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Solvability of a system of an operator equation and finitely many equality constraints

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Abstract. The existence of solutions for the problem

$$\begin{cases} Tu = u \\ \alpha_j(u) = \theta_E, \ j = 1, 2, \ldots, r \end{cases}$$

is proved, by considering Ćirić-Jotić contraction condition, where $T, \alpha_j : E \to E$ $(j = 1, 2, \cdots, r)$ are mappings and $(E, \|\cdot\|)$ is a Banach space. A set of sufficient conditions on T, α_j is used which ensure the existence of, possibly, non-unique solution to the underlying system. A common fixed point result is derived from the obtained theorem and some illustrative examples are given in order to justify the established results. An application to nonlinear matrix equations is also presented.

1. Introduction and preliminaries

Over the past few decades, numerous authors have achieved significant advancements in the field of fixed point and common fixed point theory. These achievements have been subsequently utilized to derive solutions for a wide range of equations encountered in various mathematical contexts. Within the existing body of literature, a majority of fixed point results primarily focus on the existence of a single fixed point. However, it is worth noting that there are also other fixed point results that specifically address the presence of non-unique fixed points. It is widely acknowledged that obtaining a unique solution for all types of systems, such as differential and integral equations, is not feasible. However, in order to identify a shared fixed point, it is necessary to satisfy a compatibility criterion or a weaker condition. Nevertheless, there are no assurances that this condition will be met. An alternative method for addressing such a predicament is the utilization of numerical techniques.

In order to address such instances, it is important to engage in a comprehensive examination of the notion of non-unique fixed point, so as to effectively manage the given scenario. This study is inspired by

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the research conducted by authors [8, 10], in which they employed a distinct methodology to address a system of equations. The proposed strategy was validated through many illustrative cases and successfully determined solutions to a set of matrix equations.

In the paper [10], Rakočević and Samet discussed the solvability of the system

$$\begin{cases} Tu = u \\ \alpha_j(u) = \theta_E, \ j = 1, 2, \ldots, r, \end{cases} \tag{1}$$

where $T, \alpha_j : E \to E$ $(j = 1, 2, \cdots, r)$ are mappings, $(E, \|\cdot\|)$ is a Banach space, θ_E is the null vector of E and T satisfies Ćirić contraction [2]. Subsequently, Karapınar et al. [8] dealt with the same system by using Pachpatte contraction [9] in place of Ćirić contraction. Note that the considered Pachpatte contraction [8, Equation (3)] is not symmetric as mentioned in Remark 1.1 [8].

In the literature, there are several contractions which produce non-unique fixed points and that are more general than Ćirić and Pachpatte contractions (see, e.g., [1, 5–7]). Thus, it is interesting to see whether the underlying system of operator equations can be handled with more general contractions. Inspired by these observations, we wish to continue this study of solvability of a system of operator equations by considering Ćirić-Jotić contraction [3].

Definition 1.1. *A self-map T on a metric space (E, d) is said to be a Ćirić-Jotić contraction if there exist $\gamma \geq 0$, $r \in [0, 1)$ such that for all $u, v \in E$, $u \neq v$, the following holds:*

$$\begin{aligned} &\min\left\{ \begin{array}{c} d(Tu, Tv), d(u, v), d(u, Tu), d(v, Tv), \frac{d(u,Tu)[1+d(v,Tv)]}{1+d(u,v)}, \\ \frac{d(v,Tv)[1+d(u,Tu)]}{1+d(u,v)}, \frac{\min\{d^2(Tu,Tv), d^2(u,Tu), d^2(v,Tv)\}}{d(u,v)} \end{array} \right\} \\ &- \gamma \ \min\{d(u, Tv), d(v, Tu)\} \leq r \max\{d(u, v), d(u, Tu)\}. \end{aligned}$$

Under the Ćirić-Jotić contraction, we work on the solvability of the system (1) in a Banach space with a cone. We verify our obtained results by suitable examples in Banach spaces $\mathbb{R}$ and $H(n)$ (the set of all $n \times n$ Hermitian matrices over $\mathbb{C}$ with trace norm). In the final section, it is shown how the obtained results can be applied for proving the existence of solutions for a system of nonlinear matrix equations.

In the rest of the paper, $(E, \|\cdot\|)$ will denote a Banach space.

Recall that a nonempty subset P of E is said to be a cone if the following conditions hold:

(i) P is closed and convex;
(ii) if $u \in P$ and $\mu \geq 0$, then $\mu u \in P$;
(iii) $P \cap (-P) = \{\theta_E\}$.

As usual, when the cone P in E is given, we will consider the induced partial order $\sqsubseteq_P$ on E defined by $u \sqsubseteq_P v$ if and only if $v - u \in P$.

Definition 1.2. *(see [10]). A mapping $f : E \to E$ is said to be θ_E-level closed from the left (resp. from the right) if the set*

$$\begin{aligned} &lev\text{-}f_{\sqsupseteq_P} = \{u \in E : f(u) \sqsupseteq_P \theta_E\} \neq \emptyset \\ (resp.\ &lev\text{-}f_{\sqsubseteq_P} = \{u \in E : f(u) \sqsubseteq_P \theta_E\} \neq \emptyset) \end{aligned}$$

is closed.

2. Main results

Let $T, \alpha_j : E \to E$ $(j = 1, 2, \cdots, r)$ be mappings. Consider the system (1) where θ_E is the null vector of E and T satisfies the following Ćirić-Jotić-type contraction, with a family of control functions.

Definition 2.1. *The operator T is a Ćirić-Jotić-type contraction with respect to $\{\alpha_j\}_{j=1}^r$ if there exist $\gamma \geq 0$, $\lambda \in [0,1)$ such that*

$$M(u,v) - \gamma N(u,v) \leq \lambda \max\{\|u-v\|, \|u-Tu\|\} \tag{2}$$

holds for all $u, v \in E$, $u \neq v$, such that $\alpha_j(u) \sqsubseteq_P \theta_E$ and $\alpha_j(v) \sqsupseteq_P \theta_E$, where

$$M(u,v) = \min\left\{ \begin{array}{c} \|Tu-Tv\|, \|u-v\|, \|u-Tu\|, \|v-Tv\|, \\ \frac{\|u-Tu\|[1+\|v-Tv\|]}{1+\|u-v\|}, \frac{\|v-Tv\|[1+\|u-Tu\|]}{1+\|u-v\|}, \\ \frac{\min\left\{\|Tu-Tv\|^2, \|u-Tu\|^2, \|v-Tv\|^2\right\}}{\|u-v\|} \end{array} \right\}$$

and

$$N(u,v) = \min\{\|u-Tv\|, \|v-Tu\|\}.$$

Note that, by interchanging places of u and v, we get that also the condition

$$M(u,v) - \gamma N(u,v) \leq \lambda \max\{\|u-v\|, \|v-Tv\|\} \tag{3}$$

has to be fulfilled.

Sufficient conditions for the solvability of system (1) are presented in the following theorem.

Theorem 2.2. *Let $(E, \|\cdot\|)$ be a Banach space with a cone P and let $T, \alpha_j : E \to E$ $(j = 1, 2, \cdots, r)$. Assume that:*

(i) T is orbitally continuous on $\bigcap_{j=1}^r$ lev-$\alpha_{j_{\sqsubseteq_P}}$;
(ii) T is a Ćirić-Jotić-type contraction with respect to $\{\alpha_j\}_{j=1}^r$;
(iii) α_j is θ_E-level closed from the left for $j = 1, 2, \cdots, r$;
(iv) there exists $u_0 \in E$ such that $\alpha_j(u_0) \sqsubseteq_P \theta_E$ for $j = 1, 2, \cdots, r$;
(v) for $u \in E$, $\alpha_j(u) \sqsubseteq_P \theta_E$, $j = 1, 2, \cdots, r$, implies $\alpha_j(Tu) \sqsupseteq_P \theta_E$, $j = 1, 2, \cdots, r$, and $\alpha_j(u) \sqsupseteq_P \theta_E$, $j = 1, 2, \cdots, r$, implies $\alpha_j(Tu) \sqsubseteq_P \theta_E$, $j = 1, 2, \cdots, r$.

Then the sequence $\{T^n u_0\}$ converges to a solution of the system (1).

Proof. Start with defining a sequence $\{u_n\}$ in E, where $u_n = T^n u_0$ for all $n \in \mathbb{N}$. By the conditions (iv) and (v), we have $\alpha_j(u_n) \sqsubseteq_P \theta_E$ if n is even and $\alpha_j(u_n) \sqsupseteq_P \theta_E$ if n is odd for all $j = 1, 2, \cdots, r$.

We can assume, WLOG, that $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$. Since T is a Ćirić-Jotić-type contraction with respect to $\{\alpha_j\}_{j=1}^r$, for any $n \in \mathbb{N}$, putting $u = u_{n-1}$ and $v = u_n$ in (2) we have

$$M(u_{n-1}, u_n) - \gamma N(u_{n-1}, u_n) \leq \lambda \max\{\|u_{n-1} - u_n\|, \|u_{n-1} - Tu_{n-1}\|\},$$

that is,

$$M(u_{n-1}, u_n) - \gamma N(u_{n-1}, u_n) \leq \lambda \|u_{n-1} - u_n\|, \tag{4}$$

where

$$\begin{aligned} &M(u_{n-1}, u_n) \\ &= \min\left\{ \begin{array}{c} \|Tu_{n-1} - Tu_n\|, \|u_{n-1} - u_n\|, \|u_{n-1} - Tu_{n-1}\|, \|u_n - Tu_n\|, \\ \frac{\|u_{n-1}-Tu_{n-1}\|[1+\|u_n-Tu_n\|]}{1+\|u_{n-1}-u_n\|}, \frac{\|u_n-Tu_n\|[1+\|u_{n-1}-Tu_{n-1}\|]}{1+\|u_{n-1}-u_n\|}, \\ \frac{\min\left\{\|Tu_{n-1}-Tu_n\|^2, \|u_{n-1}-Tu_{n-1}\|^2, \|u_n-Tu_n\|^2\right\}}{\|u_{n-1}-u_n\|} \end{array} \right\} \\ &= \min\left\{ \begin{array}{c} \|u_n - u_{n+1}\|, \|u_{n-1} - u_n\|, \frac{\|u_{n-1}-u_n\|[1+\|u_n-u_{n+1}\|]}{1+\|u_{n-1}-u_n\|}, \\ \frac{\min\left\{\|u_n-u_{n+1}\|^2, \|u_{n-1}-u_n\|^2\right\}}{\|u_{n-1}-u_n\|} \end{array} \right\} \\ &= \min\left\{ \|u_n - u_{n+1}\|, \|u_{n-1} - u_n\|, \frac{\|u_{n-1}-u_n\|[1+\|u_n-u_{n+1}\|]}{1+\|u_{n-1}-u_n\|}, \frac{\|u_n-u_{n+1}\|^2}{\|u_{n-1}-u_n\|} \right\} \end{aligned}$$

and

$$N(u_{n-1}, u_n) = \min\{\|u_{n-1} - u_{n+1}\|, \|u_n - u_n\|\} = 0.$$

Now the following four cases arise:

- If $M(u_{n-1}, u_n) = \|u_n - u_{n+1}\|$, then from (4), we get
$$\|u_n - u_{n+1}\| \leq \lambda \|u_{n-1} - u_n\|.$$
- If $M(u_{n-1}, u_n) = \|u_{n-1} - u_n\|$, then from (4), we get
$$\|u_{n-1} - u_n\| \leq \lambda \|u_{n-1} - u_n\|,$$
which is impossible, since $\lambda < 1$ and $u_{n-1} \neq u_n$.
- If $M(u_{n-1}, u_n) = \frac{\|u_{n-1}-u_n\|[1+\|u_n-u_{n+1}\|]}{1+\|u_{n-1}-u_n\|}$, then from (4), we get
$$\frac{\|u_{n-1} - u_n\|[1 + \|u_n - u_{n+1}\|]}{1 + \|u_{n-1} - u_n\|} \leq \lambda\, \|u_{n-1} - u_n\|,$$
implying that
$$\|u_n - u_{n+1}\| \leq \lambda \|u_{n-1} - u_n\|.$$
- If $M(u_{n-1}, u_n) = \frac{\|u_n-u_{n+1}\|^2}{\|u_{n-1}-u_n\|}$, then from (4), we get
$$\frac{\|u_n - u_{n+1}\|^2}{\|u_{n-1} - u_n\|} \leq \lambda\, \|u_{n-1} - u_n\|,$$
implying that
$$\|u_n - u_{n+1}\| \leq \lambda^{1/2} \|u_{n-1} - u_n\|.$$

Since, in all possible cases, the above relations are true for all $n \in \mathbb{N}$, the sequence $\{u_n\}$ is a Cauchy sequence in E and hence there exists $u \in E$ such that $\lim_{n\to\infty} u_n = u$. By the assumption (i), we have $\lim_{n\to\infty} u_{n+1} = Tu$. So we have $Tu = u$.

Now, since $u_{2n+1} \in \bigcap_{j=1}^{r} lev\text{-}\alpha_{j_{\sqsupseteq_P}}$ for all $n \in \mathbb{N}$ and α_j is θ_E-level closed from the left, we have $\alpha_j(u) \sqsupseteq_P \theta_E$. Therefore by the assumption (v), we get $\alpha_j(Tu) \sqsubseteq_P \theta_E$, that is, $\alpha_j(u) \sqsubseteq_P \theta_E$. Since P is a cone, we have $\alpha_j(u) = \theta_E$. This fact is true for all $j = 1, 2, \cdots, r$. This completes the proof. □

From the above theorem, we now deduce a common fixed point result for a family of mappings.

Theorem 2.3. *Let $(E, \|\cdot\|)$ be a Banach space and let P be a cone in E. Suppose that $T, F_j : E \to E$ ($j = 1, 2, \cdots, r$) be a family of mappings such that there exist $\gamma \geq 0$, $\lambda \in (0, 1)$ satisfying*

$$M(u, v) - \gamma\, N(u, v) \leq \lambda \max\{\|u - v\|, \|u - Tu\|\}$$

for all $u, v \in E$, $u \neq v$, for which $F_j(u) \sqsubseteq_P u$ and $F_j(v) \sqsupseteq_P v$, where

$$M(u,v) = \min\left\{ \begin{array}{c} \|Tu - Tv\|, \|u - v\|, \|u - Tu\|, \|v - Tv\|, \frac{\|u-Tu\|[1+\|v-Tv\|]}{1+\|u-v\|}, \\ \frac{\|v-Tv\|[1+\|u-Tu\|]}{1+\|u-v\|}, \frac{\min\{\|Tu-Tv\|^2, \|u-Tu\|^2, \|v-Tv\|^2\}}{\|u-v\|} \end{array} \right\}$$

and

$$N(u, v) = \min\{\|u - Tv\|, \|v - Tu\|\}.$$

Further, assume that

(i) *T is orbitally continuous on $\{u \in E : F_j u \sqsubseteq_P u, \forall\ j = 1, 2, \cdots, r\}$;*
(ii) *$\{u \in E : F_j u \sqsubseteq_P u\}$ is closed for all $j = 1, 2, \cdots, r$;*
(iii) *there exists $u_0 \in E$ such that $F_j(u_0) \sqsubseteq_P u_0$ for $j = 1, 2, \cdots, r$;*
(iv) *for $u \in E$, $F_j(u) \sqsubseteq_P u$, $j = 1, 2, \cdots, r$, implies $F_j(Tu) \sqsupseteq_P Tu$, $j = 1, 2, \cdots, r$, and $F_j(u) \sqsupseteq_P u$, $j = 1, 2, \cdots, r$ implies $F_j(Tu)_j \sqsubseteq_P Tu$, $j = 1, 2, \cdots, r$.*

Then the family of mappings considered here, has a common fixed point.

Proof. Start with defining r mappings $\alpha_j : E \to E$ $(j = 1, 2, \cdots, r)$ by

$$\alpha_j(u) = F_j u - u$$

for all $u \in E$. Then for $u \in E$, $\alpha_j(u) \sqsubseteq_P \theta_E$ if any only if $F_j(u) \sqsubseteq_P u$. Hence from the contraction condition satisfied by T, we see that T is a Ćirić-Jotić-type contraction with respect to $\{\alpha_j\}_{j=1}^r$. Further, we have $\{u \in E : F_j u \sqsubseteq_P u, \forall\ j = 1, 2, \cdots, r\} = \bigcap_{j=1}^r lev\text{-}\alpha_{j_{\sqsubseteq_P}}$, so T is orbitally continuous on $\bigcap_{j=1}^r lev\text{-}\alpha_{j_{\sqsubseteq_P}}$. Since $\{u \in E : F_j u \sqsubseteq_P u\}$ is nonempty and closed, it follows that $\{u \in E : \alpha_j(u) \sqsubseteq_P \theta_E\}$ is nonempty and closed. So α_j is θ_E-level closed from the left for all $j = 1, 2, \cdots, r$. Further, from the assumptions (iii) and (iv), we have $\alpha_j(u_0) \sqsubseteq_P \theta_E$ for some $u_0 \in E$, and $\alpha_j(u) \sqsubseteq_P \theta_E$ implies $\alpha_j(Tu) \sqsupseteq_P \theta_E$ and $\alpha_j(u) \sqsupseteq_P \theta_E$ implies $\alpha_j(Tu) \sqsubseteq_P \theta_E$ for $j = 1, 2, \cdots, r$.

This implies that all conditions of Theorem 2.2 hold here. Hence, there exists $u \in E$ such that $Tu = u$ and $\alpha_j(u) = \theta_E$ for $j = 1, 2, \cdots, r$, that is, the family of mappings T, F_j has a common fixed point. □

Example 2.4. *Let us consider the Banach space $(\mathbb{R}, \|\cdot\|)$, where $\|u\| = |u|$, and take the cone $P = [0, \infty)$ in $\mathbb{R}$. Next, we define three mappings $T, \alpha_j : \mathbb{R} \to \mathbb{R}$ $(j = 1, 2)$ by*

$$Tu = \begin{cases} -1, & \text{if } u \geq 1 \text{ or } u = -1; \\ -u^3, & \text{if } 0 \leq u < 1; \\ 0, & \text{otherwise}; \end{cases}$$

$$\alpha_1(u) = \begin{cases} 0, & \text{if } u = -1; \\ -u, & \text{otherwise}; \end{cases} \qquad \alpha_2(u) = \begin{cases} u^2, & \text{if } u < 0, u \neq -1, \\ 0, & \text{if } u = -1; \\ -u/2, & \text{if } u \geq 0. \end{cases}$$

Then we have $lev\text{-}\alpha_{1_{\sqsubseteq_P}} \cap lev\text{-}\alpha_{2_{\sqsubseteq_P}} = [0, +\infty) \cup \{-1\}$. Therefore, T is orbitally continuous on $lev\text{-}\alpha_{1_{\sqsubseteq_P}} \cap lev\text{-}\alpha_{2_{\sqsubseteq_P}}$. Also it is clear that the mappings α_j are θ_E-level closed from the left.

Note that if $\alpha_j(u) \sqsubseteq_P \theta_E$ for $j = 1, 2$, then $u \geq 0$ or $u = -1$ and, in both cases, $Tu \leq 0$, hence $\alpha_j(Tu) \sqsupseteq \theta_E$, $j = 1, 2$. Also, if $\alpha_j(u) \sqsupseteq \theta_E$ for $j = 1, 2$, then $u \leq 0$, implying that $Tu = 0$ (for $u \neq -1$) or $Tu = -1$ (for $u = -1$); in both cases $\alpha_j(Tu) = 0 \sqsubseteq_P \theta_E$, $j = 1, 2$.

Next, let $u, v \in \mathbb{R}$ be such that $\alpha_j(u) \sqsubseteq_P \theta_E$ and $\alpha_j(v) \sqsupseteq_P \theta_E$, $j = 1, 2$. Then $u \in [0, +\infty) \cup \{-1\}$ and $v \in (-\infty, 0]$. We are going to check that condition (2) is fulfilled for $\gamma = 0$ and $\lambda = 1/2$ by considering all possible cases. Suppose $u \neq v$.

(I) *When $u = -1$, condition (2) becomes*

$$\min\left\{1, |v+1|, 0, |v|, 0, \frac{|v|}{1+|v+1|}, \frac{\min\{1, 0, v^2\}}{|v+1|}\right\} \leq \frac{1}{2}\max\{|v+1|, 0\},$$

which is obviously true.

(II) *When $u = 0$, $v \neq -1$, condition (2) becomes*

$$\min\left\{0, |v|, 0, |v|, 0, \frac{|v|}{1+|v|}, \frac{\min\{0, 0, v^2\}}{|v|}\right\} \leq \frac{1}{2}\max\{|v|, 0],$$

which is true for any v.

(III) When $u = 0$, $v = -1$, condition (2) is trivially fulfilled.

(IV) When $0 < u \le 1$, $v = 0$, condition (2) becomes

$$\min\left\{u^3, u, u + u^3, 0, \frac{u+u^3}{1+u}, 0, \frac{\min\{u^6, (u+u^3)^2, 0\}}{u}\right\} \le \frac{1}{2}\max\{u, u^3\},$$

which is obviously true.

(V) When $0 < u \le 1$, $v = -1$, condition (2) becomes

$$\min\left\{|-u^3+1|, u+1, u+u^3, 0, \frac{u+u^3}{1+u}, 0, \frac{\min\{|-u^3+1|^2, (u+u^3)^2, 0\}}{u}\right\}$$
$$\le \frac{1}{2}\max\{u+1, u+u^3\},$$

which is obviously true.

(VI) When $0 < u \le 1$, $v \notin \{-1, 0\}$, condition (2) becomes

$$\min\left\{\begin{array}{c} u^3, |u-v|, u+u^3, |v|, \frac{(u+u^3)(1+|v|)}{1+|u-v|}, \\ \frac{|v|(1+u+u^3)}{1+|u-v|}, \frac{\min\{u^6, (u+u^3)^2, v^2\}}{|u-v|} \end{array}\right\} \le \frac{1}{2}\max\{|u-v|, u+u^3\}.$$

Since the left-hand side of the previous inequality is certainly at most equal u^3, and the right-hand side is at least $\frac{1}{2}(u+u^3)$, and since, for $0 < u \le 1$, it is $u^3 \le \frac{1}{2}(u+u^3)$, the condition is fulfilled.

(VII) When $u > 1$, $v \ne -1$, condition (2) becomes

$$\min\left\{\begin{array}{c} 1, |u-v|, u+1, |v|, \frac{(u+1)(1+|v|)}{1+|u-v|}, \\ \frac{|v|(1+(u+1))}{1+|u-v|}, \frac{\min\{1, (u+1)^2, v^2\}}{|u-v|} \end{array}\right\} \le \frac{1}{2}\max\{|u-v|, u+1\}.$$

Since the left-hand side of the previous inequality is at most equal to 1, and the right-hand side is at least $\frac{1}{2}\cdot 2 = 1$, the condition is fulfilled.

(VIII) When $u > 1$, $v = -1$, condition (2) becomes

$$\min\{0, \dots\} \le \frac{1}{2}\max\{u+1, u+1\},$$

and trivially holds true.

By a similar discussion, it can be shown that the condition

$$M(u,v) - \gamma\, N(u,v) \le \frac{1}{2}\ \max\{\|u-v\|, \|v-Tv\|\},$$

is also true for all admissible values of u, v. Hence all the conditions of Theorem 2.2 hold true. So, by this theorem, it follows that, for each $u_0 \in [0, +\infty) \cup \{-1\}$, the sequence $\{T^n u_0\}$ converges to a solution of the system $Tu = u$ and $\alpha_j(u) = 0$ ($j = 1, 2$). There are two solutions to the given system: $u = -1$ and $u = 0$ (the first one is the limit of the previous sequence if $u_0 = -1$ or $u_0 \ge 1$, and the second is the limit if $0 \le u_0 < 1$).

Example 2.5. *Let $H(n)$ stand for the set of all $n \times n$ Hermitian matrices over $\mathbb{C}$, $K(n)$ $\big(\subset H(n)\big)$ stand for the set of all $n \times n$ positive semi-definite matrices, $M(n)$ stand for the set of all $n \times n$ matrices over $\mathbb{C}$.*

For a matrix $B \in H(n)$, we will denote by $\|B\|_{tr}$ its trace norm, i.e., the sum of all of its singular values. Then $(H(n), \|\cdot\|_{tr})$ is a Banach space and $K(n)$ is a cone in $H(n)$. For $C, D \in H(n)$, $C \ge D$ (resp. $C > D$) will mean that the matrix $C - D$ is positive semi-definite (resp. positive definite). The zero matrix in $H(n)$ will be denoted by O_n, and the unit matrix by I_n.

Denote $E = K(3) \cup (-K(3))$. We define three mappings $T, \alpha_j : E \to E$ ($j = 1,2$) by

$$T(X) = \begin{cases} -\frac{1}{5}X^5, & \text{if } O_3 \preceq X \preceq I_3, \\ -I_3, & \text{if } X \succ I_3, \\ -\frac{1}{k}I_3, & \text{if } X = -\frac{1}{k}I_3, k \in \mathbb{N}_+, \\ O_3, & \text{otherwise;} \end{cases}$$

$$\alpha_1(X) = \begin{cases} -\frac{X^3}{2}, & \text{if } X \succeq O_3, \\ O_3, & \text{if } X = -\frac{1}{k}I_3, k \in \mathbb{N}_+, \\ -X, & \text{if } X \prec O_3, X \neq -\frac{1}{k}I_3, k \in \mathbb{N}_+; \end{cases}$$

$$\alpha_1(X) = \begin{cases} -\frac{X}{7}, & \text{if } X \succeq O_3, \\ O_3, & \text{if } X = -\frac{1}{k}I_3, k \in \mathbb{N}_+, \\ -X, & \text{if } X \prec O_3, X \neq -\frac{1}{k}I_3, k \in \mathbb{N}_+. \end{cases}$$

It is easy to check that lev-$\alpha_{1\sqsubseteq_P} \cap$ lev-$\alpha_{2\sqsubseteq_P} = K(3) \cup \{-\frac{1}{k}I_3 \mid k \in \mathbb{N}_+\}$ and that if $\alpha_j(X) \sqsubseteq_{K(3)} O_3$ for $j = 1,2$, then $\alpha_j(TX) \sqsupseteq_{K(3)} O_3$, $j = 1,2$, and if $\alpha_j(X) \sqsupseteq_{K(3)} O_3$ for $j = 1,2$, then $\alpha_j(TX) \sqsubseteq_{K(3)} O_3$, $j = 1,2$.

Also, T is a Ćirić-Jotić-type contraction with $\gamma = 0$ and $\lambda = 0.6$. This can be checked by opting any $X, Y \in E$ with $\alpha_j(X) \sqsubseteq_{K(3)} O_3$ and $\alpha_j(Y) \sqsupseteq_{K(3)} O_3$, $j = 1,2$, i.e., $X \in K(3) \cup \{-\frac{1}{k}I_3 \mid k \in \mathbb{N}_+\}$ and $Y \in -K(3)$.

In what follows, we consider as an illustration the only two nontrivial cases (when the value of $M(X,Y)$ in (2) is not zero), and check particular values for matrices, to verify numerically the conditions of Theorem 2.2 for $X \neq Y$.

(I) Suppose that $O_3 \prec X \preceq I_3$, $Y \neq O_3$ and $Y \neq -\frac{1}{k}I_3$, $k \in \mathbb{N}_+$. Then the condition (2) reduces to

$$\min\left\{\begin{array}{c} \|\frac{1}{5}X^5\|_{tr}, \|X-Y\|_{tr}, \|X+\frac{1}{5}X^5\|_{tr}, \|Y\|_{tr}, \\ \frac{\|X+\frac{1}{5}X^5\|_{tr}(1+\|Y\|_{tr})}{1+\|X-Y\|_{tr}}, \frac{\|Y\|_{tr}(1+\|X+\frac{1}{5}X^5\|_{tr})}{1+\|X-Y\|_{tr}}, \\ \frac{\min\left\{\|\frac{1}{5}X^5\|_{tr}^2, \|X-Y\|_{tr}^2, \|Y\|_{tr}^2\right\}}{\|X-Y\|_{tr}} \end{array}\right\}$$
$$\leq 0.6 \max\left\{\|X-Y\|_{tr}, \|X+\frac{1}{5}X^5\|_{tr}\right\}.$$

If this condition is tested numerically for

$$X = \begin{pmatrix} 0.0863 & 0.1003 & 0.1167 \\ 0.1003 & 0.1781 & 0.1628 \\ 0.1167 & 0.1628 & 0.1699 \end{pmatrix}, \quad Y = \begin{pmatrix} -0.0164 & -0.0506 & -0.0043 \\ -0.0506 & -0.2399 & -0.0181 \\ -0.0043 & -0.0181 & -0.0014 \end{pmatrix},$$

it becomes $0.0000076925 \leq 0.4152$.

(II) Suppose that $X \succ I_3$, $Y \neq -\frac{1}{k}I_3$, $k \in \mathbb{N}_+$. Then the condition (2) reduces to

$$\min\left\{\begin{array}{c} \|I_3\|_{tr}, \|X-Y\|_{tr}, \|X+I_3\|_{tr}, \|Y\|_{tr}, \\ \frac{\|X+I_3\|_{tr}(1+\|Y\|_{tr})}{1+\|X-Y\|_{tr}}, \frac{\|Y\|_{tr}(1+\|X+I_3\|_{tr})}{1+\|X-Y\|_{tr}}, \\ \frac{\min\left\{\|I_3\|_{tr}^2, \|X-Y\|_{tr}^2, \|Y\|_{tr}^2\right\}}{\|X-Y\|_{tr}} \end{array}\right\}$$
$$\leq 0.6 \max\left\{\|X-Y\|_{tr}, \|X+I_3\|_{tr}\right\}.$$

If it is tested numerically for

$$X = \begin{pmatrix} 15.8650 & 37.7850 & 12.6500 \\ 37.7850 & 95.1900 & 31.8500 \\ 12.6500 & 31.8500 & 11.5400 \end{pmatrix}, \quad Y = \begin{pmatrix} -0.0126 & -0.0070 & -0.0093 \\ -0.0070 & -0.0067 & -0.0081 \\ -0.0093 & -0.0081 & -0.0107 \end{pmatrix},$$

it becomes $0.000007348 \leq 75.3570$.

Hence, all conditions of Theorem 2.2 are satisfied for $\gamma = 0$ and $\lambda = 0.6$. So, it follows that for $U_0 \in \{-\frac{1}{k}I_3 \mid k \in \mathbb{N}_+\} \cup K(3)$ we have that $\{T^n(U_0)\}$ converges to a solution of the system $T(X) = X$ and $\alpha_j(X) = O_3$, $j = 1,2$. In particular, if $U_0 = -\frac{1}{k}I_3$, $k \in \mathbb{N}_+$ we get $T^n(U_0) \to -\frac{1}{k}I_3$, if $O_3 \preceq U_0 \preceq I_3$ we get $T^n(U_0) \to O_3$, and if $U \succ I_3$, we get $T^n(U_0) \to -I_3$. Thus, there are infinite many solutions to the underlying system.

3. Application

In the next application to nonlinear matrix equations, we will use notation introduced in Example 2.5.

Theorem 3.1. *Consider the pair of nonlinear matrix equations:*

$$\begin{aligned} X &= Q_1 + \sum_{i=1}^{m} A_i^* U(X) A_i, \\ X &= Q_2 + \sum_{i=1}^{m} A_{\mathrm{i}}^* V(X) A_i, \end{aligned} \tag{5}$$

where $Q_1, Q_2 \in K(n)$, $A_i \in M(n)$ with $\sum_{i=1}^m \|A_i^* A_i\|_{tr} = \eta$, and $U, V : H(n) \to H(n)$ are two functions, continuous in the trace norm of $H(n)$. Define two subsets of $H(n)$ as

$$\begin{aligned} W_1 &= \{X \in H(n) : Q_2 + \sum_{i=1}^{m} A_{\mathrm{i}}^* V(X) A_i \preceq X\}, \\ W_2 &= \{X \in H(n) : Q_2 + \sum_{i=1}^{m} A_{\mathrm{i}}^* V(X) A_i \succeq X\}. \end{aligned}$$

Assume that there exist a positive real number γ and a real number $\lambda \in [0, 1/\eta)$ such that

(i) for every $X, Y \in H(n)$ such that $X \in W_1$, $Y \in W_2$ and $X \neq Y$,

$$\begin{aligned} \|U(X) - U(Y)\|_{tr} \leq \lambda \, \max\left\{ \|X - Y\|_{tr}, \left\|X - Q_1 - \textstyle\sum_{i=1}^m A_i^* U(X) A_i\right\|_{tr} \right\} \\ + \gamma \, \min\left\{ \begin{array}{l} \left\|X - Q_1 - \sum_{i=1}^m A_i^* U(Y) A_i\right\|_{tr}, \\ \left\|Y - Q_1 - \sum_{i=1}^m A_i^* U(X) A_i\right\|_{tr} \end{array} \right\} \end{aligned}$$

holds true;

(ii) W_2 is a closed subset of $H(n)$;

(iii) there exists $X_0 \in H(n)$ such that $X_0 \in W_1$;

(iv) for every $X \in H(n)$, we have

$$\begin{aligned} X \in W_1 &\Rightarrow Q_1 + \sum_{i=1}^{m} A_i^* U(X) A_i \in W_2 \text{ and} \\ X \in W_2 &\Rightarrow Q_1 + \sum_{i=1}^{m} A_i^* U(X) A_i \in W_1. \end{aligned}$$

Then the system (5) *has a positive semi-definite solution. Moreover, the solution can be derived as the limit of the iterative sequence $\{X_n\}$, where*

$$X_{n+1} = Q_1 + \sum_{i=1}^{m} A_i^* U(X_n) A_i,$$

with convergence, in the sense of trace norm $\|\cdot\|_{tr}$, to a solution of the system (5).

Proof. Define mappings $T, F : H(n) \to H(n)$ by

$$\begin{aligned} T(X) &= Q_1 + \sum_{i=1}^{m} A_i^* U(X) A_i, \\ F(X) &= Q_2 + \sum_{i=1}^{m} A_{\mathrm{i}}^* V(X) A_i, \end{aligned}$$

and consider the cone $P = K(n)$ in $H(n)$. From the definition of W_1, W_2, and since the mapping U is continuous, the conditions (i)–(iv) of Theorem 2.3 are satisfied.

Now, let $X \in W_1$ and $Y \in W_2$, with $X \neq Y$, that is, $F(X) \sqsubseteq_P X$, $F(Y) \sqsupseteq_P Y$. Then

$$
\begin{aligned}
&\|T(X) - T(Y)\|_{tr} = \|\sum_{i=1}^{m} A_i^*(U(X) - U(Y))A_i\|_{tr} \\
&\leq \sum_{i=1}^{m} \|A_i^*(U(X) - U(Y))A_i\|_{tr} = \sum_{i=1}^{m} \|A_iA_i^*(U(X) - U(Y))\|_{tr} \\
&\leq \sum_{i=1}^{m} \|A_i^*A_i\|_{tr}\|(U(X) - U(Y))\|_{tr} = \|(U(X) - U(Y))\|_{tr} \sum_{i=1}^{m} \|A_i^*A_i\|_{tr} \\
&\leq \eta\Big[\lambda \, \max\{\|X - Y\|_{tr}, \|X - TX\|_{tr}\} + \gamma \min\{\|X - TY\|_{tr}, \|Y - TX\|_{tr}\}\Big] \\
&= \lambda\eta \max\{\|X - Y\|_{tr}, \|X - TX\|_{tr}\} + \gamma\eta \min\{\|X - TY\|_{tr}, \|Y - TX\|_{tr}\}.
\end{aligned}
$$

Hence (since $\lambda\eta < 1$), the conditions of Theorem 2.3 are satisfied for the mappings T and F, so there exists a positive semi-definite solution X of the system (5). □

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