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New generalized $\phi-$Atangana Baleanu Caputo fractional derivative on fuzzy Darboux problem

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Abstract. In this study, we initiate a new concept of generalized $\phi-$Caputo Atangana Baleanu fractional derivative by combining the $\phi-$Caputo and Atangana Baleanu fractional derivatives with respect to the generalized Mittag-Leffler kernel which preserves information and solves a variant of Darboux's Problem for fuzzy implicit fractional differential equations. Using fixed point approach, we prove various existence and uniqueness results. The practical significance of our findings is further proven by an application. The presented results enrich, extend and generalize several prior findings in the literature.

1. Introduction

Fractional derivatives are mathematical concepts employed in mathematical analysis to describe non-integer powers of derivation and integration operators. Fractional derivatives are applied in a number of areas of physics, including electromagnetism, acoustics, mechanical modeling of rubber and materials with viscoelastic properties that retain the memory of past deformations (see e.g. [9]).

The concept of fuzziness [17] is incorporated into the classical fractional calculus to develop fuzzy fractional calculus. Particularly, an extension of the Caputo derivative used in fuzzy fractional calculus is the Atangana [6]. The Atangana-Baleanu-Caputo derivative is distinguished by its non-local and non-singular kernel that is connected to various applications [2, 14]. Comparatively, the Caputo derivative, a local fractional derivative that is often employed in fractional calculus [13]. The generalized Casson fluid model including heat generation and chemical reaction is one example of how it is used to describe numerous phenomena. Recent studies investigated the FFDEs using the generalized Atangana-Baleanu fractional derivative [1, 3, 15, 16].

In this paper, we initiate a new concept of generalized $\phi-$Caputo Atangana Baleanu fractional derivative by combining the $\phi-$Caputo and Atangana Baleanu fractional derivatives with respect to the generalized Mittag-Leffler kernel which preserves information and solves a variant of Darboux's Problem for fuzzy implicit fractional differential equations. Using fixed point approach, we prove various existence and

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uniqueness results. First, we study the existence and uniqueness of solutions to fuzzy hyperbolic partial differential equations with non-local conditions using generalized $\phi - ABC$ by applying the Banach contraction principle to the following fractional differential equations with initial conditions:

$$^{ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi}[\zeta(u,v) - g(u,v,\zeta(u,v))] = f\left(u,v,\zeta(u,v), {}^{ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi}\zeta(u,v)\right); \tag{1}$$

$$(u,v) \in J = [0,a] \times [0,b]$$

$$\begin{cases} \zeta(u,0) = \varphi(u); u \in [0,a], \\ \zeta(0,v) = \psi(v); v \in [0,b] \\ \varphi(0) = \psi(0) \end{cases} \tag{2}$$

where $a, b > 0$, $^{ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi}$ is the generalized $\phi-$ABC fractional derivative of order $\vartheta = (\vartheta_1, \vartheta_2) \in (0,1] \times (0,1]$, $f : J \times \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ and $g : J \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ are a given continuous function, $\varphi : [0,a] \to \mathbb{R}_{\mathcal{F}}, \psi : [0,b] \to \mathbb{R}_{\mathcal{F}}$ are given absolutely continuous functions with $\varphi(u) = \phi(u,0), \psi(v) = \phi(0,v)$ for each $u \in [0,a], v \in [0,b]$ and $\zeta(0,0) = \psi(0) = \varphi(0)$.

This paper is structured as follows. In section 2, we present some basic definitions of fractional integrals and derivatives. The Basic model formulation of the generalized $\phi-$ABC fractional derivative and application of fractional calculus is demonstrated in Section 3. The existence the uniqueness and uniqueness of the problem solution are established in Section 4. Finally, section 5 specifically provides some illustrative remarks and examples.

2. Preliminaries

Here we briefly discuss some basic concepts and main properties of fuzzy sets and functions as well as fuzzy calculus. we refer to $\mathbb{R}_{\mathcal{F}}$ the set of all fuzzy numbers on $\mathbb{R}$.

Definition 2.1. *[10] we define the fuzzy set Υ in $\mathbb{R}_{\mathcal{F}}$ by a function*

$$\Upsilon : \mathbb{R}_{\mathcal{F}} \to [0,1],$$

where $\Upsilon(\tau)$ is the membership degree of τ to Υ. Υ is also supported by

$$\text{supp}(\Upsilon) = \{\tau \in \mathbb{R} \mid \Upsilon(\tau) > 0\},$$

Definition 2.2. *[11] Let us consider $\Upsilon : \mathbb{R} \to [0,1]$, Υ is called a fuzzy number if it satisfies the following properties*

1. *Υ is normal,*
2. *Υ is upper semi-continuous,*
3. *Υ is fuzzy convex,*
4. *$\overline{\{t \in \mathbb{R} \mid \Upsilon(\tau) > 0\}}$ is compact.*

A fuzzy number can be represented as an r-level as follows

$$\Upsilon^r = \{t \in \mathbb{R}, \Upsilon(t) \geq r\}.$$

We can write in interval form as follows

$$[\Upsilon]^r = \left[\underline{\Upsilon}(r), \overline{\Upsilon}(r)\right].$$

which satisfies for $r \in [0,1]$ the following conditions:

- $\underline{\Upsilon}(r)$ is a increasing left continuous function.

- $\overline{\Upsilon}(r)$ is a decreasing right continuous function.
- $\underline{\Upsilon}(r) \leq \overline{\Upsilon}(r)$.

Definition 2.3. *[5, 7] Let* $\gamma \in \mathbb{R}, \zeta = [\underline{\zeta}, \overline{\zeta}]$ *and* $\kappa = [\underline{\kappa}, \overline{\kappa}]$. *Then*

1. $\zeta \oplus \kappa = [\underline{\zeta} + \underline{\kappa}, \overline{\zeta} + \overline{\kappa}]$.
2. $\gamma \odot \zeta = \begin{cases} [\gamma\underline{\zeta}, \gamma\overline{\zeta}], \gamma \geq 0, \\ [\gamma\overline{\zeta}, \gamma\underline{\zeta}], \gamma < 0. \end{cases}$
3. *We noted* $\ominus$ *the H-difference,*
$$\omega \ominus \zeta \text{ makes sense if it exists } \varpi \in \mathbb{R}_{\mathcal{F}} \text{ such that } \omega = \zeta \oplus \varpi \text{ for all } \omega, \zeta \in \mathbb{R}_{\mathcal{F}}.$$
4. *The Hausdorff distance is defined by*
$$d_H : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}^+$$
$$d_H(\zeta, \kappa) = \sup_{r\in[0,1]} \max\{|\underline{\zeta} - \underline{\kappa}|, |\overline{\zeta} - \overline{\kappa}|\}.$$
5. *The uniform distance is given by*
$$d(\zeta, \kappa) = \sup_{\tau\in[0,1]} d_H(\zeta(\tau), \kappa(\tau)).$$

Definition 2.4. *[8] The generalized Hukuhara difference* (gH) *of two fuzzy numbers is defined as follows*

$$\omega \ominus_{gH} \varpi = \zeta \Longleftrightarrow \begin{cases} (i) \quad \omega = \varpi + \zeta \text{ or} \\ (ii) \quad \varpi = \omega + (-1)\zeta. \end{cases}$$

For all, $\omega, \varpi \in \mathbb{R}_{\mathcal{F}}$.

Definition 2.5. *[8] Let* $\ell \in (0, 1)$ *and* h *be such that* $\ell + h \in (0, 1)$, *then the generalized Hukuhara derivative of a fuzzy valued function* $\zeta : (0, 1) \to \mathbb{R}_{\mathcal{F}}$ *at* ℓ *is defined as*

$$\zeta'(\ell) = \lim_{h\to 0} \frac{\zeta(\ell + h) \ominus_{gH} \zeta(\ell)}{h}.$$

Let $\zeta = [\underline{\zeta}, \overline{\zeta}]$.

- If ζ is Caputo $(i) - gH$ differentiable then, $\zeta' = \left[\underline{\zeta}', \overline{\zeta}'\right]$.
- If ζ is Caputo $(ii) - gH$ differentiable then, $\zeta' = \left[\overline{\zeta}', \underline{\zeta}'\right]$.

3. Generalized Fuzzy $\phi-$ABC Fractional Derivative

To introduce the concept of generalized fuzzy fractional derivatives $\phi-$ABC in a fuzzy environment, we will provide basic information about fuzzy fractional derivatives. In the remainder of this section, we introduce the basic theory and identify some of the necessary results used in this paper.

Definition 3.1. *Let $\tau \in [0,a]$, and $\zeta \in L_{\mathbb{R}_{\mathcal{F}}}[0,a]$ is a fuzzy-valued function. Then The fuzzy RL-integral of order ϑ is defined as*

$$^{RL}{I_\tau}^{\vartheta}\zeta(\tau) = \frac{1}{\Gamma(\vartheta)}\int_0^\tau (\tau - s)^{\vartheta-1}\zeta(s)ds, \tag{3}$$

where, $\Gamma(\cdot)$ is the Gamma function defined by

$$\Gamma(\vartheta) = \int_0^\infty \tau^{\vartheta-1}e^{-\tau}d\tau.$$

Definition 3.2. *Let $\zeta \in L_{\mathbb{R}_{\mathcal{F}}}[0,a]$ is a fuzzy-valued function, we define the fuzzy partial fractional integral of order $\vartheta = (\vartheta_1, \vartheta_2)$ by*

$$^{RL}{I_{uv}}^{\vartheta}\zeta(u,v) = \frac{1}{\Gamma(\vartheta_1)\Gamma(\vartheta_2)}\int_0^u\int_0^v (u-s)^{\vartheta_1-1}(v-t)^{\vartheta_2-1}\zeta(s,t)dtds, \tag{4}$$

Assume that $\zeta(\tau,r) = [\underline{\zeta}(\tau,r), \overline{\zeta}(\tau,r)]$ with $r \in [0,1]$.

- If $\zeta(\tau,r)$ is Caputo $(i)-gH$ differentiable then,

$$^{RL}I^{\vartheta}_{uv}\zeta(\tau,r) = \left[^{RL}I^{\vartheta}_{\tau}\underline{\zeta}(\tau,r), {}^{RL}I^{\vartheta}_{\tau}\overline{\zeta}(\tau,r)\right].$$

- If $\zeta(\tau,r)$ is Caputo $(ii)-gH$ differentiable then,

$$^{RL}I^{\vartheta}_{uv}\zeta(\tau,r) = \left[^{RL}I^{\vartheta}_{\tau}\overline{\zeta}(\tau,r), {}^{RL}I^{\vartheta}_{\tau}\underline{\zeta}(\tau,r)\right].$$

Definition 3.3. *Let us consider $\phi \in C([0,a];\mathbb{R}_+)$ and $\zeta \in L_{\mathbb{R}_{\mathcal{F}}}[0,a]$. The fuzzy $\phi-$Riemann Liouville fractional integral of level ϑ of the function ζ is written asy*

$$^{RL}{I_\tau}^{\vartheta;\phi}\zeta(\tau) = \frac{1}{\Gamma(\vartheta)}\int_0^\tau \phi'(s)(\phi(\tau)-\phi(s))^{\vartheta-1}\zeta(s)ds. \tag{5}$$

Definition 3.4. *Let us consider $\phi \in C([0,a];\mathbb{R}_+)$ and $\zeta \in L_{\mathbb{R}_{\mathcal{F}}}[0,a]$. The mixed $\phi-$Riemann-Liouville integral of order $\vartheta = (\vartheta_1,\vartheta_2)$ of u is denied by*

$$\begin{aligned} ^{RL}{I_{uv}}^{\vartheta;\phi}\zeta(u,v) &= \frac{1}{\Gamma(\vartheta_1)\Gamma(\vartheta_2)}\int_0^u\int_0^v \phi'(s)\phi'(t) \\ &\times(\phi(u)-\phi(s))^{\vartheta_1-1}(\phi(v)-\phi(t))^{\vartheta_2-1}\zeta(s,t)dtds. \end{aligned} \tag{6}$$

In what follows, we give the AB definition in Caputo's sense and the AB integration. Let $AB(\vartheta)$ be a normalization function such that

$$AB(0) = AB(1) = 1 \text{ and } AB(\vartheta) = 1 - \vartheta + \frac{\vartheta}{\Gamma(\vartheta)}.$$

Definition 3.5. *The fuzzy derivative ABC and the integral AB of a function $\zeta \in L_{\mathbb{R}_{\mathcal{F}}}[0,a]$ of order ϑ are given respectively as follows*

$$^{ABC}D^{\vartheta}_{\tau}\zeta(\tau) = \frac{AB(\vartheta)}{1-\vartheta}\int_0^\tau E_\vartheta\left(-\frac{\vartheta}{1-\vartheta}(\tau-s)^\vartheta\right)\zeta'(s)ds, \tag{7}$$

and

$$^{AB}I^{\vartheta}_{\tau}\zeta(\tau) = \alpha_\vartheta\zeta(\tau) + \beta_\vartheta\Gamma(\vartheta+1)^{RL}I^{\vartheta}_{\tau}\zeta(\tau), \tag{8}$$

where $E_\vartheta(\tau) = \sum_{i=0}^{\infty}\frac{\tau^i}{\Gamma(\vartheta i+1)}$ is the Mittag-Leffler function and

$$\alpha_\vartheta = \frac{1-\vartheta}{AB(\vartheta)} \text{ and } \beta_\vartheta = \frac{1}{AB(\vartheta)\Gamma(\vartheta)}.$$

The following definition is a generalisation of the previous definition.

Definition 3.6. *Let $\vartheta \in (0,1)$ and $\phi \in C([0,a],\mathbb{R}_+)$. The generalized $\phi - ABC$ derivative of a fuzzy number valued function $\zeta(t)$ on interval is defined in the the following form,*

$$^{ABC}D_{\tau}^{\vartheta,\mu,\gamma;\phi}\zeta(\tau) = \frac{AB(\vartheta)}{1-\vartheta} \odot \int_0^{\tau} \zeta'_{gH}(s) \odot E_{\vartheta,\mu}^{\gamma}\left(\frac{-\vartheta}{1-\vartheta}\left(\phi(\tau)-\phi(s)\right)^{\vartheta}\right)ds, \tag{9}$$

where

$$E_{\vartheta,\mu}^{\gamma}(t) = \sum_{i=0}^{\infty} \frac{(\gamma)_i t^i}{i!\Gamma(i\vartheta+\mu)}, \quad \vartheta > 0, \quad \mu > 0, \tag{10}$$

with $(\gamma)_i$ is the Pochhammer symbol standing for

$$(\gamma)_i = \gamma(\gamma+1)(\gamma+2)\ldots(\gamma+i-1), \quad (\gamma)_0 = 1, \quad \gamma \neq 0.$$

Remark 3.7. *Here γ can also take on negative integers. We can also write $(\gamma)_i$ in the form of a gamma function, as follows*

$$(\gamma)_i = \frac{\Gamma(\gamma+i)}{\Gamma(\gamma)}, \quad \gamma > 0.$$

Definition 3.8. *Let us consider $\vartheta \in (0,1)$, $\phi \in C([0,a;\mathbb{R}_+])$ and $\zeta \in L_{\mathbb{R}_{\mathcal{F}}}[0,a]$. The generalized fractional integral operator $\phi -$ AB is defined as*

$$^{AB}\boldsymbol{I}_{\tau}^{\vartheta,\mu,\gamma;\phi}\zeta(\tau) = \sum_{k=0}^{\gamma} \begin{pmatrix} \gamma \\ k \end{pmatrix} \frac{\vartheta^k}{AB(\vartheta)(1-\vartheta)^{k-1}} {}^{RL}\boldsymbol{I}_{\tau}^{k\vartheta;\phi}\zeta(\tau). \tag{11}$$

Theorem 3.9. *The generalized $\phi - ABC$ fractional derivative of given 9 can be expressed as:*

$$^{ABC}D_{\tau}^{\vartheta,\mu,\gamma;\phi}\zeta(\tau) = \frac{AB(\vartheta)}{1-\vartheta} \sum_{k=0}^{\infty} \frac{(\gamma)_k (k\vartheta)}{(\vartheta k)_{\mu} k!} \left(-\frac{\vartheta}{1-\vartheta}\right)^k I_{\tau}^{k\vartheta+1;\phi} \frac{\zeta'(\tau)}{\phi'(\tau)}. \tag{12}$$

Proof. Based on the Mittag-Leffler concept $E_{\vartheta,\mu}^{\gamma}(\tau)$ given by (10) and the formula of $\phi-$Riemann Liouville, we get

$$\begin{aligned}
^{ABC}D_{\tau}^{\vartheta,\mu,\gamma;\phi}\zeta(\tau) &= \frac{AB(\vartheta)}{1-\vartheta} \odot \int_0^{\tau} \zeta'_{gH}(s) \odot E_{\vartheta,\mu}^{\gamma}\left(\frac{-\vartheta}{1-\vartheta}\left(\phi(\tau)-\phi(s)\right)^{\vartheta}\right)ds, \\
&= \frac{AB(\vartheta)}{1-\vartheta} \odot \int_0^{\tau} \zeta'_{gH}(s) \odot \sum_{k=0}^{\infty} \frac{(\gamma)_k}{k!\Gamma(k\vartheta+\mu)} \left(\frac{-\vartheta}{1-\vartheta}\right)^k \left(\phi(\tau)-\phi(s)\right)^{k\vartheta} ds, \\
&= \frac{AB(\vartheta)}{1-\vartheta} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{k!\Gamma(k\vartheta+\mu)} \left(\frac{-\vartheta}{1-\vartheta}\right)^k \odot \int_0^{\tau} \frac{\zeta'(s)}{\phi'(s)} \phi'(s) \odot \left(\phi(\tau)-\phi(s)\right)^{k\vartheta} ds, \\
&= \frac{AB(\vartheta)}{1-\vartheta} \sum_{k=0}^{\infty} \frac{(\gamma)_k (k\vartheta)}{(k\vartheta)_{\mu} k!} \left(\frac{-\vartheta}{1-\vartheta}\right)^k \frac{1}{\Gamma(k\vartheta+1)} \odot \int_0^{\tau} \frac{\zeta'(s)}{\phi'(s)} \phi'(s) \odot \left(\phi(\tau)-\phi(s)\right)^{k\vartheta} ds, \\
&= \frac{AB(\vartheta)}{1-\vartheta} \sum_{k=0}^{\infty} \frac{(\gamma)_k (k\vartheta)}{(\vartheta k)_{\mu} k!} \left(-\frac{\vartheta}{1-\vartheta}\right)^k I_{\tau}^{k\vartheta+1;\phi} \frac{\zeta'(\tau)}{\phi'(\tau)}.
\end{aligned}$$

□

Theorem 3.10. *Let us consider* $\zeta \in L_{\mathbb{R}_{\mathcal{F}}}[0,a]$, $\vartheta \in (0,1)$ *and* $\phi \in C([0,a;\mathbb{R}_+])$ *then, the following assertion is satisfied:*

$$ {}^{ABC}I_{\tau}^{\vartheta,\mu,\gamma;\phi}\left({}^{ABC}D_{\tau}^{\vartheta,\mu,\gamma;\phi}\right)\zeta(t)) = \zeta(\tau) \ominus_{gh} \zeta(0). \tag{13}$$

Proof. Based on the concepts of generalized $\phi - AB$ fractional integrals (11) and generalized $\phi - ABC$ fractional derivatives (12), we have

$$\begin{aligned}
&{}^{ABC}I_{\tau}^{\vartheta,\mu,\gamma;\phi ABC}D_{\tau}^{\vartheta,\mu,\gamma;\phi}\zeta(\tau)),\\
&= \sum_{k=0}^{\gamma}\begin{pmatrix}\gamma\\k\end{pmatrix}\frac{\vartheta^k}{AB(\vartheta)(1-\vartheta)^{k-1}}{}^{RL}\boldsymbol{I}^{k\vartheta;\phi ABC}D_{\tau}^{\vartheta,\mu,\gamma;\phi}\zeta(\tau),\\
&= \frac{1-\vartheta}{AB(\vartheta)}{}^{ABC}D_{\tau}^{\vartheta,\mu,\gamma;\phi}\zeta(\tau) + \sum_{k=1}^{\gamma}\begin{pmatrix}\gamma\\k\end{pmatrix}\frac{\vartheta^k}{AB(\vartheta)(1-\vartheta)^{k-1}}{}^{RL}\boldsymbol{I}^{k\vartheta;\phi ABC}D_{\tau}^{\vartheta,\mu,\gamma;\phi}\zeta(\tau),\\
&= \frac{1-\vartheta}{AB(\vartheta)}\frac{AB(\vartheta)}{1-\vartheta}\sum_{j=0}^{\infty}\frac{(\gamma)_j\,(j\vartheta)}{(j\vartheta)_\mu j!}\left(-\frac{\vartheta}{1-\vartheta}\right)^j I_{\tau}^{j\vartheta+1;\phi}\frac{\zeta'(\tau)}{\phi'(\tau)}\\
&\quad+\sum_{k=1}^{\gamma}\begin{pmatrix}\gamma\\k\end{pmatrix}\frac{\vartheta^k}{AB(\vartheta)(1-\vartheta)^{k-1}}{}^{RL}\boldsymbol{I}^{k\vartheta;\phi}\frac{AB(\vartheta)}{1-\vartheta}\sum_{j=0}^{\infty}\frac{(\gamma)_j\,(j\vartheta)}{(j\vartheta)_\mu j!}\left(-\frac{\vartheta}{1-\vartheta}\right)^j I_{\tau}^{j\vartheta+1;\phi}\frac{\zeta'(\tau)}{\phi'(\tau)},\\
&= \sum_{j=0}^{\infty}\frac{(\gamma)_j\,(j\vartheta)}{(j\vartheta)_\mu j!}\left(-\frac{\vartheta}{1-\vartheta}\right)^j I_{\tau}^{j\vartheta+1;\phi}\frac{\zeta'(\tau)}{\phi'(\tau)}\\
&\quad+\sum_{k=1}^{\gamma}\begin{pmatrix}\gamma\\k\end{pmatrix}\left(\frac{\vartheta}{1-\vartheta}\right)^k{}^{RL}\boldsymbol{I}^{k\vartheta;\phi}\sum_{j=0}^{\infty}\frac{(\gamma)_j\,(j\vartheta)}{(j\vartheta)_\mu j!}\left(-\frac{\vartheta}{1-\vartheta}\right)^j I_{\tau}^{j\vartheta+1;\phi}\frac{\zeta'(\tau)}{\phi'(\tau)},\\
&= \sum_{j=0}^{\infty}\frac{(\gamma)_j\,(j\vartheta)}{(j\vartheta)_\mu j!}\left(-\frac{\vartheta}{1-\vartheta}\right)^j I_{\tau}^{j\vartheta+1;\phi}\frac{\zeta'(\tau)}{\phi'(\tau)}\\
&\quad+\sum_{k=1}^{\gamma}\begin{pmatrix}\gamma\\k\end{pmatrix}\left(\frac{\vartheta}{1-\vartheta}\right)^k{}^{RL}\boldsymbol{I}^{k\vartheta;\phi}\sum_{j=0}^{\infty}\frac{(\gamma)_j\,(j\vartheta)}{(j\vartheta)_\mu j!}\left(-\frac{\vartheta}{1-\vartheta}\right)^j I_{\tau}^{j\vartheta+1;\phi}\frac{\zeta'(\tau)}{\phi'(\tau)},\\
&= \sum_{j=0}^{\infty}\frac{(\gamma)_j\,(j\vartheta)}{(j\vartheta)_\mu j!}\left(-\frac{\vartheta}{1-\vartheta}\right)^j I_{\tau}^{j\vartheta+1;\phi}\frac{\zeta'(\tau)}{\phi'(\tau)}\\
&\quad+\sum_{k=1}^{\gamma}\begin{pmatrix}\gamma\\k\end{pmatrix}(-1)^k\sum_{j=0}^{\infty}\frac{(\gamma)_j\,(j\vartheta)}{(j\vartheta)_\mu j!}\left(-\frac{\vartheta}{1-\vartheta}\right)^{j+k} I_{\tau}^{(j+k)\vartheta+1;\phi}\frac{\zeta'(\tau)}{\phi'(\tau)},\\
&= {}^{RL}I^{1;\phi}\frac{\zeta'(t)}{\phi'(t)}+\sum_{j=1}^{\infty}\frac{(\gamma)_j\,(j\vartheta)}{(j\vartheta)_\mu j!}\left(-\frac{\vartheta}{1-\vartheta}\right)^j I_{\tau}^{j\vartheta+1;\phi}\frac{\zeta'(\tau)}{\phi'(\tau)}\\
&\quad+\sum_{k=1}^{\gamma}\begin{pmatrix}\gamma\\k\end{pmatrix}(-1)^k\sum_{j=0}^{\infty}\frac{(\gamma)_j\,(j\vartheta)}{(j\vartheta)_\mu j!}\left(-\frac{\vartheta}{1-\vartheta}\right)^{j+k} I_{\tau}^{(j+k)\vartheta+1;\phi}\frac{\zeta'(\tau)}{\phi'(\tau)},\\
&= \zeta(\tau)\ominus_{gh}\zeta(0).
\end{aligned}$$

□

In the following theorem, we review Krasnosel'skii's fixed point theorem, which is used to verify the existence and uniqueness of solutions to the problem (1)-(2).

Theorem 3.11. *(Krasnosel'skii fixed point theorem for fuzzy metric spaces) Allow $\mathcal{M}$ to be a non empty, closed and convex subset of $C(I, \mathcal{E}^c)$ and assume that $\mathcal{Q}$ and $\mathcal{H}$ map $\mathcal{M}$ into S and*

i) $\mathcal{Q}$ is continuous and compact,

ii) $\mathcal{Q}t + \mathcal{H}s \in \mathcal{M}$, for every $t, s \in \mathcal{M}$,

iii) $\mathcal{H}$ is a contraction mapping.

Then, there exists a fixed point for $\mathcal{Q} + \mathcal{H}$ in $\mathcal{M}$, that is, there is $s \in \mathcal{M}$ for which $\mathcal{Q}s + \mathcal{H}s = s$.

4. Fuzzy differential equations involving $\phi-$ABC fractional derivative

In this section, we consider non-local conditional fuzzy hyperbolic partial differential equations involving fractional derivatives $\phi-$ABC as follows

$$^{ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi}[\zeta(u,v) - g(u,v,\zeta(u,v))] = f\left(u,v,\zeta(u,v), {}^{ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi}\zeta(u,v)\right); \tag{14}$$

$$(u,v) \in J = [0,a] \times [0,b]$$

$$\begin{cases} \zeta(u,0) = \varphi(u); u \in [0,a], \\ \zeta(0,v) = \psi(v); v \in [0,b] \\ \varphi(0) = \psi(0) \end{cases} \tag{15}$$

where $a, b > 0$, $^{ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi}$ is the generalized $\phi-$ABC fractional derivative of order $\vartheta = (\vartheta_1, \vartheta_2) \in (0,1] \times (0,1]$, $f : J \times \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ and $g : J \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ are a given continuous function, $\varphi : [0,a] \to \mathbb{R}_{\mathcal{F}}, \psi : [0,b] \to \mathbb{R}_{\mathcal{F}}$ are given absolutely continuous functions with $\varphi(u) = \phi(u,0), \psi(v) = \phi(0,v)$ for each $u \in [0,a], v \in [0,b]$ and $\zeta(0,0) = \psi(0) = \varphi(0)$.

For the existence of solutions for the problem (14)-(15) we need the following lemma.

Lemma 4.1. *Let the function $f : J \times \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ and $g : J \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ be continuous. Then problem (14)-(15) is equivalent to the problem of the solution of the equation*

$$\omega(u,v) = {}^{ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi} g\left(u,v,\eta(u,v) + {}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi}\omega(u,v)\right) + f\left(u,v,\eta(u,v) + {}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi}\omega(u,v), \omega(u,v)\right),$$

and if $\omega \in C(J)$ is the solution of this equation, then

$$\zeta(u,v) = \eta(u,v) + {}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi}\omega(u,v), \tag{16}$$

where

$$\eta(u,v) = \varphi(u) + \psi(v) - \varphi(0).$$

Proof. let's set

$$^{ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi}\zeta(u,v) = \omega(u,v) \tag{17}$$

Using the generalized $\phi-$ABC fractional derivative definition, applying the operator $^{ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi}$ to both sides of (17) and the relation (13) of theorem 3.10, we get

$$\zeta(u,v) - \zeta(u,0) - \zeta(0,v) + \zeta(0,0) = {}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi}\omega(u,v),$$
$$\zeta(u,v) = \eta(u,v) + {}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi}\omega(u,v),$$

with

$$\eta(u,v) = \zeta(u,0) + \zeta(0,v) - \zeta(0,0) = \varphi(u) + \psi(v) - \varphi(0).$$

So, by taking the value of $\zeta(u,v)$ in (14), we find

$$\omega(u,v) = {}^{ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi} g\left(u,v,\eta(u,v) + {}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi}\omega(u,v)\right) + f\left(u,v,\eta(u,v) + {}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi}\omega(u,v),\omega(u,v)\right).$$

This means that the solution to problem (14)-(15) is equivalent to problem (16), provided that $\omega(u,v)$ is the solution to equation (4.1). □

Further, we present conditions for the existence and uniqueness of a solution of problem (14)-(15).

Theorem 4.2. *Assume that the following hypotheses hold:*

(H_1) $f : J \times \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ *and* $g : J \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ *are a continuous function;*

(H_2) *For any* $\zeta,\xi,\sigma,\omega \in C(J;\mathbb{R}_{\mathcal{F}})$ *verifying* $\zeta(u,0) = \xi(u,0) = \varphi(u)$ *and* $\zeta(0,v) = \xi(0,v) = \psi(v)$ *and there exists* $\lambda > 0$, $0 < \delta;\rho < 1$ *such that*

$$d(f(u,v,\zeta,\sigma); f(u,v,\xi,\omega)) \le \lambda d(\zeta;\xi) + \delta d(\sigma;\omega)$$

and

$$d(g(u,v,\zeta); g(u,v,\xi)) \le \rho d(\zeta;\xi).$$

let's set

$$\Lambda = \delta\rho \frac{AB(\vartheta_1)}{1-\vartheta_1}\frac{AB(\vartheta_2)}{1-\vartheta_2}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{k!\Gamma(k\vartheta_1+\mu)}\left(-\frac{\vartheta_1}{1-\vartheta_1}\right)^k \times \sum_{j=0}^{\infty}\frac{(\gamma)_j}{j!\Gamma(j\vartheta_2+\mu)}\left(-\frac{\vartheta_2}{1-\vartheta_2}\right)^j$$

and

$$\Theta = \sum_{k=0}^{\gamma}\begin{pmatrix}\gamma\\ k\end{pmatrix}\frac{\vartheta_1^k(\phi(a)-\phi(0))^{k\vartheta_1}}{AB(\vartheta_1)(1-\vartheta_1)^{k-1}\Gamma(k\vartheta_1+1)} \times \sum_{j=0}^{\gamma}\begin{pmatrix}\gamma\\ j\end{pmatrix}\frac{\vartheta_2^j(\phi(b)-\phi(0))^{j\vartheta_2}}{AB(\vartheta_2)(1-\vartheta_2)^{j-1}\Gamma(j\vartheta_2+1)}.$$

If

$$\Theta \le \frac{(1-\rho)(1-\delta)}{\lambda+\Lambda}, \tag{18}$$

then there exists a unique solution for IVP (14)-(15) on J.

Proof. Transform the problem (14)-(15) into a fixed point problem.

Consider the operator $\mathcal{N} : C(J) \longrightarrow C(J)$ defined by,

$$\mathcal{N}(\zeta)(u,v) = \eta(u,v) + {}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi}\omega(u,v),$$

where $\omega \in C(J)$ such that

$$\omega(u,v) = {}^{ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi} g(u,v,\zeta(u,v)) + f(u,v,\zeta(u,v),\omega(u,v)).$$

By Lemma 4.1, the problem of finding the solutions of the IVP (14)-(15) is reduced to finding the solutions of the operator equation $\mathcal{N}(\zeta) = \zeta$.
We proceed as follows

Case Caputo $(i)-gH$ differentiable: The problem (14)-(15) is equivalent to the following fractional differential system

$$\begin{cases} {}^{ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi}\left[\underline{\zeta}^r(u,v)-g_r\left(u,v,\underline{\zeta}^r(u,v),\overline{\zeta}^r(u,v)\right)\right]=f_r\left(u,v,\underline{\zeta}^r(u,v),\overline{\zeta}^r(u,v)\right);(u,v)\in J \\ \underline{\zeta}^r(u,0)=\underline{\varphi}^r(u);\underline{\zeta}^r(0,v)=\underline{\psi}^r(v),\quad u\in[0,a],v\in[0,b] \end{cases} \tag{19}$$

and

$$\begin{cases} {}^{ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi}\left[\overline{\zeta}^r(u,v)-h_r\left(u,v,\underline{\zeta}^r(u,v),\overline{\zeta}^r(u,v)\right)\right]=k_r\left(u,v,\underline{\zeta}^r(u,v),\overline{\zeta}^r(u,v)\right);(u,v)\in J \\ \overline{\zeta}^r(u,0)=\overline{\varphi}^r(u);\overline{\zeta}^r(0,v)=\overline{\psi}^r(v),\quad u\in[0,a],v\in[0,b] \end{cases} \tag{20}$$

We define the operator $\mathscr{N}$ by

$$\mathscr{N}(\underline{\zeta}^r)(u,v)=\underline{\eta}^r(u,v)+{}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi}\omega(u,v),$$

with

$$\underline{\eta}^r(u,v)=\underline{\varphi}^r(u)+\underline{\psi}^r(v)-\underline{\varphi}(0).$$

Let $\zeta,\xi\in C(J)$. Then, for $(u,v)\in J$, we have

$$d\left(\mathscr{N}(\underline{\zeta}^r)(u,v);\mathscr{N}(\underline{\xi}^r)(u,v)\right)=d\left({}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi}\omega(u,v);{}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi}\sigma(u,v)\right),$$

where $\omega,\sigma\in C(J)$ such that

$$\omega(u,v)={}^{ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi}g\left(u,v,\zeta(u,v)\right)+f\left(u,v,\zeta(u,v),\omega(u,v)\right)$$

and

$$\sigma(u,v)={}^{ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi}g\left(u,v,\xi(u,v)\right)+f\left(u,v,\xi(u,v),\sigma(u,v)\right).$$

So,

$$\begin{aligned} d\left(\mathscr{N}(\underline{\zeta}^r)(u,v);\mathscr{N}(\underline{\xi}^r)(u,v)\right)\le{}& d\left({}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi\,ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi}g\left(u,v,\zeta(u,v)\right));{}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi\,ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi}g\left(u,v,\xi(u,v)\right)\right)\\ &+d\left({}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi}f\left(u,v,\zeta(u,v),\omega(u,v)\right);{}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi}f\left(u,v,\xi(u,v),\sigma(u,v)\right)\right). \end{aligned}$$

By (H_2), we have

$$\begin{aligned} &d\left({}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi\,ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi}g\left(u,v,\zeta(u,v)\right);{}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi\,ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi}g\left(u,v,\xi(u,v)\right)\right)\\ &\le d\left(g\left(u,v,\zeta(u,v)\right);g\left(u,v,\xi(u,v)\right)\right)+d\left(g\left(u,0,\zeta(u,0)\right);g\left(u,0,\xi(u,0)\right)\right)\\ &\quad+d\left(g\left(0,v,\zeta(0,v)\right);g\left(0,v,\xi(0,v)\right)\right)+d\left(g\left(0,0,\zeta(0,0)\right);g\left(0,0,\xi(0,0)\right)\right),\\ &\le\rho d\left(\zeta(u,v);\xi(u,v)\right)+0+0+0. \end{aligned}$$

Thus,

$$d\left({}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi\,ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi}g\left(u,v,\zeta(u,v)\right);{}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi\,ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi}g\left(u,v,\xi(u,v)\right)\right)\le\rho d\left(\underline{\zeta}^r(u,v);\underline{\xi}^r(u,v)\right) \tag{21}$$

and

$$\begin{aligned} &d\left({}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi}f\left(u,v,\zeta(u,v),\omega(u,v)\right);{}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi}f\left(u,v,\xi(u,v),\sigma(u,v)\right)\right)\\ &\le\sum_{k=0}^{\gamma}\binom{\gamma}{k}\frac{\vartheta_1^k}{AB(\vartheta_1)(1-\vartheta_1)^{k-1}\Gamma(k\vartheta_1)}\sum_{j=0}^{\gamma}\binom{\gamma}{j}\frac{\vartheta_2^j}{AB(\vartheta_2)(1-\vartheta_2)^{j-1}\Gamma(j\vartheta_2)}\\ &\quad\times\int_0^u\int_0^v\phi'(s)\phi'(t)(\phi(u)-\phi(s))^{k\vartheta_1-1}(\phi(v)-\phi(t))^{j\vartheta_2-1}\\ &\quad\times d\left(f\left(u,v,\zeta(s,t),\omega(s,t)\right);f\left(s,t,\xi(s,t),\sigma(s,t)\right)\right)dtds, \end{aligned}$$

with

$$\begin{aligned}
&d\left(f\left(u,v,\zeta(s,t),\omega(s,t)\right);f\left(s,t,\xi(s,t),\sigma(s,t)\right)\right)\\
&\quad\leq \lambda d\left(\zeta(s,t);\xi(s,t)\right)+\delta d\left(\omega(s,t);\sigma(s,t)\right)\\
(1-\delta)&d\left(f\left(u,v,\zeta(s,t),\omega(s,t)\right);f\left(s,t,\xi(s,t),\sigma(s,t)\right)\right),\\
&\quad\leq \lambda d\left(\zeta(s,t);\xi(s,t)\right)+\delta d\left({}^{ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi}g\left(u,v,\xi(u,v)\right);{}^{ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi}g\left(u,v,\zeta(u,v)\right)\right),\\
&\quad\leq \lambda d\left(\zeta(s,t);\xi(s,t)\right)+\delta\frac{AB(\vartheta_1)}{1-\vartheta_1}\frac{AB(\vartheta_2)}{1-\vartheta_2}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{k!\Gamma(k\vartheta_1+\mu)}\left(-\frac{\vartheta_1}{1-\vartheta_1}\right)^k\\
&\qquad\times\sum_{j=0}^{\infty}\frac{(\gamma)_j}{j!\Gamma(j\vartheta_2+\mu)}\left(-\frac{\vartheta_2}{1-\vartheta_2}\right)^j d\left(g\left(u,v,\xi(u,v)\right);g\left(u,v,\zeta(u,v)\right)\right),\\
&\quad\leq \lambda d\left(\zeta(s,t);\xi(s,t)\right)+\delta\rho\frac{AB(\vartheta_1)}{1-\vartheta_1}\frac{AB(\vartheta_2)}{1-\vartheta_2}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{k!\Gamma(k\vartheta_1+\mu)}\left(-\frac{\vartheta_1}{1-\vartheta_1}\right)^k\\
&\qquad\times\sum_{j=0}^{\infty}\frac{(\gamma)_j}{j!\Gamma(j\vartheta_2+\mu)}\left(-\frac{\vartheta_2}{1-\vartheta_2}\right)^j d\left(\xi(u,v)\right);\zeta(u,v)\right).
\end{aligned}$$

Consequently

$$d\left(f\left(u,v,\zeta(s,t),\omega(s,t)\right);f\left(s,t,\xi(s,t),\sigma(s,t)\right)\right)\leq\left(\frac{\lambda+\Lambda}{1-\delta}\right)d\left(\xi(u,v)\right);\zeta(u,v)\right),$$

where

$$\Lambda=\delta\rho\frac{AB(\vartheta_1)}{1-\vartheta_1}\frac{AB(\vartheta_2)}{1-\vartheta_2}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{k!\Gamma(k\vartheta_1+\mu)}\left(-\frac{\vartheta_1}{1-\vartheta_1}\right)^k\times\sum_{j=0}^{\infty}\frac{(\gamma)_j}{j!\Gamma(j\vartheta_2+\mu)}\left(-\frac{\vartheta_2}{1-\vartheta_2}\right)^j.$$

Which implies that

$$\begin{aligned}
&d\left({}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi}f\left(u,v,\zeta(u,v),\omega(u,v)\right);{}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi}f\left(u,v,\xi(u,v),\sigma(u,v)\right)\right)\\
&\leq\sum_{k=0}^{\gamma}\binom{\gamma}{k}\frac{\vartheta_1^k}{AB(\vartheta_1)(1-\vartheta_1)^{k-1}\Gamma(k\vartheta_1)}\sum_{j=0}^{\gamma}\binom{\gamma}{j}\frac{\vartheta_2^j}{AB(\vartheta_2)(1-\vartheta_2)^{j-1}\Gamma(j\vartheta_2)}\\
&\quad\times\left(\frac{\lambda+\Lambda}{1-\delta}\right)\int_0^u\int_0^v\phi'(s)\phi'(t)(\phi(u)-\phi(s))^{k\vartheta_1-1}(\phi(v)-\phi(t))^{j\vartheta_2-1}d\left(\xi(u,v)\right);\zeta(u,v)\right)dtds,\\
&\leq\sum_{k=0}^{\gamma}\binom{\gamma}{k}\frac{\vartheta_1^k(\phi(a)-\phi(0))^{k\vartheta_1}}{AB(\vartheta_1)(1-\vartheta_1)^{k-1}\Gamma(k\vartheta_1+1)}\sum_{j=0}^{\gamma}\binom{\gamma}{j}\frac{\vartheta_2^j(\phi(b)-\phi(0))^{j\vartheta_2}}{AB(\vartheta_2)(1-\vartheta_2)^{j-1}\Gamma(j\vartheta_2+1)}\\
&\quad\times\left(\frac{\lambda+\Lambda}{1-\delta}\right)d\left(\xi(u,v)\right);\zeta(u,v)\right).
\end{aligned}$$

Thus,

$$d\left({}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi}f\left(u,v,\zeta(u,v),\omega(u,v)\right);{}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi}f\left(u,v,\xi(u,v),\sigma(u,v)\right)\right)\leq\Theta\left(\frac{\lambda+\Lambda}{1-\delta}\right)d\left(\xi(u,v)\right);\zeta(u,v)\right),\tag{22}$$

where

$$\Theta=\sum_{k=0}^{\gamma}\binom{\gamma}{k}\frac{\vartheta_1^k(\phi(a)-\phi(0))^{k\vartheta_1}}{AB(\vartheta_1)(1-\vartheta_1)^{k-1}\Gamma(k\vartheta_1+1)}\sum_{j=0}^{\gamma}\binom{\gamma}{j}\frac{\vartheta_2^j(\phi(b)-\phi(0))^{j\vartheta_2}}{AB(\vartheta_2)(1-\vartheta_2)^{j-1}\Gamma(j\vartheta_2+1)}.$$

Finally from (21) and (22), we find

$$d\left(\mathcal{N}(\underline{\zeta}^r)(u,v);\mathcal{N}(\underline{\xi}^r)(u,v)\right)\leq\left[\rho+\Theta\left(\frac{\lambda+\Lambda}{1-\delta}\right)\right]d\left(\underline{\zeta}^r(u,v));\underline{\xi}^r(u,v)\right).$$

By $\mathcal{N}$ is a contraction, and hence $\mathcal{N}$ has a unique fixed point by Banach's contraction principle.

Case Caputo $(ii)-gH$ differentiability: The problem (14)-(15) is equivalent to the following fractional differential system

$$\begin{cases}{}^{ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi}\left[\underline{\zeta}^r(u,v)-g_r\left(u,v,\underline{\zeta}^r(u,v),\overline{\zeta}^r(u,v)\right)\right]=f_r\left(u,v,\underline{\zeta}^r(u,v),\overline{\zeta}^r(u,v)\right);(u,v)\in J\\ \underline{\zeta}^r(u,0)=\underline{\varphi}^r(u);\underline{\zeta}^r(0,v)=\underline{\psi}^r(v),\quad u\in[0,a],v\in[0,b]\end{cases}\tag{23}$$

and

$$\begin{cases}{}^{ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi}\left[\overline{\zeta}^r(u,v)-h_r\left(u,v,\underline{\zeta}^r(u,v),\overline{\zeta}^r(u,v)\right)\right]=k_r\left(u,v,\underline{\zeta}^r(u,v),\overline{\zeta}^r(u,v)\right);(u,v)\in J\\ \overline{\zeta}^r(u,0)=\overline{\varphi}^r(u);\overline{\zeta}^r(0,v)=\overline{\psi}^r(v),\quad u\in[0,a],v\in[0,b].\end{cases}\tag{24}$$

Transform the problem (23) into fixed point problem, so we consider the operator:

$$\mathcal{N}(\underline{\zeta}^r)(u,v)=\underline{\eta}^r(u,v)\ominus{}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi}\omega(u,v),$$

with

$$\underline{\eta}^r(u,v)=\underline{\varphi}^r(u)+\underline{\psi}^r(v)-\underline{\varphi}(0).$$

Similarly, we can show that there exists at least one solution to the problem (14)-(15).

□

The following result is based on a Krasnosel'skii fixed point theorem for fuzzy metric spaces.

Theorem 4.3. *Assume $f:J\times\mathbb{R}_{\mathcal{F}}\times\mathbb{R}_{\mathcal{F}}\to\mathbb{R}_{\mathcal{F}}$ and $g:J\times\mathbb{R}_{\mathcal{F}}\to\mathbb{R}_{\mathcal{F}}$ satisfying assumptions (H_1) and that the following hypotheses hold*

(H_3) *There exist $D;E;F;G\in C(J;\mathbb{R}_+)$ and $\zeta\in C(J;\mathbb{R}_{\mathcal{F}})$ which checks $\zeta(u,0)=\varphi(u)$ and $\zeta(0,v)=\psi(v)$ such that*

$$d(f(u,v,\zeta,\xi);\hat{0})\leq D(u,v)+E(u,v)d(\zeta;\hat{0})+F(u,v)d(\xi;\hat{0})$$

and

$$d(g(u,v,\zeta);\hat{0})\leq G(u,v)d(\zeta;\hat{0}).$$

(H_4) *For all $(u,v)\in J$ the set*

$$\mathcal{M}=\{\zeta\in C(J,\mathbb{R}_{\mathcal{F}}):d(\zeta(u,v),\eta(u,v))\leq R,(u,v)\in J\},$$

is a closed convex subset that is not empty, with

$$R=4\|G\|_\infty d\left(\zeta(u,v);\hat{0}\right)+\frac{\Theta}{1-\|F\|_\infty}\left(\|D\|_\infty+\Lambda_1 d\left(\zeta(u,v);\hat{0}\right)\right),$$

where

$$\Theta=\sum_{k=0}^{\gamma}\binom{\gamma}{k}\frac{\vartheta_1^k(\phi(a)-\phi(0))^{k\vartheta_1}}{AB(\vartheta_1)(1-\vartheta_1)^{k-1}\Gamma(k\vartheta_1+1)}\sum_{j=0}^{\gamma}\binom{\gamma}{j}\frac{\vartheta_2^j(\phi(b)-\phi(0))^{j\vartheta_2}}{AB(\vartheta_2)(1-\vartheta_2)^{j-1}\Gamma(j\vartheta_2+1)},$$

and

$$\Lambda_1=\|E\|_\infty+\|F\|_\infty\|G\|_\infty\frac{AB(\vartheta_1)}{1-\vartheta_1}\frac{AB(\vartheta_2)}{1-\vartheta_2}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{k!\Gamma(k\vartheta_1+\mu)}\left(-\frac{\vartheta_1}{1-\vartheta_1}\right)^k\sum_{j=0}^{\infty}\frac{(\gamma)_j}{j!\Gamma(j\vartheta_2+\mu)}\left(-\frac{\vartheta_2}{1-\vartheta_2}\right)^j.$$

If $\frac{\Theta\Lambda_1}{1-\|F\|_\infty} \leq 1$*, then using Krasnoselskii's fixed point theorem, the problem (14)-(15) has at least one solution.*

Proof.

Case Caputo $(i)-gH$ differentiability: Let us consider two operators $\mathscr{T}_1$ and $\mathscr{T}_2$ defined on $\mathscr{M}$ as follows

$$\mathscr{T}_1\zeta(u,v) = \eta(u,v) + {}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi}\,{}^{ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi} g\left(u,v,\zeta(u,v)\right)$$

and

$$\mathscr{T}_2\zeta(u,v) = {}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi} f\left(u,v,\zeta(u,v),\omega(u,v)\right),$$

where

$$\omega(u,v) = {}^{ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi} g\left(u,v,\zeta(u,v)\right) + f\left(u,v,\zeta(u,v),\omega(u,v)\right).$$

Thus, $\zeta(u,v)$ being fixed point of the operator $\mathscr{T}\zeta(u,v) = \mathscr{T}_1\zeta(u,v) + \mathscr{T}_2\zeta(u,v)$ is a solution of problem (14)-(15).

In the first step, we prove that $\mathscr{T}$ maps $\mathscr{M}$ into $\mathscr{M}$ i.e. for any $\zeta,\xi \in \mathscr{M}$, we have to show that $\mathscr{T}_1\zeta(u,v) + \mathscr{T}_2\xi(u,v) \in \mathscr{M}$:

$$\begin{aligned} d\left(\mathscr{T}_1\zeta(u,v) + \mathscr{T}_2\xi(u,v);\eta(u,v)\right) \leq\; & d\left({}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi}\,{}^{ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi} g\left(u,v,\zeta(u,v)\right);\hat{0}\right) \\ & + d\left({}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi} f\left(u,v,\zeta(u,v),\xi(u,v)\right);\hat{0}\right), \end{aligned}$$

where $\xi \in C(J;\mathbb{R}_{\mathcal{F}})$ such that

$$\xi(u,v) = {}^{ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi} g\left(u,v,\zeta(u,v)\right) + f\left(u,v,\zeta(u,v),\xi(u,v)\right).$$

By (H_3), we get

$$\begin{aligned} d\left({}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi}\;{}^{ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi} g\left(u,v,\zeta(u,v)\right);\hat{0}\right) &\leq d\left(g(u,v,\zeta(u,v));\hat{0}\right) + d\left(g(u,0,\zeta(u,0));\hat{0}\right) \\ &\quad + d\left(g(0,v,\zeta(0,v));\hat{0}\right) + d\left(g(0,0,\zeta(0,0));\hat{0}\right), \\ &\leq \|G\|_\infty\left(d\left(\zeta(u,v);\hat{0}\right) + d\left(\varphi(v);\hat{0}\right) + d\left(\psi(v);\hat{0}\right) + d\left(\varphi(0);\hat{0}\right)\right), \\ &\leq 4\|G\|_\infty d\left(\zeta(u,v);\hat{0}\right). \end{aligned}$$

Thus

$$d\left({}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi}\,{}^{ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi} g\left(u,v,\zeta(u,v)\right);\hat{0}\right) \leq 4\|G\|_\infty d\left(\zeta(u,v);\hat{0}\right). \tag{25}$$

And

$$\begin{aligned} & d\left({}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi} f\left(u,v,\zeta(u,v),\xi(u,v)\right);\hat{0}\right) \\ & \leq \sum_{k=0}^{\gamma}\binom{\gamma}{k}\frac{\vartheta_1^k}{AB(\vartheta_1)(1-\vartheta_1)^{k-1}\Gamma(k\vartheta_1)}\sum_{j=0}^{\gamma}\binom{\gamma}{j}\frac{\vartheta_2^j}{AB(\vartheta_2)(1-\vartheta_2)^{j-1}\Gamma(j\vartheta_2)} \\ & \quad \times\int_0^u\int_0^v \phi'(s)\phi'(t)(\phi(u)-\phi(s))^{k\vartheta_1-1}(\phi(v)-\phi(t))^{j\vartheta_2-1} d\left(f\left(u,v,\zeta(s,t),\xi(s,t)\right);\hat{0}\right)dtds, \end{aligned}$$

where

$$\begin{aligned} d\left(f\left(u,v,\zeta(s,t),\xi(s,t)\right);\hat{0}\right) &\leq D(u,v) + E(u,v)d(\zeta(u,v);\hat{0}) + F(u,v)d(\xi(u,v);\hat{0}), \\ &\leq \|D\|_\infty + \|E\|_\infty d(\zeta(u,v);\hat{0}) \\ &\quad + \|F\|_\infty d\left({}^{ABC}D_{\tau}^{\vartheta,\mu,\gamma;\phi} g\left(u,v,\zeta(u,v)\right);\hat{0}\right) \\ &\quad + \|F\|_\infty d\left(f\left(u,v,\zeta(u,v),\xi(u,v)\right);\hat{0}\right), \end{aligned}$$

Consequently,

$$\begin{aligned}
&(1-\|F\|_\infty) d\left(f\left(u,v,\zeta(s,t),\xi(s,t)\right);\hat{0}\right)\\
&\quad\le \|D\|_\infty + \|E\|_\infty d(\zeta(u,v);\hat{0}) + \|F\|_\infty d\left({}^{ABC}D_\tau^{\vartheta,\mu,\gamma;\phi} g\left(u,v,\zeta(u,v)\right);\hat{0}\right),\\
&\quad\le \|D\|_\infty + \|E\|_\infty d(\zeta(u,v);\hat{0}) + \|F\|_\infty \|G\|_\infty \frac{AB(\vartheta_1)}{1-\vartheta_1}\frac{AB(\vartheta_2)}{1-\vartheta_2}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{k!\Gamma(k\vartheta_1+\mu)}\left(-\frac{\vartheta_1}{1-\vartheta_1}\right)^k\\
&\quad\quad\times\sum_{j=0}^{\infty}\frac{(\gamma)_j}{j!\Gamma(j\vartheta_2+\mu)}\left(-\frac{\vartheta_2}{1-\vartheta_2}\right)^j d\left(\zeta(u,v);\hat{0}\right).
\end{aligned}$$

Thus,

$$d\left(f\left(u,v,\zeta(s,t),\xi(s,t)\right);\hat{0}\right) \le \frac{1}{1-\|F\|_\infty}\left(\|D\|_\infty + \Lambda_1 d\left(\zeta(u,v);\hat{0}\right)\right),$$

with

$$\Lambda_1 = \|E\|_\infty + \|F\|_\infty\|G\|_\infty \frac{AB(\vartheta_1)}{1-\vartheta_1}\frac{AB(\vartheta_2)}{1-\vartheta_2}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{k!\Gamma(k\vartheta_1+\mu)}\left(-\frac{\vartheta_1}{1-\vartheta_1}\right)^k\sum_{j=0}^{\infty}\frac{(\gamma)_j}{j!\Gamma(j\vartheta_2+\mu)}\left(-\frac{\vartheta_2}{1-\vartheta_2}\right)^j.$$

Which implies that

$$\begin{aligned}
&d\left({}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi} f\left(u,v,\zeta(u,v),\xi(u,v)\right);\hat{0}\right)\\
&\le \sum_{k=0}^{\gamma}\binom{\gamma}{k}\frac{\vartheta_1^k(\phi(a)-\phi(0))^{k\vartheta_1}}{AB(\vartheta_1)(1-\vartheta_1)^{k-1}\Gamma(k\vartheta_1+1)}\sum_{j=0}^{\gamma}\binom{\gamma}{j}\frac{\vartheta_2^j(\phi(b)-\phi(0))^{j\vartheta_2}}{AB(\vartheta_2)(1-\vartheta_2)^{j-1}\Gamma(j\vartheta_2+1)}\\
&\quad\times\frac{1}{1-\|F\|_\infty}\left(\|D\|_\infty + \Lambda_1 d\left(\zeta(u,v);\hat{0}\right)\right),\\
&\le \frac{\Theta}{1-\|F\|_\infty}\left(\|D\|_\infty + \Lambda_1 d\left(\zeta(u,v);\hat{0}\right)\right).
\end{aligned}$$

Thus

$$d\left({}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi} f\left(u,v,\zeta(u,v),\xi(u,v)\right);\hat{0}\right) \le \frac{\Theta}{1-\|F\|_\infty}\left(\|D\|_\infty + \Lambda_1 d\left(\zeta(u,v);\hat{0}\right)\right). \tag{26}$$

Therefore from (25) and (26), we get

$$d\left(\mathscr{T}_1\zeta(u,v) + \mathscr{T}_2\xi(u,v);\eta(u,v)\right) \le 4\|G\|_\infty d\left(\zeta(u,v);\hat{0}\right) + \frac{\Theta}{1-\|F\|_\infty}\left(\|D\|_\infty + \Lambda_1 d\left(\zeta(u,v);\hat{0}\right)\right) := R.$$

Hence, $\mathscr{T}\mathscr{M} \subset \mathscr{M}$.

Now to prove that $\mathscr{T}_1$ is continuous, let us consider a sequence ζ_n such that $\zeta_n \longrightarrow \zeta$,

$$\begin{aligned}
d\left(\mathscr{T}_1\zeta_n(u,v);\mathscr{T}_1\zeta(u,v)\right) &\le d\left({}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi\,ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi} g\left(u,v,\zeta_n(u,v)\right);{}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi\,ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi} g\left(u,v,\zeta(u,v)\right)\right),\\
&\le \|G\|_\infty d\left(\zeta_n;\zeta\right).
\end{aligned}$$

Then, for $\zeta_n \longrightarrow \zeta$, $\mathscr{T}_1\zeta_n(u,v) \longrightarrow \mathscr{T}_1\zeta(u,v)$. Hence, $\mathscr{T}_1$ is continuous.

Now, we show that $\mathscr{T}_1(\mathscr{M})$ resides in a relatively compact set. Taking $(u_1; v_1); (u_2; v_2) \in (0; a] \times (0; b]$, $u_1 < u_2; v_1 < v_2$ and $\zeta \in \mathscr{M}$, we have

$$\begin{aligned}
d(\mathscr{T}_1\zeta(u_1,v_1); \mathscr{T}_1\zeta(u_2,v_2)) &\le d(\eta(u_1,v_1); \eta(u_2,v_2)) \\
&\quad + d(g(u_1,v_1,\zeta(u_1,v_1)); g(u_2,v_2,\zeta(u_2,v_2))) \\
&\quad + d(g(u_1,0,\zeta(u_1,0)); g(u_2,0,\zeta(u_2,0))) \\
&\quad + d(g(0,v_1,\zeta(0,v_1)); g(0,v_2,\zeta(0,v_2))) \\
&\quad + d(g(0,0,\zeta(0,0)); g(0,0,\zeta(0,0))), \\
&\le d(\eta(u_1,v_1); \eta(u_2,v_2)) \\
&\quad + \|G\|_\infty d(\zeta(u_1,v_1); \zeta(u_2,v_2)) \\
&\quad + \|G\|_\infty d(\varphi(u_1); \varphi(u_2)) \\
&\quad + \|G\|_\infty d(\psi(v_1); \psi(v_2)) \\
&\quad + \|G\|_\infty d(\varphi(0); \varphi(0)).
\end{aligned}$$

As $u_1 \longrightarrow u_2$, $v1 \longrightarrow v_2$ the right-hand side of the above inequality tends to zero. Hence $\mathscr{T}_1(\mathscr{M})$ resides in a relatively compact.

Now, we show that $\mathscr{T}_2$ is contraction. Letting $\zeta, \xi \in \mathscr{M}$, we have

$$d(\mathscr{T}_2\zeta(u,v); \mathscr{T}_2\xi(u,v)) = d\left({}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi} f(u,v,\zeta(u,v),\omega(u,v)); {}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi} f(u,v,\xi(u,v),\sigma(u,v))\right)$$

where $\omega, \sigma \in C(J)$ such that

$$\omega(u,v) = {}^{ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi} g(u,v,\zeta(u,v)) + f(u,v,\zeta(u,v),\omega(u,v)).$$

and

$$\sigma(u,v) = {}^{ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi} g(u,v,\xi(u,v)) + f(u,v,\xi(u,v),\sigma(u,v)).$$

Thus,

$$d\left({}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi} f(u,v,\zeta(u,v),\omega(u,v)); \ {}^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi} f(u,v,\xi(u,v),\sigma(u,v))\right) \le \frac{\Theta\Lambda_1}{1 - \|F\|_\infty} d(\zeta(u,v); \xi(u,v)).$$

For $\dfrac{\Theta\Lambda_1}{1 - \|F\|_\infty} \le 1$ then $\mathscr{T}_2$ is contraction. Hence according to Theorem 3.11, we can thus conclude that the problem (14)-(15) has at least one solution in$\mathscr{M}$.

Case Caputo $(ii) - gH$ differentiability: we defined $\tilde{\mathscr{T}}_1$ and $\tilde{\mathscr{T}}_2$ on $\mathscr{M}$ as follows

$$\tilde{\mathscr{T}}_1\zeta(u,v) = \eta(u,v) \ominus (-1)^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi\, ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi} g(u,v,\zeta(u,v))$$

and

$$\tilde{\mathscr{T}}_2\zeta(u,v) = \ominus(-1)^{AB}I_{uv}^{\vartheta,\mu,\gamma;\phi} f(u,v,\zeta(u,v),\omega(u,v)),$$

where

$$\omega(u,v) = {}^{ABC}D_{uv}^{\vartheta,\mu,\gamma;\phi} g(u,v,\zeta(u,v)) \ominus (-1) f(u,v,\zeta(u,v),\omega(u,v)).$$

In the same way as above, we shown that $\zeta(u,v)$ being fixed point of the operator $\tilde{\mathscr{T}}\zeta(u,v) = \tilde{\mathscr{T}}_1\zeta(u,v) + \tilde{\mathscr{T}}_2\zeta(u,v)$ is a solution of problem (14)-(15).

□

5. Application

As an application of our results, we consider the following fuzzy hyperbolic partial functional equations with $\phi(\tau) = 2\tau$ and $\vartheta = \left(\frac{1}{2}, \frac{1}{2}\right)$ of the form

$$ {}^{ABC}D_{uv}^{\vartheta,1,5;2\tau} \left[\zeta(u,v) - \frac{1}{4}\cos(\zeta(u,v))\right] = \frac{1}{(5e^{u+v+2})\left(1 + d(\zeta(u,v),\hat{0}) + d\left({}^{ABC}D_{uv}^{\vartheta,1,5;2\tau}\zeta(u,v),\hat{0}\right)\right)}, \quad (27) $$

$$ \text{if } ((u,v) \in [0;1]\times[0;1] $$

$$ \zeta(u,0) = u, u \in [0,1], \zeta(0,v) = v^2, v \in [0,1]. \quad (28) $$

Let's set, for all $(u,v) \in [0;1]\times[0;1]$

$$ g(u,v,\zeta(u,v)) = \frac{1}{4}\cos(\zeta(u,v)) $$

and

$$ f\left(u,v,\zeta(u,v), {}^{ABC}D_{uv}^{\vartheta,1,5;2\tau}\zeta(u,v)\right) = \frac{1}{(5e^{u+v+2})\left(1 + d(\zeta(u,v),\hat{0}) + d\left({}^{ABC}D_{uv}^{\vartheta,1,5;2\tau}\zeta(u,v),\hat{0}\right)\right)}. $$

It is clear that for each $\zeta, \xi, \omega, \sigma \in \mathbb{R}_{\mathcal{F}}$ and $(u,v) \in [0;1]\times[0;1]$, we have

$$ d(f(u,v,\zeta,\omega); f(u,v,\xi,\sigma)) \leq \frac{1}{5e^2}(d(\zeta,\xi) + d(\omega;\sigma)) $$

and

$$ d(g(u,v,\zeta); g(u,v,\xi)) \leq \frac{1}{4}d(\zeta;\xi). $$

Hence condition (H_2) is satisfied with $\lambda = \delta = \frac{1}{5e^2}$ and $\rho = \frac{1}{4}$. In addition, a simple calculation can be used to verify that condition (18) holds with $a = b = 1$.

Consequently Theorem 4.2 implies that problem (27)-(28) has a unique solution defined on $[0;1]\times[0;1]$.

Remark 5.1. *In the example given in this article, it can be noted that the solution to the problem given by the generalized ABC method using fractional derivatives admits the following special case:*

- *If $\phi(\tau) = \tau$ and $\gamma = 1$, we get solutions to the ABC concept fractional derivatives.*
- *If we take $\phi(\tau) = \log(\tau)$ and $\gamma = 1$, we get fractional order in the Caputo-Hadamard sense Solution under the concept of derivative AB.*
- *If we consider $\phi(\tau) = \frac{\tau^\beta}{\beta}$, where $\beta > 0$ is a real parameter, $\gamma = 1$, we get the solution of the following formula The concept of fractional derivative AB under the concept of fractional derivative AB in the sense of Caputo-Katugampola and so on.*

6. Conclusion

In this study, we introduced the new concepts of fractional AB calculus in fuzzy environments, including the generalized Mittag-Leffler kernel function, which we use to study FFDE. Fuzzy fractional derivative $\phi-$ABC is an important concept in fuzzy fraction calculation. Recent research has shown that this derivative can be used to solve FFDEs. Studies have been conducted using fixed point theory and methods such as subsolution and supersolution methods. Therefore, it makes sense to study fuzzy problems under the new concept of fractional derivative ABC, instead of using fractional derivative AB to study a large number of well-known fuzzy problems. The results obtained demonstrate that ABC fuzzy fractional derivatives are a powerful tool for solving problems in various physical fields such as electromagnetism, acoustics, and viscoelastic material mechanics. Research has also shown that this derivative can be used to develop mathematical models of complex systems such as the human liver. In summary, ABC fuzzy fractional derivatives are an important mathematical tool that can be used to solve problems in various fields of physics and engineering.

Conflicts of Interest. The authors declare that there is no conflict of interest regarding the publication of this paper.

Informed consent. Informed consent was obtained from all individuals participants who participated in the study.

References

[1] A. L. Lupaş and A. Cătaş, Fuzzy differential subordination of the Atangana–Baleanu fractional integral, *Symmetry* **13** (2021), no. 10, 1929.

[2] A. T. Allahviranloo and B. Ghanbari, On the fuzzy fractional differential equation with interval Atangana–Baleanu fractional derivative approach, *Chaos Solitons Fractals* **130** (2020), 109397.

[3] A. M. Al-Smadi, O. A. Arqub, and D. Zeidan, Fuzzy fractional differential equations under the Mittag-Leffler kernel differential operator of the ABC approach: Theorems and applications, *Chaos Solitons Fractals* **146** (2021), 110891.

[4] A. M. Al-Smadi, et al., On numerical approximation of Atangana-Baleanu-Caputo fractional integro-differential equations under uncertainty in Hilbert Space, *Math. Model. Nat. Phenom.* **16** (2021), 41.

[5] A. G. Anastassiou, *Fuzzy mathematics: Approximation theory*, Vol. 251, Springer, Berlin, 2010.

[6] A. Atangana and D. Baleanu, *New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model*, arXiv preprint arXiv:1602.03408 (2016).

[7] B. Barnabás and G. S. Gal, Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations, *Fuzzy Sets Syst.* **151** (2005), no. 3, 581–599.

[8] B. Barnabás and S. Luciano, Generalized differentiability of fuzzy-valued functions, *Fuzzy Sets Syst.* **230** (2013), 119–141.

[9] D. F. Dubois, G. A. Cristina, and P. Nelly, Introduction à la dérivation fractionnaire-Théorie et Applications, 2010.

[10] G. J. A. Goguen, LA Zadeh. Fuzzy sets. Information and control, vol. 8 (1965), pp. 338–353.-LA Zadeh. Similarity relations and fuzzy orderings. Information sciences, vol. 3 (1971), pp. 177–200, *J. Symb. Log.* **38** (1973), no. 4, 656–657.

[11] K. O. Kaleva, A note on fuzzy differential equations, *Nonlinear Anal.* **64** (2006), no. 5, 895–900.

[12] K. F. S. Khan, et al., Freelance model with Atangana–Baleanu Caputo fractional derivative, *Symmetry* **14** (2022), no. 11, 2424.

[13] S. N. A. Sheikh, et al., Comparison and analysis of the Atangana–Baleanu and Caputo–Fabrizio fractional derivatives for generalized Casson fluid model with heat generation and chemical reaction, *Results Phys.* **7** (2017), 789–800.

[14] S. M. I. Syam and R. Mohammed Al-Refai, Fractional differential equations with Atangana–Baleanu fractional derivative: analysis and applications, *Chaos Solitons Fractals: X* **2** (2019), 100013.

[15] V. L. Verma and M. R. Ramakanta, Study on generalized fuzzy fractional human liver model with Atangana–Baleanu–Caputo fractional derivative, *Eur. Phys. J. Plus* **137** (2022), no. 11, 1–20.

[16] V. H. Ho, B. Ghanbari, and H. Ngo Van, Fuzzy fractional differential equations with the generalized Atangana-Baleanu fractional derivative, *Fuzzy Sets Syst.* **429** (2022), 1–27.

[17] Z. L. A. Lotfi Asker, K. G. J., and Y. B. Yuan, *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers*, Vol. 6, World Scientific, 1996.