Filomat 38:27 (2024), 9643–9654
https://doi.org/10.2298/FIL2427643H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Existence of positive solutions for mixed fractional differential equation with p-Laplacian operator

Nuket Aykut Hamal[a,*], Ilayda Ozbasak[a]

[a] *Department of Mathematics, Ege University, 35100 Bornova, Izmir-Turkey*

Abstract. In this paper, by using Avery-Peterson fixed point theorem, we establish the existence of at least three positive solutions for fractional order differential equation involving the Caputo fractional derivative and the Riemann-Liouville fractional derivative. An example is also presented to illustrate our main result.

1. Introduction

The existence of solutions for fractional-order boundary value problems have become an important area of investigation in recent years. We refer the readers to [1, 3, 5, 9, 13, 17] and the references there in. These studies use the fixed-point theory in cones. The differential equations with p-Laplacian operator have been background in physics. Therefore, boundary value problems of fractional differential equations with p-Laplacian operator have been greatly studied, see [2, 6, 7, 8, 16, 18]. Some authors investigated the existence of solutions for a class of mixed fractional differential equations by using different methods. For some interesting results on mixed fractional boundary value problems in literature [4, 6, 8, 10, 11, 19, 20].

In [6], Li worked on multiple positive solutions for nonlinear mixed fractional differential equation with p-Laplacion operator:

$$\begin{cases} {}^cD^{\beta}_{0^+}\left[\phi_p\left(D^{\alpha}_{0^+}u(t)\right)\right]+f(t,u(t))=0, t\in(0,1),\\ \left[\phi_p\left(D^{\alpha}_{0^+}u(0)\right)\right]^{(i)}=0,\ i=1,2,\cdots,m,\\ \phi_p\left(D^{\alpha}_{0^+}u(0)\right)=\sum\limits_{i=1}^{l-2}b_i\left[\phi_p\left(D^{\alpha}_{0^+}u(\xi_i)\right)\right],\\ (u(0))^{(j)}=0,\ j=0,1,2,\cdots,n-1,\\ D^{\alpha-1}_{0^+}u(1)=\sum\limits_{i=1}^{l-2}a_iD^{\alpha}_{0^+}u(\xi), \end{cases}$$

where $2\le n<\alpha\le n+1,\ 1\le m<\beta\le m+1$ and $m+n+1<\alpha+\beta<m+n+2$, $\phi_p(u)=|u|^{p-2}u$, $p>1$.

2020 *Mathematics Subject Classification*. Primary 34B15 ; Secondary 34B18, 34B10.
Keywords. Caputo fractional derivative, Riemann-Liouville fractional derivative, Avery-Peterson fixed point theorem, positive solutions, p-Laplacian.
Received: 03 July 2023; Accepted: 15 July 2024
Communicated by Maria Alessandra Ragusa
* Corresponding author: Nuket Aykut Hamal
Email addresses: nuket.aykut@ege.edu.tr (Nuket Aykut Hamal), ilaydaozbasak@gmail.com (Ilayda Ozbasak)

In [13], Min et al. investigated the uniqueness of positive solutions for the singular fractional differential equations involving integral boundary conditions:

$$\begin{cases} D_{0^+}^{\alpha}u(t) + f\left(t, u(t), D_{0^+}^{\alpha_1}u(t), D_{0^+}^{\alpha_2}u(t), \cdots, D_{0^+}^{\alpha_{n-2}}u(t)\right) = 0, \quad t \in (0,1), \\ u(0) = D_{0^+}^{\gamma_1}u(0) = D_{0^+}^{\gamma_2}u(0) = \cdots = D_{0^+}^{\gamma_{n-2}}u(0) = 0, \\ D_{0^+}^{\beta_1}u(1) = \int_0^{\eta} h(s)D_{0^+}^{\beta_2}u(s)dA(s) + \int_0^1 a(s)D_{0^+}^{\beta_3}u(s)dA(s), \end{cases}$$

where $D_{0^+}^{\alpha}u$, $D_{0^+}^{\alpha_k}u$, $D_{0^+}^{\gamma_k}u$, $(k = 1, 2, \cdots, n-2)$ and $D_{0^+}^{\beta_i}u$ $(i = 1, 2, 3)$ are the Riemann-Liouville derivatives and $n-1 < \alpha \leq n$, $k-1 < \alpha_k$, $\gamma_k \leq k$, $(k = 1, 2, \cdots, n-2)$, $n-k-1 < \alpha - \alpha_k \leq n-k, n-k-1 < \alpha - \gamma_k \leq n-k$, $(k = 1, 2, \cdots, n-2)$, $\gamma_{n-2} - \alpha_{n-2} \geq 0$, $\beta_1 \geq \beta_2$, $\beta_1 \geq \beta_3$, $\alpha - \beta_i > 1$, $\beta_i - \alpha_{n-2} \geq 0$, $(i = 1, 2, 3)$, $f : (0,1) \times (0, +\infty)^{n-1} \to [0, +\infty)$ is a continuos function and $a, h \in C(0,1)$, A is a function of bounded variation, $\int_0^{\eta} h(s)D_{0^+}^{\alpha-1}u(s)dA(s)$, $\int_0^1 a(s)D_{0^+}^{\alpha-1}u(s)dA(s)$ denote the Riemann-Stieltjes integral with resprect to A.

Inspired by the above papers, we discuss the existence of positive solutions for the mixed fractional differential equation with p-Laplacian operator:

$$\begin{cases} {}^cD_{0^+}^{\beta}\left[\phi_p(D_{0^+}^{\alpha}u(t))\right] + f(t, u(t)) = 0, \quad t \in (0,1), \\ \left[\phi_p\left(D_{0^+}^{\alpha}u(0)\right)\right]^{(i)} = 0, \quad i = 1, 2, \cdots, m, \\ \phi_p\left(D_{0^+}^{\alpha}u(0)\right) = \sum_{i=1}^{l-2} b_i\left[\phi_p\left(D_{0^+}^{\alpha}u(\xi_i)\right)\right], \\ (u(0))^{(j)} = 0, \quad j = 0, 1, 2, \cdots, n-1, \\ D_{0^+}^{\alpha-1}u(1) = \int_0^{\eta} h(s)D_{0^+}^{\alpha-1}u(s)dA(s) + \int_0^1 a(s)D_{0^+}^{\alpha-1}u(s)dA(s), \end{cases} \tag{1}$$

where $2 \leq n < \alpha \leq n+1$, $1 \leq m < \beta \leq m+1$, $0 < \eta < 1$, $D_{0^+}^{\alpha}$ is the standart Riemann-Liouville derivative and ${}^cD_{0^+}^{\beta}$ is the standart Caputo derivative, $f : [0,1] \times [0, \infty) \to [0, \infty)$ is a continuous function and $a, h \in C[0,1]$ and $\phi_p(u) = |u|^{p-2}u$, $p > 1$. A is a function of bounded variation and $\int_0^{\eta} h(s)D_{0^+}^{\alpha-1}u(s)dA(s)$, $\int_0^1 a(s)D_{0^+}^{\alpha-1}u(s)dA(s)$ denote the Riemann-Stieltjes integral with resprect to A.

In this study, we assume that following conditions are satisfied:

$(H1)$ $0 < \xi_1 < \xi_2 < \cdots < \xi_{l-2} < 1, 0 < b_i < 1$, $i = 1, 2, \cdots, l-2$ are constants and $\sum_{i=1}^{l-2} bi < 1$,

$(H2)$ $\int_0^{\eta} h(t)dA(t) + \int_0^1 a(t)dA(t) < 1$,

$(H3)$ $f(t, u) : [0,1] \times [0, +\infty) \to [0, +\infty)$ is a continuous function.

By using Avery-Peterson fixed point theorem, we get the existence of positive solutions for the boundary value problem (1). The organization of this paper is as follows. In section 2, we provide some definitions and preliminary lemmas which are key tools for our main result. In section 3, we give and prove our main result. Finally, we give an example to illustrate how the main result can be used in practice.

2. Preliminaries and lemmas

In this section, we present some neccessary definitons and related lemmas, which can be found in [12, 14, 15].

Definition 2.1. *The Riemann-Liouville fracitonal integral of order α for a function $y : (0,+\infty) \to \mathbb{R}$ is given by*

$$I_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds, \ \alpha > 0$$

provided that such integral exists, where

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-x} x^{\alpha-1} dx.$$

Definition 2.2. *For a continuous function $y : (0,+\infty) \to \mathbb{R}$, the Caputo derivative of fractional order $\alpha > 0$ is defined as*

$$D_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} y(s) ds,$$

where $n = [\alpha] + 1$, provided that the right side is pointwise defined on $(0,+\infty)$.

Definition 2.3. *For a continuous function $y : (0,+\infty) \to \mathbb{R}$, the Riemann-Liouville derivative of fractional order $\alpha > 0$ is defined as*

$$D_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} y(s) ds,$$

where $n = [\alpha] + 1$, provided that the rigt side is pointwise defined on $(0,+\infty)$.

Lemma 2.4. *Let $\alpha > 0$, then*

$$I_{0^+}^{\alpha} \, {}^cD_{0^+}^{\alpha} u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$$

for some $c_i \in \mathbb{R}, \quad i = 0, \cdots, n-1, \quad n = [\alpha] + 1.$

Lemma 2.5. *Let $\alpha > 0$, then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$$

for some $c_i \in \mathbb{R}, i = 1, \cdots, n, \quad n = [\alpha] + 1.$

Lemma 2.6. *Let $y \in C[0,1]$ be a given function, then the solution of the boundary value problem*

$$\begin{cases} {}^cD_{0^+}^{\beta} \left[\phi_p\left(D_{0^+}^{\alpha} u(t)\right)\right] + y(t) = 0, \quad t \in (0,1), \\ \left[\phi_p\left(D_{0^+}^{\alpha} u(0)\right)\right]^{(i)} = 0, \quad i = 1,2,\cdots,m, \\ \phi_p\left(D_{0^+}^{\alpha} u(0)\right) = \sum_{i=1}^{l-2} b_i \left[\phi_p\left(D_{0^+}^{\alpha} u(\xi_i)\right)\right], \\ (u(0))^{(j)} = 0, \quad j = 0,1,2,\cdots,n-1, \\ D_{0^+}^{\alpha-1} u(1) = \int_0^{\eta} h(s) D_{0^+}^{\alpha-1} u(s) dA(s) + \int_0^1 a(s) D_{0^+}^{\alpha-1} u(s) dA(s), \end{cases} \tag{2}$$

is given by

$$u(t) = c_1 t^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} w(s)ds, \tag{3}$$

where

$$c_1 = \frac{\int_0^1 w(s)ds - \int_0^\eta h(s) \int_0^t w(s)dsdA(t) - \int_0^1 a(s) \int_0^t w(s)dsdA(t)}{(1-\delta_1-\delta_2)\,\Gamma(\alpha)}, \tag{4}$$

in which

$$\delta_1 = \int_0^\eta h(t)dA(t), \tag{5}$$

$$\delta_2 = \int_0^1 a(t)dA(t), \tag{6}$$

$$w(s) = \phi_q \left(\frac{\sum_{i=1}^{l-2} b_i \int_0^{\xi_i} (\xi_i-\tau)^{\beta-1} y(\tau)d\tau}{(1-\sum_{i=1}^{l-2} b_i)\Gamma(\beta)} + \frac{\int_0^s (s-\tau)^{\beta-1} y(\tau)d\tau}{\Gamma(\beta)} \right), \tag{7}$$

ϕ_q is the universe function of $\phi_p(u)$, i.e. 1/p+1/q=1.

Proof. Firstly, we take the fractional integral of both sides of the equation. From Lemma 2.4, we get

$$\phi_p\left(D_{0^+}^\alpha u(t)\right) = d_0 + d_1 t + \cdots + d_m t^m - \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} y(\tau)d\tau$$

From the condition $\left[\phi_p\left(D_{0^+}^\alpha u(0)\right)\right]^{(i)} = 0, \quad i = 1, 2, \cdots, m$, we can get

$$d_1 = d_2 = \cdots = d_m = 0.$$

Thus, we have

$$\phi_p\left(D_{0^+}^\alpha u(t)\right) = d_0 - \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} y(\tau)d\tau, \tag{8}$$

and by using the condition $\phi_p\left(D_{0^+}^\alpha u(0)\right) = \sum_{i=1}^{l-2} b_i[\phi_p\left(D_{0^+}^\alpha u(\xi_i)\right)]$, we get

$$d_0 = -\frac{\sum_{i=1}^{l-2} bi \int_0^{\xi_i} (\xi_i-\tau)^{\beta-1} y(\tau)d\tau}{\left(1-\sum_{i=1}^{l-2} bi\right)\Gamma(\beta)}$$

Then,

$$D_{0^+}^{\alpha}u(t)=\phi_q\left(d_0-\frac{\int_0^t(t-\tau)^{\beta-1}y(\tau)d\tau}{\Gamma(\beta)}\right).$$

Taking integral from 0 to t, we have

$$u(t)=c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_{n+1}t^{\alpha-n-1}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\phi_q\left(d_0-\frac{\displaystyle\int_0^s(s-\tau)^{\beta-1}y(\tau)d\tau}{\Gamma(\beta)}\right)ds.$$

On the other hand, together with $(u(0))^j=0$, $j=0,1,2,\cdots,n-1$, we can get $c_2=c_3=\cdots=c_{n+1}=0$. Thus we obtain,

$$\begin{aligned}u(t)&=c_1t^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\phi_q\left(-\frac{\displaystyle\sum_{i=1}^{l-2}bi\int_0^{\xi_i}(\xi_i-\tau)^{\beta-1}y(\tau)d\tau}{\left(1-\displaystyle\sum_{i=1}^{l-2}bi\right)\Gamma(\beta)}-\frac{\displaystyle\int_0^s(s-\tau)^{\beta-1}y(\tau)d\tau}{\Gamma(\beta)}\right)ds\\&=c_1t^{\alpha-1}-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}w(s)ds.\end{aligned}$$

Then, boundary condition $D_{0^+}^{\alpha-1}u(1)=\int_0^{\eta}h(s)D_{0^+}^{\alpha-1}u(s)dA(s)+\int_0^1a(s)D_{0^+}^{\alpha-1}u(s)dA(s)$ implies that

$$c_1=\frac{\displaystyle\int_0^1w(s)ds-\int_0^{\eta}h(t)\int_0^tw(s)dsdA(t)-\int_0^1a(t)\int_0^tw(s)dsdA(t)}{(1-\delta_1-\delta_2)\Gamma(\alpha)}.$$

Substituting c_1 into $u(t)$, we have

$$u(t)=\frac{\displaystyle\int_0^1w(s)ds-\int_0^{\eta}h(t)\int_0^tw(s)dsdA(t)-\int_0^1a(t)\int_0^tw(s)dsdA(t)}{(1-\delta_1-\delta_2)\Gamma(\alpha)}t^{\alpha-1}-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}w(s)ds.$$

This completes the proof. □

Lemma 2.7. *Suppose that the condition* $(H1)$, $(H2)$ *and* $(H3)$ *hold, then* $u(t)$ *is a non-negative and non-decreasing function.*

Proof. It is obvious that $w(s)\geq 0$,

$$\begin{aligned}u(t)&=c_1t^{\alpha-1}-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}w(s)ds\\&\geq\frac{\displaystyle\int_0^1w(s)ds-\int_0^{\eta}h(t)\int_0^tw(s)dsdA(t)-\int_0^1a(t)\int_0^tw(s)dsdA(t)}{(1-\delta_1-\delta_2)\Gamma(\alpha)}t^{\alpha-1}-\frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^tw(s)ds\\&=0.\end{aligned}$$

Similarly, we can say that $u'(t)\geq 0$. Then, we get that $u(t)$ is a nonnegative and nondecreasing function. This completes the proof. □

Let us introduce the Banach Space $\mathbb{E} = C[0,1]$ with the norm $\|u\| = \max\limits_{t\in[0,1]}|u(t)|$ and we define the cone:

$$P = \{u \in \mathbb{E} : u(t) \text{ is nonnegative and nondecreasing function for } t \in [0,1]\}$$

and operator $T : P \to \mathbb{E}$ given by

$$Tu(t) = c_1 t^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} w(s)ds$$

where c_1 and $w(s)$ are defined by Lemma 2.6.

Lemma 2.8. *$T : P \to P$ is a completely continuous operator.*

Proof. From Lemma 2.7, we can get that the operator $Tu(t)$ is non-negative and non-decreasing. Thus, $T : P \to P$. In view of continuity of $f(t,u)$, we have T is a continuous operator for $t \in [0,1]$.

Let $\Omega \subset P$ be bounded. By (H_3), we get that there exists a constant $L > 0$ such that $f(t,u) \leq L$, $t \in$ [0,1], $u \in \Omega$, we have

$$\begin{aligned} w(s) &\leq \phi_q\left(\frac{L\sum_{i=1}^{l-2} bi\int_0^{\xi_i}(\xi_i-\tau)^{\beta-1}d\tau}{\left(1-\sum_{i=1}^{l-2} bi\right)\Gamma(\beta)} + \frac{L\int_0^1 (1-\tau)^{\beta-1}d\tau}{\Gamma(\beta)}\right) \\ &\leq \left(\frac{L}{\left(1-\sum_{i=1}^{l-2} bi\right)\Gamma(\beta+1)}\right)^{q-1} \end{aligned}$$

So, we get

$$\begin{aligned} Tu(t) &= c_1 t^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} w(s)ds \\ &\leq \frac{\int_0^1 w(s)ds}{(1-\delta_1-\delta_2)\,\Gamma(\alpha)} \\ &\leq \frac{L^{q-1}}{(1-\delta_1-\delta_2)\,\Gamma(\alpha)\left[(1-\sum_{i=1}^{l-2} bi)\Gamma(\beta+1)\right]^{q-1}} \end{aligned}$$

Consequently,

$$\|Tu\| \leq \frac{L^{q-1}}{(1-\delta_1-\delta_2)\Gamma(\alpha)\left[(1-\sum_{i=1}^{l-2} bi)\Gamma(\beta+1)\right]^{q-1}}.$$

In the following we will proof that T(Ω) is equicontinuous. For $t_1, t_2 \in [0,1]$. $t_1 < t_2$, u ∈ Ω, we have

$$\begin{aligned}
|(Tu)(t_2) - (Tu)(t_1)| &= \left| c_1 t_2^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} w(s) ds - c_1 t_1^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} w(s) ds \right| \\
&\leq c_1 \left| t_2^{\alpha-1} - t_1^{\alpha-1} \right| + \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_2} (t_2 - s)^{\alpha-1} w(s) ds - \int_0^{t_1} (t_1 - s)^{\alpha-1} w(s) ds \right| \\
&\leq \frac{1}{(1 - \delta_1 - \delta_2)\Gamma(\alpha)} \left(\frac{L}{(1 - \sum_{i=1}^{l-2} bi)\Gamma(\beta + 1)} \right)^{q-1} (t_2^{\alpha-1} - t_1^{\alpha-1}) \\
&\quad + \frac{1}{\Gamma(\alpha + 1)} \left(\frac{L}{(1 - \sum_{i=1}^{l-2} bi)\Gamma(\beta + 1)} \right)^{q-1} (t_2^{\alpha} - t_1^{\alpha}).
\end{aligned}$$

We have the right-hand side of the above inequalities tends to zero if $t_2 \to t_1$. Using Arzela-Ascoli Theorem, we have T is a completely continuous operator. □

Let P be a cone in real Banach space $\mathbb{E}$; γ, θ be two non negative continuous convex functionals on P; ω be a non-negative continuous concave functionals on P and ψ be non-negative continuous functionals on P. Then, for positive real numbers h, r, c and d, we define the following sets:

$$\begin{aligned}
P(\gamma, d) &= \{x \in P : \gamma(x) < d\}, \\
P(\gamma, \omega, r, d) &= \{x \in P : r \leq \omega(x), \gamma(x) \leq d\}, \\
P(\gamma, \theta, \omega, r, c, d) &= \{x \in P : r \leq \omega(x), \theta(x) \leq c, \gamma(x) \leq d\}, \\
Q(\gamma, \psi, h, d) &= \{x \in P : h \leq \psi(x), \gamma(x) \leq d\}.
\end{aligned}$$

Theorem 2.9. *([9]) Let P be a cone in real Banach space $\mathbb{E}$. Let γ and θ be non-negative continuous convex functionals on P, ω be a non-negative continuous concave functionals on P, and ψ be a non-negative continuous functionals on P satisfying $\psi(\lambda x) \leq \lambda\psi(x)$ for $0 \leq \lambda \leq 1$ such that, for some positive numbers d and M,*

$$\omega(x) \leq \psi(x) \quad \text{and} \quad \|x\| \leq M\gamma(x) \quad \text{for all} \quad x \in \overline{P(\gamma, d)}.$$

Suppose further that $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ is completely continuous and there exist positive numbers h, r and c with $h < r$ such that:

(C1) *$\{x \in P(\gamma, \theta, \omega, r, c, d) : \omega(x) > r\} \neq \emptyset$ and $\omega(Tx) > r$ for $x \in P(\gamma, \theta, \omega, r, c, d)$,*

(C2) *$\omega(Tx) > r$ for $x \in P(\gamma, \omega, r, d)$ with $\theta(Tx) > c$,*

(C3) *$0 \notin Q(\gamma, \psi, h, d)$ and $\psi(Tx) < h$ and $x \in Q(\gamma, \psi, h, d)$ with $\psi(x) = h$.*

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$ such that

$$\gamma(x_i) \leq d, \quad for\ i = 1, 2, 3$$

and

$$r < \omega(x_1), \quad h < \psi(x_2), \quad \gamma(x_2) < r, \quad \psi(x_3) < h.$$

3. Main Result

To prove that (1) has three positive solutions, the following convex and concave functionals are defined by

$$\gamma(u) = \theta(u) = \psi(u) = \max_{t\in[0,1]} |u(t)|, \ \omega(u) = \min_{t\in[\xi_{l-2},1]} |u(t)|.$$

Theorem 3.1. *Assume that there exist positive numbers h, r, c, d with $h < r < \frac{r}{\xi_{l-2}^{\alpha-1}} \leq c < d$ and f holds the following conditions:*

$(H4)$ $f(t,u) \leq (dM_1)^{p-1}$, *for* $(t,u) \in [0,1]\times[0,d]$,

$(H5)$ $f(t,u) > (rM_2)^{p-1}$, *for* $(t,u) \in [0,1]\times[r,c]$,

$(H6)$ $f(t,u) < (hM_1)^{p-1}$, *for* $(t,u) \in [0,1]\times[0,h]$,

where

$$M_1 = (1-\delta_1-\delta_2)\,\Gamma(\alpha)\left[\left(1-\sum_{i=1}^{l-2} bi\right)\Gamma(\beta+1)\right]^{q-1},$$

and

$$M_2 = \frac{(1-\delta_1-\delta_2)\,\Gamma(\alpha)\left[(1-\sum_{i=1}^{l-2} bi)\Gamma(\beta+1)\right]^{q-1}}{\left(\sum_{i=1}^{l-2} bi\xi_i^{\beta}\right)^{q-1}(1-\xi_{l-2})\,\xi_{l-2}^{\alpha-1}},$$

Then the problem (1) has at least three positive solutions u_1, u_2, u_3 satisfying

$$\gamma(u_i) \leq d \quad \text{for } i = 1,2,3$$

and

$$r < \omega(u_1), \quad h < \psi(u_2), \quad \gamma(u_2) < r, \quad \psi(u_3) < h.$$

Proof. Now, we prove that all the conditions of Theorem 2.9 are satisfied. We will show that $T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$. For $u \in \overline{P(\gamma,d)}$, by assumption $(H4)$, we get

$$w(s) \leq \frac{dM_1}{\left[(1-\sum_{i=1}^{l-2} bi)\Gamma(\beta+1)\right]^{q-1}}.$$

Then,

$$\begin{aligned} Tu(t) &= c_1 t^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\omega(s)ds \\ &\leq \frac{\int_0^1 w(s)ds}{(1-\delta_1-\delta_2)\,\Gamma(\alpha)} \\ &\leq \frac{dM_1}{(1-\delta_1-\delta_2)\,\Gamma(\alpha)\left[\left(1-\sum_{i=1}^{l-2} bi\right)\Gamma(\beta+1)\right]^{q-1}} \\ &= d. \end{aligned}$$

We obtain, $T:\overline{P(\gamma,d)}\to\overline{P(\gamma,d)}$.

To prove that (C_1) condition holds, taking $u_0(t)=(r+c)/2$, we get

$$\gamma(u_0)=\theta(u_0)=(r+c)/2,\quad \omega(u_0)=(r+c)/2,$$

and

$$\gamma(u_0)=(r+c)/2<d,\quad \theta(u_0)=(r+c)/2<c,\quad \omega(u_0)=(r+c)/2>r.$$

As a result, $\{x\in P(\gamma,\theta,\omega,r,c,d)|\omega(x)>r\}\neq\emptyset$. For all $u\in P(\gamma,\theta,\omega,r,c,d)$, by (H_5), we have

$$\begin{aligned}
\omega(Tu) &= \min_{t\in[\xi_{l-2},1]}|Tu(t)|=|Tu(\xi_{l-2})|\\
&= c_1\xi_{l-2}^{\alpha-1}-\frac{1}{\Gamma(\alpha)}\int_0^{\xi_{l-2}}(\xi_{l-2}-s)^{\alpha-1}w(s)ds\\
&\geq \frac{\int_0^1 w(s)ds-\int_0^\eta h(t)\int_0^t w(s)dsdA(t)-\int_0^1 a(t)\int_0^t w(s)dsdA(t)}{(1-\delta_1-\delta_2)\Gamma(\alpha)}\xi_{l-2}^{\alpha-1}\\
&\quad -\frac{\xi_{l-2}^{\alpha-1}}{\Gamma(\alpha)}\int_0^{\xi_{l-2}}w(s)ds\\
&= \frac{\int_{\xi_{l-2}}^1 w(s)ds}{(1-\delta_1-\delta_2)\Gamma(\alpha)}\xi_{l-2}^{\alpha-1}\\
&> \frac{\left[\frac{(rM_2)^{p-1}\sum_{i=1}^{l-2}bi\xi_i^\beta}{(1-\sum_{i=1}^{l-2}b_i)\Gamma(\beta+1)}\right]^{q-1}\int_{\xi_{l-2}}^1 ds}{(1-\delta_1-\delta_2)\Gamma(\alpha)}\xi_{l-2}^{\alpha-1}\\
&= r.
\end{aligned}$$

This shows that the condition (C_1) is satisfied. Let $u\in P(\gamma,\omega,r,d)$ with $\theta(Tu(t))>c$, we have

$$\begin{aligned}
\omega(Tu) &= \min_{[\xi_{l-2},1]}|Tu(t)|=|Tu(\xi_{l-2})|\\
&= c_1\xi_{l-2}^{\alpha-1}-\frac{1}{\Gamma(\alpha)}\int_0^{\xi_{l-2}}(\xi_{l-2}-s)^{\alpha-1}w(s)ds,
\end{aligned}$$

and

$$\begin{aligned}
\theta(Tu) &= \max_{t\in[0,1]}|Tu(t)|=Tu(1)\\
&= c_1-\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}w(s)ds.
\end{aligned}$$

Then

$$\begin{aligned}\omega(Tu)-\xi_{l-2}^{\alpha-1}\theta(Tu)&=c_1\xi_{l-2}^{\alpha-1}-\frac{1}{\Gamma(\alpha)}\int_0^{\xi_{l-2}}(\xi_{l-2}-s)^{\alpha-1}w(s)ds-c_1\xi_{l-2}^{\alpha-1}\\&\quad+\frac{\xi_{l-2}^{\alpha-1}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}w(s)ds\\&\geq 0.\end{aligned}$$

So we can get $\omega(Tu)\geq\xi_{l-2}^{\alpha-1}\theta(Tu)>\xi_{l-2}^{\alpha-1}c\geq r$, that is, $\omega(Tu)>r$ which shows that the condition (C_2) is satisfied.

Next we will prove that the condition (C_3) holds. Assume that $u\in Q(\gamma,\psi,h,d)$ with $\psi(u)=h$. Then by (H_6), we have

$$w(s)<\frac{hM_1}{\left[(1-\sum\limits_{i=1}^{l-2}bi)\Gamma(\beta+1)\right]^{q-1}}$$

Thus, we have

$$\begin{aligned}\psi(Tu)&=\max_{t\in[0,1]}|Tu(t)|=Tu(1)\\&=\frac{\int_0^1w(s)ds-\int_0^\eta h(t)\int_0^t w(s)dsdA(t)-\int_0^1 a(t)\int_0^t w(s)dsdA(t)}{(1-\delta_1-\delta_2)\Gamma(\alpha)}-\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}w(s)ds\\&\leq\frac{\int_0^1 w(s)ds}{(1-\delta_1-\delta_2)\Gamma(\alpha)}\\&<\frac{hM_1}{(1-\delta_1-\delta_2)\Gamma(\alpha)\left[(1-\sum\limits_{i=1}^{l-2}bi)\Gamma(\beta+1)\right]^{q-1}}\\&=h.\end{aligned}$$

Consequently, the condition (C_3) is satisfied. We get that the boundary value problem (1) has at least three positive solutions u_1, u_2 and u_3 such that

$$\gamma(u_i)\leq d\quad\text{for } i=1,2,3$$

and

$$r<\omega(u_1),\quad h<\psi(u_2),\quad \gamma(u_2)<r,\quad \psi(u_3)<h.$$

This completes the proof. □

4. Example

Consider the following boundary value problem :

$$\begin{cases} {}^cD_{0^+}^{\beta}\left[\phi_p\left(D_{0^+}^{\alpha}u(t)\right)\right] + f(t,u(t)) = 0, \quad t \in (0,1), \\ \left[\phi_p\left(D_{0^+}^{\alpha}u(0)\right)\right]^{(i)} = 0, \quad i = 1,2,\cdots,m, \\ \phi_p\left(D_{0^+}^{\alpha}u(0)\right) = \sum_{i=1}^{l-2} b_i\left[\phi_p\left(D_{0^+}^{\alpha}u(\xi_i)\right)\right], \\ (u(0))^{(j)} = 0, \ j = 0,1,2,\cdots,n-1, \\ D_{0^+}^{\alpha-1}u(1) = \int_0^{\eta} h(s)D_{0^+}^{\alpha-1}u(s)dA(s) + \int_0^1 a(s)D_{0^+}^{\alpha-1}u(s)dA(s) \end{cases}$$

where $n = 2$, $m = 1$, $l = 4$, $\alpha = 2.5$, $\beta = 1.5$, $b_1 = 0.2$, $b_2 = 0.3$, $p = 3$, $\eta = 0.1$, $\xi_1 = 0.1$, $\xi_2 = 0.2$ and

$$f(t,u) = \begin{cases} t^4, & 0 \le t \le 1, \quad 0 \le u \le 2, \\ t^4 + 400(u-2), & 0 \le t \le 1, \quad 2 < u \le 3, \\ t^4 + 400 + 10(u-3), & 0 \le t \le 1, \quad 3 < u \le 35, \\ t^4 + 720, & 0 \le t \le 1, \quad u > 35. \end{cases}$$

Additively, if we choose $A(t) = t$, $a(t) = \frac{t}{2}$, $h(t) = 1$, $h = 2$, $r = 3$, $c = 35$ and $d = 40$, then $f(t,u)$ satisfies the following inequalities:

$$\begin{aligned} f(t,u) &\le (dM_1)^{p-1} \approx 793.3994663, \quad (t,u) \in [0,1] \times [0,40], \\ f(t,u) &> (rM_2)^{p-1} \approx 486.45, \quad (t,u) \in [0,1] \times [3,35], \\ f(t,u) &< (hM_1)^{p-1} \approx 1.983498666, \quad (t,u) \in [0,1] \times [0,2]. \end{aligned}$$

In this case all the conditions of Theorem 2.9 are satisfied. Hence, by Theorem 2.9, we prove that the boundary value problem has at least three positive solutions u_1, u_2 and u_3 such that

$$\gamma(u_i) \le 40 \quad \text{for } i = 1,2,3$$

and

$$3 < \omega(u_1), \quad 2 < \psi(u_2), \quad \gamma(u_2) < 3, \quad \psi(u_3) < 2.$$

References

[1] M. I. Abbas, M. A. Ragusa, *Solvability of langevin equations with two hadamard fractional derivatives via mittag–leffler functions*, Applicable Analysis 101 (9) (2022), 3231–3245.

[2] N. A. Hamal, F. Erkan, *Existence of positive solutions for singular fractional boundary value problems with p-Laplacian*, Filomat (37) (2023), 6867-6876.

[3] J. Henderson, R. Luca, *Boundary value problems for systems of differential, difference and fractional equations: Positive Solutions*, (1st Edition), Elsevier, Amsterdam, 2015.

[4] A. G. Lakoud, R. Khaldi, A. Kılıçman, *Existence of solutions for a mixed fractional boundary value problem*, Adv. Differ. Equ. (2017) 2017:164.

[5] B. Li, S. Sun, P. Zhao, Z. Han, *Existence and multiplicity of positive solutions for a class of fractional differential equations with three-point boundary value conditions*, Adv. Differ. Equ. 1 (2015),1-19.

[6] Y. Li, *Multiple positive solutions for nonlinear mixed fractional differential equation with p-Laplacian operator*, Adv. Differ. Equ. 2019 (112) (2019).

[7] S. Liang, S. Shi, *Existence of multiple positive solutions for m-point fractional boundary value problems with p-Laplacian operator on infinite interval*, Journal of Applied Mathematics and Computing (38) (2012) 687-707.

[8] X. Liu, M. Jia, *The Positive Solutions for Integral Boundary Value Problem of Fractional p-Laplacian Equation with Mixed Derivatives*, Mediterr. J. Math. (2017) 14: 94 (2017).

[9] Y. Liu, *Application of Avery–Peterson fixed point theorem to nonlinear boundary value problem of fractional differential equation with the Caputo's derivative*, Communications in Nonlinear Science and Numerical Simulation (17) (2012), 4576-4584.

[10] Y. Liu, C. Yan, W. Jiang, *Existence of the positive solutions for boundary value problems of mixed differential equations involving the Caputo and Riemann–Liouville fractional derivatives*, Bound Value Probl 2023 (9)(2023), 1-15.

[11] Y. Liu, C. Yan, W. Jiang, *Existence of the unique nontrivial solution for mixed fractional differential equations*, Journal of Function Spaces 2021 (2021), 1-9, 2021.

[12] K. S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John-Wiley and Sons Inc., New York, 1993.

[13] D. Min, L. Liu, Y. Wu, *Uniqueness of positive solutions for the singular fractional differential equations involving integral boundary value conditions*, Bound Value Probl 2018 23 (2018), 1-18.

[14] S. A. Murad, Z. A. Ameen, *Existence and ulam stability for fractional differential equations of mixed caputo-riemann derivatives*, AIMS Mathematics, 7 (4) (2022), 6404–6419.

[15] I. Podlubny, *Fractional Differential Equations, Mathematics in Science and Engineering*, New York: Academic Press, 2019.

[16] S. G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[17] S. Song, Y. Cui, *Existence of solutions for integral boundary value problems of mixed fractional differential equations under resonance*, Boundary Value Problems 2020 (1) (2020), 1-23.

[18] J. Xu, D. O'Regan, *Positive Solutions for a Fractional p-Laplacian Boundary Value Problem*, Filomat, 31 (6) (2017) , 1549-1558.

[19] J. Yang, J. C. Ma, S. Zhao, Y. Ge, *Fractional multi-point boundary value problems of fractional differential equation*, Mathematics in Practice and Theory 41 (11) (2011), 188-194.

[20] Y. Yang, Q. Wang, *Multiple positive solutions for p-Laplacian equations with integral boundary conditions*, Journal of Mathematical Analysis and Applications (453) (2017), 558-571.