, \mathbb{Y}_1)\xi - v(\mathbb{Y}_1)A_N\mathbb{X}_1$,

2 $(\nabla_{\mathbb{X}_1}v)\mathbb{Y}_1 = g(A_N\mathbb{X}_1, \mathbb{Y}_1)b - g(A_N\mathbb{X}_1, \mathfrak{I}\mathbb{Y}_1)$,

3 $\nabla_{\mathbb{X}_1}\xi = -\mathfrak{I}A_N\mathbb{X}_1 + bA_N\mathbb{X}_1$,

4 $\mathbb{X}_1(b) = -2v(A_N\mathbb{X}_1)$

where $-\bar{\mathfrak{I}}N = \xi \in \Gamma(T\mathcal{M}^*)$, $\mathfrak{I}$ is a $(1,1)$ tensor field in $\mathcal{M}^*$ and $\mathbb{X}_1, \mathbb{Y}_1 \in \Gamma(T\mathcal{M}^*)$.

Proof. By using $\bar{\mathfrak{I}}\mathbb{X}_1 = \mathfrak{I}\mathbb{X}_1 + v(\mathbb{X}_1)N$, $\nabla\bar{\mathfrak{I}} = 0$ and the following equality

$$(\overline{\nabla}_{\mathbb{X}_1}\bar{\mathfrak{I}})\mathbb{Y}_1 = \overline{\nabla}_{\mathbb{X}_1}\bar{\mathfrak{I}}\mathbb{Y}_1 - \bar{\mathfrak{I}}\overline{\nabla}_{\mathbb{X}_1}\mathbb{Y}_1,$$

we have, $\overline{\nabla}_{\mathbb{X}_1}\bar{\mathfrak{I}}\mathbb{Y}_1 = \bar{\mathfrak{I}}\overline{\nabla}_{\mathbb{X}_1}\mathbb{Y}_1$. Then if we consider Gauss and Weingarten formulas, we find

$$\bar{\mathfrak{I}}\overline{\nabla}_{\mathbb{X}_1}\mathbb{Y}_1 = \overline{\nabla}_{\mathbb{X}_1}(\mathfrak{I}\mathbb{Y}_1 + v(\mathbb{Y}_1)N),$$

$$\begin{aligned}\bar{\mathfrak{I}}(\nabla_{\mathbb{X}_1}\mathbb{Y}_1 + h(\mathbb{X}_1, \mathbb{Y}_1)N) = {} & \nabla_{\mathbb{X}_1}\mathfrak{I}\mathbb{Y}_1 + h(\mathbb{X}_1, \mathfrak{I}\mathbb{Y}_1)N \\ & + v(\mathbb{Y}_1)(-A_N\mathbb{X}_1) + N(\nabla_{\mathbb{X}_1}v(\mathbb{Y}_1)).\end{aligned} \tag{26}$$

Using (13), (14) in (26) , we get

$$\begin{aligned}\mathfrak{I}\nabla_{\mathbb{X}_1}\mathbb{Y}_1 + v(\nabla_{\mathbb{X}_1}\mathbb{Y}_1)N + h(\mathbb{X}_1, \mathbb{Y}_1)\xi + h(\mathbb{X}_1, \mathbb{Y}_1)bN = {} & \nabla_{\mathbb{X}_1}\mathfrak{I}\mathbb{Y}_1 + h(\mathbb{X}_1, \mathfrak{I}\mathbb{Y}_1)N \\ & - v(\mathbb{Y}_1)A_N\mathbb{X}_1 + \nabla_{\mathbb{X}_1}v(\mathbb{Y}_1)N.\end{aligned}$$

By equating the tangent and normal components of the above equation, we find

$$(\nabla_{\mathbb{X}_1}\mathfrak{I})\mathbb{Y}_1 = h(\mathbb{X}_1, \mathbb{Y}_1)\xi + v(\mathbb{Y}_1)A_N\mathbb{X}_1 \tag{27}$$

and

$$(\nabla_{\mathbb{X}_1}v)(\mathbb{Y}_1) = h(\mathbb{X}_1, \mathbb{Y}_1)b - h(\mathbb{X}_1, \mathfrak{I}\mathbb{Y}_1). \tag{28}$$

If we use (10) and $\nabla\bar{\mathfrak{I}} = 0$, we have

$$\overline{\nabla}_{\mathbb{X}_1}\xi + \mathbb{X}_1(b)N + b\overline{\nabla}_{\mathbb{X}_1}N = -\bar{\mathfrak{I}}A_N\mathbb{X}_1.$$

From here, we obtain

$$\nabla_{\mathbb{X}_1}\xi + h(\mathbb{X}_1, \xi)N + \mathbb{X}_1(b)N + b(-A_N\mathbb{X}_1) = -\mathfrak{I}A_N\mathbb{X}_1 - v(A_N\mathbb{X}_1)N.$$

Thus, identifying the tangent and normal components, respectively, we find

$$\nabla_{\mathbb{X}_1}\xi - bA_N\mathbb{X}_1 = -\mathfrak{I}A_N\mathbb{X}_1$$

and

$$g(A_N\mathbb{X}_1, \xi) = v(A_N\mathbb{X}_1),$$

$$h(\mathbb{X}_1, \xi) + \mathbb{X}_1(b) = -v(A_N\mathbb{X}_1).$$

Hence, we have

$$\nabla_{\mathbb{X}_1}\xi = -\Im A_N\mathbb{X}_1 + bA_N\mathbb{X}_1 \tag{29}$$

and

$$\mathbb{X}_1(b) = -2v(A_N\mathbb{X}_1). \tag{30}$$

□

Remark 3.5. *If we take the function $b = 0$ on $\mathcal{M}^*$, in this case we find $v \circ \Im = 0$, $\Im\xi = \bar{\beta}\xi - \xi$ and $\|\xi\|^2 = 1$. More generally, if $\mathcal{M}^*$ is a non-invariant hypersurface of meta-Golden Riemannian manifold $\bar{\mathcal{M}}^*$, then*

1 *for the eigen value $ȼ$ of the Golden structure $\bar{\beta}$,*

$$\mathrm{Im}(b) = \left(-\frac{ȼ + \sqrt{4+ȼ^2}}{2} \text{ and } -\frac{ȼ - \sqrt{4+ȼ^2}}{2}\right)$$

2 *for the eigen value $1 - ȼ$ of the Golden structure $\bar{\beta}$*

$$\mathrm{Im}(b) = \left(G_m = -\frac{1-ȼ + \sqrt{4+(1-ȼ)^2}}{2} \text{ and } \overline{G}_m = -\frac{1-ȼ - \sqrt{4+(1-ȼ)^2}}{2}\right)$$

and

$$\|\xi\| = \sqrt{cb - b + 1 - b^2}.$$

Remark 3.6. *Let $(\bar{\mathcal{M}}^*, \bar{\beta}, \bar{\Im}, \bar{g})$ be a MGR manifold, for $\xi = 0$, which is equivalent to $v = 0$, this indicates that $\bar{\Im}\mid_{\mathcal{M}^*} = \Im$ and $\bar{\Im}N = bN$. In another saying, $\mathcal{M}^*$ is an invariant hypersurface of the $(\bar{\mathcal{M}}^*, \bar{\beta}, \bar{\Im}, \bar{g})$ MGR manifold if and only if the normal vector N, whose eigenvalue is the function b, is the eigenvector of the meta-Golden structure on the hypersurface $\mathcal{M}^*$.*

Proposition 3.7. *$\mathcal{M}^*$ be an oriantable and invariant hypersurface of MGR manifold $(\bar{\mathcal{M}}^*, \bar{\beta}, \bar{\Im}, \bar{g})$ if and only if the structure $\Pi = (\Im, \bar{\beta}, g, v, \xi, b)$ induced on $\mathcal{M}^*$ by the meta-Golden structure $(\bar{\beta}, \bar{\Im}, \bar{g})$ has the function b which is equal to either*

$$b = -\frac{ȼ \pm \sqrt{ȼ^2+4}}{2} \tag{31}$$

or

$$b = -\frac{(1-ȼ) \pm \sqrt{(1-ȼ)^2+4}}{2}. \tag{32}$$

Proposition 3.8. *If $\Pi = (\Im, \bar{\beta}, g, v, \xi, b)$ is the induced structure on a hypersurface $\mathcal{M}^*$ isometrically immersed in a MGR manifold $(\bar{\mathcal{M}}^*, \bar{\beta}, \bar{\Im}, \bar{g})$ then for every $\mathbb{X}_1 \in \Gamma(T\mathcal{M}^*)$,*

$$(\Im\mathbb{X}_1 + b\mathbb{X}_1) \perp \xi.$$

Proof. From $v(\Im\mathbb{X}_1) = (c - b - 1)v(\mathbb{X}_1)$, we have

$$g(\Im\mathbb{X}_1 + b\mathbb{X}_1, \xi) = 0 \Leftrightarrow \Im\mathbb{X}_1 + (c - b - 1)\mathbb{X}_1 \perp \xi.$$

Especially, if $b = c - 1$, then we have

$$\Im\mathbb{X}_1 \in \xi^{\perp} = \{\mathbb{X}_1 \in \Gamma(T\bar{\mathcal{M}}^*) \mid \mathbb{X}_1 \perp \xi\}.$$

Therefore, we get

$$T\mathcal{M}^* = Ker\Im \oplus \xi^{\perp}.$$

□

Remark 3.9. *If we consider*

$$(\nabla_{\mathbb{X}_1} v)\Upsilon_1 = g(A_N\mathbb{X}_1, \Upsilon_1)b - g(A_N\mathbb{X}_1, \Im\Upsilon_1),$$

and from (24), we obtain

$$(\nabla_{\mathbb{X}_1} v)\Upsilon_1 = g(bA_N\mathbb{X}_1 - \Im A_N\mathbb{X}_1, \Upsilon_1)$$

here using $\nabla_{\mathbb{X}_1}\xi = -\Im A_N\mathbb{X}_1 + bA_N\mathbb{X}_1$, *we get*

$$(\nabla_{\mathbb{X}_1} v)\Upsilon_1 = g(\nabla_{\mathbb{X}_1}\xi, \Upsilon_1).$$

Proposition 3.10. *Let* $\mathcal{M}^*$ *be an invariant hypersurface with* $\overline{\nabla}\bar{\Im} = 0$ *on* $(\bar{\mathcal{M}}^*, \bar{\beta}, \bar{\Im}, \bar{g})$ *MGR manifold and* $\Pi = (\Im, \bar{\beta}, g, \xi, v, b)$ *is induced structure on* $\mathcal{M}^*$. *In this case* $\nabla\Im = 0$.

Proof. From the following equality

$$(\nabla_{\mathbb{X}_1}\Im)(\Upsilon_1) = \nabla_{\mathbb{X}_1}\Im\Upsilon_1 - \Im\nabla_{\mathbb{X}_1}\Upsilon_1,$$

since $\mathcal{M}^*$ is invariant, we get

$$(\nabla_{\mathbb{X}_1}\Im)(\Upsilon_1) = \nabla_{\mathbb{X}_1}\bar{\Im}\Upsilon_1 - \bar{\Im}\nabla_{\mathbb{X}_1}\Upsilon_1.$$

Via Gauss formula, we find

$$(\nabla_{\mathbb{X}_1}\Im)(\Upsilon_1) = (\overline{\nabla}_{\mathbb{X}_1}\bar{\Im})\Upsilon_1,$$

$$(\nabla_{\mathbb{X}_1}\Im)(\Upsilon_1) = \overline{\nabla}_{\mathbb{X}_1}\bar{\Im}\Upsilon_1 - h(\mathbb{X}_1, \bar{\Im}\Upsilon_1)N - \bar{\Im}\overline{\nabla}_{\mathbb{X}_1}\Upsilon_1 + h(\mathbb{X}_1, \Upsilon_1)\bar{\Im}N.$$

Here, $\mathcal{M}^*$ is a invariant hypersurface, then we get

$$(\nabla_{\mathbb{X}_1}\Im)(\Upsilon_1) = \overline{\nabla}_{\mathbb{X}_1}\bar{\Im}\Upsilon_1 - h(\mathbb{X}_1, \Upsilon_1)N - \bar{\Im}\overline{\nabla}_{\mathbb{X}_1}\Upsilon_1 + h(\mathbb{X}_1, \Upsilon_1)N = (\overline{\nabla}_{\mathbb{X}_1}\bar{\Im})\Upsilon_1.$$

Therefore, the proof is completed. □

For the following theorem, let us remind that if $h = 0$, equivalently the Weingarten operator $A = 0$, then $\mathcal{M}^*$ is said to be completely geodesic.

Theorem 3.11. *Let* $(\bar{\mathcal{M}}^*, \bar{\beta}, \bar{\Im}, \bar{g})$ *be a MGR manifold such that the meta-Golden structure* $\bar{\Im}$ *is parallel with respect to the* $\overline{\nabla}$ *connection. Let* $\mathcal{M}^*$ *be a non-invariant hypersurface of* $\bar{\mathcal{M}}^*$ *and* $\Pi = (\Im, \bar{\beta}, g, \xi, v, b)$ *be a induced structure on* $\mathcal{M}^*$ *from* $\bar{\mathcal{M}}^*$. *In this case, the following expressions are equivalent;*

1. $\mathcal{M}^*$ *is totally geodesic,*
2. $\nabla\Im = 0$,
3. $\nabla\xi = 0$,
4. $\nabla v = 0$.

Proof. If $\mathcal{M}^*$ is totally geodesic. In this case from (27), we have

$$(\nabla_{\mathbb{X}_1}\Im)\Upsilon_1 = g(A_N\mathbb{X}_1, \Upsilon_1) - v(\Upsilon_1)A_N\mathbb{X}_1.$$

Then, since $\mathcal{M}^*$ is totally geodesic, $A = 0$ and $\nabla_{\mathbb{X}_1}\Im = 0$.

Also, for

$$(\nabla_{\mathbb{X}_1} v)\Upsilon_1 = g(A_N\mathbb{X}_1, \Upsilon_1)b - g(A_N\mathbb{X}_1, \Im\Upsilon_1),$$

if $A = 0$, we obtain $\nabla v = 0$. As well as, for

$$\nabla_{\mathbb{X}_1}\xi = -\mathfrak{I}A_N\mathbb{X}_1 + bA_N\mathbb{X}_1,$$

we get

$$A = 0 \Rightarrow \nabla\xi = 0.$$

Therefore (1)≡(2),(3),(4).

Now let show that (2)≡(1),(3),(4).

If $\nabla\mathfrak{I} = 0$, from (27), we have

$$g(A_N\mathbb{X}_1, \Upsilon_1)\xi + v(\Upsilon_1)A_N\mathbb{X}_1 = 0 \quad \forall\mathbb{X}_1, \Upsilon_1 \in \Gamma(T\mathcal{M}^*).$$

Taking the inner product of above equation with $\Upsilon_1 \in \Gamma(T\mathcal{M}^*)$, we find

$$2g(A_N\mathbb{X}_1, \Upsilon_1)v(\Upsilon_1) = 0.$$

In this equation, if we get $\Upsilon_1 = \xi$, we have

$$g(A_N\mathbb{X}_1, \xi)||\xi||^2 = 0, \xi \in \Gamma(T\mathcal{M}^*).$$

Since $||\xi||^2 \neq 0$ and $\mathcal{M}^*$ is non-invariant hypersurface of $\bar{\mathcal{M}}^*$, then we have

$$g(A_N\mathbb{X}_1, \xi) = 0, \quad \forall\mathbb{X}_1 \in \Gamma(T\mathcal{M}^*). \tag{33}$$

On the other side, in equation (27), if we take $\Upsilon_1 = A_N\mathbb{X}_1$, we get

$$g(A_N\mathbb{X}_1, A_N\mathbb{X}_1)\xi + v(A_N\mathbb{X}_1)A_N\mathbb{X}_1 = 0.$$

Then, from (33), we can write

$$g(A_N\mathbb{X}_1, A_N\mathbb{X}_1) = 0,$$

hence, we find that, if $A_N\mathbb{X}_1 = 0$, then $\mathcal{M}^*$ is totally geodesic. Also from $A = 0$, we have $\nabla\xi = 0$ and $\nabla v = 0$.

Now we need to show that (3)≡(1),(2),(4).
If $\nabla\xi = 0$, then we have $\nabla_{\mathbb{X}_1}\xi = -\mathfrak{I}A_N\mathbb{X}_1 + bA_N\mathbb{X}_1 = 0$, namely,

$$\mathfrak{I}A_N\mathbb{X}_1 = bA_N\mathbb{X}_1.$$

Applying $\mathfrak{I}$ to the both side of the above equation, we have

$$\mathfrak{I}^2A_N\mathbb{X}_1 = b\mathfrak{I}A_N\mathbb{X}_1 = b^2A_N\mathbb{X}_1.$$

From equation (20), we find

$$\bar{\beta}\mathfrak{I}A_N\mathbb{X}_1 - \mathfrak{I}A_N\mathbb{X}_1 + A_N\mathbb{X}_1 - v(A_N\mathbb{X}_1)\xi = b^2A_N\mathbb{X}_1.$$

Considering equation (30) and $||\xi|| = 1$, we obtain

$$A_N\mathbb{X}_1 = \frac{1}{2}\left(\frac{\mathbb{X}_1(b)\xi}{b^2 - cb + b - 1}\right), \quad \forall\mathbb{X}_1 \in \Gamma(T\mathcal{M}^*). \tag{34}$$

If we apply $\Im$ to the both side of $\Im A_N \mathbb{X}_1 = bA_N \mathbb{X}_1$, for $\bar{\beta}\xi = c\xi$, $c \in \mathbb{R}$, we find

$$b^2 A_N X = \frac{X(b)}{2\|\xi\|^2}(\bar{\beta}\xi - (l+b)\xi)$$

and

$$\frac{\mathbb{X}_1(b)}{2\|\xi\|^2}(2b^2 - bc - b)\xi = 0.$$

Since $\|\xi\| \neq 0$, we have $\mathbb{X}_1(b) = 0$ or $b = \frac{c+1}{2}$, therefore, from (34), we obtain $A_N \mathbb{X}_1 = 0$ for every $\mathbb{X}_1 \in \Gamma(T\mathcal{M}^*)$. Namely $\mathcal{M}^*$ is totally geodesic so $\nabla\Im = 0$ and $\nabla v = 0$.
If $\nabla v = 0$, then $\nabla\Im = 0$ so $\mathcal{M}^*$ is totally geodesic and $\nabla\xi = 0$. □

Recall that a hypersurface $\mathcal{M}^*$ in the meta-Golden Riemannian manifold $(\bar{\mathcal{M}}^*, \bar{\beta}, \bar{\Im}, \bar{g})$ is said to be minimal if

$$trace(A_N) = \sum_{j=1}^{n} g(A_N e_j, e_j)$$

vanishes identically, where $\{e_1, e_2, ..., e_n\}$ is an orthonormal basis of the tangent space $T_p\mathcal{M}^*$ in every point $p \in \mathcal{M}^*$.

Theorem 3.12. *Let $\mathcal{M}^*$ be an invariant hypersurface of a MGR manifold $(\bar{\mathcal{M}}^*, \bar{\beta}, \bar{\Im}, \bar{g})$, $\bar{\Im}$ is parallel according to $\overline{\nabla}$ Levi-Civita connection on $\bar{\mathcal{M}}^*$ and $\Pi = (\Im, \bar{\beta}, g, \xi, v, b)$ is induced structure with $(\Im, g)$ on $\mathcal{M}^*$. If*

$$\sum_{j=1}^{n} (\nabla e_j \Im) e_j = \sum_{j=1}^{n} v(e_j) A(e_j),$$

then, $\mathcal{M}^*$ is minimal.

Corollary 3.13. *If $\Pi = (\Im, \bar{\beta}, g, \xi, v, b)$ is the induced structure from an umbilical hypersurface $\mathcal{M}^*$ in a meta-Golden Riemannian manifold $(\bar{\mathcal{M}}^*, \bar{\beta}, \bar{\Im}, \bar{g})$ with $\overline{\nabla}\bar{\Im} = 0$, then we obtain*

$$(\nabla_{\mathbb{X}_1}\Im)\Upsilon_1 = \lambda[g(\mathbb{X}_1, \Upsilon_1)\xi - g(\xi, \Upsilon_1)X],$$
$$(\nabla_{\mathbb{X}_1} v)\Upsilon_1 = \lambda[g(\mathbb{X}_1, \Upsilon_1)b - g(\mathbb{X}_1, \Im\Upsilon_1)],$$
$$\nabla_{\mathbb{X}_1}\xi = -\Im\lambda\mathbb{X}_1 + b\lambda\mathbb{X}_1 = \lambda(b\mathbb{X}_1 - \Im\mathbb{X}_1),$$
$$\nabla_\xi\xi = -\lambda(\beta\xi - (1+2b)\xi),$$
$$\mathbb{X}_1(b) = -2\lambda g(\mathbb{X}_1, \xi),$$

for any $\mathbb{X}_1, \Upsilon_1 \in \Gamma(T\mathcal{M}^*)$.

Theorem 3.14. *Let $\mathcal{M}^*$ be an invariant umbilical ($\lambda \neq 0$) hypersurface in a MGR manifold $(\bar{\mathcal{M}}^*, \bar{\beta}, \bar{\Im}, \bar{g})$ with $\overline{\nabla}\bar{\Im} = 0$ and $\Pi = (\Im, \bar{\beta}, g, \xi, v, b)$ be the induced structure on $\mathcal{M}^*$ by $(\bar{\beta}, \bar{\Im}, \bar{g})$. Therefore, $\Im = -(c+b)I$ where c and b are constant function on $\mathcal{M}^*$ equal with the golden number $c = ₵$ and $b = -\frac{₵ \pm \sqrt{₵^2+4}}{2}$ or $c = 1 - ₵$ and $b = -\frac{(1-₵) \pm \sqrt{(1-₵)^2+4}}{2}$.*

Corollary 3.15. *Let $\mathcal{M}^*$ be a hypersurface in a MGR manifold $(\bar{\mathcal{M}}^*, \bar{\beta}, \bar{\Im}, \bar{g})$ with $\overline{\nabla}\bar{\Im} = 0$ and $\Pi = (\Im, \bar{\beta}, g, \xi, v, b)$ is the induced structure on $\mathcal{M}^*$ by $(\bar{\beta}, \bar{\Im}, \bar{g})$ with $\Im = bI$, then $\nabla\xi = 0$.*

Thus we can give the following theorem.

Theorem 3.16. *Let $\mathcal{M}^*$ be a hypersurface in a MGR manifold $(\bar{\mathcal{M}}^*, \bar{\beta}, \bar{\mathfrak{I}}, \bar{g})$ with $\overline{\nabla}\bar{\mathfrak{I}} = 0$ and $\Pi = (\mathfrak{I}, \bar{\beta}, g, \xi, v, b)$ be the induced structure on $\mathcal{M}^*$ by $(\bar{\beta}, \bar{\mathfrak{I}}, \bar{g})$ with $\beta = bI$. Then we obtain only one of the following conclusions;*

i *$\mathcal{M}^*$ is invariant hypersurface and b is a golden number where b is given with (31) or (32).*

ii *$\mathcal{M}^*$ is non-invariant totally geodesic hypersurface in the MGR manifold $(\bar{\mathcal{M}}^*, \bar{\beta}, \bar{\mathfrak{I}}, \bar{g})$.*

Example 3.17. *Let $\mathbb{E}^5$ be an Euclidean space. Then, $\mathbb{E}^5$ is an almost Golden manifold with golden structure $\bar{\beta}$ given as*

$$\bar{\beta} : \mathbb{E}^5 \to \mathbb{E}^5$$
$$(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \mathbb{Y}_1, \mathbb{Y}_2) \to (\not{c}\mathbb{X}_1, \not{c}\mathbb{X}_2, \not{c}\mathbb{X}_3, (1-\not{c})\mathbb{Y}_1, (1-\not{c})\mathbb{Y}_2).$$

Now, we define an endomorphism $\bar{\mathfrak{I}}$ an $(\mathbb{E}^5, \bar{\beta})$ by

$$\bar{\mathfrak{I}} : \mathbb{E}^5 \to \mathbb{E}^5$$
$$(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \mathbb{Y}_1, \mathbb{Y}_2) \to (\dot{\chi}\mathbb{X}_1, \dot{\chi}\mathbb{X}_2, \dot{\chi}\mathbb{X}_3, -\check{\chi}\mathbb{Y}_1, -\check{\chi}\mathbb{Y}_2).$$

Here $\check{\chi} = \frac{\not{c} + \sqrt{\not{c}^2 + 4}}{2}$, is called "the silver mean of golden mean" satisfies the

$$\check{\chi}^2 = \not{c}\check{\chi} + 1$$

identity and $\dot{\chi}$ is the meta-Golden Chi ratio. Therefore $\bar{\mathfrak{I}}$ is a meta-Golden structure. Thus $(\mathbb{E}^5, \bar{\beta}, \bar{\mathfrak{I}})$ is an almost meta-Golden manifold. Now we consider

$$S^4(r) = \left\{ (\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \mathbb{Y}_1, \mathbb{Y}_2) : \sum_{i=1}^{3} (\mathbb{X}_i)^2 + \sum_{j=1}^{2} (\mathbb{Y}_j)^2 = r^2 \right\},$$

which is a submanifold of codimension 1 in $\mathbb{E}^5$.

In every point $(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \mathbb{Y}_1, \mathbb{Y}_2) \in S^4(r)$, we take into account the normal vector field to $S^4(r)$ given by $N = \frac{1}{r}(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \mathbb{Y}_1, \mathbb{Y}_2)$.

In every point $(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \mathbb{Y}_1, \mathbb{Y}_2) \in \mathbb{E}^5$, we find a tangent vector on $S^4(r)$.

$(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \mathbb{Y}_1, \mathbb{Y}_2) \in T_{(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \mathbb{Y}_1, \mathbb{Y}_2)}(S^4(r))$ if and only if

$$\mathbb{X}_1\mathbb{X}_1 + \mathbb{X}_2\mathbb{X}_2 + \mathbb{X}_3\mathbb{X}_3 + \mathbb{Y}_1\mathbb{Y}_1 + \mathbb{Y}_2\mathbb{Y}_2 = 0.$$

From the decompositions of $\bar{\mathfrak{I}}(N)$ and $\bar{\mathfrak{I}}(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \mathbb{Y}_1, \mathbb{Y}_2)$ respectively, in tangential and normal components on $T_{(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \mathbb{Y}_1, \mathbb{Y}_2)}(S^4(r))$, we find

$$\bar{\mathfrak{I}}(N) = \xi + bN,$$

$$\bar{\mathfrak{I}}(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \mathbb{Y}_1, \mathbb{Y}_2) = \mathfrak{I}(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \mathbb{Y}_1, \mathbb{Y}_2) + v(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \mathbb{Y}_1, \mathbb{Y}_2)N$$

where $\mathbb{X} = (\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \mathbb{Y}_1, \mathbb{Y}_2)$ is a tangent vector field on $S^4(r)$, $\mathfrak{I}$ is an $(1,1)$ tensor field on $S^4(r)$, $\xi \in \bar{\mathfrak{I}}(S^4(r))$, v is a $1-$form on $S^4(r)$ and b is a smooth real function on $S^4(r)$.

Using $b = \left\langle \bar{\mathfrak{I}}N, N \right\rangle$, $\xi = \bar{\mathfrak{I}}N - bN$,

$$v(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \mathbb{Y}_1, \mathbb{Y}_2) = \langle (\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \mathbb{Y}_1, \mathbb{Y}_2), \xi \rangle$$

and

$$\mathfrak{I}(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \mathbb{Y}_1, \mathbb{Y}_2) = \bar{\mathfrak{I}}(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \mathbb{Y}_1, \mathbb{Y}_2) - v(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \mathbb{Y}_1, \mathbb{Y}_2)N,$$

the elements of the induced structure $\Pi = (\mathfrak{I}, \beta, \langle , \rangle, \xi, v, b)$ *on* $S^4(r)$ *by the meta-Golden structure* $(\mathfrak{I}, \langle , \rangle)$ *on* $\mathbb{E}^5$ *are given as follows;*

$$b = \frac{\dot{\chi}(\sum_{i=1}^{3}(\mathbb{X}_i)^2) - \check{\chi}(\sum_{j=1}^{2}(\Upsilon_j)^2)}{r^2},$$

$$\xi = \frac{1}{r}(\dot{\chi}\mathbb{X}_1, \dot{\chi}\mathbb{X}_2, \dot{\chi}\mathbb{X}_3, -\check{\chi}\Upsilon_1, -\check{\chi}\Upsilon_2) - \frac{\dot{\chi}(\sum_{i=1}^{3}(\mathbb{X}_i)^2) - \check{\chi}(\sum_{j=1}^{2}(\Upsilon_j)^2)}{r^3}(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \Upsilon_1, \Upsilon_2),$$

$$v(\mathbb{X}) = \frac{1}{r}g(\mathfrak{I}\mathbb{X}, \mathbb{X}) - bg(\mathbb{X}, \mathbb{X}),$$

$$\mathfrak{I}\mathbb{X} = \bar{\mathfrak{I}}\mathbb{X} - \frac{1}{r}g(\mathfrak{I}\mathbb{X}, \mathbb{X})N + bg(\mathbb{X}, \mathbb{X})N$$

where $\mathbb{X} = (\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \Upsilon_1, \Upsilon_2)$ *is a tangent vector field on* $S^4(r)$*. In conclusion,* $S^4(r)$ *is a non-invariant hypersurface.*

Example 3.18. *Let* $\bar{\beta}$ *be an almost golden structure;*

$$\bar{\beta} : \mathbb{E}^5 \to \mathbb{E}^5$$
$$(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \Upsilon_1, \Upsilon_2) \to (\not{c}\mathbb{X}_1, \not{c}\mathbb{X}_2, \not{c}\mathbb{X}_3, (1-\not{c})\Upsilon_1, (1-\not{c})\Upsilon_2).$$

where $\mathbb{E}^5$ *be a Euclidean space.*

Then, we define $(1, 1)$ *tensor field* $\breve{\mathfrak{I}}$ *on* $(\mathbb{E}^5, \bar{\beta})$ *by*

$$\breve{\mathfrak{I}} : \mathbb{E}^5 \to \mathbb{E}^5$$
$$(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \Upsilon_1, \Upsilon_2) \to (\dot{\chi}\mathbb{X}_1, \dot{\chi}\mathbb{X}_2, \dot{\chi}\mathbb{X}_3, -\check{\chi}\Upsilon_1, -\check{\chi}\Upsilon_2),$$

where $\check{\chi} = \frac{\not{c}+\sqrt{\not{c}^2+4}}{2}$.
It is easy to see that $\breve{\mathfrak{I}}$ *is a meta-golden structure on* $\mathbb{E}^5$ *and so* $(\mathbb{E}^5, \bar{\beta}, \breve{\mathfrak{I}}, \bar{g})$ *is an almost meta-golden Riemannian manifold, where* $\bar{g}$ *is the usual Euclid metric on* $\mathbb{E}^5$*.*

Now, we consider a hypersurface $\mathcal{M}$ *of* $\mathbb{E}^5$ *given by* $\Upsilon_1 = \Upsilon_2$*. Then* $T\mathcal{M}$ *is spanned by*

$$\mathbb{Z}_1 = \frac{\partial}{\partial x_1}, \quad \mathbb{Z}_2 = \frac{\partial}{\partial x_2}, \quad \mathbb{Z}_3 = \frac{\partial}{\partial x_3}, \quad \mathbb{Z}_4 = \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}, \quad \mathbb{N} = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2}).$$

In this case since $\breve{\mathfrak{I}}\bar{\beta}(T\mathcal{M}) \subset T\mathcal{M}$ *then* $\mathcal{M}$ *is an invariant hypersurface of* $\mathbb{E}^5$*.*
Then let calculate the function b as follow; from $b = \bar{g}(\breve{\mathfrak{I}}\mathbb{N}, \mathbb{N})$*, we get*

$$b = -\frac{\not{c} + \sqrt{\not{c}^2 + 4}}{2}.$$

Example 3.19. *Let* $\bar{\beta}$ *be an almost golden structure;*

$$\bar{\beta} : \mathbb{E}^5 \to \mathbb{E}^5$$
$$(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \Upsilon_1, \Upsilon_2) \to (\not{c}\mathbb{X}_1, \not{c}\mathbb{X}_2, \not{c}\mathbb{X}_3, (1-\not{c})\Upsilon_1, (1-\not{c})\Upsilon_2).$$

where $\mathbb{E}^5$ *be a Euclidean space.*

Then, we define $(1, 1)$ *tensor field* $\breve{\mathfrak{I}}$ *on* $(\mathbb{E}^5, \bar{\beta})$ *by*

$$\breve{\mathfrak{I}} : \mathbb{E}^5 \to \mathbb{E}^5$$
$$(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \Upsilon_1, \Upsilon_2) \to (\dot{\chi}\mathbb{X}_1, \dot{\chi}\mathbb{X}_2, \dot{\chi}\mathbb{X}_3, -\check{\chi}\Upsilon_1, -\check{\chi}\Upsilon_2),$$

where $\check{\chi} = \frac{\not{c} + \sqrt{\not{c}^2 + 4}}{2}$.

It is easy to see that $\bar{\mathfrak{I}}$ *is a meta-Golden structure on* $\mathbb{E}^5$ *and so* $(\mathbb{E}^5, \bar{\beta}, \bar{\mathfrak{I}}, \bar{g})$ *is an AMGR manifold, where* $\bar{g}$ *is the usual Euclid metric on* $\mathbb{E}^5$.

Now, we consider a hypersurface $\mathcal{M}$ *of* $\mathbb{E}^5$ *given by*
$\mathbb{X}_1 = cos(\mathbb{X}_2) + sin(\mathbb{X}_3)$, $\mathbb{X}_2 = cos(\mathbb{X}_3) - sin(\mathbb{X}_2)$.
Then $T\mathcal{M}$ *is spanned by*

$$\mathbb{Z}_1 = -sin(\mathbb{X}_2)\frac{\partial}{\partial x_1} - cos(\mathbb{X}_2)\frac{\partial}{\partial x_2}, \quad \mathbb{Z}_2 = cos(\mathbb{X}_3)\frac{\partial}{\partial x_1} - sin(\mathbb{X}_3)\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3},$$

$$\mathbb{Z}_3 = \frac{\partial}{\partial y_1}, \quad \mathbb{Z}_4 = \frac{\partial}{\partial y_2}$$

and

$$\mathbb{N} = -cos(\mathbb{X}_3)\frac{\partial}{\partial x_1} + sin(\mathbb{X}_2)\frac{\partial}{\partial x_2} + cos(\mathbb{X}_2 - \mathbb{X}_3)\frac{\partial}{\partial x_3}, \quad \mathbb{X}_2 > \mathbb{X}_3.$$

In this case since $\bar{\mathfrak{I}}\bar{\beta}(T\mathcal{M}) \subset T\mathcal{M}$ *then* $\mathcal{M}$ *is a non-totally geodesic invariant hypersurface of* $\mathbb{E}^5$.

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