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On the wave equation with conformable operator

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Abstract. In this paper, we focus on the "fractional conformable wave equation". By some new techniques, we obtained the explicit formula of the mild solution. In the linear case, we study the convergence of the mild solution when the fractional order derivative tends to 1^-. As for the nonlinear case, we show the global existence of the mild solution.

1. Introduction

Fractional derivatives can be consider as extensions of the classical derivative where the order of derivatives is not an integer but a real number. From this new perspective a new field has arisen called fractional calculus. Many famous mathematicians have mentioned and contributed to this theory over the years, such as Laplace, Euler, Leibniz, Fourier and Lacroix, etc. Although it emerged as early as in 1695 from the discussion between Leibniz and L' Hospital, for a long time the theory of fractional calculus developed only as a purely theoretical area of mathematics.

The first application of fractional calculus was by Niels Hendrik Abel in 1823 with his work on integral equation that arises in the formulation of the isochrone problem. After many controversy both contribution, in the last decade of the nineteenth century, Oliver Heaviside used fractional differential operators (generalized operator that time) to showed how certain linear differential equations can be solved. His study have proved to be useful to engineers in the theory of the tranmission of electrical currents in cables. [17]

Over time, the role and importance of fractional derivatives have been increasingly affirmed. One of the advantages of fractional derivative is to model unusual phenomena as well as quick updates while data changes. In the last decades, it was found that fractional derivatives and fractional integrals can provide a better tool for understanding some physical phenomena, especially when processes with memory are considered. That is the reason why the number of studies on fractional differential equations have increased dramatically recently. Nowaday, the applications of fractional calculus mainly include in the modeling of viscoelastic and viscoplastic materials, chemical sciences, biology, economics, engineering problems ... [8, 9, 18, 20, 21].

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It should be noted that most of the known fractional derivatives do not satisfy all the properties associated with classical integer derivatives. For instance, except the Caputo definition, the others do not vanish the constant function with arbirary order. Riemann-Liouville and Caputo derivatives do not satisfy the derivative of the product and quotient of two functions. Neither the Riemann-Liouville derivative nor the Caputo derivative satisfy the chain rule nor the index rule. The only property that fractional derivative definitions all satisfy is the linearity. In addition, the Caputo derivative assumes that the function f is differentiable. Since they do not satisfy some basic principles of known integer order derivatives, it is not possible to solve some fractional derivatives using those definitions.

Due to the limitations of known definitions as indicated above, more and more new definitions of fractional derivatives are proposed to better satisfy the basic principles. Recently, the conformable fractional derivative suggested by Khalil et al. [13] which is a simple and efficient definition.

Let $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ be a bounded domain with a smooth boundary $\partial\Omega$, and $T > 0$ be a given positive number. In this paper, we are interested in studying the following problem

$$\begin{cases} \dfrac{{}^C\partial^\alpha}{\partial t^\alpha}\left(\dfrac{{}^C\partial^\alpha y(x,t)}{\partial t^\alpha}\right) - \Delta y(x,t) = F(x,t,y(x,t)), & x \in \Omega, \quad t \in (0,T), \\ \qquad y(x,t) = 0, & x \in \partial\Omega, t \in (0,T), \\ \qquad y(x,0) = f(x), \quad \frac{{}^C\partial^\alpha y(x,0)}{\partial t^\alpha} = g(x), & x \in \Omega. \end{cases} \tag{1}$$

where the operator $\dfrac{{}^C\partial^\alpha}{\partial t^\alpha}$, $(0 < \alpha \leq 1)$ is time fractional conformable derivative defined by

$$\frac{{}^C\partial^\alpha}{\partial t^\alpha} y(t) := \lim_{h \to 0} \frac{y(t + ht^{1-\alpha}) - y(t)}{h}, \tag{2}$$

We name the problem under consideration as "fractional conformable wave equation" since when $\alpha = 1$ the problem (1) becomes the wave equation with classical derivative. Some components in the model such as the source function on the right side F, the two initial data functions f, g or properties of the fractional conformable derivative will be introduced in more detail in following sections.

The definition of a conformable derivative is based on limit approach and provides many computational simplifications in the application of chain rules. That is why various researchers have started research in the development of the theory of systems with conformable derivatives. Like other generalizations, conformable fractional derivatives were initially questioned about the applicability and practical significance. Some researchers even argued that this generalization was not sufficiently new mathematically. However, they also accepted that conformable fractional derivatives would play a role in constructing some valuable new mathematical models to study certain physical phenomena in practice [14]. Mathematically, these could be differential equations conformable derivatives and deviating arguments (delayed or neutral type).

Recently, Cenesiz and Kurt in [10] have investigated the heat equation with fractional conformable derivative in both cases: the time equation and space equation. They showed that the conformable fractional derivatives has many advantages in solving the fractional differential equations, such it can easily and efficiently transform fractional differential equations into classical usual differential equations without the need of complicated methods to find the analytical solutions. Evenly, it can be expanded for others partial fractional differential equations of higher dimensional systems.

In [22] Zhou has shown that the conformable derivative is better fitted to data in anomalous diffusion and so that the modelling is improved. The anomalous diffusion is complex transport which is easily encounter in many physical, biological phenomena. For the diffusion equation with conformable operator in Hilbert spaces, we refer to [4]. In that paper, they studied completely the existence and regularity for the conformable diffusion equations with linear and nonlinear source. Based on the ideas of [4], recently, Tuan-Tien-Chao [19] focused on the pseudo-parabolic equation with conformable derivative. They used some new techniques for considering the existence and regularity of the mild solution.

In the work [16] the authors focused on conformable stochastic functional differential equations of neutral type. They obtained the existence and uniqueness theorem of a solution. The moment estimation and exponential stability results are given. Recently, in [2], Ahmed-Ragusa considered the Sobolev-type conformable fractional stochastic evolution inclusions. They investigated a nonlocal controllability of nonlocal problem. In addition, new sufficient conditions for nonlocal controllability of their considered system are investigated. In the paper [3], the authors derived the existence and approximation solutions of the forward and backward problem for conformable diffusion equation. In [5], the authors considered the initial inverse problem for a diffusion equation with a conformable derivative. They proved that the backward problem is ill-posed, and they also proposed a regularizing scheme using a fractional Landweber regularization method. In [6] the authors used semigroup theory combined with Schaefer fixed point theorem to prove the existence of mild solutions for a class of nonlocal conformable fractional differential equations. In [15] the author focused on the diffusion equation with conformable derivative. He also proved the convergence of the mild solution when the order of fractional Laplacian tends to 1^-.

Very recently, in the paper [12] the authors studied a nonlinear Volterra equation with conformable derivative. They showed that the problem have a mild soltution which exists locally in time. Then they also showed that the convergence of the mild solution when the parameter tends to zero.

In the interesting paper [7], the authors also proved the convergence of the mild solution to conformable diffusion to the solution of classical problem. Our current paper is a good continuation of that paper for the wave equation. We learn many techniques in the paper [7] but we have many different modifications. The main contribution of the paper is organized as follows

- At first, we show that the convergence of the mild solution to linear conformable wave equation when α tends to 1^-.
- As for the nonlinear case, we prove the global existence of the mild solution and give some estimation in appropriate space with particular condition of initial data.

The rest of this paper is organized as follows. In section 2, we recall some needed preliminaries concerning the fractional conformable calculus along with some related functional spaces. In section 3, we provide some results on the formula of mild solution. Section 4 give result on the continuity problem for the linear conformable wave equation. In final section, we derive the global existence for the mild solution to the "fractional conformable wave equation" for the nonlinear case.

2. Preliminaries and notations

In this section, we review some basic results about fractional conformable calculus and some common used function spaces. Let's start with the definition of fractional derivative and fractional integral in the sense of conformable.

Definition 2.1. *(see [13]) The conformable fractional derivative of $y(t)$ at $t > 0$ of order α is defined as follows:*

$$(T^\alpha y)(t) = \frac{{}^C\partial^\alpha}{\partial t^\alpha} y(t) := \lim_{h\to 0} \frac{y(t + ht^{1-\alpha}) - y(t)}{h}, \tag{3}$$

where $0 < \alpha \le 1$ and for any functional y taking values in Banach space $\mathcal{B}$.

If $\lim_{t\to 0^+} \frac{{}^C\partial^\alpha}{\partial t^\alpha} y(t)$ exists then we define the conformable fractional derivative at $t = 0$ as follow

$$\frac{{}^C\partial^\alpha}{\partial t^\alpha} y(0) := \lim_{t\to 0^+} \frac{{}^C\partial^\alpha}{\partial t^\alpha} y(t)$$

The left fractional derivative starting from a of the function $y : [a, \infty) \to \mathcal{B}$ of order $0 < \alpha \le 1$ is:

$$(T_a^\alpha y)(t) := \lim_{h\to 0} \frac{y(t + h(t-a)^{1-\alpha}) - y(t)}{h}, \tag{4}$$

Definition 2.2. *The conformable fractional integral $I_\alpha(y)$ of a function $y(.)$ is defined by*

$$I_\alpha(y)(t) = \int_0^t \nu^{\alpha-1} y(\nu)d\nu = \int_0^t \frac{y(\nu)}{\nu^{1-\alpha}} d\nu. \tag{5}$$

Remark 2.3. *Some properties of fractional conformable derivative are listed below.*
Let $\alpha \in (0,1]$ and f, g are two $\alpha-$differentiable functions at a point $t > 0$, then:

- $T^\alpha(af + bg) = aT^\alpha(f) + bT^\alpha(g)$, *for all* $a, b \in \mathbb{R}$
- $T^\alpha(t^p) = p.t^{p-\alpha}$, *for all* $p \in \mathbb{R}$
- $T^\alpha(\lambda) = 0$, *for all constant function*
- $T^\alpha(fg) = fT^\alpha(g) + gT^\alpha(f)$
- $T^\alpha(f(g)) = \dfrac{df}{dg} T^\alpha g$,
- *In addition, if f(t) is diffentiable then* $T^\alpha f = t^{1-\alpha} \dfrac{df}{dt}$

In [1], the author gave some properties about the conformable fractional derivative and integral operators as below

Theorem 2.4. *Let $y(.)$ be differentiable, then we have*

$$I_\alpha \left(\frac{{}^C\partial^\alpha y}{\partial t^\alpha} \right)(t) = y(t) - y(0). \tag{6}$$

Let $y(\cdot)$ be a continuous function in the domain of I_α, then we have

$$\frac{{}^C\partial^\alpha \left(I_\alpha(y)(t)\right)}{\partial t^\alpha} = y(t) \tag{7}$$

Definition 2.5. *(see [6]) The conformable fractional Laplace transform of order $\alpha \in (0,1]$ of a function $y(.)$ is defined by*

$$\mathcal{L}_\alpha(y(t))(\lambda) = \int_0^{+\infty} t^{\alpha-1} e^{-\lambda \frac{t^\alpha}{\alpha}} y(t)dt, \qquad \lambda > 0$$

Theorem 2.6. *If $y(.)$ be differentiable, then we have*

$$\mathcal{L}_\alpha \left(\frac{{}^C\partial^\alpha y(t)}{\partial t^\alpha} \right)(\lambda) = \lambda \mathcal{L}_\alpha(y(t))(\lambda) - y(0),$$

Remark 2.7. *For two functions $x(\cdot)$ and $y(\cdot)$, we have*

a) $\mathcal{L}_\alpha \left(y(\frac{t^\alpha}{\alpha}) \right)(\lambda) = \mathcal{L}_1 \left(y(t) \right)(\lambda)$,

b) $\mathcal{L}_\alpha \left(\int_0^t s^{\alpha-1} x\left(\frac{t^\alpha - s^\alpha}{\alpha} \right) y(s)ds \right)(\lambda) = \mathcal{L}_1 \left(x(t) \right)(\lambda) \mathcal{L}_\alpha \left(y(t) \right)(\lambda)$

provided that the both terms of each equality exist.

Now, we recall some definitions and results concerning the functional spaces which shall be used in our paper. It is well known that the spectral problem:

$$\begin{cases} -\Delta\psi_n(x) &= \lambda_n\psi_n(x), \quad x \in \Omega \\ \psi_n(x) &= 0, \quad x \in \partial\Omega \end{cases}$$

has the eigenvalues λ_n and corresponding eigenfunctions $\psi_n \in H_0^1(\Omega) \cap H^2(\Omega)$. Note that $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \ldots$ and $\lim_{n\to\infty} \lambda_n = \infty$.
For all $s \geq 0$, the Hilbert scale space $\mathbb{H}^s(\Omega)$ includes,

$$\mathbb{H}^s(\Omega) = \left\{ \theta \in L^2(\Omega) : \sum_{n=1}^{\infty} \left|\langle \theta, \psi_n\rangle\right|^2 \lambda_n^{2s} < \infty \right\}. \tag{8}$$

The space $\mathbb{H}^s(\Omega)$ is also a Banach space equipped with the norm

$$\|\theta\|_{\mathbb{H}^s(\Omega)} := \left(\sum_{n=1}^{\infty} \left|\langle \theta, \psi_n\rangle\right|^2 \lambda_n^{2s} \right)^{\frac{1}{2}}, \quad \theta \in \mathbb{H}^s(\Omega).$$

Let $C([0,T];\mathcal{B})$ be the set of all continuous functions which map $[0,T]$ into a Banach space $\mathcal{B}$. This is a Banach space endowed with the usual supremum norm.

For all $0 < \alpha \leq 1$ and $d, \theta > 0$, we denote $\mathbf{Z}_{d,\theta,\alpha}((0,T];X)$ the weighted space of all functions $\psi \in C((0,T];X)$ such that

$$\|\psi\|_{\mathbf{Z}_{d,\theta,\alpha}((0,T];X)} := \sup_{t\in(0,T]} t^d e^{-\theta t^\alpha} \|\psi(t,\cdot)\|_X < \infty, \tag{9}$$

3. Linear conformable parabolic equation

In this section we focus on the linear problem given as below

$$\begin{cases} \dfrac{{}^C\partial^\alpha}{\partial t^\alpha}\left(\dfrac{{}^C\partial^\alpha y(x,t)}{\partial t^\alpha}\right) - \Delta y(x,t) = F(x,t), & x \in \Omega, \quad t \in (0,T), \\ y(x,t) = 0, & x \in \partial\Omega, t \in (0,T), \\ y(x,0) = f(x), \quad \dfrac{{}^C\partial^\alpha y(x,0)}{\partial t^\alpha} = g(x), & x \in \Omega. \end{cases} \tag{10}$$

Theorem 3.1. *The mild solution to linear problem* (10) *is defined by*

$$\begin{aligned} y(x,t) = &\sum_{n=1}^{\infty} \cos\Big(\sqrt{\lambda_n}\frac{t^\alpha}{\alpha}\Big)\langle f, \psi_n\rangle\psi_n(x) + \sum_{n=1}^{\infty} \frac{\sin\Big(\sqrt{\lambda_n}\frac{t^\alpha}{\alpha}\Big)}{\sqrt{\lambda_n}}\langle g, \psi_n\rangle\psi_n(x) \\ &+ \sum_{n=1}^{\infty}\left[\int_0^t s^{\alpha-1}\frac{\sin\Big(\sqrt{\lambda_n}(\frac{t^\alpha - s^\alpha}{\alpha})\Big)}{\sqrt{\lambda_n}} F_n(s)ds\right]\psi_n(x). \end{aligned} \tag{11}$$

Proof. In order to construct the explicit mild solution, the main technique is based on the Fourier representation, separable of variables and semi group method. Firstly, we consider the searching solution has the Fourier series form of:

$$y(x,t) = \sum_{n=1}^{\infty} \langle y(.,t), \psi_n\rangle\psi_n(x) = \sum_{n=1}^{\infty} y_n(t)\psi_n(x). \tag{12}$$

After substituting in to the equation and combining with the fact that $\psi_n(x)$ is the eigenfunction of $-\Delta$ corresponding to eigenvalue λ_n we obtain:

$$\frac{{}^C\partial^\alpha}{\partial t^\alpha}\left(\frac{{}^C\partial^\alpha y_n(t)}{\partial t^\alpha}\right) = -\lambda_n y_n(t) + F_n(t), \tag{13}$$

with corresponding initial conditions: $y_n(0) = \langle f, \psi_n\rangle$, $\frac{{}^C\partial^\alpha y_n(0)}{\partial t^\alpha} = \langle g, \psi_n\rangle$.

Note that $F_n(t)$ is the Fourier coefficient of the source function $F(x,t)$.
As many fractional derivatives, the conformable derivative does not satisfy the index rule. That fact leads to many difficulties in solving the wave equation which has 2 times differentiate of order α. Therefore, we have to treat each time derivative separately.
For the first derivation, we apply the integral conformable fractional operator $I_\alpha(t)$ into both sides of above equality

$$I_\alpha\Big(\frac{{}^C\partial^\alpha}{\partial t^\alpha}\left(\frac{{}^C\partial^\alpha y_n(t)}{\partial t^\alpha}\right)\Big) = -\lambda_n I_\alpha\Big(y_n(t)\Big) + I_\alpha\Big(F_n(t)\Big).$$

Thus using theorem (2.4), we have the following equality

$$I_\alpha\Big(\frac{{}^C\partial^\alpha}{\partial t^\alpha}\left(\frac{{}^C\partial^\alpha y_n(t)}{\partial t^\alpha}\right)\Big) = \frac{{}^C\partial^\alpha y_n(t)}{\partial t^\alpha} - \frac{{}^C\partial^\alpha y_n(0)}{\partial t^\alpha} = t^{1-\alpha}\frac{dy_n(t)}{dt} - \langle g, \psi_n\rangle \tag{14}$$

where we note that

$$\langle g, \psi_n\rangle = \lim_{t\to 0^+}\Big(t^{1-\alpha}\frac{dy_n(t)}{dt}\Big).$$

Combining with definition (6) we have that

$$t^{1-\alpha}\frac{dy_n(t)}{dt} = \langle g, \psi_n\rangle - \lambda_n\int_0^t \nu^{\alpha-1}y_n(\nu)d\nu + \int_0^t \nu^{\alpha-1}F_n(\nu)d\nu. \tag{15}$$

Taking the first derivative to bothsides above, we get that

$$\frac{d}{dt}\Big(t^{1-\alpha}\frac{dy_n(t)}{dt}\Big) = -\lambda_n t^{\alpha-1}y_n(t) + t^{\alpha-1}F_n(t). \tag{16}$$

For convenient in solving the equation above, we set the variable $\omega(t) = \frac{t^\alpha}{\alpha}$ and $z_n(\omega) = y_n(t)$.
It is obvious to see that

$$\frac{dz_n(\omega)}{d\omega} = \frac{dy_n(t)}{dt}\frac{dt}{d\omega} = t^{1-\alpha}\frac{dy_n(t)}{dt}. \tag{17}$$

This implies that

$$\frac{d}{dt}\Big(t^{1-\alpha}\frac{dy_n(t)}{dt}\Big) = \frac{d^2 z_n(\omega)}{d\omega^2}\frac{d\omega}{dt} = t^{\alpha-1}\frac{d^2 z_n(\omega)}{d\omega^2}. \tag{18}$$

Combining (16) and (18), one has

$$\frac{d^2 z_n(\omega)}{d\omega^2} = -\lambda_n z_n(\omega) + F_n(t) = -\lambda_n z_n(\omega) + \overline{F}_n(w), \tag{19}$$

where we set the following function

$$\overline{F}_n(w) = \overline{F}_n(\frac{t^\alpha}{\alpha}) = F_n(t), \quad 0 \le t \le T. \tag{20}$$

Since the fact that $\langle g, \psi_n \rangle = \lim_{t\to 0^+} \Big(t^{1-\alpha} \dfrac{dy_n(t)}{dt} \Big)$ and $z_n(\omega) = y_n(t)$, $\omega(t) = \frac{t^\alpha}{\alpha}$, we know that

$$z_n(0) = y_n(0) = \langle f, \psi_n \rangle, \quad \frac{dz_n(0)}{d\omega} = \langle g, \psi_n \rangle. \tag{21}$$

By solving the Cauchy problem of second order differential equations (19), (21) above, it is well-known that

$$z_n(\omega) = \cos\Big(\sqrt{\lambda_n}\omega\Big) z_n(0) + \frac{\sin\Big(\sqrt{\lambda_n}\omega\Big)}{\sqrt{\lambda_n}} \frac{dz_n(0)}{d\omega} + \int_0^\omega \frac{\sin\Big(\sqrt{\lambda_n}(\omega - r)\Big)}{\sqrt{\lambda_n}} \overline{F}_n(r) dr. \tag{22}$$

It is obvious to see that

$$\int_0^\omega \frac{\sin\Big(\sqrt{\lambda_n}(\omega - r)\Big)}{\sqrt{\lambda_n}} \overline{F}_n(r) dr = \int_0^{\frac{t^\alpha}{\alpha}} \frac{\sin\Big(\sqrt{\lambda_n}(\frac{t^\alpha}{\alpha} - r)\Big)}{\sqrt{\lambda_n}} \overline{F}_n(r) dr. \tag{23}$$

We consider the following integral term

$$\mathbb{T} = \int_0^t s^{\alpha-1} \frac{\sin\Big(\sqrt{\lambda_n}(\frac{t^\alpha - s^\alpha}{\alpha})\Big)}{\sqrt{\lambda_n}} F_n(s) ds. \tag{24}$$

Let $r = \frac{s^\alpha}{\alpha}$ then $dr = s^{\alpha-1}ds$. Since the fact that (20), we know the fact that

$$\overline{F}_n(r) = \overline{F}_n(\frac{s^\alpha}{\alpha}) = F_n(s).$$

This implies that

$$\mathbb{T} = \int_0^t s^{\alpha-1} \frac{\sin\Big(\sqrt{\lambda_n}(\frac{t^\alpha - s^\alpha}{\alpha})\Big)}{\sqrt{\lambda_n}} F_n(s) ds = \int_0^{\frac{t^\alpha}{\alpha}} \frac{\sin\Big(\sqrt{\lambda_n}(\frac{t^\alpha}{\alpha} - r)\Big)}{\sqrt{\lambda_n}} \overline{F}_n(r) dr. \tag{25}$$

Combining (22), (23) and (25), we find that the following equality

$$\begin{aligned} y_n(t) = \cos\Big(\sqrt{\lambda_n}\frac{t^\alpha}{\alpha}\Big)\langle f, \psi_n \rangle + \frac{\sin\Big(\sqrt{\lambda_n}\frac{t^\alpha}{\alpha}\Big)}{\sqrt{\lambda_n}} \langle g, \psi_n \rangle \\ + \int_0^t s^{\alpha-1} \frac{\sin\Big(\sqrt{\lambda_n}(\frac{t^\alpha - s^\alpha}{\alpha})\Big)}{\sqrt{\lambda_n}} F_n(s) ds. \end{aligned} \tag{26}$$

This fomula implies that (11). □

Lemma 3.2. *Let $\alpha_0 \le \alpha \le 1$. Then we get*

$$\Big| \frac{t^\alpha}{\alpha} - t \Big| \le C(\alpha_0, \mu) t^{\alpha-\mu} \mathbf{M}(\alpha, \mu), \tag{27}$$

where any $\mu > 0$ and

$$\mathbf{M}(\alpha, \mu) = (1-\alpha)^\mu + (1-\alpha) + |T^{1-\alpha} - 1|.$$

The proof of Lemma (3.2) can be found in the proof of Theorem 3 [7].

Lemma 3.3. *Let $\alpha_0 \le \alpha \le 1$. Then, for all $0 < \varepsilon < 1$ we get*

$$\Big|\cos\Big(\sqrt{\lambda_n}\frac{t^\alpha}{\alpha}\Big) - \cos\Big(\sqrt{\lambda_n} t\Big)\Big| \le C(\varepsilon, \alpha_0, \mu)\lambda_n^{\frac{\varepsilon}{2}} t^{(\alpha-\mu)\varepsilon}\Big|\mathbf{M}(\alpha, \mu)\Big|^\varepsilon. \tag{28}$$

and

$$\Big|\frac{\sin\Big(\sqrt{\lambda_n}\frac{t^\alpha}{\alpha}\Big)}{\sqrt{\lambda_n}} - \frac{\sin\Big(\sqrt{\lambda_n} t\Big)}{\sqrt{\lambda_n}}\Big| \le C(\varepsilon, \alpha_0, \mu)\lambda_n^{\frac{\varepsilon-1}{2}} t^{(\alpha-\mu)\varepsilon}\Big|\mathbf{M}(\alpha, \mu)\Big|^\varepsilon. \tag{29}$$

Proof. For convenience, from now on, we only consider $\varepsilon \in (0, 1)$. Using the inequality $|\cos(a) - \cos(b)| \le C(\varepsilon)|a - b|^\varepsilon$ and in view of Lemma (3.2), we get the following inequality

$$\Big|\cos\Big(\sqrt{\lambda_n}\frac{t^\alpha}{\alpha}\Big) - \cos\Big(\sqrt{\lambda_n} t\Big)\Big| \le C(\varepsilon)\lambda_n^{\frac{\varepsilon}{2}}\Big|\frac{t^\alpha}{\alpha} - t\Big|^\varepsilon \le C(\varepsilon, \alpha_0, \mu)\lambda_n^{\frac{\varepsilon}{2}} t^{(\alpha-\mu)\varepsilon}\Big|\mathbf{M}(\alpha, \mu)\Big|^\varepsilon. \tag{30}$$

By a similar explanation as above, we get that

$$\Big|\sin\Big(\sqrt{\lambda_n}\frac{t^\alpha}{\alpha}\Big) - \sin\Big(\sqrt{\lambda_n} t\Big)\Big| \le C(\varepsilon, \alpha_0, \mu)\lambda_n^{\frac{\varepsilon}{2}} t^{(\alpha-\mu)\varepsilon}\Big|\mathbf{M}(\alpha, \mu)\Big|^\varepsilon. \tag{31}$$

By dividing both sides of the above expression by $\sqrt{\lambda_n}$, one has

$$\Big|\frac{\sin\Big(\sqrt{\lambda_n}\frac{t^\alpha}{\alpha}\Big)}{\sqrt{\lambda_n}} - \frac{\sin\Big(\sqrt{\lambda_n} t\Big)}{\sqrt{\lambda_n}}\Big| \le C(\varepsilon, \alpha_0, \mu)\lambda_n^{\frac{\varepsilon-1}{2}} t^{(\alpha-\mu)\varepsilon}\Big|\mathbf{M}(\alpha, \mu)\Big|^\varepsilon \tag{32}$$

The proof of our lemma is showed. □

4. Continuous dependence of the mild solution in the linear case

Theorem 4.1. *Let $0 < \alpha_0 \le \alpha < 1$. Let y_α be the mild solution to problem* (10)*. Let y^* be the solution of classical problem*

$$\begin{cases} y_{tt} - \Delta y = F(x, t), \quad (x, t) \in \Omega \times (0, T), \\ y(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, T), \\ y(x, 0) = f(x), \qquad y_t(x, 0) = g(x), \quad x \in \Omega, \end{cases} \tag{33}$$

Assume that $f \in \mathbb{H}^{s+\frac{\varepsilon}{2}}(\Omega)$, $g \in \mathbb{H}^{s+\frac{\varepsilon-1}{2}}(\Omega)$ and $F \in L^\infty(0, T; \mathbb{H}^{s+\frac{\varepsilon-1}{2}}(\Omega))$ where $s > 0$ and $2s + \varepsilon > 1$. Then we get

$$\begin{aligned} \Big\|y^*(., t) - y_\alpha(., t)\Big\|_{\mathbb{H}^s(\Omega)} &\le C\Big|\mathbf{M}(\alpha, \mu)\Big|^\varepsilon\bigg(\Big\|f\Big\|_{\mathbb{H}^{s+\frac{\varepsilon}{2}}(\Omega)} + \Big\|g\Big\|_{\mathbb{H}^{s+\frac{\varepsilon-1}{2}}(\Omega)} + \Big\|F\Big\|_{L^\infty(0,T;\mathbb{H}^{s+\frac{\varepsilon-1}{2}}(\Omega))}\bigg) \\ &\quad + T^\varepsilon C(\alpha_0, \theta, T)\mathbf{M}(\alpha, \theta)\Big\|F\Big\|_{L^\infty(0,T;\mathbb{H}^{s+\frac{\varepsilon-1}{2}}(\Omega))}. \end{aligned} \tag{34}$$

Proof. Let us denote y_α be the mild solution to problem (10). Then one has

$$\begin{aligned} y_\alpha(x, t) &= \sum_{n=1}^\infty \cos\Big(\sqrt{\lambda_n}\frac{t^\alpha}{\alpha}\Big)\langle f, \psi_n\rangle\psi_n(x) + \sum_{n=1}^\infty \frac{\sin\Big(\sqrt{\lambda_n}\frac{t^\alpha}{\alpha}\Big)}{\sqrt{\lambda_n}}\langle g, \psi_n\rangle\psi_n(x) \\ &\quad + \sum_{n=1}^\infty \bigg[\int_0^t r^{\alpha-1}\frac{\sin\Big(\sqrt{\lambda_n}(\frac{t^\alpha - r^\alpha}{\alpha})\Big)}{\sqrt{\lambda_n}} F_n(r)dr\bigg]\psi_n(x). \end{aligned} \tag{35}$$

Let y^* be the mild solution of classical problem (33).

$$\begin{aligned} y^*(x,t) &= \sum_{n=1}^{\infty} \cos\Big(\sqrt{\lambda_n}t\Big)\langle f,\psi_n\rangle\psi_n(x) + \sum_{n=1}^{\infty} \frac{\sin\Big(\sqrt{\lambda_n}t\Big)}{\sqrt{\lambda_n}}\langle g,\psi_n\rangle\psi_n(x) \\ &+ \sum_{n=1}^{\infty}\Bigg[\int_0^t \frac{\sin\Big(\sqrt{\lambda_n}(t-r)\Big)}{\sqrt{\lambda_n}}F_n(r)dr\Bigg]\psi_n(x). \end{aligned} \tag{36}$$

From (11) and (36), we have the following equality

$$\begin{aligned} y^*(x,t) - y_\alpha(x,t) &= \sum_{n=1}^{\infty}\Bigg[\cos\Big(\sqrt{\lambda_n}\frac{t^\alpha}{\alpha}\Big) - \cos\Big(\sqrt{\lambda_n}t\Big)\Bigg]\langle f,\psi_n\rangle\psi_n(x) \\ &+ \sum_{n=1}^{\infty}\Bigg[\frac{\sin\Big(\sqrt{\lambda_n}\frac{t^\alpha}{\alpha}\Big)}{\sqrt{\lambda_n}} - \frac{\sin\Big(\sqrt{\lambda_n}t\Big)}{\sqrt{\lambda_n}}\Bigg]\langle g,\psi_n\rangle\psi_n(x) \\ &+ \sum_{n=1}^{\infty}\Bigg[\int_0^t r^{\alpha-1}\Big[\frac{\sin\Big(\sqrt{\lambda_n}(\frac{t^\alpha-r^\alpha}{\alpha})\Big)}{\sqrt{\lambda_n}} - \frac{\sin\Big(\sqrt{\lambda_n}(t-r)\Big)}{\sqrt{\lambda_n}}\Big]F_n(r)dr\Bigg]\psi_n(x) \\ &+ \sum_{n=1}^{\infty}\Bigg[\int_0^t \Big(r^{\alpha-1}-1\Big)\frac{\sin\Big(\sqrt{\lambda_n}(t-r)\Big)}{\sqrt{\lambda_n}}F_n(r)dr\Bigg]\psi_n(x) = \sum_{j=1}^{4}\mathbb{B}_j(x,t). \end{aligned} \tag{37}$$

Let us now divide into some following steps.
Step 1. Estimation of $\mathbb{B}_1$. Let us now to treat the first term $\mathbb{B}_1(x,t)$. Using (28), we provide the following estimate

$$\begin{aligned} \Big\|\mathbb{B}_1(.,t)\Big\|^2_{\mathbb{H}^s(\Omega)} &= \sum_{n=1}^{\infty}\lambda_n^{2s}\Bigg[\cos\Big(\sqrt{\lambda_n}\frac{t^\alpha}{\alpha}\Big) - \cos\Big(\sqrt{\lambda_n}t\Big)\Bigg]^2\langle f,\psi_n\rangle^2 \\ &\le |C(\varepsilon,\alpha_0,\mu)|^2 t^{2(\alpha-\mu)\varepsilon}\Big|\mathbf{M}(\alpha,\mu)\Big|^{2\varepsilon}\sum_{n=1}^{\infty}\lambda_n^{2s+\varepsilon}\langle f,\psi_n\rangle^2. \end{aligned} \tag{38}$$

Using Parseval's equality, we derive that

$$\begin{aligned} \Big\|\mathbb{B}_1(.,t)\Big\|_{\mathbb{H}^s(\Omega)} &\le C(\varepsilon,\alpha_0,\mu)t^{(\alpha-\mu)\varepsilon}\Big|\mathbf{M}(\alpha,\mu)\Big|^{\varepsilon}\Big\|f\Big\|_{\mathbb{H}^{s+\frac{\varepsilon}{2}}(\Omega)} \\ &\le C(\varepsilon,\alpha_0,\mu)T^{(\alpha-\mu)\varepsilon}\Big|\mathbf{M}(\alpha,\mu)\Big|^{\varepsilon}\Big\|f\Big\|_{\mathbb{H}^{s+\frac{\varepsilon}{2}}(\Omega)} \end{aligned} \tag{39}$$

Step 2. Estimation of $\mathbb{B}_2$. Now, we consider the second term $\mathbb{B}_2(x,t)$. Indeed, we have

$$\begin{aligned} \Big\|\mathbb{B}_2(.,t)\Big\|^2_{\mathbb{H}^s(\Omega)} &= \sum_{n=1}^{\infty}\lambda_n^{2s}\Bigg[\frac{\sin\Big(\sqrt{\lambda_n}\frac{t^\alpha}{\alpha}\Big)}{\sqrt{\lambda_n}} - \frac{\sin\Big(\sqrt{\lambda_n}t\Big)}{\sqrt{\lambda_n}}\Bigg]^2\langle g,\psi_n\rangle^2 \\ &\le |C(\varepsilon,\alpha_0,\mu)|^2 t^{2(\alpha-\mu)\varepsilon}\Big|\mathbf{M}(\alpha,\mu)\Big|^{2\varepsilon}\sum_{n=1}^{\infty}\lambda_n^{2s+\varepsilon-1}\langle g,\psi_n\rangle^2. \end{aligned} \tag{40}$$

Thus, we find that the following bound

$$\Big\|\mathbb{B}_2(.,t)\Big\|_{\mathbb{H}^s(\Omega)} \le C(\varepsilon,\alpha_0,\mu)t^{(\alpha-\mu)\varepsilon}\Big|\mathbf{M}(\alpha,\mu)\Big|^{\varepsilon}\Big\|g\Big\|_{\mathbb{H}^{s+\frac{\varepsilon-1}{2}}(\Omega)}. \tag{41}$$

Step 3. Estimation of $\mathbb{B}_3$. By Parseval's equality and Hölder inequality, we have that

$$\begin{aligned}\Big\|\mathbb{B}_3(.,t)\Big\|^2_{\mathbb{H}^s(\Omega)} &= \sum_{n=1}^{\infty}\lambda_n^{2s}\Big(\int_0^t r^{\alpha-1}\Big[\frac{\sin\big(\sqrt{\lambda_n}(\frac{t^\alpha-r^\alpha}{\alpha})\big)}{\sqrt{\lambda_n}}-\frac{\sin\big(\sqrt{\lambda_n}(t-r)\big)}{\sqrt{\lambda_n}}\Big]F_n(r)dr\Big)^2\\ &\le \sum_{n=1}^{\infty}\lambda_n^{2s}\Big(\int_0^t r^{\alpha-1}dr\Big)\Big(\int_0^t r^{\alpha-1}\Big[\frac{\sin\big(\sqrt{\lambda_n}(\frac{t^\alpha-r^\alpha}{\alpha})\big)}{\sqrt{\lambda_n}}-\frac{\sin\big(\sqrt{\lambda_n}(t-r)\big)}{\sqrt{\lambda_n}}\Big]^2|F_n(r)|^2dr\Big)\\ &\le \frac{t^\alpha}{\alpha}\sum_{n=1}^{\infty}\lambda_n^{2s}\Big(\int_0^t r^{\alpha-1}\Big[\frac{\sin\big(\sqrt{\lambda_n}(\frac{t^\alpha-r^\alpha}{\alpha})\big)}{\sqrt{\lambda_n}}-\frac{\sin\big(\sqrt{\lambda_n}(t-r)\big)}{\sqrt{\lambda_n}}\Big]^2|F_n(r)|^2dr\Big).\end{aligned} \tag{42}$$

Using the inequality $|\sin(a)-\sin(b)|\le C(\varepsilon)|a-b|^\varepsilon$, for any $0<\varepsilon<1$, we obtain:

$$\begin{aligned}&\Big|\sin\Big(\sqrt{\lambda_n}(\frac{t^\alpha-r^\alpha}{\alpha})\Big)-\sin\Big(\sqrt{\lambda_n}(t-r)\Big)\Big|\\ &\qquad\le C(\varepsilon)\lambda_n^{\frac{\varepsilon}{2}}\Big|\frac{t^\alpha-r^\alpha}{\alpha}-(t-r)\Big|^\varepsilon\le C(\varepsilon)\lambda_n^{\frac{\varepsilon}{2}}\Big[\Big|\frac{t^\alpha}{\alpha}-t\Big|^\varepsilon+\Big|\frac{r^\alpha}{\alpha}-r\Big|^\varepsilon\Big].\end{aligned} \tag{43}$$

In view of (27) and noting that $(a+b)^\varepsilon\le C(\varepsilon)\,(a^\varepsilon+b^\varepsilon)$ for any $a,b\ge 0$, we know that

$$\Big|\frac{t^\alpha}{\alpha}-t\Big|^\varepsilon+\Big|\frac{r^\alpha}{\alpha}-r\Big|^\varepsilon\le C(\alpha_0,\mu,\varepsilon)\Big|\mathbf{M}(\alpha,\mu)\Big|^\varepsilon\Big(t^{(\alpha-\mu)\varepsilon}+r^{(\alpha-\mu)\varepsilon}\Big). \tag{44}$$

Some above observations implies that

$$\begin{aligned}&\Big|\frac{\sin\big(\sqrt{\lambda_n}(\frac{t^\alpha-r^\alpha}{\alpha})\big)}{\sqrt{\lambda_n}}-\frac{\sin\big(\sqrt{\lambda_n}(t-r)\big)}{\sqrt{\lambda_n}}\Big|^2\\ &\qquad\le 2|C(\alpha_0,\mu,\varepsilon)|^2\lambda_n^{\varepsilon-1}\Big|\mathbf{M}(\alpha,\mu)\Big|^{2\varepsilon}\Big(t^{2(\alpha-\mu)\varepsilon}+r^{2(\alpha-\mu)\varepsilon}\Big).\end{aligned} \tag{45}$$

Thus, we get immediately that

$$\begin{aligned}&\int_0^t r^{\alpha-1}\Big[\frac{\sin\big(\sqrt{\lambda_n}(\frac{t^\alpha-r^\alpha}{\alpha})\big)}{\sqrt{\lambda_n}}-\frac{\sin\big(\sqrt{\lambda_n}(t-r)\big)}{\sqrt{\lambda_n}}\Big]^2|F_n(r)|^2dr\\ &\le 2|C(\alpha_0,\mu,\varepsilon)|^2\Big|\mathbf{M}(\alpha,\mu)\Big|^{2\varepsilon}t^{2(\alpha-\mu)\varepsilon}\int_0^t r^{\alpha-1}\lambda_n^{\varepsilon-1}|F_n(r)|^2dr\\ &+2|C(\alpha_0,\mu,\varepsilon)|^2\Big|\mathbf{M}(\alpha,\mu)\Big|^{2\varepsilon}\int_0^t r^{\alpha-1+2(\alpha-\mu)\varepsilon}\lambda_n^{\varepsilon-1}|F_n(r)|^2dr.\end{aligned} \tag{46}$$

This inequality implies that

$$\begin{aligned}&\sum_{n=1}^{\infty}\lambda_n^{2s}\Big(\int_0^t r^{\alpha-1}\Big[\frac{\sin\big(\sqrt{\lambda_n}(\frac{t^\alpha-r^\alpha}{\alpha})\big)}{\sqrt{\lambda_n}}-\frac{\sin\big(\sqrt{\lambda_n}(t-r)\big)}{\sqrt{\lambda_n}}\Big]^2F_n(r)|^2dr\Big)\\ &\le 2|C(\alpha_0,\mu,\varepsilon)|^2\Big|\mathbf{M}(\alpha,\mu)\Big|^{2\varepsilon}t^{2(\alpha-\mu)\varepsilon}\int_0^t r^{\alpha-1}\Big(\sum_{n=1}^{\infty}\lambda_n^{2s+\varepsilon-1}|F_n(r)|^2\Big)dr\\ &+2|C(\alpha_0,\mu,\varepsilon)|^2\Big|\mathbf{M}(\alpha,\mu)\Big|^{2\varepsilon}\int_0^t r^{\alpha-1+2(\alpha-\mu)\varepsilon}\Big(\sum_{n=1}^{\infty}\lambda_n^{2s+\varepsilon-1}|F_n(r)|^2\Big)dr.\end{aligned} \tag{47}$$

Using Parseval's equality, we get that the following bound

$$\sum_{n=1}^{\infty}\lambda_n^{2s}\Big(\int_0^t r^{\alpha-1}\Big[\frac{\sin\Big(\sqrt{\lambda_n}(\frac{t^\alpha-r^\alpha}{\alpha})\Big)}{\sqrt{\lambda_n}}-\frac{\sin\Big(\sqrt{\lambda_n}(t-r)\Big)}{\sqrt{\lambda_n}}\Big]^2 F_n(r)|^2dr\Big)$$
$$\leq 2|C(\alpha_0,\mu,\varepsilon)|^2\Big|\mathbf{M}(\alpha,\mu)\Big|^{2\varepsilon}t^{2(\alpha-\mu)\varepsilon}\int_0^t r^{\alpha-1}\Big\|F(.,r)\Big\|^2_{\mathbb{H}^{s+\frac{\varepsilon-1}{2}}(\Omega)}dr$$
$$+2|C(\alpha_0,\mu,\varepsilon)|^2\Big|\mathbf{M}(\alpha,\mu)\Big|^{2\varepsilon}\int_0^t r^{\alpha-1+2(\alpha-\mu)\varepsilon}\Big\|F(.,r)\Big\|^2_{\mathbb{H}^{s+\frac{\varepsilon-1}{2}}(\Omega)}dr \tag{48}$$

It is obvious to see that

$$\int_0^t r^{\alpha-1}\Big\|F(.,r)\Big\|^2_{\mathbb{H}^{s+\frac{\varepsilon-1}{2}}(\Omega)}dr\leq\Big\|F\Big\|^2_{L^\infty(0,T;\mathbb{H}^{s+\frac{\varepsilon-1}{2}}(\Omega))}\Big(\int_0^t r^{\alpha-1}dr\Big)$$
$$=\frac{t^\alpha}{\alpha}\Big\|F\Big\|^2_{L^\infty(0,T;\mathbb{H}^{s+\frac{\varepsilon-1}{2}}(\Omega))} \tag{49}$$

and

$$\int_0^t r^{\alpha-1+2(\alpha-\mu)\varepsilon}\Big\|F(.,r)\Big\|^2_{\mathbb{H}^{s+\frac{\varepsilon-1}{2}}(\Omega)}dr\leq\Big\|F\Big\|^2_{L^\infty(0,T;\mathbb{H}^{s+\frac{\varepsilon-1}{2}}(\Omega))}\Big(\int_0^t r^{\alpha-1+2(\alpha-\mu)\varepsilon}dr\Big)$$
$$=\frac{t^{\alpha+2(\alpha-\mu)\varepsilon}}{\alpha+2(\alpha-\mu)\varepsilon}\Big\|F\Big\|^2_{L^\infty(0,T;\mathbb{H}^{s+\frac{\varepsilon-1}{2}}(\Omega))}. \tag{50}$$

Combining (42), (48), (49) and (50), we find that

$$\Big\|\mathbb{B}_3(.,t)\Big\|^2_{\mathbb{H}^s(\Omega)}\leq 2|C(\alpha_0,\mu,\varepsilon)|^2\Big|\mathbf{M}(\alpha,\mu)\Big|^{2\varepsilon}$$
$$t^{2\alpha+2(\alpha-\mu)\varepsilon}\Big[\frac{1}{\alpha^2}+\frac{1}{\alpha^2+2\alpha\varepsilon(\alpha-\mu)}\Big]\Big\|F\Big\|^2_{L^\infty(0,T;\mathbb{H}^{s+\frac{\varepsilon-1}{2}}(\Omega))}. \tag{51}$$

Therefore, by taking the square root of the above two sides, we immediately have

$$\Big\|\mathbb{B}_3(.,t)\Big\|_{\mathbb{H}^s(\Omega)}\leq 2C(\alpha_0,\mu,\varepsilon)\Big|\mathbf{M}(\alpha,\mu)\Big|^{\varepsilon}$$
$$t^{\alpha+(\alpha-\mu)\varepsilon}\sqrt{\frac{1}{\alpha^2}+\frac{1}{\alpha^2+2\alpha\varepsilon(\alpha-\mu)}}\Big\|F\Big\|_{L^\infty(0,T;\mathbb{H}^{s+\frac{\varepsilon-1}{2}}(\Omega))}. \tag{52}$$

Step 4. Estimation of $\mathbb{B}_4$. Using Parseval's indentity and Hölder inequality, we find that

$$\Big\|\mathbb{B}_4(.,t)\Big\|^2_{\mathbb{H}^s(\Omega)}=\sum_{n=1}^{\infty}\lambda_n^{2s}\Big[\int_0^t\Big(r^{\alpha-1}-1\Big)\frac{\sin\Big(\sqrt{\lambda_n}(t-r)\Big)}{\sqrt{\lambda_n}}F_n(r)dr\Big]^2$$
$$\leq\sum_{n=1}^{\infty}\lambda_n^{2s}\Big(\int_0^t|r^{\alpha-1}-1|dr\Big)\Big[\int_0^t|r^{\alpha-1}-1|\Big|\frac{\sin\Big(\sqrt{\lambda_n}(t-r)\Big)}{\sqrt{\lambda_n}}\Big|^2|F_n(r)|^2dr\Big]. \tag{53}$$

Using the inequality $|\sin(z)|\leq C_\beta z^\beta$ for $0<\beta<1$, we have the following bound

$$\Big|\frac{\sin\Big(\sqrt{\lambda_n}(t-r)\Big)}{\sqrt{\lambda_n}}\Big|\leq C_\beta\lambda_n^{\frac{\beta-1}{2}}(t-r)^\beta. \tag{54}$$

This implies that

$$\int_0^t \Big|r^{\alpha-1}-1\Big|\Big|\frac{\sin\Big(\sqrt{\lambda_n}(t-r)\Big)}{\sqrt{\lambda_n}}\Big|^2|F_n(r)|^2dr \le C_\beta^2\lambda_n^{\beta-1}\int_0^t \Big|r^{\alpha-1}-1\Big|(t-r)^{2\beta}|F_n(r)|^2dr. \tag{55}$$

Hence, we find that the following bound

$$\begin{aligned}\sum_{n=1}^{\infty}\lambda_n^{2s}\Big[\int_0^t \Big|r^{\alpha-1}-1\Big|\Big|\frac{\sin\Big(\sqrt{\lambda_n}(t-r)\Big)}{\sqrt{\lambda_n}}\Big|^2|F_n(r)|^2dr\Big]&\\ \lesssim \int_0^t \Big|r^{\alpha-1}-1\Big|(t-r)^{2\beta}\Big\|F(r)\Big\|^2_{\mathbb{H}^{s+\frac{\beta-1}{2}}(\Omega)}dr&\\ \le \Big(\int_0^t \Big|r^{\alpha-1}-1\Big|(t-r)^{2\beta}dr\Big)\Big\|F\Big\|^2_{L^\infty(0,T;\mathbb{H}^{s+\frac{\beta-1}{2}}(\Omega))}&\\ \le T^{2\beta}\Big(\int_0^t \Big|r^{\alpha-1}-1\Big|dr\Big)\Big\|F\Big\|^2_{L^\infty(0,T;\mathbb{H}^{s+\frac{\beta-1}{2}}(\Omega))}.&\end{aligned} \tag{56}$$

By applying Theorem 4 (see in [7]), we obtain that the following result

$$\begin{aligned}\int_0^t |r^{\alpha-1}-1|dr &\le C(\alpha_0,\theta)t^{\alpha-\theta}\mathbf{M}(\alpha,\theta)+\frac{1}{\alpha}-1\\ &\le C(\alpha_0,\theta)T^{\alpha-\theta}\mathbf{M}(\alpha,\theta)+\frac{1-\alpha}{\alpha_0}\le C(\alpha_0,\theta,T)\mathbf{M}(\alpha,\theta)\end{aligned} \tag{57}$$

for any $0<\theta\le\alpha_0$. Here we recall that

$$\mathbf{M}(\alpha,\theta)=1-\alpha+(1-\alpha)^\theta+|T^{1-\alpha}-1|.$$

Combining (53), (56) and (57), we deduce that the following estimate

$$\begin{aligned}\Big\|\mathbb{B}_4(.,t)\Big\|_{\mathbb{H}^s(\Omega)} &\le T^\beta\Big(\int_0^t \Big|r^{\alpha-1}-1\Big|dr\Big)\Big\|F\Big\|_{L^\infty(0,T;\mathbb{H}^{s+\frac{\beta-1}{2}}(\Omega))}\\ &\le T^\beta C(\alpha_0,\theta,T)\mathbf{M}(\alpha,\theta)\Big\|F\Big\|_{L^\infty(0,T;\mathbb{H}^{s+\frac{\beta-1}{2}}(\Omega))}.\end{aligned} \tag{58}$$

By the choice $\beta=\varepsilon$, we know that the following estimate

$$\Big\|\mathbb{B}_4(.,t)\Big\|_{\mathbb{H}^s(\Omega)} \le T^\varepsilon C(\alpha_0,\theta,T)\mathbf{M}(\alpha,\theta)\Big\|F\Big\|_{L^\infty(0,T;\mathbb{H}^{s+\frac{\varepsilon-1}{2}}(\Omega))}. \tag{59}$$

Combining (37), (39), (41), (52) and (59), we find that

$$\begin{aligned}\Big\|y^*(.,t)-y_\alpha(.,t)\Big\|_{\mathbb{H}^s(\Omega)} &\le \Big\|\mathbb{B}_1(.,t)\Big\|_{\mathbb{H}^s(\Omega)}+\Big\|\mathbb{B}_2(.,t)\Big\|_{\mathbb{H}^s(\Omega)}+\Big\|\mathbb{B}_3(.,t)\Big\|_{\mathbb{H}^s(\Omega)}+\Big\|\mathbb{B}_4(.,t)\Big\|_{\mathbb{H}^s(\Omega)}\\ &\le C\Big|\mathbf{M}(\alpha,\mu)\Big|^\varepsilon\Big(\Big\|f\Big\|_{\mathbb{H}^{s+\frac{\varepsilon}{2}}(\Omega)}+\Big\|g\Big\|_{\mathbb{H}^{s+\frac{\varepsilon-1}{2}}(\Omega)}+\Big\|F\Big\|_{L^\infty(0,T;\mathbb{H}^{s+\frac{\varepsilon-1}{2}}(\Omega))}\Big)\\ &\quad+T^\varepsilon C(\alpha_0,\theta,T)\mathbf{M}(\alpha,\theta)\Big\|F\Big\|_{L^\infty(0,T;\mathbb{H}^{s+\frac{\varepsilon-1}{2}}(\Omega))}.\end{aligned} \tag{60}$$

Note that $\lim_{\alpha\to1^-}\mathbf{M}(\alpha,.)=0$, so we obtain the convergence of mild solution. □

5. Conformable wave equation with nonlinear source

In this section, we study the global existence of Problem (1).

Theorem 5.1. *Let Ω be an open, bounded, sufficiently smooth domain in $\mathbb{R}^N$, $N \geq 1$. Let $F : \mathbb{H}^q(\Omega) \to \mathbb{H}^p(\Omega)$ such that $F(\mathbf{0}) = \mathbf{0}$ and*

$$\|F(y_1) - F(y_2)\|_{\mathbb{H}^p(\Omega)} \leq L_F \|y_1 - y_2\|_{\mathbb{H}^q(\Omega)}, \tag{61}$$

for any $y_1, y_2 \in H^q(\Omega)$ and L_F is a positive constant. Here the two numbers p, q above are chosen such that $1 \leq p \leq q$ and $q - p < \frac{1}{2}$. Let the initial datum $f, g \in \mathbb{H}^q(\Omega)$. Then problem (1) has a global unique solution

$$y_\alpha \in \mathbf{Z}_{d,\theta_0,\alpha}((0,T]; \mathbb{H}^q(\Omega))$$

for θ_0 large enough. Here d is a constant which satisfies $0 < d < \alpha$. Moreover, one has

$$\left\|y_\alpha\right\|_{L^p(0,T;\mathbb{H}^q(\Omega))} \lesssim \left(\left\|f\right\|_{\mathbb{H}^q(\Omega)} + \left\|g\right\|_{\mathbb{H}^q(\Omega)}\right) \tag{62}$$

for $1 \leq p < \frac{1}{d}$.

The following lemma is introduced to play an important role in later proofs (see [11], Lemma 8).

Lemma 5.2. *Let $a > -1$, $b > -1$ such that $a + b \geq -1$, $\theta > 0$ and $t \in [0, T]$. For $h > 0$, the following limit holds*

$$\lim_{\theta\to\infty}\left(\sup_{t\in[0,T]} t^h \int_0^1 \nu^a (1-\nu)^b e^{-\theta t^\alpha (1-\nu)} \mathrm{d}\nu\right) = 0.$$

Proof. Let us define the following operator

$$\mathbf{P}(t)(f) = \sum_{n=1}^{\infty} \cos\left(\sqrt{\lambda_n} t\right) \langle f, \psi_n \rangle \psi_n(x), \quad \mathbf{Q}(t)(f) = \sum_{n=1}^{\infty} \frac{\sin\left(\sqrt{\lambda_n} t\right)}{\sqrt{\lambda_n}} \langle f, \psi_n \rangle \psi_n(x). \tag{63}$$

Using the inequality $|\sin(z)| \leq C_\rho z^\rho$ for any $0 < \rho < 1$, we get that

$$\left|\sin\left(\sqrt{\lambda_n} t\right)\right| \leq C_\rho \lambda_n^{\frac{\rho}{2}} t^\rho.$$

This implies that

$$\left\|\mathbf{Q}(t)(f)\right\|^2_{\mathbb{H}^s(\Omega)} = \sum_{n=1}^{\infty} \lambda_n^{2s} \left(\frac{\sin\left(\sqrt{\lambda_n} t\right)}{\sqrt{\lambda_n}}\right)^2 \langle f, \psi_n \rangle^2 \leq C_\rho^2 t^{2\rho} \sum_{n=1}^{\infty} \lambda_n^{2s+\rho-1} \langle f, \psi_n \rangle^2. \tag{64}$$

Thus, we find that

$$\left\|\mathbf{Q}(t)(f)\right\|_{\mathbb{H}^s(\Omega)} \leq C_\rho t^\rho \left\|f\right\|_{\mathbb{H}^{s+\frac{\rho-1}{2}}(\Omega)}, \quad \rho > 0. \tag{65}$$

By a fact that

$$\left|\cos\left(\sqrt{\lambda_n} t\right)\right| \leq 1.$$

We obtain

$$\left\|\mathbf{P}(t)(f)\right\|^2_{\mathbb{H}^s(\Omega)} = \sum_{n=1}^{\infty} \lambda_n^{2s} \left|\cos\left(\sqrt{\lambda_n} t\right)\right|^2 \langle f, \psi_n \rangle^2 \leq \sum_{n=1}^{\infty} \lambda_n^{2s} \langle f, \psi_n \rangle^2. \tag{66}$$

Hence, we can verify that

$$\Big\|\mathbf{P}(t)(f)\Big\|_{\mathbb{H}^s(\Omega)} \le \Big\|f\Big\|_{\mathbb{H}^s(\Omega)}. \tag{67}$$

Let us start by proving the existence and uniqueness of the mild solution to Problem (1).
Let $\mathbf{Z}_{d,\theta,\alpha}((0,T];\mathbb{H}^q(\Omega))$ be the weighted space as stated in (9). We define the operator $\mathbf{M}$ as below:

$$\mathbf{M} : \mathbf{Z}_{d,\theta,\alpha}((0,T];\mathbb{H}^q(\Omega)) \to \mathbf{Z}_{d,\theta,\alpha}((0,T];\mathbb{H}^q(\Omega))$$

with $\theta > 0$ and

$$\mathbf{M}y(t) := \mathbf{P}\Big(\frac{t^\alpha}{\alpha}\Big)(f) + \mathbf{Q}\Big(\frac{t^\alpha}{\alpha}\Big)(g) + \int_0^t \nu^{\alpha-1}\mathbf{Q}\Big(\frac{t^\alpha - \nu^\alpha}{\alpha}\Big)F(y(\nu))d\nu. \tag{68}$$

If $y = 0$ then one has

$$\mathbf{M}\left(y(t) = 0\right) := \mathbf{P}\Big(\frac{t^\alpha}{\alpha}\Big)(f) + \mathbf{Q}\Big(\frac{t^\alpha}{\alpha}\Big)(g).$$

This together with (65) and (66) with the choide $\rho = 1$ in (65), we find that

$$\Big\|\mathbf{P}\Big(\frac{t^\alpha}{\alpha}\Big)(f)\Big\|_{\mathbb{H}^q(\Omega)} + \Big\|\mathbf{Q}\Big(\frac{t^\alpha}{\alpha}\Big)(g)\Big\|_{\mathbb{H}^q(\Omega)} \lesssim C(\alpha)\Big(\Big\|f\Big\|_{\mathbb{H}^q(\Omega)} + \Big\|g\Big\|_{\mathbb{H}^q(\Omega)}\Big), \tag{69}$$

where we note that $f \in \mathbb{H}^q(\Omega)$. The above inequality allows us to deduce that

$$\mathbf{M}\left(y(t) = 0\right) \in \mathbf{Z}_{d,\theta,\alpha}((0,T];\mathbb{H}^q(\Omega))$$

for any $\theta > 0$.
Let us to continue to provide upper bound for the term $\Big\|\mathbf{M}y_1(t) - \mathbf{M}y_2(t)\Big\|_{\mathbb{H}^s(\Omega)}$.
For any $y_1, y_2 \in \mathbf{Z}_{d,\theta,\alpha}((0,T];\mathbb{H}^q(\Omega))$, it is obvious to see that

$$\begin{aligned}\Big\|\mathbf{M}y_1(t) - \mathbf{M}y_2(t)\Big\|_{\mathbb{H}^q(\Omega)} &= \Big\|\int_0^t \nu^{\alpha-1}\mathbf{Q}\Big(\frac{t^\alpha - \nu^\alpha}{\alpha}\Big)\Big(F(y_1(\nu)) - F(y_2(\nu))\Big)d\nu\Big\|_{\mathbb{H}^q(\Omega)} \\ &\le C_\rho \int_0^t \nu^{\alpha-1}\Big(\frac{t^\alpha - \nu^\alpha}{\alpha}\Big)^\rho \Big\|F(y_1(\nu)) - F(y_2(\nu))\Big\|_{\mathbb{H}^{q+\frac{\rho-1}{2}}(\Omega)}d\nu,\end{aligned} \tag{70}$$

where ρ is chosen such that

$$0 < \rho \le 1 - 2(q-p) \quad \text{and} \quad q + \frac{\rho - 1}{2} < p$$

Since the above, we know that the Sobolev embedding $\mathbb{H}^p(\Omega) \hookrightarrow \mathbb{H}^{q+\frac{\rho-1}{2}}(\Omega)$, so we obtain:

$$\begin{aligned}\Big\|F(y_1(\nu)) - F(y_2(\nu))\Big\|_{\mathbb{H}^{q+\frac{\rho-1}{2}}(\Omega)} &\le C(p,q)\Big\|F(y_1(\nu)) - F(y_2(\nu))\Big\|_{\mathbb{H}^p(\Omega)} \\ &\le C(p,q)L_F\Big\|y_1(\nu) - y_2(\nu)\Big\|_{\mathbb{H}^q(\Omega)},\end{aligned} \tag{71}$$

where we have used the globally Lipschitz of F.
This estimate together with (70) yields to

$$\begin{aligned}t^d e^{-\theta t^\alpha}\Big\|\mathbf{M}y_1(t) - \mathbf{M}y_2(t)\Big\|_{\mathbb{H}^q(\Omega)} &\le C_1 L_F t^d e^{-\theta t^\alpha}\int_0^t \nu^{\alpha-1}\Big(t^\alpha - \nu^\alpha\Big)^\rho \Big\|y_1(\nu) - y_2(\nu)\Big\|_{\mathbb{H}^q(\Omega)}d\nu \\ &= C_1 L_F t^d \int_0^t \nu^{\alpha-1-d}e^{-\theta(t^\alpha - \nu^\alpha)}\Big(t^\alpha - \nu^\alpha\Big)^\rho \nu^d e^{-\theta\nu^\alpha}\Big\|y_1(\nu) - y_2(\nu)\Big\|_{\mathbb{H}^q(\Omega)}d\nu\end{aligned} \tag{72}$$

where C_1 depends on p, q, ρ, α. Thus, we find that

$$\int_0^t \nu^{\alpha-1-d} e^{-\theta(t^\alpha-\nu^\alpha)} \Big(t^\alpha - \nu^\alpha\Big)^\rho \nu^d e^{-\theta\nu^\alpha} \Big\| y_1(\nu) - y_2(\nu) \Big\|_{\mathbb{H}^q(\Omega)} d\nu$$
$$\leq \left(\int_0^t \nu^{\alpha-1-d} e^{-\theta(t^\alpha-\nu^\alpha)} \Big(t^\alpha - \nu^\alpha\Big)^\rho d\nu \right) \Big\| y_1 - y_2 \Big\|_{\mathbf{Z}_{d,\theta,\alpha}((0,T];\mathbb{H}^q(\Omega))}. \tag{73}$$

Since two latter estimates, we obtain

$$t^d e^{-\theta t^\alpha} \Big\| \mathbf{M} y_1(t) - \mathbf{M} y_2(t) \Big\|_{\mathbb{H}^q(\Omega)}$$
$$\leq C_1 L_F t^d \Big(\int_0^t \nu^{\alpha-1-d} e^{-\theta(t^\alpha-\nu^\alpha)} \Big(t^\alpha - \nu^\alpha\Big)^\rho d\nu \Big) \Big\| y_1 - y_2 \Big\|_{\mathbf{Z}_{d,\theta,\alpha}((0,T];\mathbb{H}^q(\Omega))}. \tag{74}$$

By set the variable $\nu = t\mu^{\frac{1}{\alpha}}$, we get $d\nu = \frac{1}{\alpha} t \mu^{\frac{1}{\alpha}-1} d\mu$. Thus, after a simple caculation, one has

$$t^d \Big(\int_0^t \nu^{\alpha-1-d} e^{-\theta(t^\alpha-\nu^\alpha)} \Big(t^\alpha - \nu^\alpha\Big)^\rho d\nu \Big) = \frac{1}{\alpha} t^{\alpha+\alpha\rho} \int_0^1 \mu^{\frac{-d}{\alpha}} (1-\mu)^\rho e^{-\theta t^\alpha (1-\mu)} d\mu. \tag{75}$$

Since $d < \alpha$ we know that

$$-\frac{d}{\alpha} > -1, \quad -\frac{d}{\alpha} + \rho > -1.$$

Hence, all conditions of Lemma (5.2) is true. By the virtue of Lemma (5.2), we provide that

$$\lim_{\theta\to\infty} \left[\sup_{t\in[0,T]} t^d \Big(\int_0^t \nu^{\alpha-1-d} e^{-\theta(t^\alpha-\nu^\alpha)} \Big(t^\alpha - \nu^\alpha\Big)^\rho d\nu \Big) \right] = 0. \tag{76}$$

From the above equality, we know that there exists a positive constant θ_0 such as

$$\sup_{t\in[0,T]} t^d \Big(\int_0^t \nu^{\alpha-1-d} e^{-\theta_0(t^\alpha-\nu^\alpha)} \Big(t^\alpha - \nu^\alpha\Big)^\rho d\nu \Big) \leq \frac{1}{2C_1 L_F}. \tag{77}$$

Combining (74) and (77), we give the following confirmation

$$\sup_{t\in[0,T]} t^d e^{-\theta_0 t^\alpha} \Big\| \mathbf{M} y_1(t) - \mathbf{M} y_2(t) \Big\|_{\mathbb{H}^q(\Omega)} \leq \frac{1}{2} \Big\| y_1 - y_2 \Big\|_{\mathbf{Z}_{d,\theta_0,\alpha}((0,T];\mathbb{H}^q(\Omega))}. \tag{78}$$

Hence

$$\Big\| \mathbf{M} y_1 - \mathbf{M} y_2 \Big\|_{\mathbf{Z}_{d,\theta_0,\alpha}((0,T];\mathbb{H}^q(\Omega))} \leq \frac{1}{2} \Big\| y_1 - y_2 \Big\|_{\mathbf{Z}_{d,\theta_0,\alpha}((0,T];\mathbb{H}^q(\Omega))}. \tag{79}$$

Since the fact that $\mathbf{M}\,(y(t) = 0) \in \mathbf{Z}_{d,\theta_0,\alpha}((0,T];\mathbb{H}^q(\Omega))$, we come to the conclusion

$$\mathbf{M} y \in \mathbf{Z}_{d,\theta_0,\alpha}((0,T];\mathbb{H}^q(\Omega)), \qquad \forall y \in \mathbf{Z}_{d,\theta_0,\alpha}((0,T];\mathbb{H}^q(\Omega))$$

It follows from (79) that $\mathbf{M}$ is a contraction mapping on $\mathbf{Z}_{d,\theta_0,\alpha}((0,T];\mathbb{H}^q(\Omega))$. By applying Banach fixed point theory, we deduce that $\mathbf{M}$ has a fixed point $y_\alpha \in \mathbf{Z}_{d,\theta_0,\alpha}((0,T];\mathbb{H}^q(\Omega))$ which satisfies that

$$y_\alpha(t) = \mathbf{P}\Big(\frac{t^\alpha}{\alpha}\Big)(f) + \mathbf{Q}\Big(\frac{t^\alpha}{\alpha}\Big)(g) + \int_0^t \nu^{\alpha-1} \mathbf{Q}\Big(\frac{t^\alpha - \nu^\alpha}{\alpha}\Big) F(y_\alpha(\nu)) d\nu. \tag{80}$$

Using (79), we obtain that

$$\begin{aligned}\left\|y_\alpha\right\|_{\mathbf{Z}_{d,\theta_0,\alpha}((0,T];\mathbb{H}^q(\Omega))} &= \left\|\mathbf{M}y_\alpha\right\|_{\mathbf{Z}_{d,\theta_0,\alpha}((0,T];\mathbb{H}^q(\Omega))}\\ &\le \frac{1}{2}\left\|y_\alpha\right\|_{\mathbf{Z}_{d,\theta_0,\alpha}((0,T];\mathbb{H}^q(\Omega))} + C(\alpha)\left(\left\|f\right\|_{\mathbb{H}^q(\Omega)} + \left\|g\right\|_{\mathbb{H}^q(\Omega)}\right) \end{aligned} \tag{81}$$

This implies that

$$\left\|y_\alpha\right\|_{\mathbf{Z}_{d,\theta_0,\alpha}((0,T];\mathbb{H}^q(\Omega))} \le 2C(\alpha)\left(\left\|f\right\|_{\mathbb{H}^q(\Omega)} + \left\|g\right\|_{\mathbb{H}^q(\Omega)}\right). \tag{82}$$

Hence, for all $t \in (0,T]$, we have:

$$\left\|y_\alpha(t)\right\|_{\mathbb{H}^q(\Omega)} \le 2C(\alpha)\left(\left\|f\right\|_{\mathbb{H}^q(\Omega)} + \left\|g\right\|_{\mathbb{H}^q(\Omega)}\right)t^{-d}e^{\theta_0 T^\alpha}. \tag{83}$$

Since $1 \le p < \frac{1}{d}$, we know that $y_\alpha \in L^p(0,T;\mathbb{H}^q(\Omega))$ and

$$\left\|y_\alpha\right\|_{L^p(0,T;\mathbb{H}^q(\Omega))} \lesssim \left(\left\|f\right\|_{\mathbb{H}^q(\Omega)} + \left\|g\right\|_{\mathbb{H}^q(\Omega)}\right). \tag{84}$$

From there, we complete the proof of Theorem (5.1). □

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