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Generalized solutions for a nonlinear elliptic problems with measure as data in $\mathbb{L}_0^{1,p(\cdot)}(\Omega)$

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Abstract. In this work, we generalize the notions of $\mathbb{T}$-sets to a larger framework, and we establish the existence of a generalized solution for nonlinear elliptic equations, involving variable exponents and measure data.

1. Introduction

The present work, is devoted to the study of a nonlinear elliptic problems with variable exponents and measure data, motivated by theirs applications in the description of many phenomena in applied sciences (physics in nonhomogeneous materials, electro-rheological fluids and image processing [7]).

Let Ω be a bounded open subset of $\mathbb{R}^N (N \geq 2)$ with Lipschitz boundary $\partial\Omega$. We consider the following elliptic problem

$$\begin{cases} -\text{div}\,(a(x,u,\nabla u)) = \mu & \text{in } \ \mathcal{D}'(\Omega), \\ u = 0 & \text{on } \ \partial\Omega, \end{cases} \tag{1}$$

where $\mu \in M(\Omega)$ is a bounded Radon measure. Here, we suppose that $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, is a Carathéodory function and satisfies, for a.e. $x \in \Omega$, $\forall s \in \mathbb{R}$ and $\forall \xi, \xi' \in \mathbb{R}^N$, the following assumptions

$$a(x,s,\xi).\xi \geq c_1|\xi|^{p(x)}, \tag{2}$$

$$|a(x,s,\xi)| \leq c_2\left(l(x) + |s|^{p(x)-1} + |\xi|^{p(x)-1}\right), \quad l \in L^{p'(x)}(\Omega), \tag{3}$$

$$(a(x,s,\xi) - a(x,s,\xi')) \cdot (\xi - \xi') > 0, \quad \xi \neq \xi', \tag{4}$$

where c_1, c_2 are strictly positive real numbers, and $p : \overline{\Omega} \longrightarrow (1, +\infty)$ be a continuous function satisfying the following condition

$$1 < p^- = \min_{x\in\overline{\Omega}} p(x) < p^+ = \max_{x\in\overline{\Omega}} p(x) < N. \tag{5}$$

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To find a solution in the Sobolev space $W_0^{1,q(\cdot)}(\Omega)$, then the function $q(\cdot)$ must satisfy the condition $1 \leq q(x) < \frac{N(p(x)-1)}{N-1}$ for all $x \in \overline{\Omega}$. This condition is discussed in [5] and implies that $p^- > 2 - \frac{1}{N}$. Therefore, when $p^- \in (1, 2 - \frac{1}{N})$ one cannot anticipate solutions to be part of $W^{1,1}(\Omega)$. As a result, the notions of weak derivatives and distributional solutions become problematic. This problem is studied in the literature using the notion of entropy/renormalized solutions.

In the constants case (i.e $p(x) = p > 2 - \frac{1}{N}$, The authors in [6] proved the existence of solution $u \in W_0^{1,q}(\Omega)$ with $1 \leq q < \frac{N(p-1)}{N-1}$. The general case where $1 < p < N$ was treated by Rakotoson in [20–22], the author have shown the existence of generalized solutions to (1) by introducing the notion of $L_0^{1,p}$- sets.

Our aim in this paper is to extend the notion of $L_0^{1,p}$- sets and define a new class of solution in which the problem (1) is well posed. The main difficulty in solving problem with measures lies in obtaining an a priori estimate in Lebesgue space $L^{s(\cdot)}(\Omega)$. However, we overcome this difficulty by using some properties achieved by a new type of sets.

2. Variable Lebesgue and Sobolev Spaces

In this section we recall some facts about the generalized Lebesgue– Sobolev spaces $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot}(\Omega)$. For further details, we refer to the following references: [2, 3, 8, 9, 11, 12, 15–19, 23, 24], as well as the references cited therein.

Consider a continuous function $p : \Omega \to [1, \infty)$, where Ω is an open subset of $\mathbb{R}^N (N \geq 2)$. The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ represent the space of measurable functions $f(x)$ on Ω satisfying

$$\rho_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)} dx < +\infty.$$

The norm on the space $L^{p(\cdot)}(\Omega)$ is defined as follows

$$\|f\|_{p(\cdot)} := \|f\|_{L^{p(\cdot)}(\Omega)} = \inf\{\lambda > 0 \mid \ \rho_{p(\cdot)}(f/\lambda) \leq 1\}.$$

We set

$$p^- = \min_{x\in\overline{\Omega}} p(x), \quad \text{and} \quad p^+ = \max_{x\in\overline{\Omega}} p(x). \tag{6}$$

If $p^- > 1$, then $L^{p(\cdot)}(\Omega)$ forms a Banach space. Additionally, it's reflexive, and its dual space is associated with $L^{p'(\cdot)}(\Omega)$ through $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$. For every $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, the Hölder inequality is defined as

$$\|uv\|_{L^1(\Omega)} \leq 2\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)}.$$

We also define the Banach space $W_0^{1,p(\cdot)}(\Omega)$ as follows

$$W_0^{1,p(\cdot)}(\Omega) = \left\{f \in L^{p(\cdot)}(\Omega) \mid |\nabla f| \in L^{p(\cdot)}(\Omega) \ \text{ and } \ f = 0 \ \text{ on } \partial\Omega\right\},$$

equipped with the norm $\|f\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla f\|_{p(\cdot)}$. When $p \in C(\overline{\Omega}, [1, +\infty))$ and $1 < p^- < p^+ < \infty$, the space $W_0^{1,p(\cdot)}(\Omega)$ is both separable and reflexive.

For any $u \in W_0^{1,p(\cdot)}(\Omega)$ with $p \in C(\overline{\Omega}, [1, +\infty))$, there exists a constant $C > 0$ such that the Poincaré inequality holds (we refer to [13] for more details)

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq C\|\nabla u\|_{L^{p(\cdot)}(\Omega)}. \tag{7}$$

The analysis of generalized Lebesgue and Sobolev spaces essentially relies on the fundamental role of the modular $\rho_{p(\cdot)}(u)$ connected with the space $L^{p(\cdot)}(\Omega)$. In this context we present the following result.

Lemma 2.1. *Let $p : \overline{\Omega} \to [1, +\infty[$ be a continuous function and $p^+ < +\infty$, then the following properties hold:*

$$\min\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\right) \le \|u\|_{L^{p(\cdot)}(\Omega)} \le \max\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\right),$$
$$\min\left(\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}\right) \le \rho_{p(\cdot)}(u) \le \max\left(\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}\right),$$

and

$$\|u\|_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) + 1. \tag{8}$$

Definition 2.2. *The variable exponent $p : \overline{\Omega} \to [1, +\infty)$ is said to be satisfy the log-continuity condition, and we denote by $p \in C_{log}(\Omega)$ if there exists a positive constant C such that*

$$\forall x, y \in \overline{\Omega}, \ \ |x - y| \le 1/2; \ \ \ |p(x) - p(y)| < \frac{C}{-\log(|x - y|)}. \tag{9}$$

Lemma 2.3. *([20]) If $p \in C_{log}(\Omega)$, then the set $C^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega) = W^{1,p(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega)$.*

If $q \in C(\overline{\Omega})$ and for all $x \in \overline{\Omega}$, $\ q(x) < p^*(x)$, then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact (see [12]). Moreover, if p satisfies the log-Holder continuity assumption (9) and $p^+ < N$, then the Sobolev embedding holds also for the critical case $q(\cdot) = p^*(\cdot)$ i.e. the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$ is continuous.

Remark 2.4. *In the case where $|\Omega| < \infty$ the inclusion between Lebesgue spaces generalizes naturally, i.e. if r_1, r_2 are variable exponents such that $r_1(\cdot) \le r_2(\cdot)$ almost everywhere in Ω, then the continuous embedding $L^{r_2(\cdot)}(\Omega) \hookrightarrow L^{r_1(\cdot)}(\Omega)$ holds.*

Lemma 2.5. *[4] (Differentiation of a composition) Let $u \in W_0^{1,p(\cdot)}(\Omega)$ with $p : \overline{\Omega} \longrightarrow (1, +\infty)$ and $p \in C(\overline{\Omega})$. Assume that $f \in C^1(\mathbb{R})$ be such that $f(0) = 0$ and $\left|f'(s)\right| \le M$, $\forall s \in \mathbb{R}$ for some constant M. Then we have*

$$f \circ u \in W_0^{1,p(\cdot)}(\Omega),$$
$$\frac{\partial}{\partial x_i}(f \circ u) = (f' \circ u)\,\frac{\partial u}{\partial x_i}, \quad i = 1, 2, \ldots, N. \tag{10}$$

3. Definition and Properties of the Space $\mathbb{L}_0^{1,p(\cdot)}(\Omega)$

Let Ω be a bounded open set of $\mathbb{R}^N$ and $p : \overline{\Omega} \to [1, +\infty[$ be a continuous function. We introduce the following set

$$\mathbb{L}\mathrm{ip}_{p(\cdot)}(\mathbb{R}) = \left\{\Phi \in W^{1,\infty}(\mathbb{R}) \text{ such that } \Phi' \in L^{p(\cdot)}(\mathbb{R}), \dot{\Phi}(0) = 0\right\}.$$

For $k > 0$, we set $T_k(\sigma) = [k - (k - |\sigma|)_+]\,\mathrm{sign}(\sigma), \sigma \in \mathbb{R}$. We define the space $\mathbb{L}_0^{1,p(\cdot)}(\Omega)$ as follow

$$\mathbb{L}_0^{1,p(\cdot)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \;\middle|\; \begin{array}{l} \text{measurable such that} \\ \forall\, \Phi \in \mathbb{L}\mathrm{ip}_{p(\cdot)}(\mathbb{R}), \Phi(u) \in W_0^{1,p(\cdot)}(\Omega), \text{ and} \\ \sup\limits_{k>0} \displaystyle\int_\Omega \frac{|\nabla T_k(u)|^{p(x)}}{(1 + |T_k(u)|)^{1+\delta}} dx \text{ is finite for all } \delta > 0 \end{array} \right\}.$$

Remark 3.1. *if $p(x) = p$ for all $x \in \Omega$, then the space $\mathbb{L}_0^{1,p(\cdot)}(\Omega)$ is called $\mathbb{T}-$ sets or $L_0^{1,p}$-sets (we refer to [20]).*

Proposition 3.2. *Let $v \in \mathbb{L}_0^{1,p(\cdot)}(\Omega)$ and $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function. Then $\nabla v(x)$ exists almost everywhere in Ω. Moreover, we have*
i) $\nabla(f \circ v) = (f' \circ v)(x) \cdot \nabla v(x)$ for a.e. $x \in \Omega$.
ii) For all $k > 0$, the function $T_k(v), k > 0$ satisfies

$$\nabla T_k(v) = \begin{cases} 0 & if\, |v| > k \\ \nabla v & otherwise \end{cases} \qquad a.e.\ in\ \Omega.$$

iii) For $p^- > 1$, one has the inclusion $W_0^{1,p(\cdot)}(\Omega) \subset \mathbb{L}_0^{1,p(\cdot)}(\Omega)$.

Proof. i) Let $v \in \mathbb{L}_0^{1,p(\cdot)}(\Omega)$, we consider the C^1-function $\Phi(v) = \arctan v$; $\Phi \in \mathbb{L}\mathrm{ip}_{p(\cdot)}(\mathbb{R})$, thus $w = \Phi(v) \in W_0^{1,p(\cdot)}(\Omega)$, we deduce from Deny-Lions' theorem (see [10]) that ∇w exists a.e. in Ω. Moreover, if we denote by $(e_1, e_2, \cdots, e_N)$ the canonical basis of $\mathbb{R}^{\mathbb{N}}$ then the maps

$$t \in \mathbb{R} \longrightarrow w(x + te_i), \quad \text{are continous for a.e } x \in \Omega.$$

Now, we can write for all $t \in \mathbb{R}$ and for all $i = 1, \cdots, N$

$$v(x + te_i) - v(x) = \tan[w(x + te_i)] - \tan[w(x)] = (1 + \tan^2 C_{x,t})[w(x + te_i) - w(x)], \quad a.e\ x \in \Omega,$$

with $C_{x,t}$ is a point between $w(x + te_i)$ and $w(x)$. The continuity of w on the segment passing through x in the direction e_i shows that $C_{x,t} \to w(x)$, as t goes to zero, so that

$$\frac{\partial v}{\partial x_i}(x) = \lim_{t \to 0} \frac{v(x + te_i) - v(x)}{t} = (1 + \tan^2 w(x))\frac{\partial w}{\partial x_i}(x). \tag{11}$$

Hence, ∇v exists a.e. in Ω, this result combined with Lemma 2.5 gives i).

ii) For $k > 0$, it's not difficult to check that $T_k(v) = \tan[T_{\Phi(k)}(w)]$ a.e in Ω. We apply the chain rule (10) that gives

$$\begin{aligned}\frac{\partial T_k v}{\partial x_i}(x) &= (1 + \tan^2(T_{\Phi(k)}w)\frac{\partial T_{\Phi(k)}w}{\partial x_i}(x) \\ &= (1 + \tan^2(T_{\Phi(k)}w)\begin{cases} \frac{\partial w}{\partial x_i}(x), & if\ |w(x)| < \Phi(k), \\ 0, & otherwise, \end{cases} \\ &= (1 + v^2)\begin{cases} \frac{1}{1+v^2}\frac{\partial v}{\partial x_i}(x), & if\ |v(x)| < k, \\ 0, & otherwise. \end{cases}\end{aligned}$$

Thus, the statement ii) is proved.

iii) Let $u \in W_0^{1,p(\cdot)}(\Omega)$, by the lemma 2.5 we have for all $T \in \mathbb{L}\mathrm{ip}_{p(\cdot)}(\mathbb{R})$, $T(u) \in W_0^{1,p(\cdot)}(\Omega)$ and for all $k > 0$, $\delta > 0$ we can write

$$\int_\Omega \frac{|\nabla T_k(u)|^{p(x)}}{(1 + |T_k(u)|)^{1+\delta}} dx \le \int_\Omega \frac{|\nabla u|^{p(x)}}{(1 + |u|)^{1+\delta}} dx \le \rho_{p(\cdot)}(\nabla u) < \infty,$$

this finished the proof of iii). □

In the sequel, we will denote by C serval constants whose value may change from line to line and, sometimes, on the same line. These values will only depend on the data, but the will never depend on the indexes of the sequences we will often introduce.

Proposition 3.3. *Let $p(\cdot), q(\cdot) \in C(\overline{\Omega})$, suppose that*

$$2 - \frac{1}{N} < p^- < N, \quad 1 \le q(x) < \frac{N}{N-1}(p(x) - 1), \quad for\ all\ x \in \overline{\Omega},$$

then we have $\mathbb{L}_0^{1,p(\cdot)}(\Omega) \subset W_0^{1,q(\cdot)}(\Omega)$.

Proof. Let $v \in \mathbb{L}_0^{1,p(\cdot)}(\Omega)$. By Proposition 3.2 and Beppo-Levi's theorem, we have for all $\delta > 0$

$$\sup_{k>0} \int_\Omega \frac{|T_k(v)|^{p(x)}}{(1+|T_k(v)|)^{1+\delta}}dx = \sup_{k>0} \int_{\{|v|\leq k\}} \frac{|\nabla v|^{p(x)}}{(1+|v|)^{1+\delta}}dx \leq \int_\Omega \frac{|\nabla v|^{p(x)}}{(1+|v|)^{1+\delta}}dx \leq C. \tag{12}$$

We divide the proof into two steps:

Step 1: let us consider the case

$$1 \leq q^+ < \frac{N}{(N-1)}(p^- - 1) < 1.$$

By Hölder's inequality, the estimate (12) and the fact that $|T_k(v)|^{p^-} \leq |T_k(v)|^{p(x)} + 1$, we obtain

$$\begin{aligned}
&\|\nabla T_k(v)\|^{q^+}_{L^{q^+}(\Omega)}\\
&= \int_\Omega \frac{|\nabla T_k(v)|^{q^+}}{(1+|T_k(v)|)^{(1+\delta)\frac{q^+}{p^-}}} (1+|T_k(v)|)^{(1+\delta)\frac{q^+}{p^-}}\, dx\\
&\leq \left(\int_\Omega \frac{|\nabla T_k(v)|^{p^-}}{(1+T_k(v))^{1+\delta}}dx\right)^{\frac{q^+}{p^-}} \left(\int_\Omega (1+T_k(v))^{(1+\delta)\frac{q^+}{p^- - q^+}}\, dx\right)^{\frac{p^- - q^+}{p^-}}\\
&\leq C + C\left(\int_\Omega |T_k(v)|^{(1+\delta)\frac{q^+}{p^- - q^+}}\, dx\right)^{\frac{p^- - q^+}{p^-}}.
\end{aligned} \tag{13}$$

Now, we chose $\delta > 0$ such that

$$(1+\delta)\frac{q^+}{p^- - q^+} = q^{+*},$$

the previous equality is equivalent to

$$\delta = \frac{N(p^- - q^+)}{N - q^+} - 1 > 0 \Leftrightarrow q^+ < \frac{N}{(N-1)}(p^- - 1).$$

From Sobolev's inequality applied to $T_k(v) \in W_0^{1,p(\cdot)}(\Omega)$, the estimate (13) yield to

$$\|\nabla T_k(v)\|^{q^+}_{L^{q^+}(\Omega)} \leq C + C\|\nabla T_k(v)\|^{\frac{q^{+*}(p^- - q^+)}{p^-}}_{L^{q^+}(\Omega)}, \tag{14}$$

the choice of q^+ implies $\frac{q^{+*}}{q^+}\frac{p^- - q^+}{p^-} < 1$. Thus, there exists a constant $C > 0$ (independent of k) such that

$$\|\nabla T_k(v)\|^{q^+}_{L^{q^+}(\Omega)} \leq C, \quad \forall\, k > 0. \tag{15}$$

Which implies

$$\rho_{q(\cdot)}(\nabla T_k(v)) \leq C, \quad \forall k > 0. \tag{16}$$

Step 2: Let us consider a continuous variable exponent $q(\cdot)$ on $\overline{\Omega}$ satisfying

$$q(x) < \frac{N}{N-1}(p(x) - 1), \quad \text{and } q^+ \geq \frac{N}{N-1}(p^- - 1), \tag{17}$$

we slightly modify the previous proof in the first step. Since $p(\cdot), q(\cdot) \in C(\overline{\Omega})$ then there exists a constant $\rho > 0$ such that

$$\max_{t\in B(x,\rho)\cap\Omega} q(t) < \min_{t\in B(x,\rho)\cap\Omega} \frac{N(p^- - 1)}{N-1},$$

where $B(x,\rho)$ is a cube with center x and diameter ρ. Remark that $\overline{\Omega}$ is compact and therefore, we can write $\overline{\Omega} = \cup_{j=1}^{k} B_j$ where B_j, $j = 1, ..., k$ is a cube with borders parallel to the coordinate axes. Moreover, there exists a constant $\sigma > 0$ such that

$$\rho > |\Omega_j| > \sigma, \quad \Omega_j = B_j \cap \Omega, \quad \text{for all } j = 1, \cdots, k. \tag{18}$$

We denote by $\Omega_j = B_j \cap \Omega$, and q_j^+(respectively p_j^-) the local maximum of $q_j(\cdot)$ on $\overline{\Omega}_j$ (respectively the local minimum of $p_j()$ on $\overline{\Omega}_j$), such that

$$q_j^+ < \frac{N\left(p_j^- - 1\right)}{N-1} \text{ for all } j = 1, \cdots, k.$$

Using now the same arguments as before locally, we see that the inequality (14) holds on Ω_j, so

$$\|\nabla T_k(v)\|_{L^{q_j^+}(\Omega_j)}^{q_j^+} \leq C + C\|\nabla T_k(v)\|_{L^{q_j^+}(\Omega_j)}^{\frac{q_j^{+*}(p_j^- - q_j^+)}{p_j^-}}. \tag{19}$$

Denote by $\widetilde{T_k(v)}$ the average of $T_k(v)$ over Ω_j

$$\widetilde{T_k(v)} = \frac{1}{|\Omega_j|}\|T_k(v)\|_{L^1(\Omega_j)}.$$

By Poincaré-Wirtinger inequality, we obtain

$$\|T_k(v) - \widetilde{T_k(v)}\|_{L^{q_j^{+*}}(\Omega_j)} \leq C\|\nabla T_k(v))\|_{L^{q_j^+}(\Omega_j)}, \quad \forall j = 1, \cdots, k. \tag{20}$$

Using (15) and (20), we get

$$\begin{aligned} \|T_k(v)\|_{L^{q_j^{+*}}(\Omega_j)} &\leq \|T_k(v) - \widetilde{T_k(v)}\|_{L^{q_j^{+*}}(\Omega_j)} + \|\widetilde{T_k(v)}\|_{L^{q_j^{+*}}(\Omega_j)} \\ &\leq C\|\nabla T_k(v)\|_{L^{q_j^+}(\Omega_j)} + C, \quad \forall j = 1, \cdots, k. \end{aligned} \tag{21}$$

We deduce from (20) and (21)

$$\|\nabla T_k(v)\|_{L^{q_j^+}(\Omega_j)}^{q_j^+} \leq C, \quad \forall j = 1, \cdots, k. \tag{22}$$

Knowing that $q(x) \leq q_j^+$ for all $x \in \Omega_j$, and all $j = 1, \cdots, k$, so we conclude that

$$\int_{\Omega_j} |\nabla T_k(v)|^{q(x)} dx \leq C, \quad \forall j = 1, \cdots, k,$$

that is,

$$\rho_{q(\cdot)}(\nabla T_k(v)) \leq \sum_{j=1}^{k} \int_{\Omega_j} |\nabla T_k(v)|^{q(x)} dx \leq C. \tag{23}$$

By (8) , (16), (23) and Since $T_k(v)$ converges to v almost everywhere, we get that $v \in W_0^{1,q(\cdot)}(\Omega)$. □

Proposition 3.4. *Let $p(\cdot)$ be a Log-Hölder continuous function defined on $\overline{\Omega}$ satisfying $1 < p^- < p^+ < N$ and $|\nabla p| \in L^\infty(\Omega)$. Then we have the inclusion $\mathbb{L}_0^{1,p(\cdot)}(\Omega) \subset L^{s(\cdot)}(\Omega)$, for any measurable function $s : \overline{\Omega} \to (1, +\infty)$ with*

$$0 < s(x) < \frac{N(p(x)-1)}{N-p(x)} \quad \textit{and} \quad |\nabla s| \in L^\infty(\Omega). \tag{24}$$

Proof. Let $v \in \mathbb{L}_0^{1,p(\cdot)}(\Omega)$ and $s(\cdot)$ as in (24), we define for $x \in \Omega$

$$\alpha(x) = 1 - \frac{s(x)}{p^*(x)} \in \left]\frac{1}{p(x)}, 1\right[. \tag{25}$$

For $k > 0$, we introduce the function Ψ_k defined by

$$\Psi_k(x, v) = \left[(1 + |T_k(v)|)^{1-\alpha(x)} - 1\right] \text{sign}(v), \quad x \in \overline{\Omega}.$$

Thus

$$\nabla\Psi_k(x, v) = \nabla(1 - \alpha(x)) \ln(1 + |T_k(v)|)(1 + |T_k(v)|)^{1-\alpha(x)} \text{sign}(v) + (1 - \alpha(x))(1 + |T_k(v)|)^{-\alpha(x)} \nabla T_k(v). \tag{26}$$

Its not difficult to check that for all $x \in \Omega$, $\Psi_k(x, v) \in W_0^{1,p(\cdot)}(\Omega)$. By the Sobolev embedding (since $p \in C_{log}(\Omega)$) and Poincaré's inequality (7) we obtain

$$\|\Psi_k(x, v)\|_{L^{p^*(\cdot)}(\Omega)} \leq C\|\nabla\Psi_k(x, v)\|_{L^{p(\cdot)}(\Omega)}, \tag{27}$$

which yield

$$\begin{aligned}
&\|\Psi_k(x, v)\|_{L^{p^*(\cdot)}(\Omega)} \\
&\quad \leq C\|\nabla(1 - \alpha(x)) \ln(1 + |T_k(v)|)(1 + |T_k(v)|)^{1-\alpha(x)} \text{sign}(v)\|_{L^{p(\cdot)}(\Omega)} \\
&\quad + C\|(1 - \alpha(x))(1 + |T_k(v)|)^{-\alpha(x)} \nabla T_k(v)\|_{L^{p(\cdot)}(\Omega)}.
\end{aligned}$$

By lemma 2.1 and that $|\nabla\alpha| \in L^\infty(\Omega)$ (because $|\nabla p|, |\nabla s| \in L^\infty(\Omega)$), we can write

$$\|\Psi_k(x, v)\|_{L^{p^*(\cdot)}(\Omega)} \leq C \int_\Omega (1 + |T_k(v)|)^{(1-\alpha(x))p(x)} (\ln(1 + |T_k(v)|))^{p(x)} dx + C \int_\Omega \frac{|\nabla T_k(v)|^{p(x)}}{(1 + |T_k(v)|)^{\alpha(x)p(x)}} dx + C. \tag{28}$$

Since $v \in \mathbb{L}_0^{1,p(\cdot)}(\Omega)$ and remark that

$$(1 + |T_k(v)|)^{-\alpha(x)p(x)} (\ln(1 + |T_k(v)|))^{p(x)} \quad \text{is bounded for all } x \in \overline{\Omega},$$

we conclude that

$$\int_\Omega (1 + |T_k(v)|)^{p(x)} (1 + |T_k(v)|)^{-\alpha(x)p(x)} (\ln(1 + |T_k(v)|))^{p(x)} dx \leq C. \tag{29}$$

In the other hand, using (25), we have

$$\alpha(x)p(x) - 1 = \frac{N - p(x)}{N} \left(\frac{N(p(x) - 1)}{N - p(x)} - s(x)\right) > 0.$$

This implies that

$$\int_\Omega \frac{|\nabla T_k(v)|^{p(x)}}{(1 + |T_k(v)|)^{\alpha(x)p(x)}} \leq C. \tag{30}$$

Combining (28), (29) and (30), we obtain

$$\|\Psi_k(x, v)\|_{L^{p^*(\cdot)}(\Omega)} \leq C. \tag{31}$$

Remark that, for all $k > 0$ and $x \in \Omega$, $|T_k(v)|^{1-\alpha(x)} \leq |\Psi_k(x, v)| + 1$, by Lemma 2.1 yielding

$$\begin{aligned}
\rho_{p^*(\cdot)(1-\alpha(\cdot))}(T_k(v)) &\leq C\rho_{p^*(\cdot)}(\Psi_k(v)) + C \\
&\leq C \max\left\{\|\Psi_k(v)\|_{L^{p^*(\cdot)}(\Omega)}^{p^{*+}}, \|\Psi_k(v)\|_{L^{p^*(\cdot)}(\Omega)}^{p^{*-}}\right\} + C.
\end{aligned} \tag{32}$$

Hence, it follows from (31), (32) and (25) that

$$\rho_{s(\cdot)}\left(T_k(v)\right) \leq C, \tag{33}$$

where $C > 0$ is a constant independent of k, Finally we obtain by Fatou's lemma

$$\rho_{s(\cdot)}(v) \leq C.$$

This finished the proof of Proposition 3.4. □

4. Existence of Generalized Solutions

Definition 4.1. *We will say that a function u is generalized solution to problems* (1) *if*

$$u \in \mathbb{L}_0^{1,p(\cdot)}(\Omega), \quad a(x,u,\nabla u) \in (L^1(\Omega))^N,$$

and for all $\varphi \in \mathcal{D}(\Omega)$ one has

$$\int_\Omega a(x,u,\nabla u)\nabla\varphi dx = < u, \varphi >_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} .$$

The principal result of our paper is the following.

Theorem 4.2. *Let $\mu \in M(\Omega)$, $p : \overline{\Omega} \to (1,+\infty)$ be a continuous function, let us assume that* (5), (9) *and* $|\nabla p| \in L^\infty(\Omega)$ *hold true. Then, there exists at least one solution of* (1) *in the sens of definition 4.1.*

Remark 4.3. *In the case where $2 - \frac{1}{N} < p^- < N$, we have $u \in \mathbb{L}_0^{1,p(\cdot)}(\Omega) \subset W_0^{1,q(\cdot)}(\Omega$, where $q(x) < \frac{N}{N-1}(p(x) - 1)$, for all $x \in \overline{\Omega}$. Therfore, the Proposition 3.3 ensure that, the generalized solution of problem* (1) *is also destributional solution.*

Let $\mu \in M(\Omega)$, then there exists a sequence $(\mu_n) \subset \mathcal{D}(\Omega)$ such that $\mu_n \longrightarrow \mu$ in $\mathcal{D}'(\Omega)$ and satisfies $\|\mu_n\|_{L^1(\Omega)} \leq \|\mu\|_{M(\Omega)}$ for all $n \geq 1$. We consider the following approximation problems:

$$\begin{cases} u_n \in W_0^{1,p(\cdot)}(\Omega) \\ \displaystyle\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla\varphi dx = \int_\Omega \mu_n\varphi dx, \quad \forall\varphi \in \mathcal{D}(\Omega). \end{cases} \tag{34}$$

The existence of a weak solution $u_n \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ to problem (34) is guaranteed by [14](Proposition 6.1).

Lemma 4.4. *Let (u_n) the sequence of solution of problem* (34). *Then for every $n \geq 1$ we have*

$$\forall\phi \in \mathbb{L}ip_{p(\cdot)}(\mathbb{R}),\ \exists C = C(T) \quad \textit{such that} \quad \rho_{p(\cdot)}\left(\nabla\phi(u_n)\right) \leq C, \tag{35}$$

$$\forall\delta > 0,\ \exists C = C(\delta) \quad \textit{such that} \quad \int_\Omega \frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)^{1+\delta}} \leq C. \tag{36}$$

Proof. Let $\phi \in \mathbb{L}\mathrm{ip}_{p(\cdot)}(\mathbb{R})$, we choose $\Phi(u_n) = \int_0^{u_n} |\phi'(\sigma)|^{p(x)}d\sigma$ as test function in (34), one has

$$\int_\Omega a(x,u_n,\nabla u_n)|\phi'(u_n)|^{p(x)}\cdot\nabla u_n dx = \int_\Omega \mu_n\Phi(u_n)dx. \tag{37}$$

Remark that (since $\phi \in \mathbb{L}\mathrm{ip}_{p(\cdot)}(\mathbb{R})$)

$$|\Phi(u_n)| \leq \int_{-\infty}^{+\infty} |\phi'(u_n)|^{p(x)}dx \leq C,$$

using the last estimate and (2), we obtain

$$\rho_{p(\cdot)}\left(\nabla\phi(u_n)\right) \leq C\|\mu\|_{M(\Omega)},\ \forall n \geq 1.$$

Let us introduce the functions $\psi_\delta : \mathbb{R} \to \mathbb{R}$ by

$$\psi_\delta(t) = -\frac{1}{\delta}\left((1+|t|)^{-\delta} - 1\right)\text{sign}(t), \quad \forall\delta > 0.$$

Note that $\psi_\delta \in W_0^{1,p(.)}(\Omega)$ (since $\psi_\delta(0) = 0$, and $\left|\psi'_\delta(\cdot)\right| \leq 1$). We take $\psi_\delta(u_n)$ as a test function in (34) using (2) and the fact that $\left|\psi_\delta(\cdot)\right| \leq \frac{1}{\delta}$, we get

$$\begin{aligned}\int_\Omega \frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)^{1+\delta}}dx &\leq C\int_\Omega |\mu_n||\psi_\delta(u_n)|dx\\ &\leq \frac{C}{\delta}\|\mu\|_{M(\Omega)}.\end{aligned}$$

□

Lemma 4.5. *For all $q \in [1, \frac{N}{N-1})$, there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$, and for all $x \in \overline{\Omega}$*

$$\rho_{q(p(\cdot)-1)}\left(\nabla u_n\right) \leq C. \tag{38}$$

Proof. For $q \in [1, \frac{N}{N-1})$, we have $q(p(x)-1)) < \frac{N(p(x)-1)}{N-1} < p(x)$, then $q < p'(x)$. By Hölder inequality with indices $\left(\frac{p'(x)}{q}, \frac{p'(x)}{p'(x)-q}\right)$ and Lemma 2.1, we can write for all $\delta > 0$,

$$\begin{aligned}&\rho_{q(p(\cdot)-1)}\left(\nabla u_n\right)\\ &= \int_\Omega \frac{|\nabla u_n|^{q(p(x)-1)}}{(1+|u_n|)^{\frac{(1+\delta)q}{p'(x)}}}\left(1+|u_n|\right)^{\frac{(1+\delta)q}{p'(x)}}dx\\ &\leq 2\left\|\frac{|\nabla u_n|^{q(p(x)-1)}}{(1+|u_n|)^{\frac{(1+\delta)q}{p'(x)}}}\right\|_{\frac{p'(x)}{q}}\left\|\left(1+|u_n|\right)^{\frac{(1+\delta)q}{p'(x)}}\right\|_{\frac{p'(x)}{p'(x)-q}}\\ &\leq C\max\left\{\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)^{1+\delta}}dx\right)^{\frac{q}{p'^+}}, \left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)^{1+\delta}}dx\right)^{\frac{q}{p'^-}}\right\}\\ &\quad\times\max\left\{\rho_{\frac{(1+\delta)q}{p'(\cdot)-q}}\left(1+|u_n|\right)^{\frac{p'^+-q}{p'^+}}, \rho_{\frac{(1+\delta)q}{p'(\cdot)-q}}\left(1+|u_n|\right)^{\frac{p'^--q}{p'^-}}\right\}.\end{aligned}$$

Now, since $p(x) < N$ and $1 \leq q < \frac{N}{N-1}$ we choose

$$0 < \delta < \frac{p(x)(N-1)}{q(N-p(x))}\left(\frac{N}{N-1} - q\right). \tag{39}$$

By the inequality 39, we derive

$$0 < s(x) = \frac{(1+\delta)q}{p'(x)-q} < \frac{N(p(x)-1)}{N-p(x)}.$$

Using Lemma 4.4, and the fact that

$$\rho_{s(\cdot)}\left(u_n\right) \leq C, \tag{40}$$

where C is a constant independent on n. Thus, the proof of lemma 4.5 is achieved. □

Lemma 4.6. *Let* $(u_n)_n$ *the sequence of solution of problem* (34). *Then*

$$i)\ u_n \to u \text{ a.e. in } \Omega, \quad \text{and} \quad u \in \mathbb{L}_0^{1,p(.)}(\Omega) \cap L^{p(\cdot)-1}(\Omega), \tag{41}$$

$$ii)\ \nabla u_n \to \nabla u \text{ a.e. in } \Omega. \tag{42}$$

Proof. i) Consider the function $t \longmapsto \arctan t$ that belong to $\mathbb{L}\mathrm{ip}_{p(\cdot)}(\mathbb{R})$, so from Lemma 4.4, the sequence $(v_n)_n = (\arctan u_n)_n$ remain in a bounded set of $W_0^{1,p(\cdot)}(\Omega)$. Therefore, there exists a subsequence of $(v_n)_n$ still denoted by $(v_n)_n$, and a measurable function v such that

$$v_n \rightharpoonup v \quad \text{weakly in } W_0^{1,p(\cdot)}(\Omega) \text{ and a.e. in } \Omega.$$

Taking $u = \tan v$, since the function $t \longmapsto \arctan t$ is invertible, thus u_n converge to u a.e in Ω. Moreover, since $p(x) - 1 < \frac{N(p(x)-1)}{N-p(x)}$, by (40) and Fatou's Lemma, we conclude that $u \in L^{p(x)-1}(\Omega)$.

ii) Let $\delta > 0$, Egoroff's theorem states that, there exists a set Ω_δ with $|\Omega - \Omega_\delta| \le \delta$ such that $u_n \longrightarrow u$ uniformly in Ω. So let $\varepsilon > 0$ then there exists n_ε such that $\forall n \ge n_\varepsilon$ and $\forall x \in \Omega_\delta$ one has $|u_n(x) - u(x)| \le \varepsilon$. Now let us choose $T_\varepsilon\,(u_n - T_k(u))$ as a test function in (34) we obtain

$$\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla T_\varepsilon(u_n - T_k(u))dx = \int_\Omega \mu_n T_\varepsilon(u_n - T_k(u))dx \le \varepsilon\|\mu\|_{M(\Omega)}.$$

We denote by

$$\Lambda(u_n, T_k(u)) = (a(x, u_n, \nabla u_n) - a(x, u_n, \nabla T_k(u))) \cdot \nabla(u_n - T_k(u)) \ge 0.$$

Hence

$$\int_{\{|u_n - T_k(u)| \le \varepsilon\}} \Lambda(u_n, T_k(u))dx \le C\varepsilon - \int_\Omega a(x, u_n, \nabla T_k(u)) \cdot \nabla T_\varepsilon(u_n - T_k(u))dx. \tag{43}$$

Since $T_\varepsilon(u_n - T_k(u)) \rightharpoonup T_\varepsilon(u - T_k(u))$ in $L^{p(\cdot)}(\Omega)$ as n goes to infinity, we derive

$$\lim_{n\to+\infty} \int_\Omega a(x, u_n, \nabla T_k(u)) \cdot \nabla T_\varepsilon(u_n - T_k(u))dx = \int_\Omega a(x, u, \nabla T_k(u)) \cdot \nabla T_\varepsilon(u - T_k(u))dx.$$

Taking the limsup in (43), one has using the fact that : $\nabla T_k(u) = \nabla u$ if $|u| \le k$ and that $\Lambda(u_n, T_k(u)) \ge 0$

$$\begin{aligned}\limsup_n \int_{\{\Omega_\delta \cap |u| \le k\}} \Lambda(u_n, u)dx &\le \limsup_n \int_{\{|u_n - T_k(u)| \le \varepsilon\}} \Lambda(u_n, T_k(u))dx \\ &\le C\varepsilon - \int_\Omega a(x, u, \nabla T_k(u) \cdot \nabla T_\varepsilon(u - T_k(u))dx.\end{aligned} \tag{44}$$

Since $T_\varepsilon(u - T_k(u))$ rests in a bounded set of $W_0^{1,p(\cdot)}(\Omega)$ as ε goes to zero, moreover, $T_\varepsilon(u - T_k(u)) \longrightarrow 0$ a.e. in Ω, we deduce that $T_\varepsilon(u - T_k(u)) \rightharpoonup 0$ weakly in $W_0^{1,p(\cdot)}(\Omega)$). Then we have

$$\lim_{\varepsilon\to 0} \int_\Omega a(x, u, \nabla T_k(u) \cdot \nabla T_\varepsilon(u - T_k(u))dx = 0.$$

Thus, letting $\varepsilon \longrightarrow 0$ in (44), we obtain

$$\limsup_n \int_{\{\Omega_\delta \cap |u| \le k\}} \Lambda(u_n, u)dx = 0. \tag{45}$$

We derive that for a subsequence still indexed by n that ∇u_n converges to ∇u almost everywhere on $\{\Omega_\delta \cap |u| \le k\}$.

Now, let $\alpha \in (0, p^- - 1)$, we define the sequence

$$\mathcal{T}(n,\delta) = \|\nabla u_n - \nabla u\|^{\alpha}_{L^{\alpha}(\Omega_\delta \cap \{|u|\le k\})},$$

we see that all sequences $\mathcal{T}(n,\delta)$ converge to zero as n tends to infinity. We are going to prove this result on Ω, so we show that $\lim_n \|\nabla u_n - \nabla u\|^{\alpha}_{L^{\alpha}(\Omega)} = 0$. After decomposing this last integral as before, we have for $\delta > 0$:

$$\|\nabla u_n - \nabla u\|^{\alpha}_{L^{\alpha}(\Omega)} \le \mathcal{T}(n,\delta) + \|\nabla u_n - \nabla u\|^{\alpha}_{L^{\alpha}(\{|u|>k\})} + \|\nabla u_n - \nabla u\|^{\alpha}_{L^{\alpha}(\Omega-\Omega_\delta)}. \tag{46}$$

Using Hölder's inequality, Lemma 4.5 (remark that $|\nabla u_n|^{p^- -1} \le |\nabla u_n|^{p(x)-1} + 1$) and that $|\Omega - \Omega_\delta| \le \delta$ we get

$$\|\nabla u_n - \nabla u\|^{\alpha}_{L^{\alpha}(\Omega-\Omega_\delta)} \le C\delta^{1-\frac{\alpha}{p^- -1}}. \tag{47}$$

Furthermore, by Using Hölder's inequality, Lemma 4.5 and the fact that $u \in L^{p(x)-1}(\Omega)$

$$\begin{aligned}\|\nabla u_n - \nabla u\|^{\alpha}_{L^{\alpha}(\{|u|>k\})} &\le \left(\int_{\{|u|>k\}} k^{-(p^- -1)}|u|^{(p^- -1)}dx\right)^{1-\frac{\alpha}{p^- -1}}\\ &\le Ck^{-(p^- -1-\alpha)}.\end{aligned} \tag{48}$$

Combining (46), (47) and (48) we get

$$\|\nabla u_n - \nabla u\|^{\alpha}_{L^{\alpha}(\Omega)} \le I(n,\delta) + C\delta^{1-(\alpha/(p^- -1))} + Ck^{-(p^- -1-\alpha)}.$$

The last inequality holds for all $k > 0$ and $\delta > 0$, so we pass to the limite as $\delta \longrightarrow 0$ and $k \longrightarrow +\infty$, we get

$$\lim_{n\to+\infty} \|\nabla u_n - \nabla u\|^{\alpha}_{L^{\alpha}(\Omega)} = 0.$$

Hence, we deduce up subsequence (still denoted ∇u_n) that ∇u_n converges to ∇u almost everywhere in Ω.

Now, we show that $u \in \mathbb{L}_0^{1,p(\cdot)}(\Omega)$, using (36) we have

$$\int_\Omega \frac{|\nabla T_k(u_n)|^{p(x)}}{(1+|T_k(u_n)|)^{1+\delta}} \le C, \quad \forall n \ge 1.$$

By Fatou's lemma combining with (4.6) and (42) we deduce that

$$\sup_{k>0} \int_\Omega \frac{|\nabla T_k(u)|^{p(x)}}{(1+|T_k(u)|)^{1+\delta}} \le C.$$

We pass to the limit as $n \longrightarrow +\infty$ in (35) and by Fatou's lemma we obtain

$$\forall T \in \mathbb{L}\mathrm{ip}_{p(\cdot)}(\mathbb{R}),\ T(u) \in W_0^{1,p(\cdot)}(\Omega).$$

Thus, $u \in \mathbb{L}_0^{1,p(.)}(\Omega)$. □

4.1. Passage to the limit

Thanks to the result of Lemma (4.6), combining with (38), (40) and Vitali's theorem we conculude that

$$a(x,u_n,\nabla u_n) \longrightarrow a(x,u,\nabla u) \text{ strongly in } L^1(\Omega)^N.$$

Now, let $\varphi \in \mathcal{D}(\Omega)$

$$\begin{aligned}\int_\Omega a(x,u,\nabla u)\cdot\nabla\varphi dx &= \lim_{n\to+\infty}\int_\Omega a(x,u,\nabla u)\cdot\nabla\varphi dx\\ &= \lim_{n\to+\infty} < \mu_n, \varphi > = < \mu, \varphi >_{\mathcal{D}'((\Omega),\mathcal{D}(\Omega)}.\end{aligned}$$

Ethics declarations

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References

[1] Y. Atik, *T-ensembles locaux et problèmes èlliptiques quasi-linéaires dègenerès â donnèe mèsure sur un ouvert quelconque*, Thѐse de doctorat en Sciences, Soutenue en 1993. Poitiers, France.
[2] A. Aberqi, J. Bennouna, O. Benslimane, M.A. Ragusa, *Existence results for double phase problem in Sobolev-Orlicz spaces with variable exponents in complete manifold*, Mediterranean Journal of Mathematics, 19 (4), (2022);art.n.158.
[3] C. Aykol, E. Kaya, *B-maximal operators, B-singular integral operators and B-Riesz potentials in variable exponent Lorentz spaces*, Filomat, 37 (17), (2023) 5765–5774.
[4] E. Azroul, M. B. Benboubker, and M. Rhoudaf, *Entropy Solution for Some p(x)-Quasilinear Problem with Right-Hand Side Measure*, (2012) 23-44.
[5] M. Bendahmane, and P. Wittbold, *Renormalized solutions for nonlinear elliptic equations with variable exponents and L1 data*, Nonlinear Analysis: Theory, Methods Applications, 70(2), (2009) 567-583.
[6] L. Boccardo, T. Gallouöt, *Nonlinear elliptic equations with right hand side measures*, Comm. Partial Dffierential Equations, 17(3-4), (1992) 641-655.
[7] Y. Chen, S. Levine, and M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math, 66, (2006) 1383-1406.
[8] L. Diening, *Riesz potential and Sobolev embeddings of generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$*, Math. Nachr, 268(1),(2004) 31-43.
[9] L. Diening, P. Hasto, T. Harjulehto, M. Ruiziicka *Lebesque and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, Vol., Springer Verlag. Berlin, (2011).
[10] J. Deny, J. L. Lions, *Espaces de Beppo Levi et applications*, Comptes rendus hybdomadaires des seances, 239(19), (1954) 1174-1177.
[11] X.L. Fan, D. Zhao, *On the spaces $L^{p(x)}(U)$ and $W^{m,p(x)}(U)$*, J. Math. Anal. Appl. 263(2), (2001) 424-446.
[12] X.L. Fan , J. Shen, D. Zhao, *Sobolev embedding theorems for spaces $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl. 262(2), (2001) 749-760.
[13] T. Harjulehto, Hästö, P., Koskenoja, M., Varonen, S, *The Dirichlet energy integral and variable exponent Sobolev spaces with zero boundary values*, Potential Anal, 25, (2006) 205-222.
[14] A. Irshaad, A. Fiorenza, M. Rosaria Formica, A. Gogatishvili, A. El Hamidi, and J. M. Rakotoson, *Quasilinear PD Es, Interpolation spaces and Hölderian mappings*, arXiv preprint arXiv:2211.01574 (2022).
[15] H. Khelifi, M.A, Zouatini, *Nonlinear degenerate $p(x)-$Laplacian equation with singular gradient and lower order term*, Indian J. Pure Appl. Math, (2023)1-21.
[16] H. Khelifi, Y. Elhadfi, *Nonlinear elliptic equations with variable exponents involving singular nonlinearity*, Mathematical Modeling and Computing, 8(4), (2021) 705–715.
[17] H. Khelifi, K. Ait-Mahiout, *Regularity for solutions of elliptic p (x)-Laplacian type equations with lower order terms and hardy potential*, Ricerche di Matematica, (2023) 1-15.
[18] H. Khelifi, *The Obstacle Problem For Nonlinear Degenerate Elliptic Equations with Variable Exponents and L1-Data*, J. Part. Diff. Eq., 35(1), (2022)101-122.
[19] O. Kovacik, J. Rakosnik, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J, 41(116), (1991) 592-618.
[20] J. M. Rakotoson, *Generalized solutions in a new type of sets for problems with measures as data*, Differential and Integral Equations 6(1), (1993) 27-36.
[21] J. M. Rakotoson, *Resolution of the critical cases for problems with L1-data*, Asymptotic Analysis, 6(3), (1991) 285-293.
[22] J. M. Rakotoson, *T-sets and relaxed solutions for parabolic equations*, Journal of differential equations, 111(2), (1994) 458-471.
[23] M.A. Ragusa, A. Tachikawa, *Regularity of minimizers for double phase functionals of borderline case with variable exponents, Advances in Nonlinear Analysis*, 13 (1),(2024); art.n. 20240017.
[24] M.A. Zouatini, F. Mokhtari, H. Khelifi, *Degenerate elliptic problem with singular gradient lower order term and variable exponents*, Mathematical Modeling and Computing, **10**(1), (2023)133-147.