Filomat 38:27 (2024), 9633–9641
https://doi.org/10.2298/FIL2427633B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Coincidence of relatively expansive maps

Maher Berzig

Université de Tunis, École Nationale Supérieure d'Ingénieurs de Tunis, Département de Mathématiques,
5 avenue Taha Hussein Montfleury, 1008 Tunis, Tunisie

Abstract. Let C be a bounded closed convex subset of a uniformly convex Banach space, and let f and g be selfmaps of C such that f is expansive relative to g. Without assuming compactness of C, we show that f and g have coincidence points, and they have common fixed points if they commute. As a consequence, we derive the fixed point theorem of Browder-Göhde-Kirk.

1. Introduction

Let $(B, \|\cdot\|)$ be a Banach space and C be a set of B and let $f, g\colon C \to C$ be given maps. We say that f (resp. g) is expansive relative to g (resp. nonexpansive relative to f [1]) or g-expansive (resp. f-nonexpansive) if

$$\|gx - gy\| \leq \|fx - fy\| \quad \text{for all } x, y \in C.$$

An expansive or a nonexpansive map relative to the identity map is simply called expansive or nonexpansive map, respectively. After the independent publications of the important fixed point theorem for nonexpansive maps by Browder [2], Göhde [3] and Kirk [4], a particularly short and elementary proof was given by Goebel [5]. Due to the importance of this theorem, it has been generalized to handle wider classes of maps. For the most recent publications, we refer the reader to [6–8].

One of these research directions was to study the existence of coincidence points, which may be roughly classified into two main classes. The first concerns a whole family of maps as in the works of Belluce and Kirk [9] and Bachar and Khamsi [10], while the second deals with the relative nonexpansiveness as in papers of Park [11] and Latif and Tweddle [12]. In both classes, some sort of compactness and commutativity assumptions are required.

We intend here to investigate the second class using the fine geometric structure of the underlying Banach space instead of assuming compactness of the domain. More precisely, we show that f and g have coincidence points when f is g-expansive, continuous, linear or affine and C is bounded, closed, convex set of a uniformly convex Banach with some additional conditions on $f(C)$ and $g(C)$. Moreover, if we assume that f and g commute, we show that they have common fixed points. As a consequence, we derive the fixed point theorem of Browder-Göhde-Kirk.

2020 *Mathematics Subject Classification*. 47H09, 47H10, 54H25
Keywords. coincidence point, expansive map, uniformly convex space, starshaped set
Received: 05 September 2023; Revised: 10 May 2024; Accepted: 26 July 2024
Communicated by Erdal Karapınar
Email address: maher.berzig@gmail.com (Maher Berzig)

2. Preliminaries

Let B be a Banach space endowed with a norm $\|\cdot\|$ and θ be its zero element. The modulus of convexity is a function $\delta\colon [0,2] \to [0,1]$ given by

$$\delta(\varepsilon) = \inf\left\{1 - \tfrac{1}{2}\|x+y\| : \|x\| \le 1,\ \|y\| \le 1 \text{ and } \|x-y\| \ge \varepsilon\right\}.$$

The Banach space B is said to be uniformly convex if the modulus of convexity δ is positive on $(0,2]$. Note that $\delta(0) = 0$. Recall now the result of Browder-Göhde-Kirk.

Theorem 2.1. *Let C be a bounded closed convex subset of a uniformly convex Banach space. If a map $g\colon C \to C$ is nonexpansive, then it has a fixed point.*

Note that the convexity structure in Theorem 2.1 is not superfluous, since it may occur that g is fixed points free, see for instance [13]. If B is uniformly convex, then δ, which may be assumed to be increasing on $(0,2]$, satisfies

$$\left.\begin{array}{l}\|a-x\| \le r\\ \|a-y\| \le r\\ \|x-y\| \ge \varepsilon\, r\end{array}\right\} \implies \|a - \tfrac{1}{2}(x+y)\| \le (1-\delta(\varepsilon))r.$$

Remark 2.2. *If β is the inverse of δ, then $\lim_{t\to 0}\beta(t) = 0$. For further details on these functions see [14].*

Lemma 2.3 ([5]). *If $x, y, z \in B$ such that $\|z-x\| \le R$, $\|z-y\| \le R$ and $\|z - \frac{1}{2}(x+y)\| \ge r > 0$, then*

$$\|x-y\| \le R\,\beta(1 - \tfrac{r}{R}).$$

3. Coincidence points of relatively expansive maps

In order to obtain our first theorem, we start by establishing a preliminary result. Here and below, we assume that $\theta \in C$.

Proposition 3.1. *Let C be a bounded closed convex set of a Banach space $(B, \|\cdot\|)$ and let $f, g\colon C \to C$ be given maps such that $\mathrm{cl}\,(g(C)) \subseteq f(C)$. If f is linear, continuous and g-expansive, then*

$$\inf\{\|fx - gx\| : x \in C\} = 0.$$

Proof. For $\varepsilon \in (0,1)$, let $h_\varepsilon = \varepsilon\, g$. We shall show that $\mathrm{cl}\,(h_\varepsilon(C)) \subseteq f(C)$. Since $\mathrm{cl}\,(g(C)) \subseteq f(C)$, then if $\{x_n\} \subset g(C)$ is a convergent sequence, its point of convergence is in $f(C)$. Let $\varepsilon \in (0,1)$ and take a convergent sequence $\{y_n^\varepsilon\} \subset h_\varepsilon(C)$ to some point y^ε, and we shall show that $y^\varepsilon \in f(C)$. From $\{y_n^\varepsilon\} \subset h_\varepsilon(C)$, we deduce that there exists a sequence $\{x_n\} \subset g(C)$ such that $y_n^\varepsilon = \varepsilon\, x_n$. Since $\{y_n^\varepsilon\}$ is convergent, then so is $\{x_n\}$, and it converges in $f(C)$. Hence, $\{x_n\}$ converges to $x = \varepsilon^{-1} y^\varepsilon \in f(C)$, so there exists $z \in C$ such that $x = fz$. By using the linearity of f and the convexity of C, we deduce that

$$y^\varepsilon = \varepsilon\, x = \varepsilon\, fz = f(\varepsilon\, z).$$

This proves that $y^\varepsilon \in f(C)$ and therefore $\mathrm{cl}\,(h_\varepsilon(C)) \subseteq f(C)$.

Next, we have

$$\|h_\varepsilon x - h_\varepsilon y\| = \varepsilon\,\|gx - gy\| \le \varepsilon\,\|fx - fy\|,$$

then by [15, Corollary 2.2] follows that f and h_ε have a coincidence point x_ε, say. Thus

$$\|fx_\varepsilon - gx_\varepsilon\| = \|h_\varepsilon x_\varepsilon - gx_\varepsilon\| = \|\varepsilon\, gx_\varepsilon - gx_\varepsilon\| = (1-\varepsilon)\|gx_\varepsilon\| \le (1-\varepsilon)\,\rho,$$

where ρ is the diameter of C. The result follows by letting ε tends to 1. □

Remark 3.2. *The condition* $\mathrm{cl}\,(g(C)) \subseteq f(C)$*, which was used in the following short list of references [16–25], can be easily satisfied if for example* $g(C) \subseteq f(C)$ *and* $f(C)$ *is closed or* $f\colon C \to C$ *is surjective [26, 27], see also the application below. For the proof of this inclusion in the case of a fractional differential equation see [25, Lemma 1.16].*

We next present the first main result.

Theorem 3.3. *Let* C *be a bounded closed convex subset of a uniformly convex Banach space* $(B, \|\cdot\|)$ *and let* $f, g\colon C \to C$ *be two maps such that* $\mathrm{cl}\,(g(C)) \subseteq f(C)$*. If* f *is linear, continuous and* g*-expansive, then* f *and* g *have a coincidence point.*

Proof. Let $\varepsilon \in (0, 1)$. Define

$$C_\varepsilon := \Big\{x \in C : \|fx - gx\| \le \varepsilon\Big\},$$

and

$$D_\varepsilon := \Big\{x \in C_\varepsilon : \|x\| \le a + \varepsilon\Big\},$$

where $a := \lim_{\varepsilon\to 0} \inf\{\|x\| : x \in C_\varepsilon\}$. We shall show that the intersection of all sets C_ε is nonempty. Otherwise if $a > 0$, it follows by Proposition 3.1 that every C_ε is closed and nonempty. Take x and y two elements in C_ε and let $z = \frac{1}{2}(x + y)$, then for

$$R_\varepsilon := \tfrac{1}{2}\|fx - fy\| + \varepsilon,$$

and by linearity and expansiveness of f, we get

$$\|fx - fz\| \le \tfrac{1}{2}\|fx - fy\| \le R_\varepsilon, \tag{2.1a}$$
$$\|fx - gz\| \le \|fx - gx\| + \|gx - gz\| \le \varepsilon + \tfrac{1}{2}\|fx - fy\| \le R_\varepsilon. \tag{2.1b}$$

Similarly, we have

$$\|fy - fz\| \le \tfrac{1}{2}\|fx - fy\| \le R_\varepsilon, \tag{2.2a}$$
$$\|fy - gz\| \le \|fy - gy\| + \|gy - gz\| \le \varepsilon + \tfrac{1}{2}\|fx - fy\| \le R_\varepsilon. \tag{2.2b}$$

Using the triangular inequality, then one of the following inequalities holds:

$$\|fx - \tfrac{1}{2}(fz + gz)\| \ge r, \tag{2.3a}$$
$$\|fy - \tfrac{1}{2}(fz + gz)\| \ge r, \tag{2.3b}$$

where $r = \frac{1}{2}\|fx - fy\|$. We deduce, by Lemma 2.3, (2.1), (2.2) and (2.3) that,

$$\begin{aligned}\|fz - gz\| &\le R_\varepsilon \beta(1 - \tfrac{r}{R_\varepsilon})\\ &\le \sup\Big\{(t+\varepsilon)\beta\Big(\tfrac{\varepsilon}{t+\varepsilon}\Big) : 0 \le t \le \tfrac{\rho}{2}\Big\}\\ &\le \max\Big(\sup\Big\{(t+\varepsilon)\beta\Big(\tfrac{\varepsilon}{t+\varepsilon}\Big) : 0 \le t \le \sqrt{\varepsilon} - \varepsilon\Big\},\\ &\qquad \sup\Big\{(t+\varepsilon)\beta\Big(\tfrac{\varepsilon}{t+\varepsilon}\Big) : \sqrt{\varepsilon} - \varepsilon \le t \le \tfrac{\rho}{2}\Big\}\Big)\\ &\le \phi(\varepsilon) := \max\Big(2\sqrt{\varepsilon},\ (\tfrac{\rho}{2} + \varepsilon)\beta(\sqrt{\varepsilon})\Big).\end{aligned}$$

Thus if $x, y \in C_\varepsilon$, then $z \in C_{\phi(\varepsilon)}$. Clearly $\lim_{\varepsilon\to 0} \phi(\varepsilon) = 0$.

Let $x, y \in D_\varepsilon$, so $\|x\| \le a + \varepsilon$ and $\|y\| \le a + \varepsilon$, and since $z \in C_{\phi(\varepsilon)}$, we deduce that $\|z\| \ge a_\phi := \inf\{\|x\| : x \in C_{\phi(\varepsilon)}\}$. Now, using again Lemma 2.3, we obtain

$$\mathrm{diam}(D_\varepsilon) := \sup_{x,y\in D_\varepsilon} \|x - y\| \le (a + \varepsilon)\beta\Big(\tfrac{a+\varepsilon-a_\phi}{a+\varepsilon}\Big),$$

and thus $\lim_{\varepsilon\to 0} \mathrm{diam}(D_\varepsilon) = 0$. We deduce by Cantor's theorem that the intersection of all D_ε is nonempty, and so is the intersection of all C_ε. We conclude that f and g have a coincidence point. □

Remark 3.4. *From the proofs of Proposition 3.1 and Theorem 3.3 it is clear that the linearity of f can be replaced by the following property:*

$$f(\lambda x) = \lambda f(x), \text{ for all } \lambda \in (0,1) \text{ and } x \in C. \tag{P}$$

The following corollary follows immediately from Theorem 3.3.

Corollary 3.5. *Theorem 2.1 is a particular case of Theorem 3.3.*

We next show that Proposition 3.1 remains true if $f(C)$ is θ-starshaped and f is not necessarily linear, where a set $M \subset B$ is said to be θ-starshaped if $\lambda x \in M$ whenever $\lambda \in [0,1]$ and $x \in M$.

Proposition 3.6. *Let C be a bounded closed set of a Banach space $(B, \|\cdot\|)$ and let $f, g\colon C \to C$ be given maps such that $\mathrm{cl}\,(g(C)) \subseteq f(C)$ and $f(C)$ is θ-starshaped. If f is continuous and g-expansive, then*

$$\inf\{\|fx - gx\| : x \in C\} = 0.$$

Proof. We only need to show that $\mathrm{cl}\,(h_\varepsilon(C)) \subseteq f(C)$ in the proof of Proposition 3.1. This follows from the fact that

$$y^\varepsilon = \varepsilon\, x = \varepsilon\, fz \in f(C),$$

because $\varepsilon \in (0,1)$, $f(z) \in f(C)$ and $f(C)$ is θ-starshaped. □

In the next result, we assume that $f(C)$ is θ-starshaped and that f is affine, that is, $fx = Tx + f\theta$, where $T\colon C \to C$ is a linear map.

Theorem 3.7. *Let $(B, \|\cdot\|)$ be a uniformly convex Banach space and C be a bounded closed convex set of B. Let $f, g\colon C \to C$ be two maps such that $\mathrm{cl}\,(g(C)) \subseteq f(C)$ and $f(C)$ is θ-starshaped. If f is affine, continuous and g-expansive, then f and g have a coincidence point.*

Proof. Let $\varepsilon \in (0,1)$. Define

$$C_\varepsilon := \Big\{x \in C : \|fx - gx\| \le \varepsilon\Big\},$$

and

$$D_\varepsilon := \Big\{x \in C_\varepsilon : \|x\| \le a + \varepsilon\Big\},$$

where $a := \lim_{\varepsilon \to 0} \inf\{\|x\| : x \in C_\varepsilon\}$. We shall show that the intersection of all sets C_ε is nonempty. Otherwise if $a > 0$, it follows by Proposition 3.6 that every C_ε is closed. Take x and y two elements in C_ε and let $z = \frac{1}{2}(x + y)$, then for

$$R_\varepsilon := \tfrac{1}{2}\|fx - fy\| + \varepsilon,$$

and by using the linearity and the expansiveness of f, we get

$$\|fx - fz\| \le \|fx - T(\tfrac{1}{2}(x + y)) - f\theta\| \le R_\varepsilon,$$
$$\|fx - gz\| \le \|fx - gx\| + \|gx - gz\| \le \varepsilon + \|fx - fz\| \le R_\varepsilon.$$

Using the triangular inequality, then one of the following inequalities holds:

$$\|fx - \tfrac{1}{2}(fz + gz)\| \ge \tfrac{1}{2}\|fx - fy\|,$$
$$\|fy - \tfrac{1}{2}(fz + gz)\| \ge \tfrac{1}{2}\|fx - fy\|.$$

The remaining of the proof is exactly as the proof of Theorem 3.3. □

4. Common fixed points of relatively expansive maps

We start with a preliminary result.

Proposition 4.1. *Let C be a bounded closed convex set of a Banach space $(B, \|\cdot\|)$ and let $f, g\colon C \to C$ be commutative maps such that* $\operatorname{cl}(g(C)) \subseteq f(C)$. *If f is linear, continuous and g-expansive, then*

$$\inf\{\|x - fx\| + \|fx - gx\| : x \in C\} = 0.$$

Proof. For $\varepsilon \in (0,1)$ let $h_\varepsilon = \varepsilon\, g$. As in Proposition 3.1, we show that $\operatorname{cl}(h_\varepsilon(C)) \subseteq f(C)$. Next, it is not difficult to see that f and h_ε commute whenever f and g do. Moreover, we have for all $x, y \in C$,

$$\|h_\varepsilon x - h_\varepsilon y\| = \varepsilon\,\|gx - gy\| \le \varepsilon\,\|fx - fy\|.$$

Hence, by [15, Theorem 2.1], f and each h_ε have a unique common fixed point x_ε, say. Hence

$$\|x_\varepsilon - fx_\varepsilon\| + \|fx_\varepsilon - gx_\varepsilon\| = \|h_\varepsilon x_\varepsilon - gx_\varepsilon\| = (1-\varepsilon)\|gx_\varepsilon\| \le (1-\varepsilon)\,\rho,$$

where ρ is the diameter of C. Hence, the result follows as ε tends to 1. □

Theorem 4.2. *Let $(B, \|\cdot\|)$ be a uniformly convex Banach space and C be a bounded closed convex set of B. Let $f, g\colon C \to C$ be commutative maps such that* $\operatorname{cl}(g(C)) \subseteq f(C)$. *If f is linear, continuous and g-expansive, then f and g have a common fixed point.*

Proof. For a positive $\varepsilon \in (0,1)$, let

$$C_\varepsilon := \Big\{x \in C : \|x - fx\| + \|fx - gx\| \le \varepsilon\Big\},$$

and

$$D_\varepsilon := \Big\{x \in C_\varepsilon : \|x\| \le a + \varepsilon\Big\},$$

where $a := \lim_{\varepsilon\to 0}\inf\{\|x\| : x \in C_\varepsilon\}$. We shall show that the intersection of all sets C_ε is nonempty. Otherwise if $a > 0$, it follows by Proposition 4.1 that every C_ε is closed. Take x and y two elements in C_ε and let $z = \frac{1}{2}(x + y)$, then by linearity of f, we get

$$\|z - fz\| \le \tfrac{1}{2}\|x - fx\| + \tfrac{1}{2}\|y - fy\| \le \varepsilon.$$

Now, take

$$R_\varepsilon := \tfrac{1}{2}\|x - y\| + \varepsilon \quad \text{and} \quad R'_\varepsilon := \tfrac{1}{2}\|fx - fy\| + \varepsilon.$$

Hence, we obtain

$$\|x - fz\| \le \|x - z\| + \|z - fz\| \le R_\varepsilon, \tag{4.1a}$$

$$\|y - fz\| \le \|y - z\| + \|z - fz\| \le R_\varepsilon. \tag{4.1b}$$

Similarly, and by using the expansiveness of f, we obtain

$$\|fx - gz\| \le \|fx - gx\| + \|gx - gz\| \le \varepsilon + \|fx - fz\| \le R'_\varepsilon, \tag{4.2a}$$

$$\|fy - gz\| \le \|fy - gy\| + \|gy - gz\| \le \varepsilon + \|fy - fz\| \le R'_\varepsilon. \tag{4.2b}$$

Observe also that we have

$$\|x - z\| \le R_\varepsilon \quad \text{and} \quad \|y - z\| \le R_\varepsilon. \tag{4.3a}$$

$$\|fx - fz\| \le R'_\varepsilon \quad \text{and} \quad \|fy - fz\| \le R'_\varepsilon. \tag{4.3b}$$

Using the triangular inequality, then either

$$\|x - \tfrac{1}{2}(fz + z)\| \geq r \quad \text{or} \quad \|y - \tfrac{1}{2}(fz + z)\| \geq r, \tag{4.4}$$

where $r := \frac{1}{2}\|x - y\|$. Similarly, either

$$\|fx - \tfrac{1}{2}(gz + fz)\| \geq r' \quad \text{or} \quad \|fy - \tfrac{1}{2}(gz + fz)\| \geq r', \tag{4.6}$$

where $r' := \frac{1}{2}\|fx - fy\|$. So, by (4.1)–(4.5), and by applying twice Lemma 2.3, we obtain

$$\begin{aligned}\|z - fz\| + \|fz - gz\| &\leq R_\varepsilon \beta(1 - \tfrac{r}{R_\varepsilon}) + R'_\varepsilon \beta(1 - \tfrac{r'}{R'_\varepsilon}) \\ &\leq 2\sup\Big\{(t+\varepsilon)\beta\Big(\tfrac{\varepsilon}{t+\varepsilon}\Big) : 0 \leq t \leq \tfrac{\rho}{2}\Big\} \\ &\leq 2\max\Big(\sup\Big\{(t+\varepsilon)\beta\Big(\tfrac{\varepsilon}{t+\varepsilon}\Big) : 0 \leq t \leq \sqrt{\varepsilon} - \varepsilon\Big\}, \\ &\qquad\qquad \sup\Big\{(t+\varepsilon)\beta\Big(\tfrac{\varepsilon}{t+\varepsilon}\Big) : \sqrt{\varepsilon} - \varepsilon \leq t \leq \tfrac{\rho}{2}\Big\}\Big) \\ &\leq \phi(\varepsilon) := 2\max\Big(2\sqrt{\varepsilon}, \ (\tfrac{\rho}{2} + \varepsilon)\beta(\sqrt{\varepsilon})\Big).\end{aligned}$$

Therefore $z \in C_{\phi(\varepsilon)}$, if $x, y \in C_\varepsilon$ and $\lim_{\varepsilon\to 0}\phi(\varepsilon) = 0$. Finally, we conclude as in Theorem 3.3 that the intersection of all C_ε is nonempty. Hence f and g have a common fixed point. □

Proposition 4.3. *Let $(B, \|\cdot\|)$ be a uniformly convex Banach space and C be a bounded closed convex set of B. Let $f, g\colon C \to C$ be given maps such that $\mathrm{cl}(g(C)) \subseteq f(C)$ and $f(C)$ is θ-starshaped. If f is continuous and g-expansive, then*

$$\inf\{\|x - fx\| + \|fx - gx\| : x \in C\} = 0.$$

Proof. The proof is similar to that of Proposition 4.1 except for the proof of $\mathrm{cl}(h_\varepsilon(C)) \subseteq f(C)$, which is similar to that of Proposition 3.6. □

Theorem 4.4. *Let $(B, \|\cdot\|)$ be a uniformly convex Banach space and C be a bounded closed convex set of B. Let $f, g\colon C \to C$ be two maps such that $\mathrm{cl}(g(C)) \subseteq f(C)$ and $f(C)$ is θ-starshaped. If f is affine, continuous and g-expansive, then f and g have a common fixed point.*

Proof. For a positive $\varepsilon \in (0, 1)$, let

$$C_\varepsilon := \Big\{x \in C : \|x - fx\| + \|fx - gx\| \leq \varepsilon\Big\},$$

and

$$D_\varepsilon := \Big\{x \in C_\varepsilon : \|x\| \leq a + \varepsilon\Big\},$$

where $a := \lim_{\varepsilon\to 0}\inf\{\|x\| : x \in C_\varepsilon\}$. We shall show that the intersection of all sets C_ε is nonempty. Otherwise if $a > 0$, it follows by Proposition 4.3 that every C_ε is closed. Take x and y two elements in C_ε and let $z = \frac{1}{2}(x + y)$. Observe first that by convexity of C and the affinity of f, we have

$$\|z - fz\| \leq \|\tfrac{1}{2}(x + y) - T(\tfrac{1}{2}(x + y)) - f\theta\| \leq \tfrac{1}{2}\|x - fx\| + \tfrac{1}{2}\|y - fy\| \leq \varepsilon,$$

and that

$$\begin{aligned}\|fx - fz\| &\leq \|fx - T(\tfrac{1}{2}(x + y)) - f\theta\| \leq \tfrac{1}{2}\|fx - fy\|, \\ \|fy - fz\| &\leq \|fy - T(\tfrac{1}{2}(x + y)) - f\theta\| \leq \tfrac{1}{2}\|fx - fy\|.\end{aligned}$$

Now, take

$$R_\varepsilon := \tfrac{1}{2}\|x - y\| + \varepsilon \quad \text{and} \quad R'_\varepsilon := \tfrac{1}{2}\|fx - fy\| + \varepsilon.$$

Hence, we obtain

$$\|x - fz\| \leq \|x - z\| + \|z - fz\| \leq R_\varepsilon,$$
$$\|y - fz\| \leq \|y - z\| + \|z - fz\| \leq R_\varepsilon.$$

Similarly, and by using the expansiveness of f, we obtain

$$\|fx - gz\| \leq \|fx - gx\| + \|gx - gz\| \leq \varepsilon + \|fx - fz\| \leq R'_\varepsilon,$$
$$\|fy - gz\| \leq \|fy - gy\| + \|gy - gz\| \leq \varepsilon + \|fy - fz\| \leq R'_\varepsilon.$$

Also, we have

$$\|x - z\| \leq R_\varepsilon, \quad \|y - z\| \leq R_\varepsilon, \quad \|fx - fz\| \leq R'_\varepsilon \quad \text{and} \quad \|fy - fz\| \leq R'_\varepsilon.$$

Using the triangular inequality, then either

$$\|x - \tfrac{1}{2}(fz + z)\| \geq r \quad \text{or} \quad \|y - \tfrac{1}{2}(fz + z)\| \geq r,$$

where $r := \frac{1}{2}\|x - y\|$. Similarly, either

$$\|fx - \tfrac{1}{2}(gz + fz)\| \geq r' \quad \text{or} \quad \|fy - \tfrac{1}{2}(gz + fz)\| \geq r',$$

where $r' := \frac{1}{2}\|fx - fy\|$. The remaining of the proof is similar to the proof of Theorem 4.2. □

5. Application

Let X be a real Hilbert space endowed with the scalar product $(\cdot|\cdot)$. We show that a solution of the following differential equation must satisfies certain periodicity property,

$$f(x(t))' = k(t, x(t)), \tag{5.1}$$

where here the initial condition is given by

$$x(0) = x_0. \tag{5.2}$$

Theorem 5.1. *Let $(X, (\cdot|\cdot))$ be an Hilbert space and $p > 0$ is fixed. Assume that:*

(a) *The function $f\colon X \to X$ is continuous and satisfy the property* (P).

(b) *The function $k\colon [0, +\infty) \times X \to X$ satisfies:*

  (i) *$k(t + p, x) = k(t, x)$ for all $t \in [0, +\infty)$ and $x \in X$,*

  (ii) *$(k(t, x(t)) - k(t, y(t))|f(x(t)) - f(y(t))) \leq 0$ for all $t \in [0, +\infty)$ and $x, y \in X$.*

(c) *There is a real number $R > 0$ such that for all $(t, x) \in [0, +\infty) \times X$ with $\|f(x)\| = R$, we have*

$$(k(t, x(t))|f(x(t))) < 0.$$

(d) *The initial-value problem* (5.1)-(5.2) *has a solution $x\colon [0, +\infty) \to X$ for all x_0 with $\|fx_0\| \leq R$.*

Then the differential equation (5.1) *has a solution which satisfies*

$$f(x(t)) = f(x(t + p)).$$

Remark 5.2. *The condition* (d) *may be fulfilled if for example we can apply [25, Theorem 1.4].*

Proof. Let $u, v\colon \mathbb{R} \to X$ be maps into Hilbert space $(X, (\cdot|\cdot))$ which are differentiable with respect to t. Then we have the following formula:

$$\frac{d}{dt}(u(t)|v(t)) = (u'(t)|v(t)) + (u(t)|v'(t)).$$

If x and y are two solutions of (5.1) on $[0, +\infty)$, then it follows from (b)-(ii) that

$$\frac{d}{dt}(\|f(x(t)) - f(y(t))\|^2) = 2(k(t, x(t)) - k(t, y(t))|f(x(t)) - f(y(t))) \leq 0, \tag{5.3}$$

which implies

$$\|f(x(t)) - f(y(t))\| \leq \|f(x(0)) - f(y(0))\| \text{ for all } t \geq 0. \tag{5.4}$$

Hence $f(x(0)) = f(y(0))$ implies that $f(x(t)) = f(y(t))$ for all $t \geq 0$.

Define the function $L(x) = \|f(x(t))\|^2$ on X. Observe that for a solution of (5.1) on $[0, +\infty)$ with $\|f(x(t))\| = R$ and $t \in [0, p]$, we have by (c) that

$$\frac{d}{dt}L(x(t)) = 2(k(t, x(t))|f(x(t))) < 0. \tag{5.5}$$

Let $M := \{fx \in X\colon \|fx\| \leq R\}$. For $fx_0 \in M$, we define the operator g by

$$gx_0 = f(x(p)), \tag{5.6}$$

where $x(\cdot)$ is a solution of (5.1)-(5.2). We deduce from (5.5) and the definition of L that

$$\|f(x(t))\| \leq \|f(x(0))\| = \|fx_0\| \leq R,$$

and this means that $t \mapsto f(x(t))$ remains in M for $t \in [0, p]$. Observe also that M is closed and bounded, thus its image by the continuous function f will be closed, we deduce by definition of g that we have $\mathrm{cl}\,(g(M)) \subseteq f(M) \subseteq M$. The operator g is f-nonexpansive, since by (5.3) and (5.6) we have

$$\begin{aligned}\|gx_0 - gy_0\| &= \|f(x(p)) - f(y(p))\| \\ &\leq \|f(x(0)) - f(y(0))\| = \|fx_0 - fy_0\|.\end{aligned}$$

We conclude by Theorem 3.3 that f and g have a coincidence point, which is a solution of (5.1) and satisfies $f(x(0)) = f(x(p))$. Now, to see that $f(x(t)) = f(y(t))$, where $y(t) = x(t + p)$ note that by (b)-(i) if $t \mapsto x(t)$ is solution of (5.1) implies that $t \mapsto y(t)$ is also its solution. Further, $y(0) = x(0)$ so $f(y(0)) = f(x(0))$ and by (5.4), we have $f(x(t)) = f(y(t))$ for all $t \geq 0$, that is, $f(x(t)) = f(x(t + p))$ for all $t \geq 0$. □

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