Filomat 38:27 (2024), 9711–9725
https://doi.org/10.2298/FIL2427711P

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Fractals of $\mathcal{FG}$-Hutchinson Barnsley operator in metric spaces

Rajendra Pant[a], Hemant Kumar Nashine[a,b,*]

[a]*Department of Mathematics and Applied Mathematics, University of Johannesburg, Kingsway Campus, Auckland Park 2006, South Africa*
[b]*Mathematics Division, School of Advanced Sciences and Languages, VIT Bhopal University, Bhopal-Indore Highway, Kothrikalan, Sehore, Madhya Pradesh - 466114, India*

Abstract. The goal of this work is to build fractals using a finite family of $\mathcal{FG}$-contraction mappings. This class of mappings is more general than Banach contraction, F-contraction, Geraghty-contraction, rational-type contraction and many other contractions on metric spaces. We derive several consequence of our main result and discuss certain iterated function systems that fulfil a specific set of contractive requirements. Several examples are presented with accompanying graphics to substantiate the findings obtained here. Our results integrate, generalize, and extend previous work in this area.

1. Introductory notes

1.1. *Iterated Function System in Fractals Analysis*

Euclidean geometry defines regular objects such as points, curves, surfaces, and cubes using integer dimensions 0, 1, 2, and 3 correspondingly. Each dimension is associated with a measurement of the item, such as the length of a line, the area of a surface, or the volume of a cube. However, many natural things, such as coasts, rivers, lakes, and porous media, are disordered and uneven, and cannot be represented by Euclidean geometry due to scale-dependent length, area, and volume measurements. These objects are known as fractals, and their dimensions are specified as fractal dimensions [9]. Fractal theory has evolved into a distinct field of contemporary mathematics, essentially a new world view and technique.

Fractal geometry provides a broad framework for the investigation of irregular collections such as these. Fractal geometry is a branch of mathematics that is developed from classical geometry. A variety of physical configurations, from ferns to galaxies, may be accurately modelled using this kind of simulation. An example of fractal geometry is a geometric shape that is rough or fractured and may be divided into pieces, each of which is a duplicate of the whole, but with reduced size (at least approximately), a trait called self-similarity.

In 1975, Benoit Mandelbrot, coined the term “fractal”. The term fractal comes from the Latin word “fractus”, which meaning shattered or fractured, and was used to describe objects that were too irregular

2020 *Mathematics Subject Classification*. Primary 47H10; Secondary 54H25, 54E50
Keywords. Iterated function system; set-valued mapping; domain of sets; fixed point; metric space.
Received: 31 August 2023; Revised: 23 November 2023; Accepted: 04 April 2024
Communicated by Adrian Petrusel
* Corresponding author: Hemant Kumar Nashine
Email addresses: pant.rajendra@gmail.com, rpant@uj.ac.za (Rajendra Pant), hemantkumar.nashine@vitbhopal.ac.in, drhemantnashine@gmail.com (Hemant Kumar Nashine)

or complicated to fit into a conventional geometrical framework. Mandelbrot technically defined a fractal as a set whose Hausdorff dimension is strictly greater than its topological dimension [9–11].

Using the Banach fixed point theorem, Hutchinson [8] and Barnsley [3] began and developed the Hutchinson-Barnsley theory (HB theory) to define and create fractals as a compact invariant subset of a complete metric space formed by the Iterated Function System (IFS) of contractions. In other words, a fractal set is the unique fixed point of the Hutchinson-Barnsley operator (HB operator), described by the dynamical system of contraction mappings. An IFS was introduced as an application to the theory of discrete dynamical systems and is a valuable tool for the construction of fractal and other comparable sets. As a result, IFS have proved to be a very important method for creating self-similar or fractal objects.

1.2. Fixed Point Theory

Fixed point theory is a fascinating field of mathematics that is constantly evolving. It is a fusion of the disciplines of analysis, topology, and geometry. For past several decades, the theory of fixed points has been demonstrated to be a very important and useful instrument in the study of nonlinear processes. In 1922, the polish mathematician Stefan Banach established a bench mark theorem on the existence and uniqueness of a fixed point of contraction mappings in a complete metric space. This theorem is still considered to be a milestone in the history of metric fixed point theory.

Markin initiated the study of fixed points for set-valued mappings using the Pompeiu-Hausdorff metric. Nadler [13] pioneered the concept of set-valued contraction (or multivalued contraction) in 1969. He established the multivaled version of Banach's fixed point theorem. Subsequently, numerous contractions and multivalued contractions type mappings have been defined by many mathematicians [1, 4, 12, 14].

Several authors have modified and extended the HB theory by substituting alternative metric and topological fixed point theorems to create metric fractals, topological fractals, Tarski's fractals, semifractals, and multivalued fractals [16, 18–21, 24, 25, 27, 28].

1.3. Metric Fractals

We use the Hutchinson-Barnsley (HB) theory to define and create the IFS fractals in complete metric spaces.

Definition 1.1. [2]. *Let (Ξ, d) be a metric space. A self mapping f on Ξ is said to be Banach contraction (or simply contraction) on Ξ, if there exists $\lambda \in [0, 1)$ such that*

$$d(f(x), f(y)) \le \lambda\, d(x, y) \text{ for all } x, y \in \Xi,$$

where λ is known as a contraction constant.

Theorem 1.2. *A contraction mapping on a complete metric space has a unique fixed point.*

Let (Ξ, d) be a metric space and $\mathcal{K}(\Xi)$ the set of all non-empty compact subsets of Ξ. For $\mathcal{A}, \mathcal{B} \in \mathcal{K}(\Xi)$, the Pompeiu-Hausdorff metric induced by d is defined as:

$$\mathcal{H}(\mathcal{A}, \mathcal{B}) = \max\left\{\sup_{b\in\mathcal{B}} d(b, \mathcal{A}), \sup_{a\in\mathcal{A}} d(a, \mathcal{B})\right\},$$

where $d(x, \mathcal{B}) = \inf\{d(x, y) : y \in \mathcal{B}\}$ is the distance of a point x from the set $\mathcal{B}$. If (Ξ, d) is a compete metric space then $(\mathcal{K}(\Xi), \mathcal{H})$ is also a complete metric space.

Lemma 1.3. [13, 15]. *Let (Ξ, d) be a metric space. For all $\mathcal{A}, \mathcal{B}, C, \mathcal{D} \in \mathcal{K}(\Xi)$, the following hold:*

(i) *If $\mathcal{B} \subseteq C$, then $\sup_{a\in\mathcal{A}} d(a, C) \le \sup_{a\in\mathcal{A}} d(a, \mathcal{B})$.*

(ii) $\sup_{x\in\mathcal{A}\cup\mathcal{B}} d(x,C) = \max\left\{\sup_{a\in\mathcal{A}} d(a,C), \sup_{b\in B} d(b,C)\right\}$.

(iii) $\mathcal{H}(\mathcal{A}\cup\mathcal{B}, C\cup\mathcal{D}) \leq \max\{\mathcal{H}(\mathcal{A},C), \mathcal{H}(\mathcal{B},\mathcal{D})\}$.

Consider a finite family of contractions $(f_n)_{n=1}^N$ on Ξ with contraction constants $\lambda_n \in [0,1)$, respectively. Then the system $(\Xi; f_n : n = 1,2,3,\ldots,N)$ is called an IFS or finite iterated function system of contractions.

The HB operator $\mathfrak{I} : \mathcal{K}(\Xi) \to \mathcal{K}(\Xi)$ is defined by

$$\mathfrak{I}(\mathcal{A}) = \bigcup_{n=1}^{N} f_n(\mathcal{A}), \text{ for all } \mathcal{A} \in \mathcal{K}(\Xi).$$

The HB operator $\mathfrak{I}$ is a contraction on $\mathcal{K}(\Xi)$ with contraction constant $r = \max\{\lambda_n : n = 1,2,3,\ldots,N\}$.

Theorem 1.4. *Let (Ξ,d) be a metric space. Let $(\Xi; f_n : n = 1,2,3,\ldots,N)$ be an IFS of contractions. Then, the HB operator $\mathfrak{I}$ is a contraction mapping on $\mathcal{K}(\Xi)$.*

Theorem 1.5. (HB Theorem for Metric IFS [3, 8]). *Let (Ξ,d) be a complete metric space and $(\Xi; f_n : n = 1,2,3,\ldots,N)$ be an IFS of contractions. Then, $\mathfrak{I}$ has a unique fixed point namely $A^\infty \in \mathcal{K}(\Xi)$.*

Definition 1.6. (Metric Fractals [3]). *The fixed point $A^\infty \in \mathcal{K}(\Xi)$ of the HB operator $\mathfrak{I}$ described in the Theorem 1.5 is called the Attractor (Fractal) of the IFS of contractions. Sometimes $A^\infty \in \mathcal{K}(\Xi)$ is called a Metric Fractal generated by the IFS of contractions.*

Singh et al. [24] published a brief overview of fractal advances emerging from iterated map systems in 2009. Sahu et al. [21] proposed the K-iterated map system, which used the Kannan map to cover a wider variety of maps and proved the collage theorem for the K-iterated map system. Using a Banach-like fixed point theory, Xu et al. [28] established the existence of the attractors for Reich iterated map systems in 2015. As a consequence, the authors answered an open question raised by Singh et al. [24] and developed a collage theorem for Reich iterated map systems under the constraint that the Reich maps considered are continuous. By presenting several instances, Dung and Petrusel [5] pointed out the mistake in the findings of Xu et al. [28], and the result of IFS for pair of maps was also reviewed. With the aid of a finite family of F-contraction mappings, Secelean [22] build a fractal set of IFS using Wardowski's concept of F-contraction which is further generalized in [15] under generalized F-contraction. A more generalized F-IFS on product of metric spaces has been explored in [23].

1.4. Motivation

One of the most visually appealing uses of contraction mapping is found in the field of fractal theory. The question now is whether the aforementioned HB-operators can be enhanced and generalized. We explore iterated function systems comprised of $\mathcal{FG}$-contractions, extending certain fixed point findings from the conventional HB theory of IFS consisting of Banach-HB-operator, F-HB-operator, Geraghty-type-HB-operator, to provide an affirmative response. Additionally, two novel rational-type HB operators are deduced. Three examples with graphical presentations are offered to assist illustrate the topic.

1.5. Contribution

The following is the overview of the paper's structure. In Section 2, a fractal set of iterated function systems is a finite collection of mappings created on a metric space that induce compact valued mappings defined on a family of compact subsets of a metric space. We show that the HB operator is itself a generalised $\mathcal{FG}$-contraction mapping on a family of compact subsets of Ξ when defined with the assistance of a finite family of $\mathcal{FG}$-contraction mappings on a complete metric space. In Section 3, we get a final fractal by using a generalised $\mathcal{FG}$-Hutchinson operator repeatedly. Section 4 provides several nontrivial instances with graphical depiction to support the conclusion made here.

2. $\mathcal{FG}$-Hutchinson-Barnsley operator and based results

Motivated by Parvaneh *et al*. [17], we consider the following slightly modified definition. Let $\mathbb{R}$ and $\mathbb{R}_+$ denote the set of all real and positive real numbers, respectively.

Definition 2.1. *The collection of all functions $\mathcal{F} : \mathbb{R}_+ \to \mathbb{R}$ satisfying the following conditions will be denoted by $\mathbb{F}$:*

($\mathbb{F}_1$) $\mathcal{F}$ is continuous and strictly increasing;

($\mathbb{F}_2$) for each $\{\xi_n\} \subseteq \mathbb{R}_+$, $\lim_{n\to\infty} \xi_n = 0$ iff $\lim_{n\to\infty} \mathcal{F}(\xi_n) = -\infty$.

The collection of all pairs of mappings $(\mathcal{G}, \beta)$, where $\mathcal{G} : \mathbb{R}_+ \to \mathbb{R}$, $\beta : \mathbb{R}_+ \to (0, 1)$ satisfying the following conditions will be denoted by $\mathbb{G}_\beta$:

($\mathbb{F}_3$) for each $\{\xi_n\} \subseteq \mathbb{R}_+$, $\limsup_{n\to\infty} \mathcal{G}(\xi_n) \geq 0$ iff $\limsup_{n\to\infty} \xi_n \geq 1$;

($\mathbb{F}_4$) for each $\{\xi_n\} \subseteq \mathbb{R}_+$, $\limsup_{n\to\infty} \beta(\xi_n) = 1$ implies $\lim_{n\to\infty} \xi_n = 0$;

($\mathbb{F}_5$) for each $\{\xi_n\} \subseteq \mathbb{R}_+$, $\sum_{n=1}^{\infty} \mathcal{G}(\beta(\xi_n)) = -\infty$.

Definition 2.2. *Let (Ξ, d) be a metric space. A self-mapping f on Ξ is called a $\mathcal{FG}$-contraction if there exist $\mathcal{F} \in \mathbb{F}$ and $(\mathcal{G}, \beta) \in \mathbb{G}_\beta$ such that for any $x, y \in \Xi$, $d(fx, fy) > 0$ implies*

$$\mathcal{F}(d(fx, fy)) \leq \mathcal{F}(d(x, y)) + \mathcal{G}(\beta(d(x, y))). \tag{1}$$

Theorem 2.3. *Let (Ξ, d) be a metric space and $f : \Xi \to \Xi$ an $\mathcal{FG}$-contraction and continuous. Then $f : \mathcal{K}(\Xi) \to \mathcal{K}(\Xi)$ defined by $f(\mathcal{A}) = \{f(x) : x \in \mathcal{A}\}$ is also $\mathcal{FG}$-contraction mapping on $(\mathcal{K}(\Xi), \mathcal{H})$.*

Proof. By the continuity of f, the image of a compact subset under $f : \Xi \to \Xi$ is compact, so we obtain

$$\mathcal{A} \in \mathcal{K}(\Xi) \text{ implies } f(\mathcal{A}) \in \mathcal{K}(\Xi).$$

Now, let $\mathcal{A}, \mathcal{B} \in \mathcal{K}(\Xi)$ with $\mathcal{H}(f(\mathcal{A}), f(\mathcal{B})) \neq 0$. Since $f : \Xi \to \Xi$ is a $\mathcal{FG}$-contraction, by ($\mathbb{F}_5$) we have

$$0 < d(fx,\ fy) < d(x,\ y) \text{ for all } x, y \in \Xi, fx \neq fy.$$

Thus,

$$d(fx, f(\mathcal{B})) = \inf_{y\in\mathcal{B}} d(fx,\ fy) \leq \inf_{y\in\mathcal{B}} d(x, y) = d(x, \mathcal{B}).$$

Similarly,

$$d(fy, f(\mathcal{A})) = \inf_{x\in\mathcal{A}} d(fy, fx) \leq \inf_{x\in\mathcal{A}} d(y, x) = d(y, \mathcal{A}).$$

Now

$$\begin{aligned} \mathcal{H}(f(\mathcal{A}), f(\mathcal{B})) &= \max\{\sup_{x\in\mathcal{A}} d(fx, f(\mathcal{B})), \sup_{y\in\mathcal{B}} d(fy, f(\mathcal{A}))\} \\ &\leq \max\{\sup_{x\in\mathcal{A}} d(x, \mathcal{B}), \sup_{y\in\mathcal{B}} d(y, \mathcal{A})\} = \mathcal{H}(\mathcal{A}, \mathcal{B}). \end{aligned}$$

By strictly increasing of $\mathcal{F}$ implies

$$\mathcal{F}(\mathcal{H}(f(\mathcal{A}), f(\mathcal{B}))) \leq \mathcal{F}(\mathcal{H}(\mathcal{A}, \mathcal{B})).$$

By the definition of the function $(\mathcal{G}, \beta)$ there exists a $(\mathcal{G}, \beta) \in \mathbb{G}_\beta$ such that

$$\mathcal{F}(\mathcal{H}(f(\mathcal{A}), f(\mathcal{B}))) \leq \mathcal{F}(\mathcal{H}(\mathcal{A}, \mathcal{B})) + \mathcal{G}(\beta(\mathcal{H}(\mathcal{A}, \mathcal{B}))).$$

Hence $f : \mathcal{K}(\Xi) \to \mathcal{K}(\Xi)$ is a $\mathcal{FG}$-contraction (see [28, Lemma 3.1]). □

Theorem 2.4. *Let (Ξ, d) be a metric space and $\{f_n : n = 1, 2, \ldots, N\}$ a finite family of $\mathcal{FG}$-contractions, continuous self-mappings on Ξ. Define $\Im : \mathcal{K}(\Xi) \to \mathcal{K}(\Xi)$ by*

$$\Im(\mathcal{A}) = f_1(\mathcal{A}) \cup f_2(\mathcal{A}) \cup \cdots \cup f_N(\mathcal{A}) = \bigcup_{n=1}^{N} f_n(\mathcal{A}), \text{ for each } \mathcal{A} \in \mathcal{K}(\Xi).$$

Then $\Im$ is a $\mathcal{FG}$-contraction on $\mathcal{K}(\Xi)$.

Proof. We show that the assertion is true for $N = 2$. Let $f_1, f_2 : \Xi \to \Xi$ be two $\mathcal{FG}$-contractions. Take $\mathcal{A}, \mathcal{B} \in \mathcal{K}(\Xi)$ with $\mathcal{H}(\Im(\mathcal{A}), \Im(\mathcal{B})) \neq 0$. From Lemma 1.3 (iii), it follows that

$$\begin{aligned}
& \mathcal{F}(\mathcal{H}(\Im(\mathcal{A}), \Im(\mathcal{B}))) \\
= \ & \mathcal{F}(\mathcal{H}(f_1(\mathcal{A}) \cup f_2(\mathcal{A}), \ f_1(\mathcal{B}) \cup f_2(\mathcal{B}))) \\
\leq \ & \mathcal{F}(\max\{\mathcal{H}(f_1(\mathcal{A}), f_1(\mathcal{B})), \mathcal{H}(f_2(\mathcal{A}), f_2(\mathcal{B}))\}) \\
\leq \ & \mathcal{F}(\mathcal{H}(\mathcal{A}, \mathcal{B})) + \mathcal{G}(\beta(\mathcal{H}(\mathcal{A}, \mathcal{B}))).
\end{aligned}$$

□

Definition 2.5. *Let (Ξ, d) be a metric space. A mapping $\Im : \mathcal{K}(\Xi) \to \mathcal{K}(\Xi)$ is said to be a $\mathcal{FG}$-contraction if for $\mathcal{F} \in \mathbb{F}$ and $(\mathcal{G}, \beta) \in \mathbb{G}_\beta$ such that for any $\mathcal{A}, \mathcal{B} \in \mathcal{K}(\Xi)$ with $\mathcal{H}(\Im(\mathcal{A}), \ \Im(\mathcal{B})) \neq 0$, the following holds:*

$$\mathcal{F}(\mathcal{H}(\Im(\mathcal{A}), \Im(\mathcal{B}))) \leq \mathcal{F}(\Theta(\mathcal{A}, \mathcal{B})) + \mathcal{G}(\beta(\Theta(\mathcal{A}, \mathcal{B}))) \tag{2}$$

where

$$\Theta(\mathcal{A}, \mathcal{B}) = \max\left\{ \begin{array}{c} \mathcal{H}(\mathcal{A}, \mathcal{B}), \mathcal{H}(\mathcal{A}, \Im(\mathcal{A})), \mathcal{H}(\mathcal{B}, \Im(\mathcal{B})), \frac{\mathcal{H}(\mathcal{A}, \Im(\mathcal{B})) + \mathcal{H}(\mathcal{B}, \Im(\mathcal{A}))}{2}, \\ \mathcal{H}(\Im^2(\mathcal{A}), \Im(\mathcal{A})), \mathcal{H}(\Im^2(\mathcal{A}), \mathcal{B}), \mathcal{H}(\Im^2(\mathcal{A}), \Im(\mathcal{B})) \end{array} \right\}. \tag{3}$$

Theorem 2.6. *Let (Ξ, d) be a metric space and $\{f_n : n = 1, 2, \ldots, N\}$ a finite sequence of continuous, $\mathcal{FG}$-contraction mappings on Ξ. If $\Im : \mathcal{K}(\Xi) \to \mathcal{K}(\Xi)$ is defined by*

$$\Im(\mathcal{A}) = f_1(\mathcal{A}) \cup f_2(\mathcal{A}) \cup \cdots \cup f_N(\mathcal{A}) = \bigcup_{n=1}^{N} f_n(\mathcal{A}), \text{ for each } \mathcal{A} \in \mathcal{K}(\Xi),$$

then $\Im$ is a $\mathcal{FG}$-contraction mapping on $\mathcal{K}(\Xi)$.

Proof. Using Theorem 2.4 with property $(\mathbb{F}_1)$, the result follows. □

An operator $\Im$ in above Theorem is called $\mathcal{FG}$-Hutchinson-Barnsley operator or $\mathcal{FG}$-HB operator.

Definition 2.7. *Let Ξ be a complete metric space. If $f_n : \Xi \to \Xi$, $n = 1, 2, \ldots, N$ are $\mathcal{FG}$-contraction mappings, then $(\Xi; f_1, f_2, \ldots, f_N)$ is called $\mathcal{FG}$-contractive iterated function system (IFS).*

Thus $\mathcal{FG}$-contractive iterated function system consists of a complete metric space and finite family of $\mathcal{FG}$-contraction mappings on Ξ.

Definition 2.8. *A nonempty compact set $\mathcal{A} \subset \Xi$ is said to be an attractor of the $\mathcal{FG}$-contractive IFS $\Im$ if*

(i) *$\Im(\mathcal{A}) = \mathcal{A}$ and*

(ii) *there is an open set $\mathcal{U} \subset \Xi$ such that $\mathcal{A} \subset \mathcal{U}$ and $\lim_{n\to\infty} \Im^n(\mathcal{B}) = \mathcal{A}$ for any compact set $\mathcal{B} \subset \mathcal{U}$, where the limit is taken with respect to the Pompeiu-Hausdorff metric.*

The largest open set $\mathcal{V}$ satisfying (ii) is called a basin of attraction.

3. Main Results

We begin with the following main outcome, that is, attractor of $\mathcal{F}\mathcal{G}$-Hutchinson-Barnsley operator.

Theorem 3.1. *Let (Ξ, d) be a complete metric space and $\{\Xi; f_n,\ n = 1, 2, \cdots, k\}$ a continuous, $\mathcal{F}\mathcal{G}$-contractive IFS. Let $\Im : \mathcal{K}(\Xi) \to \mathcal{K}(\Xi)$ be defined by*

$$\Im(\mathcal{A}) = \bigcup_{n=1}^{k} f_n(\mathcal{A}), \text{ for all } \mathcal{A} \in \mathcal{K}(\Xi)$$

is $\mathcal{F}\mathcal{G}$-Hutchinson-Barnsley operator. Then hold the following:

(a) *Operator $\Im$ has a unique fixed point $\mathcal{U} \in \mathcal{K}(\Xi)$, that is*

$$\mathcal{U} = \Im(\mathcal{U}) = \bigcup_{n=1}^{k} f_n(\mathcal{U}).$$

(b) *For any initial set $\mathcal{A}_0 \in \mathcal{K}(\Xi)$, the sequence of compact sets $\{\mathcal{A}_0,\ \Im(\mathcal{A}_0),\ \Im^2(\mathcal{A}_0), \ldots\}$ converges to a fixed point of $\Im$.*

Proof. Let $\mathcal{A}_0$ be an arbitrary element in $\mathcal{K}(\Xi)$. If $\mathcal{A}_0 = \Im(\mathcal{A}_0)$, then the proof is complete. So, we assume that $\mathcal{A}_0 \neq \Im(\mathcal{A}_0)$. Define

$$\mathcal{A}_1 = \Im(\mathcal{A}_0), \mathcal{A}_2 = \Im(\mathcal{A}_1), \ldots, \ \mathcal{A}_{m+1} = \Im(\mathcal{A}_m)$$

for $m \in \mathbb{N}$.

We may assume that $\mathcal{A}_m \neq \mathcal{A}_{m+1}$ for all $m \in \mathbb{N}$. If not, then $\mathcal{A}_j = \mathcal{A}_{j+1}$ for some j implies $\mathcal{A}_j = \Im(\mathcal{A}_j)$ and this completes the proof. Take $\mathcal{A}_m \neq \mathcal{A}_{m+1}$ for all $m \in \mathbb{N}$. From (1), we have

$$\begin{aligned} & \mathcal{F}(\mathcal{H}(\mathcal{A}_{m+1}, \mathcal{A}_{m+2})) \\ = \ & \mathcal{F}(\mathcal{H}(\Im(\mathcal{A}_m), \Im(\mathcal{A}_{m+1}))) \\ \leq \ & \mathcal{F}(\Theta(\mathcal{A}_m, \mathcal{A}_{m+1})) + \mathcal{G}(\beta(\Theta(\mathcal{A}_m, \mathcal{A}_{m+1}))), \end{aligned}$$

where

$$\begin{aligned} & \Theta(\mathcal{A}_m, \mathcal{A}_{m+1}) \\ = \ & \max\left\{ \begin{array}{c} \mathcal{H}(\mathcal{A}_m, \mathcal{A}_{m+1}), \mathcal{H}(\mathcal{A}_m, \Im(\mathcal{A}_m)), \mathcal{H}(\mathcal{A}_{m+1}, \Im(\mathcal{A}_{m+1})), \\ \frac{\mathcal{H}(\mathcal{A}_m, \Im(\mathcal{A}_{m+1})) + \mathcal{H}(\mathcal{A}_{m+1}, \Im(\mathcal{A}_m))}{2}, \mathcal{H}(\Im^2(\mathcal{A}_m), \Im(\mathcal{A}_m)), \\ \mathcal{H}(\Im^2(\mathcal{A}_m), \mathcal{A}_{m+1}), \mathcal{H}(\Im^2(\mathcal{A}_m), \Im(\mathcal{A}_{m+1})) \end{array} \right\} \\ = \ & \max\left\{ \begin{array}{c} \mathcal{H}(\mathcal{A}_m, \mathcal{A}_{m+1}), \mathcal{H}(\mathcal{A}_m, \mathcal{A}_{m+1}), \mathcal{H}(\mathcal{A}_{m+1}, \mathcal{A}_{m+2}), \\ \frac{\mathcal{H}(\mathcal{A}_m, \mathcal{A}_{m+2}) + \mathcal{H}(\mathcal{A}_{m+1}, \mathcal{A}_{m+1})}{2}, \mathcal{H}(\mathcal{A}_{m+2}, \mathcal{A}_{m+1}), \\ \mathcal{H}(\mathcal{A}_{m+2}, \mathcal{A}_{m+1}), \mathcal{H}(\mathcal{A}_{m+2}, \mathcal{A}_{m+2}) \end{array} \right\} \\ = \ & \max\{\mathcal{H}(\mathcal{A}_m, \mathcal{A}_{m+1}), \mathcal{H}(\mathcal{A}_{m+1}, \mathcal{A}_{m+2})\}. \end{aligned}$$

In case $\Theta(\mathcal{A}_m, \mathcal{A}_{m+1}) = \mathcal{H}(\mathcal{A}_{m+1}, \mathcal{A}_{m+2})$, we have

$$\mathcal{F}(\mathcal{H}(\mathcal{A}_{m+1}, \mathcal{A}_{m+2})) \leq \mathcal{F}(\mathcal{H}(\mathcal{A}_{m+1}, \mathcal{A}_{m+2})) + \mathcal{G}(\beta(\mathcal{H}(\mathcal{A}_{m+1}, \mathcal{A}_{m+2})),$$

Therefore $\mathcal{G}(\beta(\mathcal{H}(\mathcal{A}_{m+1}, \mathcal{A}_{m+2})) \geq 0$, which yields that $\beta(\mathcal{H}(\mathcal{A}_{m+1}, \mathcal{A}_{m+2})) \geq 1$, which is a contradiction. Therefore $\Theta(\mathcal{A}_m, \mathcal{A}_{m+1}) = \mathcal{H}(\mathcal{A}_m, \mathcal{A}_{m+1})$ and we have

$$\mathcal{F}(\mathcal{H}(\mathcal{A}_{m+1}, \mathcal{A}_{m+2})) \ \leq \ \mathcal{F}(\mathcal{H}(\mathcal{A}_m, \mathcal{A}_{m+1})) + \mathcal{G}(\beta(\mathcal{H}(\mathcal{A}_m, \mathcal{A}_{m+1})))$$

for all $m \in \mathbb{N}$. We conclude that

$$\begin{aligned}\mathcal{F}(\mathcal{H}(\mathcal{A}_m,\mathcal{A}_{m+1})) &\leq \mathcal{F}(\mathcal{H}(\mathcal{A}_{m-1},\mathcal{A}_m)) + \mathcal{G}(\beta(\Theta(\mathcal{A}_{m-1},\mathcal{A}_m)))\\ &\leq \mathcal{F}(\mathcal{H}(\mathcal{A}_{m-2},\mathcal{A}_{m-1})) + \mathcal{G}(\beta(\Theta(\mathcal{A}_{m-1},\mathcal{A}_m))) + \mathcal{G}(\beta(\Theta(\mathcal{A}_{m-2},\mathcal{A}_{m-1})))\\ &\vdots\\ &\leq \mathcal{F}(\mathcal{H}(\mathcal{A}_0,\mathcal{A}_1)) + \sum_{n=1}^{m}\mathcal{G}(\beta(\Theta(\mathcal{A}_{n-1},\mathcal{A}_n)))\end{aligned}$$

that is,

$$\mathcal{F}(\mathcal{H}(\mathcal{A}_m,\mathcal{A}_{m+1})) \leq \mathcal{F}(\mathcal{H}(\mathcal{A}_0,\mathcal{A}_1)) + \sum_{m=1}^{n}\mathcal{G}(\beta(\Theta(\mathcal{A}_{m-1},\mathcal{A}_m)))$$

for all $n \in \mathbb{N}$. By the properties of $(\mathcal{G},\beta) \in \mathbb{G}_\beta$, $\lim_{n\to\infty}\mathcal{F}(\mathcal{H}(\mathcal{A}_m,\mathcal{A}_{m+1})) = -\infty$ and by $(\mathbb{F}_2)$, $\lim_{n\to\infty}\mathcal{H}(\mathcal{A}_m,\mathcal{A}_{m+1}) = 0$.

Next, we assert that the sequence $\{\mathcal{A}_n\}$ is Cauchy. Assume the opposite, that there is a $\epsilon > 0$ and two ascending sequences of numbers $\{r(\ell)\}$ and $\{s(\ell)\}$, $s(\ell) > r(\ell) > \ell$ such that $\mathcal{H}(\mathcal{A}_{r(\ell)},\mathcal{A}_{s(\ell)})$, $\mathcal{H}(\mathcal{A}_{s(\ell)+1},\mathcal{A}_{r(\ell)-1})$, $\mathcal{H}(\mathcal{A}_{s(\ell)},\mathcal{A}_{r(\ell)-1}))$, $\mathcal{H}(\mathcal{A}_{r(\ell)+1},\mathcal{A}_{s(\ell)})$, and $\mathcal{H}(\mathcal{A}_{r(\ell)+1},\mathcal{A}_{s(\ell)+1})$ tend to ϵ as $\ell \to \infty$.

Due to (2) with $\mathcal{A} = \mathcal{A}_{r(\ell)-1}$ and $\mathcal{B} = \mathcal{A}_{s(\ell)}$, we have

$$\begin{aligned}\mathcal{F}(\mathcal{H}(\mathcal{A}_{r(\ell)},\mathcal{A}_{s(\ell)+1})) &= \mathcal{F}(\mathcal{H}(\Im(\mathcal{A}_{r(\ell)-1}),\Im(\mathcal{A}_{s(\ell)})))\\ &\leq \mathcal{F}(\Theta(\mathcal{A}_{r(\ell)-1},\mathcal{A}_{s(\ell)})) + \mathcal{G}(\beta(\Theta(\mathcal{A}_{r(\ell)-1},\mathcal{A}_{s(\ell)}))),\end{aligned} \tag{4}$$

where

$$\begin{aligned}&\Theta(\mathcal{A}_{r(\ell)-1},\mathcal{A}_{s(\ell)})\\ &= \max\left\{\begin{array}{c}\mathcal{H}(\mathcal{A}_{r(\ell)-1},\mathcal{A}_{s(\ell)}),\mathcal{H}(\mathcal{A}_{r(\ell)-1},\Im(\mathcal{A}_{r(\ell)-1})),\mathcal{H}(\mathcal{A}_{s(\ell)},\Im(\mathcal{A}_{s(\ell)})),\\ \frac{\mathcal{H}(\mathcal{A}_{r(\ell)-1},\Im(\mathcal{A}_{s(\ell)}))+\mathcal{H}(\mathcal{A}_{s(\ell)},\Im(\mathcal{A}_{r(\ell)-1}))}{2},\mathcal{H}(\Im^2(\mathcal{A}_{r(\ell)-1}),\Im(\mathcal{A}_{r(\ell)-1})),\\ \mathcal{H}(\Im^2(\mathcal{A}_{r(\ell)-1}),\mathcal{A}_{s(\ell)}),\mathcal{H}(\Im^2(\mathcal{A}_{r(\ell)-1}),\Im(\mathcal{A}_{s(\ell)}))\end{array}\right\}\\ &= \max\left\{\begin{array}{c}\mathcal{H}(\mathcal{A}_{r(\ell)-1},\mathcal{A}_{s(\ell)}),\mathcal{H}(\mathcal{A}_{r(\ell)-1},\mathcal{A}_{r(\ell)}),\mathcal{H}(\mathcal{A}_{s(\ell)},\mathcal{A}_{s(\ell)+1}),\\ \frac{\mathcal{H}(\mathcal{A}_{r(\ell)-1},\mathcal{A}_{s(\ell)+1})+\mathcal{H}(\mathcal{A}_{s(\ell)},\mathcal{A}_{r(\ell)})}{2},\mathcal{H}(\mathcal{A}_{r(\ell)+1},\mathcal{A}_{r(\ell)}),\\ \mathcal{H}(\mathcal{A}_{r(\ell)+1},\mathcal{A}_{s(\ell)}),\mathcal{H}(\mathcal{A}_{r(\ell)+1},\mathcal{A}_{s(\ell)+1})\end{array}\right\}.\end{aligned} \tag{5}$$

Taking the limit as $\ell \to \infty$ in (5),

$$\lim_{k\to\infty}\Theta(\mathcal{A}_{r(\ell)-1},\mathcal{A}_{s(\ell)}) = \max\{\epsilon,0,0,\tfrac{1}{2}(\epsilon+\epsilon),0,\epsilon,\epsilon\} = \epsilon.$$

Taking the limit as $\ell \to \infty$ in (4), we get

$$\begin{aligned}\mathcal{F}(\epsilon) &\leq \mathcal{F}(\limsup_{\ell\to\infty}\mathcal{H}(\mathcal{A}_{r(\ell)},\mathcal{A}_{s(\ell)+1})\\ &\leq \limsup_{\ell\to\infty}\mathcal{F}(\Theta(\mathcal{A}_{r(\ell)-1},\mathcal{A}_{s(\ell)})) + \limsup_{\ell\to\infty}\mathcal{G}(\beta(\Theta(\mathcal{A}_{r(\ell)-1},\mathcal{A}_{s(\ell)}))),\\ &\leq \mathcal{F}(\epsilon) + \limsup_{\ell\to\infty}\mathcal{G}(\beta(\Theta(\mathcal{A}_{r(\ell)-1},\mathcal{A}_{s(\ell)}))),\end{aligned}$$

which further implies

$$\limsup_{\ell\to\infty}\mathcal{G}(\beta(\Theta(\mathcal{A}_{r(\ell)-1},\mathcal{A}_{s(\ell)}))) \geq 0.$$

Using the properties of functions $\mathcal{G}$ and β, we get $\limsup_{\ell\to\infty}\beta(\Theta(\mathcal{A}_{r(\ell)-1},\mathcal{A}_{s(\ell)})) = 1$ and $\lim_{\ell\to\infty}\Theta(\mathcal{A}_{r(\ell)-1},\mathcal{A}_{s(\ell)}) = 0$, which is a contradiction with $\epsilon > 0$. Therefore $\{\mathcal{A}_m\}$ is a Cauchy sequence in Ξ. Since $(\mathcal{K}(\Xi),d)$ is complete, we have $\mathcal{A}_m \to \mathcal{U}$ as $m \to \infty$ for some $\mathcal{U} \in \mathcal{K}(\Xi)$.

In order to show that $\mathcal{U}$ is the fixed point of $\Im$, we contrary assume that Pompeiu-Hausdorff distance between $\mathcal{U}$ and $\Im(\mathcal{U})$ is not zero. Now

$$\begin{aligned} \mathcal{F}(\mathcal{H}(\mathcal{A}_{m+1}, \Im(\mathcal{U}))) &= \mathcal{F}(\mathcal{H}(\Im(\mathcal{A}_m), \Im(\mathcal{U}))) \\ &\leq \mathcal{F}(\Theta(\mathcal{A}_m, \mathcal{U})) + \mathcal{G}(\beta(\Theta(\mathcal{A}_m, \mathcal{U}))), \end{aligned} \tag{6}$$

where

$$\begin{aligned} &\Theta(\mathcal{A}_m, \mathcal{U}) \\ &= \max\left\{ \begin{array}{c} \mathcal{H}(\mathcal{A}_m, \mathcal{U}), \mathcal{H}(\mathcal{A}_m, \Im(\mathcal{A}_m)), \mathcal{H}(\mathcal{U}, \Im(\mathcal{U})), \\ \frac{\mathcal{H}(\mathcal{A}_m, \Im(\mathcal{U})) + \mathcal{H}(\mathcal{U}, \Im(\mathcal{A}_m))}{2}, \mathcal{H}(\Im^2(\mathcal{A}_m), \Im(\mathcal{A}_m)), \\ \mathcal{H}(\Im^2(\mathcal{A}_m), \mathcal{U}), \mathcal{H}(\Im^2(\mathcal{A}_m), \Im(\mathcal{U})) \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} \mathcal{H}(\mathcal{A}_m, \mathcal{U}), \mathcal{H}(\mathcal{A}_m, \mathcal{A}_{m+1}), \mathcal{H}(\mathcal{U}, \Im(\mathcal{U})), \\ \frac{\mathcal{H}(\mathcal{A}_m, \Im(\mathcal{U})) + \mathcal{H}(\mathcal{U}, \mathcal{A}_{m+1})}{2}, \mathcal{H}(\mathcal{A}_{m+2}, \mathcal{A}_{m+1}), \\ \mathcal{H}(\mathcal{A}_{m+2}, \mathcal{U}), \mathcal{H}(\mathcal{A}_{m+2}, \Im(\mathcal{U})) \end{array} \right\}. \end{aligned}$$

On making $m \to \infty$, we have

$$\mathcal{F}(\mathcal{H}(\Im(\mathcal{U}), \mathcal{U})) \leq \mathcal{F}(\mathcal{H}(\mathcal{U}, \Im(\mathcal{U}))) + \mathcal{G}(\beta(\mathcal{H}(\mathcal{U}, \Im(\mathcal{U})))),$$

a contradiction by the properties of $(\mathcal{G}, \beta) \in \mathbb{G}_\beta$.

Thus, $\mathcal{U}$ is the fixed point of $\Im$. To show the uniqueness of fixed point of $\Im$, assume that $\mathcal{U}$ and $\mathcal{V}$ are two fixed points of $\Im$ with $\mathcal{H}(\mathcal{U}, \mathcal{V})$ is not zero. Since $\Im$ is a $\mathcal{F}\mathcal{G}$-HB operator, we obtain that

$$\begin{aligned} \mathcal{F}(\mathcal{H}(\mathcal{U}, \mathcal{V})) &= \mathcal{F}(\mathcal{H}(\Im(\mathcal{U}), \Im(\mathcal{V}))) \\ &\leq \mathcal{F}(\Theta(\mathcal{U}, \mathcal{V})) + \mathcal{G}(\beta(\Theta(\mathcal{U}, \mathcal{V}))), \end{aligned}$$

where

$$\begin{aligned} \Theta(\mathcal{U}, \mathcal{V}) &= \max\left\{ \begin{array}{c} \mathcal{H}(\mathcal{U}, \mathcal{V}), \mathcal{H}(\mathcal{U}, \Im(\mathcal{U})), \mathcal{H}(\mathcal{V}, \Im(\mathcal{V})), \frac{\mathcal{H}(\mathcal{U}, \Im(\mathcal{V})) + \mathcal{H}(\mathcal{V}, \Im(\mathcal{U}))}{2}, \\ \mathcal{H}(\Im^2(\mathcal{U}), \mathcal{U}), \mathcal{H}(\Im^2(\mathcal{U}), \mathcal{V}), \mathcal{H}(\Im^2(\mathcal{U}), \Im(\mathcal{V})) \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} \mathcal{H}(\mathcal{U}, \mathcal{V}), \mathcal{H}(\mathcal{U}, \mathcal{U}), \mathcal{H}(\mathcal{V}, \mathcal{V}), \frac{\mathcal{H}(\mathcal{U}, \mathcal{V}) + \mathcal{H}(\mathcal{V}, \mathcal{U})}{2}, \\ \mathcal{H}(\mathcal{U}, \mathcal{U}), \mathcal{H}(\mathcal{U}, \mathcal{V}), \mathcal{H}(\mathcal{U}, \mathcal{V}) \end{array} \right\} \\ &= \mathcal{H}(\mathcal{U}, \mathcal{V}), \end{aligned}$$

that is,

$$\mathcal{F}(\mathcal{H}(\mathcal{U}, \mathcal{V})) \leq \mathcal{F}(\mathcal{H}(\mathcal{U}, \mathcal{V})) + \mathcal{G}(\beta(\mathcal{H}(\mathcal{U}, \mathcal{V}))),$$

a contradiction by the properties of $(\mathcal{G}, \beta) \in \mathbb{G}_\beta$. Thus $\Im$ has a unique fixed point $\mathcal{U} \in \mathcal{K}(\Xi)$. □

We get various classes of $\mathcal{F}\mathcal{G}$-HB operator and $\mathcal{F}\mathcal{G}$-contractive IFS in a complete metric space by considering a variety of concrete functions $\mathcal{F} \in \mathbb{F}$ and $(\mathcal{G}, \beta) \in \mathbb{G}_\beta$ in the Theorem 3.1. We are listing few of them below.

Corollary 3.2. *Let (Ξ, d) be a complete metric space and $\{\Xi; f_n,\ n = 1, 2, \cdots, k\}$ a continuous, Wardowski-type-contractive IFS, that is, each f_i ($i = 1, 2, \ldots, k$) is a self-mapping on Ξ such that*

$$\tau + \mathcal{F}(d(f_i x, f_i y)) \leq \mathcal{F}(d(x, y)),$$

for all $x, y \in \Xi$, $f_i x \neq f_i y$, where $\tau > 0$. Let $\Im : \mathcal{K}(\Xi) \to \mathcal{K}(\Xi)$ be defined by

$$\Im(\mathcal{A}) = \bigcup_{n=1}^{k} f_n(\mathcal{A}), \text{ for all } \mathcal{A} \in \mathcal{K}(\Xi)$$

is Wardowski-type-HB operator, that is,

$$\tau + \mathcal{F}(\mathcal{H}(\Im(\mathcal{A}), \Im(\mathcal{B}))) \leq \mathcal{F}(\Theta(\mathcal{A}, \mathcal{B})) \tag{7}$$

where $\Theta(\mathcal{A}, \mathcal{B})$ *is given in* (3). *Then results (a)-(b) of Theorem* 3.1 *hold.*

Proof. This comes when we take $\mathcal{G}(t) = \ln t$ $(t > 0)$, $\beta(t) = \lambda \in (0, 1)$ and $\tau = -\ln \lambda > 0$ in the Theorem 3.1. □

Corollary 3.3. *Let* (Ξ, d) *be a complete metric space and* $\{\Xi; f_n,\ n = 1, 2, \cdots,\ k\}$ *a continuous, Banach-type-contractive IFS, that is, each* f_i $(i = 1, 2, \ldots, k)$ *is a self-mapping on* Ξ *such that*

$$d(f_i x, f_i y)) \leq \lambda\, d(x, y),$$

for all $x, y \in \Xi$, $f_i x \neq f_i y$, $\lambda \in (0, 1)$. *Let* $\Im : \mathcal{K}(\Xi) \to \mathcal{K}(\Xi)$ *be defined by*

$$\Im(\mathcal{A}) = \bigcup_{n=1}^{k} f_n(\mathcal{A}), \text{ for all } \mathcal{A} \in \mathcal{K}(\Xi)$$

is a Banach-type-HB operator, that is, there exists $\lambda \in (0, 1)$ *such that*

$$\mathcal{H}(\Im(\mathcal{A}), \Im(\mathcal{B})) \leq \lambda\, \Theta(\mathcal{A}, \mathcal{B}) \tag{8}$$

where $\Theta(\mathcal{A}, \mathcal{B})$ *is given in* (3). *Then results (a)-(b) of Theorem* 3.1 *hold.*

Proof. Put $\mathcal{F}(t) = \mathcal{G}(t) = \ln t$ $(t > 0)$, $\beta(t) = \lambda \in (0, 1)$ in the Theorem 3.1, we have results. □

Corollary 3.4. *Let* (Ξ, d) *be a complete metric space and* $\{\Xi; f_n,\ n = 1, 2, \cdots,\ k\}$ *a continuous, Geraghty-type-contractive IFS, that is, each* f_i $(i = 1, 2, \ldots, k)$ *is a self-mapping on* Ξ *such that*

$$d(f_i x, f_i y)) \leq \beta(d(x, y))d(x, y),$$

for all $x, y \in \Xi$, $f_i x \neq f_i y$, $(\mathcal{G}, \beta) \in \mathbb{G}_\beta$. *Let* $\Im : \mathcal{K}(\Xi) \to \mathcal{K}(\Xi)$ *be defined by*

$$\Im(\mathcal{A}) = \bigcup_{n=1}^{k} f_n(\mathcal{A}), \text{ for all } \mathcal{A} \in \mathcal{K}(\Xi)$$

is a Geraghty-type-HB operator, that is, there exists $(\mathcal{G}, \beta) \in \mathbb{G}_\beta$ *such that*

$$\mathcal{H}(\Im(\mathcal{A}), \Im(\mathcal{B})) \leq \beta(\Theta(\mathcal{A}, \mathcal{B}))\Theta(\mathcal{A}, \mathcal{B}) \tag{9}$$

where $\Theta(\mathcal{A}, \mathcal{B})$ *is given in* (3). *Then results (a)-(b) of Theorem* 3.1 *hold.*

Proof. If we take $\mathcal{F}(t) = \mathcal{G}(t) = \ln t$ $(t > 0)$ in the Theorem 3.1, we have Geraghty-type [6] results. □

Corollary 3.5. *Let* (Ξ, d) *be a complete metric space and* $\{\Xi; f_n,\ n = 1, 2, \cdots,\ k\}$ *a continuous, rational-type (I)-contractive IFS, that is, each* f_i $(i = 1, 2, \ldots, k)$ *is a self-mapping on* Ξ *such that*

$$d(f_i x, f_i y)) \leq \frac{d(x, y)}{(1 + \tau \sqrt{d(x, y)})^2},$$

for all $x, y \in \Xi$, $f_i x \neq f_i y$, $\tau > 0$. *Let* $\Im : \mathcal{K}(\Xi) \to \mathcal{K}(\Xi)$ *be defined by*

$$\Im(\mathcal{A}) = \bigcup_{n=1}^{k} f_n(\mathcal{A}), \text{ for all } \mathcal{A} \in \mathcal{K}(\Xi)$$

is a rational-type(I)-HBs operator, that is, there exists $\tau > 0$ *such that*

$$\mathcal{H}(\Im(\mathcal{A}), \Im(\mathcal{B})) \leq \frac{\Theta(\mathcal{A}, \mathcal{B})}{(1 + \tau\sqrt{\Theta(\mathcal{A}, \mathcal{B})})^2} \tag{10}$$

where $\Theta(\mathcal{A}, \mathcal{B})$ *is given in* (3)*. Then results (a)-(b) of Theorem* 3.1 *hold.*

Proof. If we take $\mathcal{F}(t) = -\frac{1}{\sqrt{t}}$, $\mathcal{G}(t) = \ln t$ $(t > 0)$ and $\beta(t) = \lambda \in (0, 1)$, $\tau = -\ln \lambda > 0$ in the Theorem 3.1, we have results. □

Corollary 3.6. *Let* (Ξ, d) *be a complete metric space and* $\{\Xi; f_n,\ n = 1, 2, \cdots, k\}$ *a continuous, rational-type (II)-contractive IFS, that is, each* f_i $(i = 1, 2, \ldots, k)$ *is a self-mapping on* Ξ *such that*

$$d(f_i x, f_i y)) \leq \frac{d(x, y)}{(1 - \tau\sqrt{d(x, y)}\ln(\beta(d(x, y))))^2},$$

for all $x, y \in \Xi$, $f_i x \neq f_i y$, $\tau > 0$ *and* $(\mathcal{G}, \beta) \in \mathbb{G}_\beta$ *Let* $\Im : \mathcal{K}(\Xi) \to \mathcal{K}(\Xi)$ *be defined by*

$$\Im(\mathcal{A}) = \bigcup_{n=1}^{k} f_n(\mathcal{A}), \text{for all } \mathcal{A} \in \mathcal{K}(\Xi)$$

is a rational-type(II)-HB operator, that is, there exist $\tau > 0$ *and* $(\mathcal{G}, \beta) \in \mathbb{G}_\beta$ *such that*

$$\mathcal{H}(\Im(\mathcal{A}), \Im(\mathcal{B})) \leq \frac{\Theta(\mathcal{A}, \mathcal{B})}{(1 - \tau\sqrt{\Theta(\mathcal{A}, \mathcal{B})}\ln(\beta(\Theta(\mathcal{A}, \mathcal{B}))))^2}, \tag{11}$$

where $\Theta(\mathcal{A}, \mathcal{B})$ *is given in* (3)*. Then results (a)-(b) of Theorem* 3.1 *hold.*

Proof. If we take $\mathcal{F}(t) = -\frac{1}{\sqrt{t}}$, $\mathcal{G}(t) = \ln t$ $(t > 0)$ in the Theorem 3.1, we have results. □

Remark 3.7. *According to Theorem 3.1, if we consider* $\Delta(\Xi)$ *to be the collection of all singleton subsets of* Ξ*, then* $\Delta(\Xi) \subseteq \mathcal{K}(\Xi)$*. Additionally, if* $f = f_i$ *for every* $i \in \{1, 2, 3, \ldots, k\}$*, the mapping* $\Im$ *becomes.*

$$\Im(x) = f(x).$$

As a result of this configuration, we receive the following fixed point result:

Corollary 3.8. *Let* (Ξ, d) *be a complete metric space and* $\{\Xi : f_n,\ n = 1, 2, \cdots, k\}$ *a continuous,* $\mathcal{FG}$*-contractive IFS. Let* $f : \Xi \to \Xi$ *be a mapping defined as in Remark* 2.2*. If there exist some* $\mathcal{F} \in \mathbb{F}$ *and* $(\mathcal{G}, \beta) \in \mathbb{G}_\beta$ *such that for any* $x, y \in \mathcal{K}(\Xi)$ *with* $d(f(x), f(y)) \neq 0$*, the following holds:*

$$\mathcal{F}(d(fx, fy)) \leq \mathcal{F}(\Psi(x, y)) + \mathcal{G}(\beta(\Psi(x, y))),$$

where

$$\Psi(x, y) = \max\left\{ \begin{array}{c} d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2}, \\ d(f^2x, y), d(f^2x, fx), d(f^2x, fy) \end{array} \right\}.$$

Then f *has a unique fixed point in* Ξ*. Moreover, for any initial* $x_0 \in \Xi$*, the sequence* $\{x_0,\ fx_0,\ f^2x_0, \ldots\}$ *converges to a fixed point of* f*.*

4. Examples

In this section, we present some examples to illustrate our results.

Example 4.1. *Let* $\Xi = [0,1] \times [0,1]$ *equipped with the standard Euclidean metric and* $f_1, f_2 : \Xi \to \Xi$ *be mappings defined by*

$$\begin{aligned} f_1(x,y) &= \left(\frac{x^2}{2}, \frac{y^2}{2}\right); \\ f_2(x,y) &= \left(\frac{\sin x}{2}, \frac{\sin y}{2}\right) \end{aligned}$$

for all $(x,y) \in \Xi$. *Further, let* $\mathcal{F}(t) = -\frac{1}{\sqrt{t}}$, $\mathcal{G}(t) = \ln t$ $(t > 0)$ *and* $\beta(t) = \lambda \in (0,1)$.

Then, for $x = (x_1, y_1)$ *and* $y = (x_2, y_2)$ *in* Ξ *with* $x \neq y$, *we have*

$$\begin{aligned} d(f_1x, f_1y) &= d\left(\left(\frac{{x_1}^2}{2}, \frac{{y_1}^2}{2}\right), \left(\frac{{x_2}^2}{2}, \frac{{y_2}^2}{2}\right)\right) \\ &= \frac{1}{2}\sqrt{({x_1}^2 - {x_2}^2)^2 + ({y_1}^2 - {y_2}^2)^2} \\ &\leq \frac{d(x,y)}{(1 + \tau\sqrt{d(x,y)})^2} \end{aligned}$$

for some $\tau > 0$. *Similarly,*

$$d(f_2x, f_2y) \leq \frac{d(x,y)}{(1 + \tau\sqrt{d(x,y)})^2}.$$

Therefore

$$\begin{aligned} &\mathcal{F}(d(f_1x, f_1y)) \leq \mathcal{F}(d(x,y)) + \mathcal{G}(\beta(d(x,y))) \text{ and} \\ &\mathcal{F}(d(f_2x, f_2y)) \leq \mathcal{F}(d(x,y)) + \mathcal{G}(\beta(d(x,y))) \end{aligned}$$

for $\mathcal{G}$ *and* β *defined above and any* $\mathcal{F}$.

Now, consider the iterated function system $(\mathbb{R}^2, f_1, f_2)$ *with the mappings* $\Im : \mathcal{K}(\Xi) \to \mathcal{K}(\Xi)$ *given by*

$$\Im(\mathcal{A}) = f_1(\mathcal{A}) \cup f_2(\mathcal{A}) \text{ for all } \mathcal{A} \in \mathcal{K}(\Xi).$$

Thus for all $\mathcal{A}, \mathcal{B} \in \mathcal{K}(\Xi)$, *we have*

$$\mathcal{F}(\mathcal{H}(f(\mathcal{A}), f(\mathcal{B}))) \leq \mathcal{F}(\mathcal{H}(\mathcal{A}, \mathcal{B})) + \mathcal{G}(\beta(\mathcal{H}(\mathcal{A}, \mathcal{B}))).$$

Figure 1 and Figure 2 show the IFS fractal as a result of the convergence of $\Im$.

Example 4.2. [7]. *Let* $\Xi = [0,1] \times [0,1]$ *and* $f_1, f_2 : \Xi \to \Xi$ *defined by*

$$\begin{aligned} f_1(x,y) &= (\cos(x^2) + \cos(y^2), \sin(x) + \sin(y)), \\ f_2(x,y) &= (-\cos(x^2) + \cos(y^2), -\sin(x) - \sin(y)). \end{aligned}$$

It is shown in [7] the mappings f_1 *and* f_2 *are contractions and therefore* $\mathcal{FG}$*-contractions. Figure 3 shows the IFS fractal as a result of the convergence of* $\Im$.

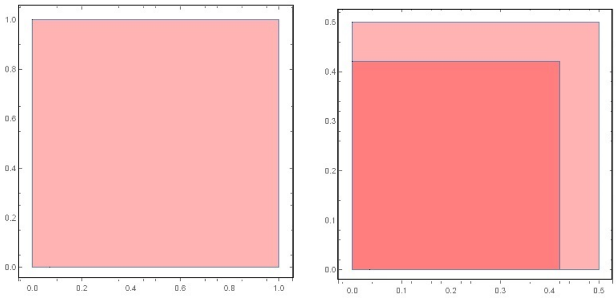

Figure 1: Left: Domain $\mathcal{A}$. Right: $\mathfrak{I}(\mathcal{A})$ operator.

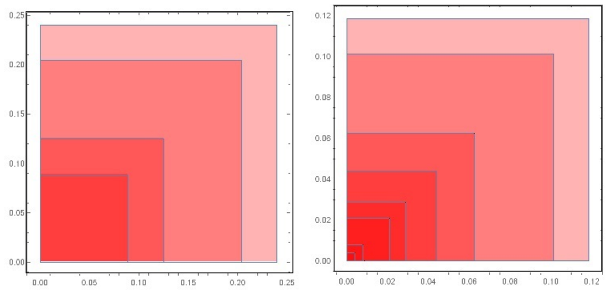

Figure 2: Left: $\mathfrak{I}^2(\mathcal{A})$ operator. Right: $\mathfrak{I}^3(\mathcal{A})$ operator.

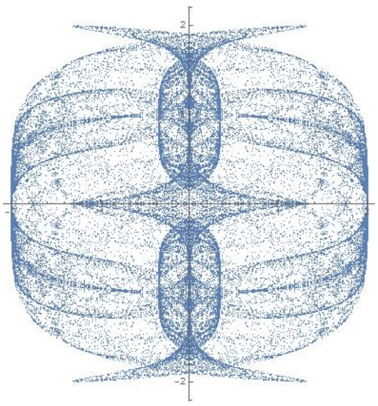

Figure 3: Construction of the fractal

Example 4.3. *Let X be the triangle made with $\{(0,0),(1,0),(0,1)\}$ and $\Xi = X \times X$ equipped with the standard Euclidean metric. Let $f_1, f_2 : \Xi \to \Xi$ be mappings defined by*

$$f_1(x,y) = \left(\frac{1-x}{2}, \frac{1-y}{2}\right);$$

$$f_2(x,y) = \left(\sin^{-1}\left(\frac{x}{4}\right), \sin^{-1}\left(\frac{y}{4}\right)\right)$$

for all $(x,y) \in \Xi$. Further, let $\mathcal{F}(t) = -\frac{1}{\sqrt{t}}$, $\mathcal{G}(t) = \ln t$ $(t > 0)$ and $\beta(t) = \lambda \in (0,1)$.

Then, for $x = (x_1, y_1)$ *and* $y = (x_2, y_2)$ *in* Ξ *with* $x \neq y$*, we have*

$$\begin{aligned} d(f_2x, f_2y) &= d\left(\left(\sin^{-1}\left(\frac{x_1}{4}\right), \sin^{-1}\left(\frac{y_1}{4}\right)\right), \left(\sin^{-1}\left(\frac{x_2}{4}\right), \sin^{-1}\left(\frac{y_2}{4}\right)\right)\right) \\ &= \sqrt{\left(\sin^{-1}\left(\frac{x_1}{4}\right) - \sin^{-1}\left(\frac{x_2}{4}\right)\right)^2 + \left(\sin^{-1}\left(\frac{y_1}{4}\right) - \sin^{-1}\left(\frac{y_2}{4}\right)\right)^2} \\ &\leq \sqrt{\left(\frac{x_1}{4} - \frac{x_2}{4}\right)^2 + \left(\frac{y_1}{4} - \frac{y_2}{4}\right)^2} \\ &\leq \frac{d(x, y)}{(1 + \tau\sqrt{d(x, y)})^2} \end{aligned}$$

for some $\tau > 0$*. Similarly, one can easily show that*

$$d(f_1x, f_1y) \leq \frac{d(x, y)}{(1 + \tau\sqrt{d(x, y)})^2}.$$

Therefore

$$\mathcal{F}(d(f_1x, f_1y)) \leq \mathcal{F}(d(x, y)) + \mathcal{G}(\beta(d(x, y))) \text{ and}$$
$$\mathcal{F}(d(f_2x, f_2y)) \leq \mathcal{F}(d(x, y)) + \mathcal{G}(\beta(d(x, y)))$$

for $\mathcal{G}$ *and* β *defined above and any* $\mathcal{F}$*.*

Now, consider the iterated function system $(\mathbb{R}^2, f_1, f_2)$ *with the mappings* $\Im : \mathcal{K}(\Xi) \to \mathcal{K}(\Xi)$ *given by*

$$\Im(\mathcal{A}) = f_1(\mathcal{A}) \cup f_2(\mathcal{A}) \text{ for all } \mathcal{A} \in \mathcal{K}(\Xi).$$

Thus for all $\mathcal{A}, \mathcal{B} \in \mathcal{K}(\Xi)$*, we have*

$$\mathcal{F}(\mathcal{H}(f(\mathcal{A}), f(\mathcal{B}))) \leq \mathcal{F}(\mathcal{H}(\mathcal{A}, \mathcal{B})) + \mathcal{G}(\beta(\mathcal{H}(\mathcal{A}, \mathcal{B}))).$$

Figure 4, Figure 5 and Figure 6 show the IFS fractal as a result of the convergence of $\Im$*.*

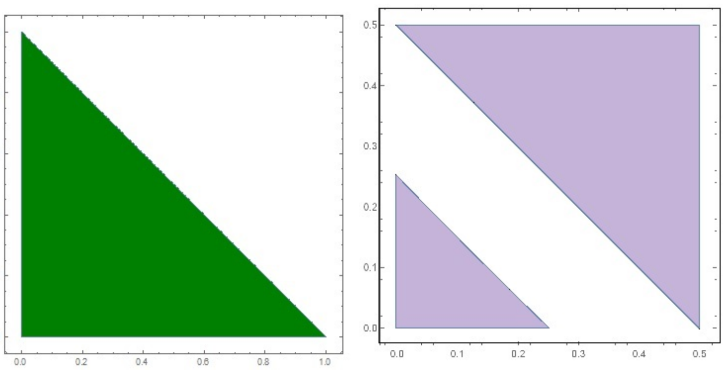

Figure 4: Left: Domain $\mathcal{A}$. Right: $\Im(\mathcal{A})$ operator.

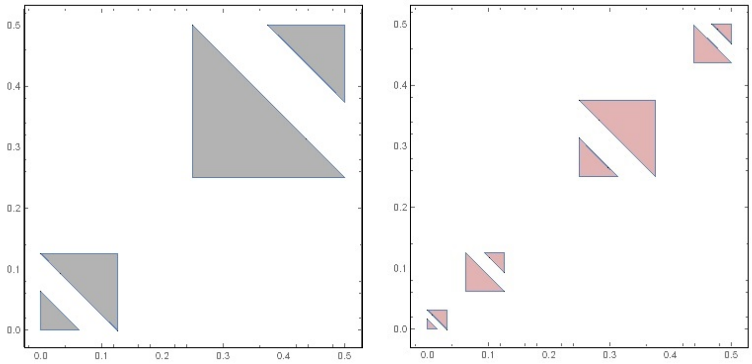

Figure 5: Left: $\mathfrak{I}^2(\mathcal{A})$ operator. Right: $\mathfrak{I}^3(\mathcal{A})$ operator.

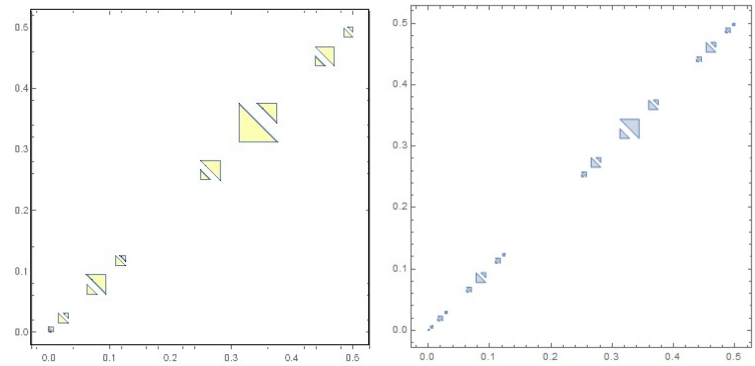

Figure 6: Left: $\mathfrak{I}^4(\mathcal{A})$ operator. Right: $\mathfrak{I}^5(\mathcal{A})$ operator.

5. Conclusions

An attempt is made in this work to create a fractal using a finite family of $\mathcal{FG}$-contraction mappings specified on a complete metric space. This leads to generate diverse iterated function systems that fulfil distinct contractive requirements. A number of instances are provided to back up the conclusions that have been made. All of our findings consolidate, broaden, and expand previous findings.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that there is not any competing interest regarding the publication of this manuscript.

Author's contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

Conflict of interest

The authors declare that there is no conflict of interest.

Acknowledgement

The authors are thankful to the reviewer for his/her constructive comments and suggestions that have been useful for the improvement of this paper.

References

[1] I. Altun, G. Minak, H. Dağ, Multivalued F-contractions on complete metric space, J. Nonlinear Convex Anal. **16**(4)(2015), 659–666.
[2] S. Banach, Sur les op*é*rations dans les ensembles abstraits et leur application aux *é*quationsint*é*grales, Fund. Math. **3** (1922), 133–181.
[3] M. F. Barnsley, Fractals Everywhere, 2nd edn. Academic Press, USA, 1993.
[4] Lj. Čirič, Multi-valued nonlinear contraction mappings, Nonlinear Anal. **71**(7-8) (2009), 2716–2723.
[5] N. V. Dung, A. Petrusel, On iterated function systems consisting of Kannan maps, Reich maps, Chatterjea type maps, and related results, J. Fixed Point Theory Appl. **192**(2017), 271–2285.
[6] M. Geraghty, On contractive mappings, Proc. Am. Math. Soc. **40**(1973), 604–608.
[7] C. Getz, J. Helmstedt, Graphics with Mathematica : Fractals, Julia Sets, Patterns and Natural Forms. Elsevier Pubication, Netherlands, 2004.
[8] J. E. Hutchinson, Fractals and self similarity. Indiana Univ. Math. J. **30**(1981), 713–747.
[9] B. B. Mandelbrot, Fractals Form, Chance and Dimension. W.H. Freeman and Company, San Francisco, 1977.
[10] B. B. Mandelbrot, Fractal geometry: what is it and what does it do?, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences **423**(1989), 3–16.
[11] B. B. Mandelbrot, Fractal and Chaos: The Mandelbrot Set and Beyond. Springer - Verlag, New York, 2004.
[12] N. Mizoguchi, W. Takahashi, Fixed point theorems for multivalued mappings on complete metric spaces, J. Math. Anal. Appl. **141**(1)(1989), 177–188.
[13] S. B. Nadler, Multi-valued contraction mappings, Pacific J. Math. **30**(1969), 475–488.
[14] H. K. Nashine, R. W. Ibrahim, B. E. Rhoades, R. Pant, Unified Feng-Liu type fixed point theorems solving control problems, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **115** (1)(5)(2021), 1–17.
[15] T. Nazir, S. Silvestrov, M. Abbas, Fractals of generalized F-Hutchinson operator, Waves, Wavelets Fractals **2**(2016), 29–40.
[16] R. Pant, Fixed point theorems for nonlinear contractions with applications to iterated function systems, Appl. Gen. Topol. **19**(1)(2018), 163–172.
[17] V. Parvaneh, N. Hussain, Z. Kadelburg, Generalized Wardowski type fixed point theorems via α-admissible FG-contractions in b-metric spaces, Acta Math. Sci. **36B**(5)(2016), 1445–1456.
[18] A. Petrusel, Generalized multivalued contractions, Nonlinear Analysis **47** (1)(2001), 649–659.
[19] A. Petrusel, Fixed point theory with applications to dynamical systems and fractals. Seminar on Fixed Point Theory, (2002), 305–316.
[20] B. Prasad, Fractals for A-iterated function and multifunction, International Journal of Applied Engineering Research, **7** (11)(2012), 2032–2036.
[21] D. R. Sahu, A. Chakraborty, R. P. Dubey, K-Iterated function system, Fractals **18** (1)(2010), 139–144.
[22] N. A. Secelean, Iterated function systems consisting of F-contractions, Fixed Point Theory Appl. 277 (2013), 1–13.
[23] N. A. Secelean, Generalized F-iterated function systems on product of metric spaces, J. Fixed Point Theory Appl. **17** (2015), 575–595.
[24] S. L. Singh, B. Prasad, A. Kumar, Fractals via iterated functions and multifunctions, Chaos, Solitons & Fractals, **39** (3)(2009), 1224–1231.
[25] S. L. Singh, S. N. Mishra, S. Jain, Orbit of an image under iterated system, Communications in Nonlinear Science and Numerical Simulation **16** (3)(2011), 1469–1482.
[26] D. Wardowski, Fixed point theory of a new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl. **2012**:94 (2012).
[27] X. Y. Wang, F. P. Li, A class of nonlinear iterated function system attractors. Nonlinear Anal. **70**(2)(2009), 830—838.
[28] S. Xu, S. Cheng, Z. Zhou, Reich's iterated function systems and well-posedness via fixed point theory, Fixed Point Theory Appl. **2015**:71(2015), 1–11.